

Task 2

Task 2: Design and implement a custom MCMC algorithm for the outlined Bayesian hierarchical model. Monitor the convergence of the MCMC chains, using diagnostic plots and summary statistics to check for any issues.

Conditional posterior distributions

1. \mathbf{B} :

$$\begin{aligned}
 & \pi(\beta_i \mid \boldsymbol{\mu}^\top, \boldsymbol{\Sigma}, \boldsymbol{\gamma}^\top, \sigma, \mathbf{Y}^\top) \\
 & \propto \exp \left[-\frac{1}{2}(\beta_i - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\beta_i - \boldsymbol{\mu}) - \frac{1}{2\sigma^2} \sum_{j=1}^{m_i} (Y_{i,j} - \mathbf{Z}_{i,j-1}^\top \beta_i - \mathbf{X}_i^\top \boldsymbol{\gamma})^2 \right] \\
 & \propto \exp \left[-\frac{1}{2} \left((\beta_i - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\beta_i - \boldsymbol{\mu}) + \frac{1}{\sigma^2} \sum_{j=1}^{m_i} (-2\beta_i^\top (Y_{i,j} - \mathbf{X}_i^\top \boldsymbol{\gamma}) \mathbf{Z}_{i,j-1} + \beta_i^\top (\mathbf{Z}_{i,j-1} \mathbf{Z}_{i,j-1}^\top) \beta_i) \right) \right] \\
 & \propto \exp \left\{ -\frac{1}{2} \left[\beta_i^\top \left(\boldsymbol{\Sigma}^{-1} + \frac{1}{\sigma^2} \sum_{j=1}^{m_i} \mathbf{Z}_{i,j-1} \mathbf{Z}_{i,j-1}^\top \right) \beta_i - 2\beta_i^\top \left(\boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} + \frac{1}{\sigma^2} \sum_{j=1}^{m_i} (Y_{i,j} - \mathbf{X}_i^\top \boldsymbol{\gamma}) \mathbf{Z}_{i,j-1} \right) \right] \right\}.
 \end{aligned}$$

Thus we have

$$\begin{aligned}
 & \beta_i \mid (\boldsymbol{\mu}^\top, \boldsymbol{\Sigma}, \boldsymbol{\gamma}^\top, \sigma, \mathbf{Y}^\top) \\
 & \sim N \left(\left(\boldsymbol{\Sigma}^{-1} + \frac{1}{\sigma^2} \sum_{j=1}^{m_i} \mathbf{Z}_{i,j-1} \mathbf{Z}_{i,j-1}^\top \right)^{-1} \left(\boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} + \frac{1}{\sigma^2} \sum_{j=1}^{m_i} (Y_{i,j} - \mathbf{X}_i^\top \boldsymbol{\gamma}) \mathbf{Z}_{i,j-1} \right), \left(\boldsymbol{\Sigma}^{-1} + \frac{1}{\sigma^2} \sum_{j=1}^{m_i} \mathbf{Z}_{i,j-1} \mathbf{Z}_{i,j-1}^\top \right)^{-1} \right).
 \end{aligned}$$

2. $\boldsymbol{\mu}$:

$$\begin{aligned}
 & \pi(\boldsymbol{\mu} \mid \mathbf{B}^\top, \boldsymbol{\Sigma}, \boldsymbol{\gamma}^\top, \sigma, \mathbf{Y}^\top) \\
 & \propto \exp \left[-\frac{1}{2} \left(\sum_{i=1}^n (\beta_i - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\beta_i - \boldsymbol{\mu}) + \boldsymbol{\mu}^\top \mathbf{V}^{-1} \boldsymbol{\mu} \right) \right] \\
 & \propto \exp \left\{ -\frac{1}{2} \left[\boldsymbol{\mu} - (n\boldsymbol{\Sigma}^{-1} + \mathbf{V}^{-1})^{-1} \boldsymbol{\Sigma}^{-1} \left(\sum_{i=1}^n \beta_i \right) \right]^\top (n\boldsymbol{\Sigma}^{-1} + \mathbf{V}^{-1}) \left[\boldsymbol{\mu} - (n\boldsymbol{\Sigma}^{-1} + \mathbf{V}^{-1})^{-1} \boldsymbol{\Sigma}^{-1} \left(\sum_{i=1}^n \beta_i \right) \right] \right\} \\
 & \propto \exp \left\{ -\frac{1}{2} \left[\boldsymbol{\mu} - \mathbf{V}(n\mathbf{V} + \boldsymbol{\Sigma})^{-1} \left(\sum_{i=1}^n \beta_i \right) \right]^\top (\mathbf{V}(n\mathbf{V} + \boldsymbol{\Sigma})^{-1})^{-1} \left[\boldsymbol{\mu} - \mathbf{V}(n\mathbf{V} + \boldsymbol{\Sigma})^{-1} \left(\sum_{i=1}^n \beta_i \right) \right] \right\}.
 \end{aligned}$$

Thus we have

$$\boldsymbol{\mu} \mid (\mathbf{B}^\top, \boldsymbol{\Sigma}, \boldsymbol{\gamma}^\top, \sigma, \mathbf{Y}^\top) \sim N \left(\mathbf{V}(n\mathbf{V} + \boldsymbol{\Sigma})^{-1} \left(\sum_{i=1}^n \beta_i \right), \mathbf{V}(n\mathbf{V} + \boldsymbol{\Sigma})^{-1} \right).$$

3. Σ :

$$\begin{aligned}\pi(\Sigma \mid \mathbf{B}^\top, \boldsymbol{\mu}^\top, \boldsymbol{\gamma}^\top, \sigma, \mathbf{Y}^\top) &\propto |\Sigma|^{-(n+\nu+6)/2} \exp \left[-\frac{1}{2} \left(\sum_{i=1}^n (\beta_i - \boldsymbol{\mu})^\top \Sigma^{-1} (\beta_i - \boldsymbol{\mu}) + \text{tr}(\mathbf{S} \Sigma^{-1}) \right) \right] \\ &\propto |\Sigma|^{-(n+\nu+6)/2} \exp \left[-\frac{1}{2} \text{tr} \left(\left(\mathbf{S} + \sum_{i=1}^n (\beta_i - \boldsymbol{\mu})(\beta_i - \boldsymbol{\mu})^\top \right) \Sigma^{-1} \right) \right].\end{aligned}$$

Thus we have

$$\Sigma \mid (\mathbf{B}^\top, \boldsymbol{\mu}^\top, \boldsymbol{\gamma}^\top, \sigma, \mathbf{Y}^\top) \sim \mathcal{W}^{-1} \left(\mathbf{S} + \sum_{i=1}^n (\beta_i - \boldsymbol{\mu})(\beta_i - \boldsymbol{\mu})^\top, n + \nu \right).$$

4. γ :

$$\begin{aligned}\pi(\gamma \mid \mathbf{B}^\top, \boldsymbol{\mu}^\top, \Sigma, \sigma, \mathbf{Y}^\top) \\ &\propto \exp \left(-200 \|\gamma\|_2^2 - \frac{1}{2\sigma^2} \sum_{i=1}^n \sum_{j=1}^{m_i} (Y_{i,j} - \mathbf{Z}_{i,j-1}^\top \beta_i - \mathbf{X}_i^\top \gamma)^2 \right) \\ &\propto \exp \left\{ -\frac{1}{2} \left[\gamma^\top \left(400\mathbf{I} + \frac{1}{\sigma^2} \sum_{i=1}^n m_i \mathbf{X}_i \mathbf{X}_i^\top \right) \gamma - 2\gamma^\top \left(-\frac{1}{\sigma^2} \sum_{i=1}^n \sum_{j=1}^{m_i} (\mathbf{Z}_{i,j-1}^\top \beta_i - Y_{i,j}) \mathbf{X}_i \right) \right] \right\}.\end{aligned}$$

Thus we have

$$\begin{aligned}\gamma \mid (\mathbf{B}^\top, \boldsymbol{\mu}^\top, \Sigma, \sigma, \mathbf{Y}^\top) \\ \sim N \left(\left(400\mathbf{I} + \frac{1}{\sigma^2} \sum_{i=1}^n m_i \mathbf{X}_i \mathbf{X}_i^\top \right)^{-1} \left(-\frac{1}{\sigma^2} \sum_{i=1}^n \sum_{j=1}^{m_i} (\mathbf{Z}_{i,j-1}^\top \beta_i - Y_{i,j}) \mathbf{X}_i \right), \left(400\mathbf{I} + \frac{1}{\sigma^2} \sum_{i=1}^n m_i \mathbf{X}_i \mathbf{X}_i^\top \right)^{-1} \right).\end{aligned}$$

5. σ :

$$\pi(\sigma \mid \mathbf{B}^\top, \boldsymbol{\mu}^\top, \Sigma, \boldsymbol{\gamma}^\top, \mathbf{Y}^\top) \propto I(\sigma > 0) \frac{\sigma^{-\sum_{i=1}^n m_i}}{1 + (\sigma/10)^2} \exp \left(-\frac{1}{2\sigma^2} \sum_{i=1}^n \sum_{j=1}^{m_i} (Y_{i,j} - \mathbf{Z}_{i,j-1}^\top \beta_i - \mathbf{X}_i^\top \gamma)^2 \right).$$