Task 2

Task 2: Design and implement a custom MCMC algorithm for the outlined Bayesian hierarchical model. Monitor the convergence of the MCMC chains, using diagnostic plots and summary statistics to check for any issues.

Conditional posterior distributions

1. $\mathbf{B}$:

$$\begin{aligned}
&\pi(\boldsymbol{\beta}_i \mid \boldsymbol{\mu}^\top, \boldsymbol{\Sigma}, \boldsymbol{\gamma}^\top, \sigma, \boldsymbol{Y}^\top) \\
&\propto \exp\left[-\frac{1}{2}(\boldsymbol{\beta}_i - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\mu}) - \frac{1}{2\sigma^2}\sum_{j=1}^{m_i}(Y_{i,j} - \mathbf{Z}_{i,j-1}^\top \boldsymbol{\beta}_i - \mathbf{X}_i^\top \gamma)^2\right] \\
&\propto \exp\left[-\frac{1}{2}\left((\boldsymbol{\beta}_i - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\mu}) + \frac{1}{\sigma^2}\sum_{j=1}^{m_i}(-2\boldsymbol{\beta}_i^\top (Y_{i,j} - \mathbf{X}_i^\top \boldsymbol{\gamma})\mathbf{Z}_{i,j-1} + \boldsymbol{\beta}_i^\top (\mathbf{Z}_{i,j-1}\mathbf{Z}_{i,j-1}^\top)\boldsymbol{\beta}_i)\right)\right] \\
&\propto \exp\left\{-\frac{1}{2}\left[\boldsymbol{\beta}_i^\top \left(\boldsymbol{\Sigma}^{-1} + \frac{1}{\sigma^2}\sum_{j=1}^{m_i}\mathbf{Z}_{i,j-1}\mathbf{Z}_{i,j-1}^\top\right)\boldsymbol{\beta}_i - 2\boldsymbol{\beta}_i^\top \left(\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \frac{1}{\sigma^2}\sum_{j=1}^{m_i}(Y_{i,j} - \mathbf{X}_i^\top \boldsymbol{\gamma})\mathbf{Z}_{i,j-1}\right)\right]\right\}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\boldsymbol{\beta}_i \mid (\boldsymbol{\mu}^\top, \boldsymbol{\Sigma}, \boldsymbol{\gamma}^\top, \sigma, \boldsymbol{Y}^\top) \\
&\sim N\left(\left(\boldsymbol{\Sigma}^{-1} + \frac{1}{\sigma^2}\sum_{j=1}^{m_i}\mathbf{Z}_{i,j-1}\mathbf{Z}_{i,j-1}^\top\right)^{-1}\left(\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \frac{1}{\sigma^2}\sum_{j=1}^{m_i}(Y_{i,j} - \mathbf{X}_i^\top \boldsymbol{\gamma})\mathbf{Z}_{i,j-1}\right), \left(\boldsymbol{\Sigma}^{-1} + \frac{1}{\sigma^2}\sum_{j=1}^{m_i}\mathbf{Z}_{i,j-1}\mathbf{Z}_{i,j-1}^\top\right)^{-1}\right).
\end{aligned}$$

2. $\boldsymbol{\mu}$:

$$\begin{aligned}
&\pi(\boldsymbol{\mu} \mid \mathbf{B}^\top, \boldsymbol{\Sigma}, \boldsymbol{\gamma}^\top, \sigma, \boldsymbol{Y}^\top) \\
&\propto \exp\left[-\frac{1}{2}\left(\sum_{i=1}^{n}(\boldsymbol{\beta}_i - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\mu}) + \boldsymbol{\mu}^\top \boldsymbol{V}^{-1}\boldsymbol{\mu}\right)\right] \\
&\propto \exp\left\{-\frac{1}{2}\left[\boldsymbol{\mu} - (n\boldsymbol{\Sigma}^{-1} + \boldsymbol{V}^{-1})^{-1}\boldsymbol{\Sigma}^{-1}\left(\sum_{i=1}^{n}\boldsymbol{\beta}_i\right)\right]^\top (n\boldsymbol{\Sigma}^{-1} + \boldsymbol{V}^{-1})\left[\boldsymbol{\mu} - (n\boldsymbol{\Sigma}^{-1} + \boldsymbol{V}^{-1})^{-1}\boldsymbol{\Sigma}^{-1}\left(\sum_{i=1}^{n}\boldsymbol{\beta}_i\right)\right]\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left[\boldsymbol{\mu} - \boldsymbol{V}(n\boldsymbol{V} + \boldsymbol{\Sigma})^{-1}\left(\sum_{i=1}^{n}\boldsymbol{\beta}_i\right)\right]^\top (\boldsymbol{V}(n\boldsymbol{V} + \boldsymbol{\Sigma})^{-1})^{-1}\left[\boldsymbol{\mu} - \boldsymbol{V}(n\boldsymbol{V} + \boldsymbol{\Sigma})^{-1}\left(\sum_{i=1}^{n}\boldsymbol{\beta}_i\right)\right]\right\}.
\end{aligned}$$

Thus we have

$$\boldsymbol{\mu} \mid (\mathbf{B}^\top, \boldsymbol{\Sigma}, \boldsymbol{\gamma}^\top, \sigma, \boldsymbol{Y}^\top) \sim N\left(\boldsymbol{V}(n\boldsymbol{V} + \boldsymbol{\Sigma})^{-1}\left(\sum_{i=1}^{n}\boldsymbol{\beta}_i\right), \boldsymbol{V}(n\boldsymbol{V} + \boldsymbol{\Sigma})^{-1}\right).$$

3. $\mathbf{\Sigma}$:

$$\begin{aligned}\pi(\mathbf{\Sigma} \mid \mathbf{B}^\top, \boldsymbol{\mu}^\top, \boldsymbol{\gamma}^\top, \sigma, \boldsymbol{Y}^\top) &\propto |\mathbf{\Sigma}|^{-(n+v+6)/2} \exp\left[-\frac{1}{2}\left(\sum_{i=1}^{n}(\boldsymbol{\beta}_i - \boldsymbol{\mu})^\top \mathbf{\Sigma}^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\mu}) + \mathrm{tr}(\boldsymbol{S}\mathbf{\Sigma}^{-1})\right)\right] \\ &\propto |\mathbf{\Sigma}|^{-(n+v+6)/2} \exp\left[-\frac{1}{2}\mathrm{tr}\left(\left(\boldsymbol{S} + \sum_{i=1}^{n}(\boldsymbol{\beta}_i - \boldsymbol{\mu})(\boldsymbol{\beta}_i - \boldsymbol{\mu})^\top\right)\mathbf{\Sigma}^{-1}\right)\right].\end{aligned}$$

Thus we have

$$\mathbf{\Sigma} \mid (\mathbf{B}^\top, \boldsymbol{\mu}^\top, \boldsymbol{\gamma}^\top, \sigma, \boldsymbol{Y}^\top) \sim \mathcal{W}^{-1}\left(\boldsymbol{S} + \sum_{i=1}^{n}(\boldsymbol{\beta}_i - \boldsymbol{\mu})(\boldsymbol{\beta}_i - \boldsymbol{\mu})^\top, n + \nu\right).$$

4. $\boldsymbol{\gamma}$:

$$\begin{aligned}&\pi(\boldsymbol{\gamma} \mid \mathbf{B}^\top, \boldsymbol{\mu}^\top, \mathbf{\Sigma}, \sigma, \boldsymbol{Y}^\top) \\ \propto\ &\exp\left(-200\|\boldsymbol{\gamma}\|_2^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\sum_{j=1}^{m_i}(Y_{i,j} - \mathbf{Z}_{i,j-1}^\top\boldsymbol{\beta}_i - \mathbf{X}_i^\top\boldsymbol{\gamma})^2\right) \\ \propto\ &\exp\left\{-\frac{1}{2}\left[\boldsymbol{\gamma}^\top\left(400\boldsymbol{I} + \frac{1}{\sigma^2}\sum_{i=1}^{n} m_i\mathbf{X}_i\mathbf{X}_i^\top\right)\boldsymbol{\gamma} - 2\boldsymbol{\gamma}^\top\left(-\frac{1}{\sigma^2}\sum_{i=1}^{n}\sum_{j=1}^{m_i}(\mathbf{Z}_{i,j-1}^\top\boldsymbol{\beta}_i - Y_{i,j})\mathbf{X}_i\right)\right]\right\}.\end{aligned}$$

Thus we have

$$\begin{aligned}&\boldsymbol{\gamma} \mid (\mathbf{B}^\top, \boldsymbol{\mu}^\top, \mathbf{\Sigma}, \sigma, \boldsymbol{Y}^\top) \\ \sim N&\left(\left(400\boldsymbol{I} + \frac{1}{\sigma^2}\sum_{i=1}^{n} m_i\mathbf{X}_i\mathbf{X}_i^\top\right)^{-1}\left(-\frac{1}{\sigma^2}\sum_{i=1}^{n}\sum_{j=1}^{m_i}(\mathbf{Z}_{i,j-1}^\top\boldsymbol{\beta}_i - Y_{i,j})\mathbf{X}_i\right), \left(400\boldsymbol{I} + \frac{1}{\sigma^2}\sum_{i=1}^{n} m_i\mathbf{X}_i\mathbf{X}_i^\top\right)^{-1}\right).\end{aligned}$$

5. σ:

$$\pi(\sigma \mid \mathbf{B}^\top, \boldsymbol{\mu}^\top, \mathbf{\Sigma}, \boldsymbol{\gamma}^\top, \boldsymbol{Y}^\top) \propto \frac{\sigma^{-\sum_{i=1}^{n} m_i}}{1 + (\sigma/10)^2}\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\sum_{j=1}^{m_i}(Y_{i,j} - \mathbf{Z}_{i,j-1}^\top\boldsymbol{\beta}_i - \mathbf{X}_i^\top\boldsymbol{\gamma})^2\right).$$