Module 2
Equilibrium of forces

Equilibrium of forces- Definition of Equilibrant; Conditions of static equilibrium for different force systems, Lami's theorem; Numerical problems on equilibrium of coplanar – concurrent and non-concurrent force systems.
Static Friction in rigid bodies in contact-Types of friction, Laws of static friction, Limiting friction, Angle of friction, angle of repose; Impending motion on horizontal and inclined planes; Numerical Problems on single and two blocks on planes, wedge friction, ladder friction, rope and pulley systems

2.1 EQUILIBRIUM OF FORCE SYSTEM

2.2 Equilibrium

Equilibrium of a rigid body is a state of balance. If a rigid body is acted upon by a system of forces and remains in the rest then the system is called static equilibrium. A body is in equilibrium if the algebraic sum of all the forces acting on the body is zero and also if the algebraic sum of moments of forces about any fixed point is zero.

$\Sigma F_x=0, \Sigma F_y=0, \Sigma M=0$

Equilibrant

Equilibrant is a single force which when added to a system of forces brings the status of equilibrium. Hence this force is of the same magnitude as the resultant but opposite in direction.

Free Body Diagram

Free Body Diagram is a diagram in which the body under consideration is freed from all the contact surfaces and is represents only by the forces (applied forces, reactions, self-weight) acting on it. Fig 2.1 shows free body diagram for different cases.

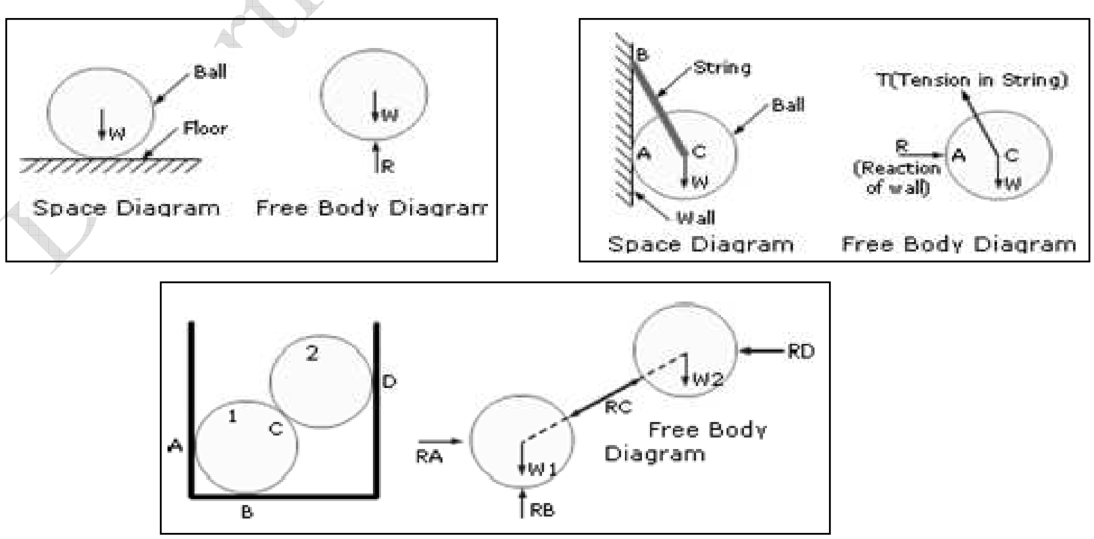


Fig 2.1 Free Body Diagram

Conditions of equilibrium of coplanar concurrent force system

1. Algebraic sum of all the horizontal components of the force system must be zero.i.e.,
   $\Sigma F_x=0.$
2. Algebraicsumofalltheverticalcomponentsoftheforcesystemmustbezero.i.e.,
   $\Sigma F_y=0.$

Since all the forces of concurrent force system act through the same point, cannot cause any rotation. Therefore $\boldsymbol{\Sigma M=0}$ is automatically satisfied.

Conditions of equilibrium of coplanar non-concurrent force system

1. Algebraic sum of all the horizontal components of the force system must be zero.i.e., $\Sigma F_x=0.$
2. Algebraicsumofalltheverticalcomponentsoftheforcesystemmustbezero.i.e., $\Sigma F_y=0.$
3. Algebraic sum of moments of all the forces about any point system must be zero.i.e., $\Sigma M=0.$

Lami's Theorem

Statement: If a body is in equilibrium under action of three forces, each force is proportional to the sine of the angle between the other two forces.

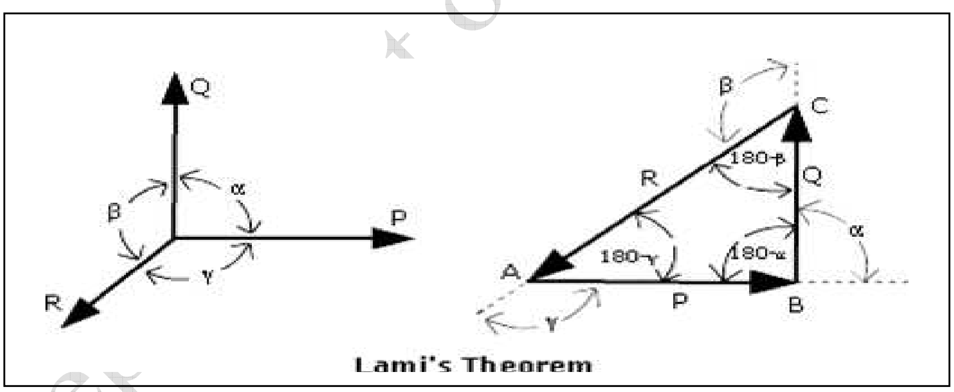


Fig 2.2 Lami's Theorem

From the figure;

$$P \propto \sin\beta;\ Q \propto \sin\gamma;\ R \propto \sin\alpha;\ \text{or}\ \frac{P}{\sin\beta} = \frac{Q}{\sin\gamma} = \frac{R}{\sin\alpha}$$

Proof: Applying sine rule to the triangle ABC

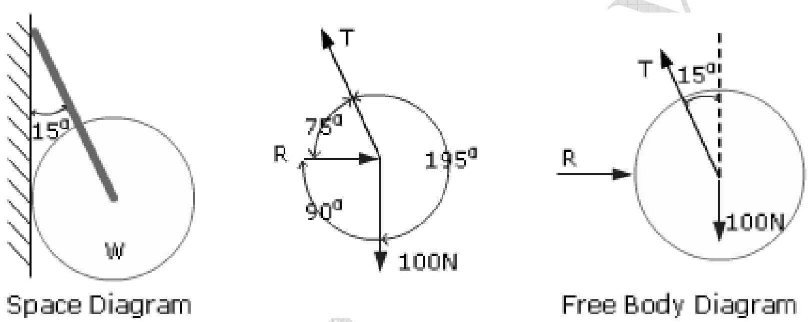
$$\frac{AB}{\sin\angle BCA} = \frac{BC}{\sin\angle CAB} = \frac{CA}{\sin\angle ABC};$$

$$\frac{P}{\sin(180-\beta)} = \frac{Q}{\sin(180-\gamma)} = \frac{R}{\sin(180-\alpha)}$$

$$\frac{P}{\sin\beta} = \frac{Q}{\sin\gamma} = \frac{R}{\sin\alpha}$$ because sin (180- θ) = sinθ

Problems:

1. A sphere of weight 100 N is tied to a smooth wall by a string as shown in Fig. Find the tension T in the string and reaction of the wall.



Solution: Free body diagram of the sphere is shown in Fig. along with force diagram. Solution is obtained by applying Lami's theorem

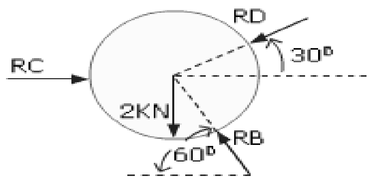
R/sin(195) = T/sin(90) = 100/sin(75)

R/sin(195) = 100/sin(75)
R = 1000*sin(195)/sin(75)
.: R – 26.79 N

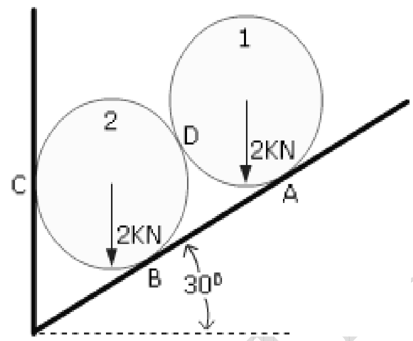
Similarly
T/sin(90) = 100/sin(75).
T = 100*sin(90)/sin(75)
T = 103.53 N

2. Two identical cylinders each weighing 2kN are supported by vertical and inclined plane as shown in Fig. Assuming smooth surfaces determine the reactions at A, B

andC



Free Body Diagram of Cylinder1



Free Body Diagram of Cylinder2

The solution is obtained by considering first the free body diagram of cylinder 2. Lami's theorem is applied on this as there are three forces in equilibrium.

RD/sin(30) = 2/sin(270) = RA/sin(60)
RD = 2*sin(30)/sin(270)
RA = 2*sin(60)/sin(270)
.: RA = 1.73 kN
RD = 1 kN

Lami's theorem cannot be applied on the free body diagram of cylinder 1, as there are four forces acting on this cylinder. Hence the solution is obtained using the method of components and equilibrium condition.

$\sum F_x = 0$
RC – RD*cos(30) – RB*cos(60) = 0 --------------- (1)
$\sum F_y = 0$
-RD*sin(30) – 2 + RB*sin(60) = 0 ----------------- (2)

In both equations (1) and (2) value of RD = 1 kN from cylinder 2 calculations
Hence substituting value of RD in (2) we get
RB = 2.88 kN
And substituting RD and RB value in (1) we get RC = 2.31 kN

2.3 FRICTION

2.4 Introduction

Whatever we have studied so far, we have always taken the force applied by one surface on an object to be normal to the surface. In doing so, we have been making an approximation i.e., we have been neglecting a very important force viz., the frictional force. In this chapter we look at the frictional force in varioussituations.
Even when a smooth surface is observed under a microscope, it will be seen that the surface has undulations with troughs and crests as illustrated in Fig 2.3.

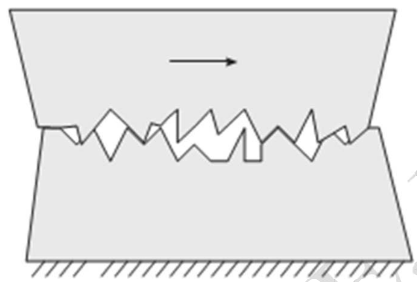


Fig 2.3 Two bodies moving over each other

In this chapter when we talk about friction, we would mean frictional force between two dry surfaces. This is known as Coulomb friction. Frictional forces also exist when there is a thin film of liquid between two surfaces or within a liquid itself. This is known as the viscous force. We will not be talking about such forces and will focus our attention on Coulomb friction i.e., frictional forces between two dry surfaces only. Frictional force always opposes the motion or tendency of an object to move against another object or against a surface. We distinguish between two kinds of frictional forces - static and kinetic - because it is observed that kinetic frictional force is slightly less than maximum static frictional force.
Let us now perform the following experiment. Put a block on a rough surface and pull it by a force *F* (see figure 1). Since the force *F* has a tendency to move the block, the frictional force acts in the opposite direction and opposes the applied force *F*. All the forces acting on the block are shown in figure 1. Note that we have shown the weight and the normal reaction acting at two different points on the block. I leave it for you to think why should the weight and the normal reaction not act along the same vertical line?

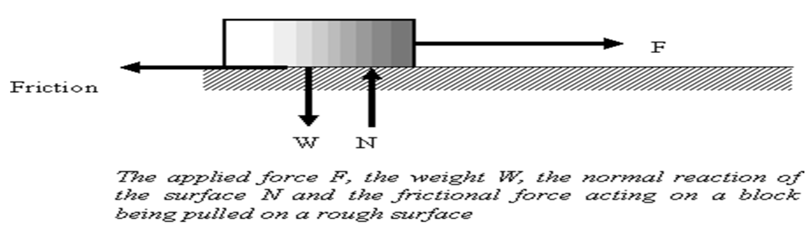


Fig 2.4 Forces acting on an object

It is observed that the block does not move until the applied force *F* reaches a maximum value F_{max}. Thus from F = 0 up to $F = F_{max}$, the frictional force adjusts itself so that it is just sufficient to stop the motion. It was observed by Coulombs that *F max* is proportional to the normal reaction of the surface on the object. You can observe all this while trying to push a table across the room; heavier the table, larger the push required to move it. Thus we can write

$$F_{max} \propto N$$

$$\text{or} \quad F_{max} = \mu_s N$$

Where μ_s is known as the coefficient of static friction. It should be emphasized again that is the maximum possible value of frictional force, applicable when the object is about to stop, otherwise frictional force could be less than, just sufficient to prevent motion. We also note that frictional force is independent of the area of contact and depends only on *N*.

As the applied force *F* goes beyond F_{max} , the body starts moving now experience slightly less force. This force is seem to be when is known as the coefficient of kinetic friction. At low velocities it is a constant but decrease slightly at high velocities. A schematic plot of frictional force *F* as a function of the applied force is as shown in fig 2.5

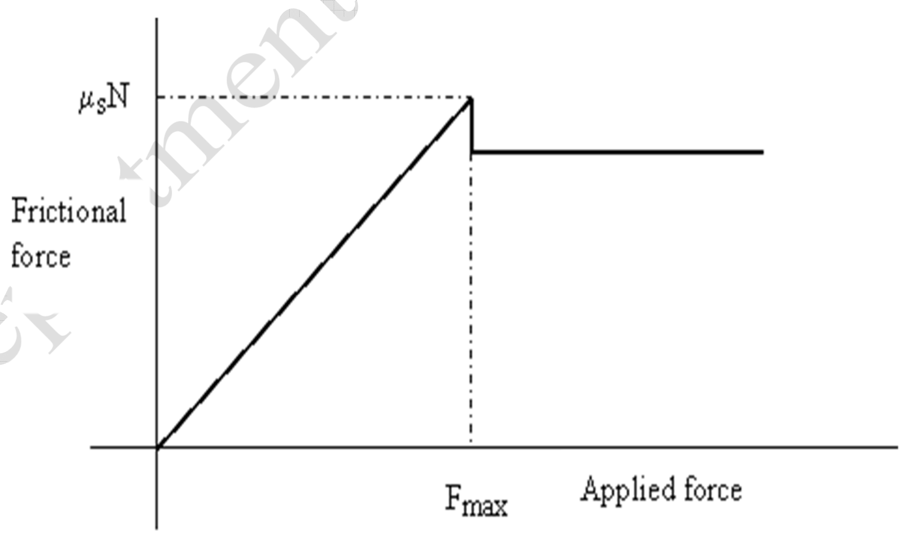
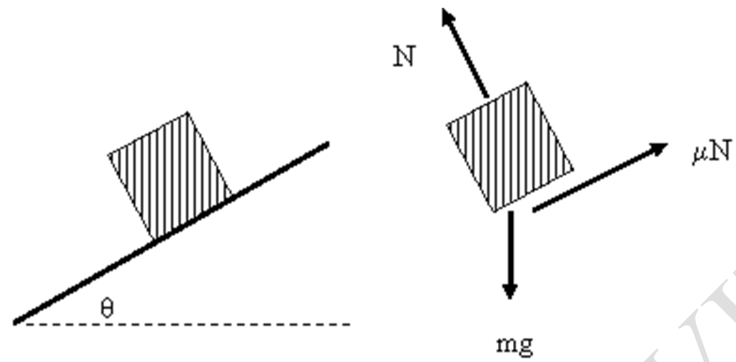


Fig 2.5 Frictional force as a function of applied force

Values of frictional coefficients for different materials vary from almost zero (ice on ice) to as large as 0.9 (rubber tire on cemented road) always remaining less than 1.

A quick way of estimating the value of static friction is to look at the motion an object on an inclined plane. Its free-body diagram is given in Fig 2.6.



A block of mass m on an inclined plane (left) and its free-body diagram (right) when it is about to slide down the ramp

Fig 2.6 Object on inclined plane

Since the block has a tendency to slide down, the frictional force points up the inclined plane.
As long as the block is in equilibrium

$$mg \sin \theta \leq maximum\ friction$$
$$mg \cos \theta = N$$

As θ is increased, mgsinθ increases and when it goes past the maximum possible value of friction $f_{max}$the block starts sliding down. Thus at the angle at which it slides down we have

$$\left.\begin{aligned} mg \sin \theta = f_{max} &= \mu_s N \\ &= \mu_s mg \cos \theta \end{aligned}\right\} \quad \Rightarrow \quad \mu_s = \tan \theta$$

2.5 LIMITING FRICTION

The self-adjusting opposing and resisting friction *F* which opposes the sliding motion of one body over another, has a limiting value and if the applied force exceeds this value, the body begins to move. This limiting value of the force is called the limiting friction and at this stage the body is in limiting equilibrium and just on the verge of motion.

Coefficient of Friction (μ)

It is the constant ratio which the limiting friction *F* bears to the normal reaction *N*, i.e. as shown in Fig 2.7, where a body of weight *W* is in equilibrium under the applied force *P* and the frictional force *F*,

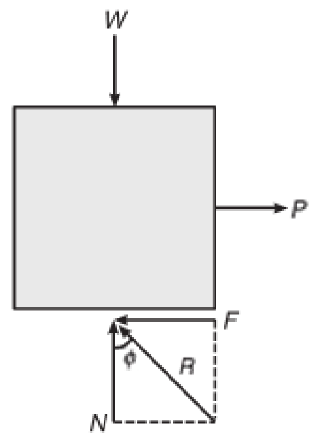


Fig 2.7 Coefficient of Friction (μ)

$\mu = \frac{F}{N}$

μ= 0, for smooth surfaces

Angle of friction (f)

Let us again consider a body of weight W which is placed over a rough surface and is subjected to an external force P as shown in Fig 2.8. The following forces are acting on the body:

Self-weight, W
External force, P
Frictional force, F
Normal reaction, N

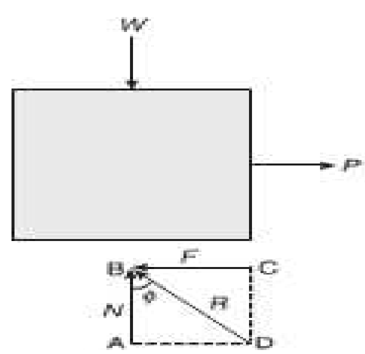


Fig 2.8 Angle of friction (f)

The angle of friction for two contacting surfaces is the angle between the resultant R (of friction force F and the normal reaction N) and the normal reaction N. It is denoted by f.

In triangle ABD,

$tan\phi = \mathrm{F/N} = \mu$

Angle of Repose (u)

When a plane is inclined to the horizontal by a certain angle, the body placed on it will remain at rest up to a certain angle of inclination, beyond which the body just begins to move. This maximum angle made by the inclined plane with the horizontal, when the body placed on that plane is just at the point of sliding down the plane, is known as the angle of repose. Repose means sleep which is disturbed at that particular angle of inclination.

Let us consider a body of weight *W* which is placed on an inclined plane as shown in Fig 2.9. The body is just at the point of sliding down the plane when the angle of inclination is u. The various forces acting on the body are self-weight, normal reaction, and frictional force.

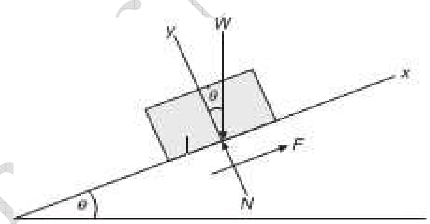


Fig 2.9 Angle of Repose (u)

Applying the conditions of equilibrium, $\Sigma F_x=0$, $\Sigma F_y=0$

Resolving forces along the x-axis,

$-F + W \sin\theta = 0$,

$\Rightarrow F = W \sin\theta$

Resolving the forces along the y-axis,

$N - W \cos\theta = 0$

$\Rightarrow N = W \cos\theta$

We know that $\mu = F/N = W \sin\theta / W \cos\theta = \tan\theta$

Angle of friction = Angle of repose

2.6 TYPES OFFRICTION

Depending on the state of rest or motion, we can categorize friction into:

(i) **Static friction**: It is the friction experienced between two bodies when both bodies are at rest.

(ii) **Dynamic friction:** It is the friction experienced between two bodies when one body moves over the other body. It can be further classified into two types

a. **Sliding friction:** It is the resisting force which opposes the sliding motion of one body over another body. This force acts in a direction opposite to the direction of impending motion.

b. **Rolling friction:** It is the friction between the two bodies when one body rolls over the other body.

Based on the surface of contact, there are two types of friction, namely:

(i) **Dry friction:** If the contact surface between the two bodies is dry, then the friction between such bodies is known as Dry friction.

(ii) **Fluid friction:** The friction between two fluid layers or the friction between a solid and a fluid layer.

2.7 LAWS OF FRICTION

(i) The laws of static friction are:

(ii) The force of friction always acts in a direction, opposite to that in which the body tends tomove.

(iii) The magnitude of the force of friction is exactly equal to the applied force which just moves thebody.

(iv) The magnitude of the limiting friction bears a constant ratio to the normal reaction between the two surfaces in contact, i.e. $F/\mathrm{N} = \mathrm{Constant}$

(v) Where F is the limiting friction and N is the normal reaction.

(vi) The force of friction is independent of the area of contact between the two surfaces.

(vii) The force of friction depends upon the roughness of the surfaces in contact.

The laws of dynamic friction are:

(i) The force of friction always acts in a direction, opposite to that in which the body is moving.

(ii) The magnitude of the kinetic friction bears a constant ratio to the normal reaction between the two surfaces in contact. But this ratio is slightly less than that in the case of limitingfriction.

(iii) The friction force remains constant for moderate speeds but decreases slightly with the increase in speed.

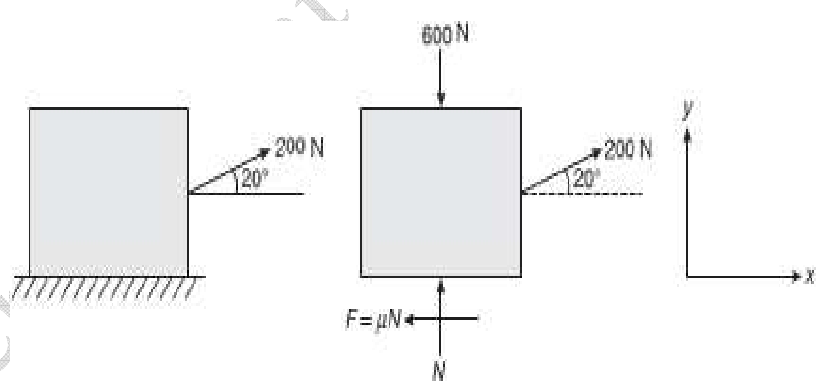
PROBLEMS

Tips to Solve the Problems

1. Draw a free body diagram.
2. Draw the reference axes. We have to choose the reference axes in such a way that one of the axes must be along the direction of motion.
3. The following forces should be considered while drawing the free body diagram.
   a. Self-weight always acts **vertically downwards**.
   b. Frictional force (tangential force) which is always opposite to the direction of motion at the contact surface and is parallel to the contact surface.
   c. Normal reaction, which is always perpendicular to the contact surface.
4. Write the algebraic sum of the forces along the x-axis, i.e. $\Sigma F_x = 0$
5. Write the algebraic sum of the forces along the y-axis, i.e. $\Sigma F_y = 0$
6. Write the limiting friction equation, i.e. $\mu = F/N$.
7. Solve for the three unknowns from the three equations.

Let us now solve a couple of simple standard examples involving static friction/kinetic friction.

1. A block shown in Figure 8.6 is just moved by a force of 200 N. The weight of the block is 600 N. Determine the coefficient of static friction between the block and the floor.



Solution: Considering the conditions of equilibrium, $\Sigma Fx = 0$

$200 \cos 20° - \mu N = 0$

$\mu N = 187.938$

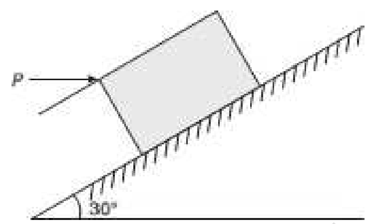
$\Sigma Fy = 0$

$N - 600 + 200 \sin 20° = 0$ $N = 531.596$ N

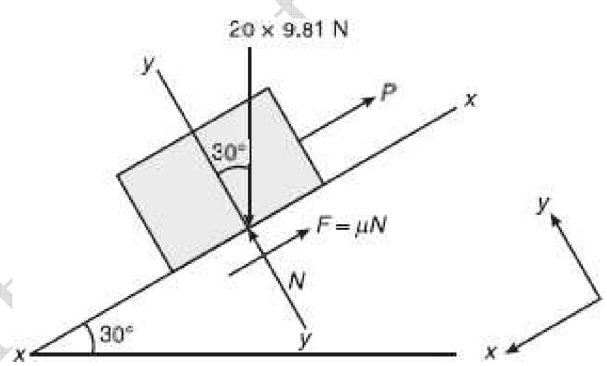
Substituting the value of N in Eq. (i), we get $\mu = 0.35$

2. A small block of weight 1000 N as shown in Figure, is placed on a 30° inclined plane with = 0.25. Determine the horizontal force to be applied for:

(i) Impending motion down the plane
(ii) Impending motion up the plane.



Solution (i) *The value of P for impending motion down the plane*



Consider the free body diagram of block $\Sigma Fy = 0$

Or $\quad 20 \times 9.81 \times \cos 30° - N = 0$

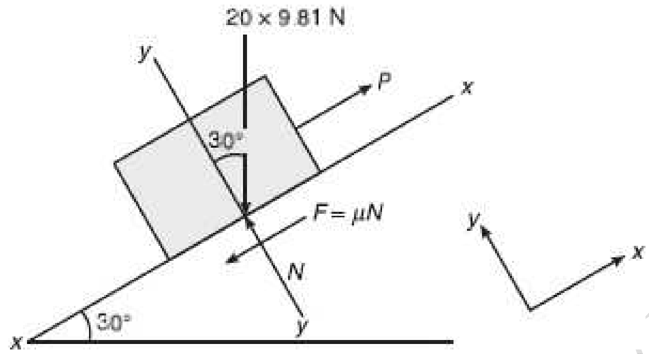
Or $\quad N = 20 \times 9.81 \cos 30° = 169.914$N

Also, $\quad \Sigma Fx = 0$

or $\quad -P - 0.24 \times 169.914 + 20 \times 9.81 \sin 30°$

∴ $P = 57.320\text{N}$

(ii) *The value of P for impending motion up the plane*



Consider the free body diagram of block $\Sigma Fy = 0$

Or $N = 20 \times 9.81 \cos 30°$

Or $N = 169.914\text{N}$

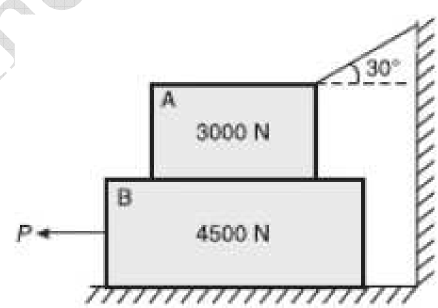
Also, $\sum Fx = 0$

Or $-20 \times 9.81 \sin 30° + P - N = 0$

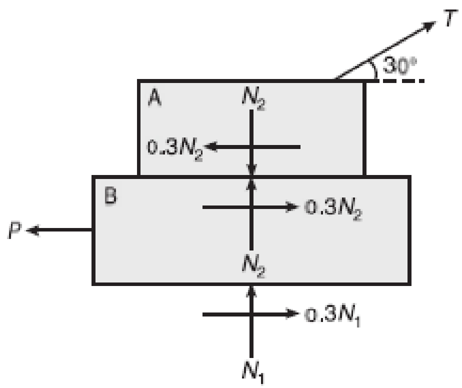
Or $P = 20 \times 9.81 \sin 30° + 0.24 \times 169.914$

$P = 138.879$ N

3. A block weighing 4500 N resting on horizontal surface supports another block of 3000 N as shown in Figure 8.22. Find the horizontal force *P* required to just move the block to the left. Take the coefficient of friction for all contact surfaces as 0.5.



Solution When one body is placed over the other body, equal and opposite reactions and also equal and opposite frictional forces will come into existence between the contact surfaces of the two bodies as shown in Figure 8.25.



From Fig

For $\Sigma Fy = 0$, we get $\quad -3000 + N2 + T \sin 30° = 0$

$N2 = 3000 - 0.5T$

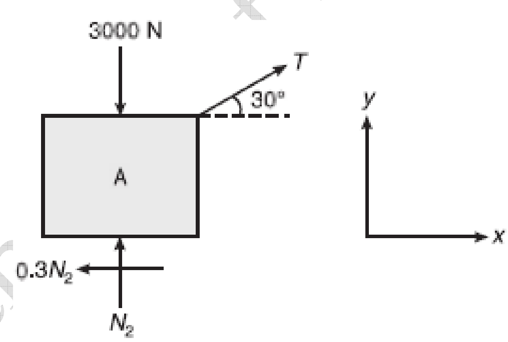
For $\Sigma Fx = 0$, we get $\quad T \cos 30° - 0.3(3000 - 0.5T) = 0$

Or $\quad T \cos 30° - 900 + 0.15T = 0$

Or $\quad 1.016T = 900$

Or $\quad T = 885.827\text{N}$

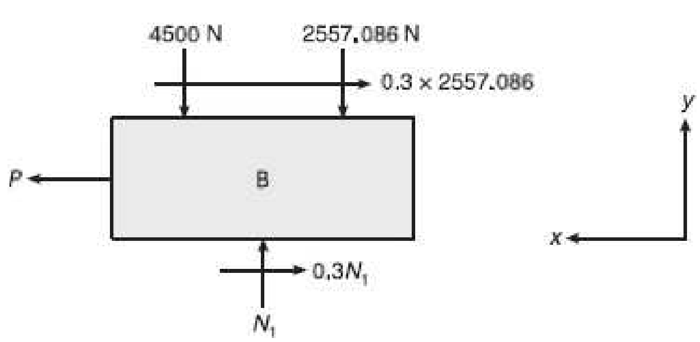
$N2 = 3000 - 0.5T = 2557.086$ N



FromFigure

For $\Sigma Fy = 0$, we get

$N1 = 4500 + 2557.086 = 7057.086$ N



And for
$\Sigma Fx = 0$,

$$P - 0.3 \times 7057.086 - 0.3 \times 2557.086 = 0$$
$$P = 2884.252 \text{ N}$$

2.8 WEDGE FRICTION

A wedge is usually a triangular or trapezoidal body in action. It is used either for lifting heavy loads or for slight adjustments in the position of a body, i.e. tightening fits or keys for a shaft. When lifting a heavy load, the wedge is placed below the load and a horizontal force P is applied as shown in Figure 2.10.

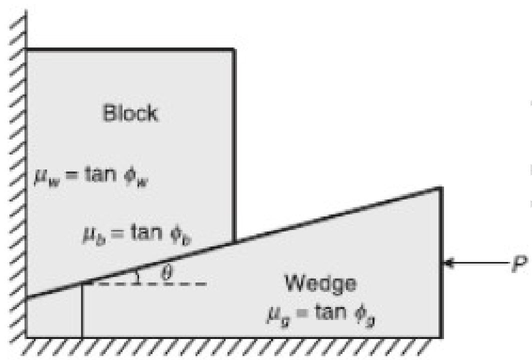


Fig 2.10 Wedge friction

The load will be resting against the wedge as shown in the above figure. As can be seen, the wedge moves from right to left lifting the block upwards, Hence the friction force reaction always opposes the impending motion and therefore the friction reaction R_w between the block and the wall is downwards, and that between the ground and the wedge R_g is acting towards left toright.

The analysis of block and wedge depends on the relative motion. In case of block, it is moving from left to right relative to the motion of wedge (right to left) and hence the friction reaction is right to left, whereas for the wedge, the friction reaction is acting from left to right. Further, the analysis of forces is easier with the Lami's theorem. It can be seen that the forces acting on both the block and wedge are a convergent system of forces and also there are three forces acting at each point O and O′ respectively. As the system is in equilibrium, the Lami's theorem can be applied in succession at both the pointsrespectively.

Consider the block first, as the number of forces acting on it is minimum (Figure 2.11). Out of the three forces, the weight of the block acts vertically downwards. The friction reaction with the wall will be acting at **w** from the normal to the contact surface of wall and block, i.e normal tothe vertical surface. Finally, the third force is the friction reaction between the block and the wedge acting at b from the normal to the contact surface, which in turn is inclined at θ (wedge angle) with thevertical.

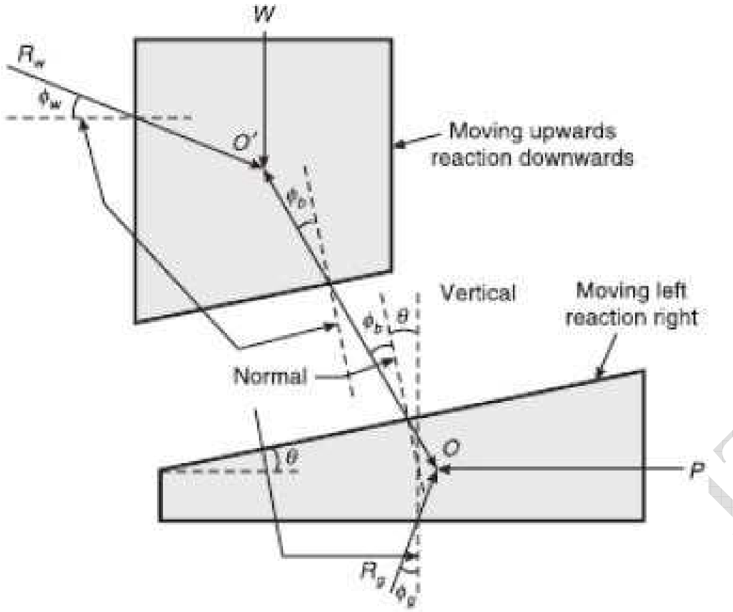


Fig 2.11Forcs acting on the wedge

Hence, the friction reaction from the wedge is inclined at an angle from the vertical. Following is the free body diagram at pointO'

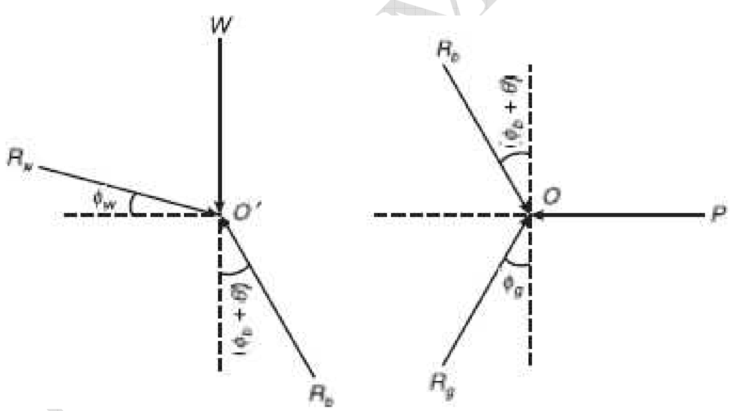
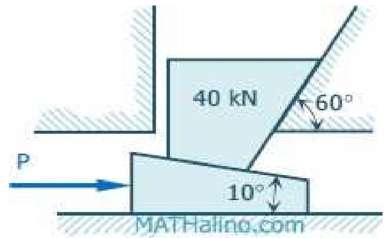


Fig 2.12 free body diagram

1. Determine the value of P just sufficient to start the 10° wedge under the 40-kN block. The angle of friction is 20° for all contact surfaces.



From the FBD of 40 kN block

$\Sigma F_H = 0$

$R_1 \sin 80^\circ = R_2 \sin 30^\circ$

$$R_1 = \frac{R_2 \sin 30^\circ}{\sin 80^\circ}$$

$R_1 = 0.5077 R_2$

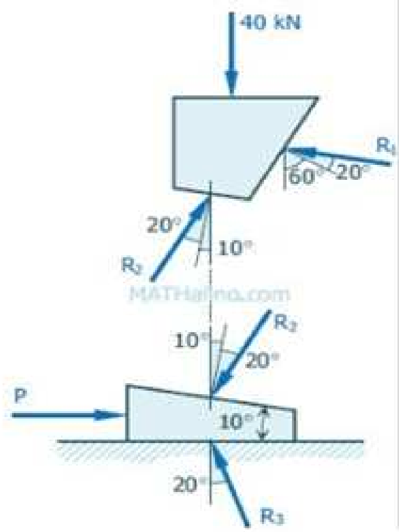
$\Sigma F_V = 0$

$R_2 \cos 30^\circ + R_1 \cos 80^\circ = 40$

$R_2 \cos 30^\circ + (0.5077 R_2) \cos 80^\circ = 40$

$0.9542 R_2 = 40$

$R_2 = 41.92 \text{ kN}$



From the FBD of lower block

$\Sigma F_V = 0$

$R_3 \cos 20^\circ = R_2 \cos 30^\circ$

$R_3 \cos 20^\circ = 41.92 \cos 30^\circ$

$R_3 = 38.634 \text{ kN}$

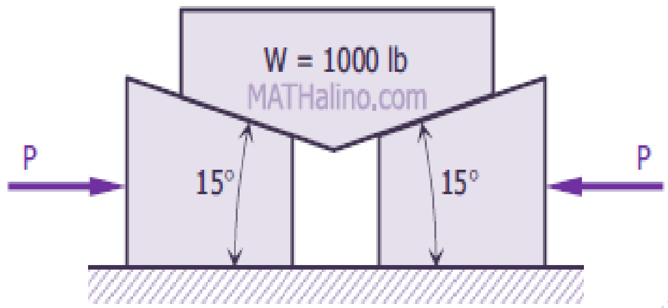
$\Sigma F_H = 0$

$P = R_2 \sin 30^\circ + R_3 \sin 20^\circ$

$P = 41.92 \sin 30^\circ + 38.634 \sin 20^\circ$

$P = 34.174 \text{ kN}$ *answer*

2. What force P must be applied to the wedges shown in Figure to start them under the block? The angle of friction for all contact surfaces is10°.

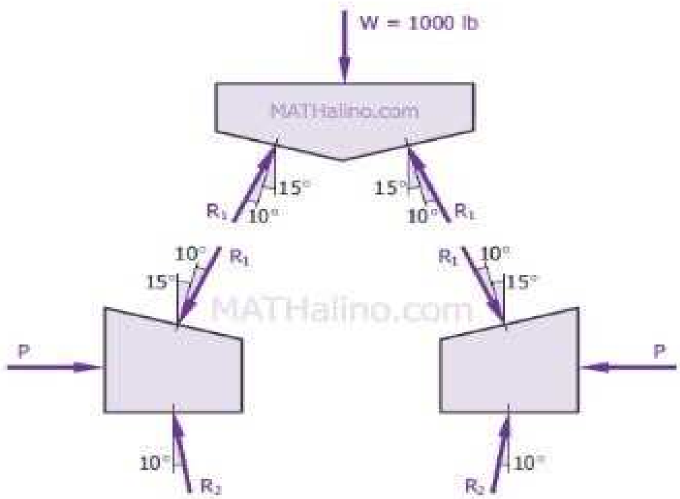


From the FBD of 1000 lb block

$\Sigma F_V = 0$

$2(R_1 \cos 25°) = 1000$

$R_1 = 551.69$ lb



From the FBD of any of the wedges

$\Sigma F_V = 0$

$R_2 \cos 10° = R_1 \cos 25°$

$R_2 \cos 10° = 551.69 \cos 25°$

$R_2 = 507.71$ lb

$\Sigma F_H = 0$

$P = R_1 \sin 25° + R_2 \sin 10°$

$P = 551.69 \sin 25° + 507.71 \sin 10°$

$P = 321.32$ lb *answer*

2.9 LADDER FRICTION

A ladder is a device used for climbing the roofs or walls. It consists of two long uprights of wood, steel or iron pipes connected by a number of cross pieces. The friction developed between the contact surfaces of ladder and floor and also between the ladder and the wall is known as ladder friction.

Consider a ladder AB resting on ground and leaning against a wall as shown in Fig 2.11

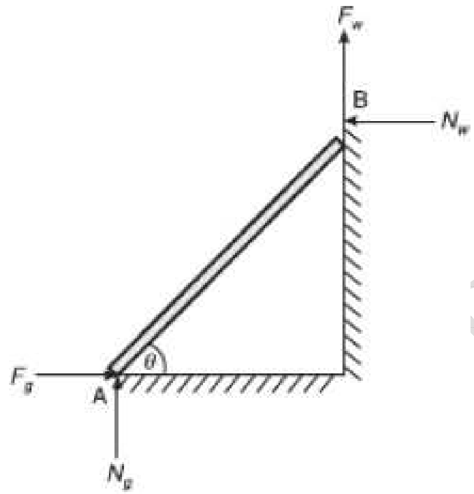
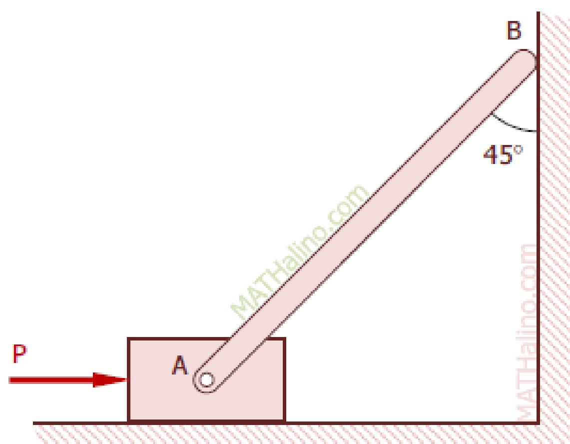


Fig 2.11 Ladder friction

As the upper end of ladder tends to slip downwards, the force of friction between the ladder and the wall will be Fw upwards. Similarly, as the lower end of the ladder tends to slip away from the wall, the force Fg should be towards the wall.
Since the system is in equilibrium, the algebraic sum of the components of forces must be zero.

1. A uniform bar AB, weighing 424 N, is fastened by a frictionless pin to a block weighing 200 N as shown in Fig. P-535. At the vertical wall, μ = 0.268 while under the block, μ = 0.20. Determine the force P needed to start motion to the right.



ΣMA=0 NB(2x)=424x+fB(2x) 2NB=
2NB=424+2(0.268NB)

1.464NB=424 NB=289.62 N

fB=0.268(289.62) fB=77.62 N

ΣFV=0 NA=fB+200+424 NA=77.62-

fA=0.20(701.62) fA=140.32 N

ΣFH=0 P=fA+NB P=140.32+289.62
P=429.94N *answer*