Task 2

Task 2: Design and implement a custom MCMC algorithm for the outlined Bayesian hierarchical model. Monitor the convergence of the MCMC chains, using diagnostic plots and summary statistics to check for any issues.

Conditional posterior distributions

1. **B**:

$$\pi(\boldsymbol{\beta}_i \mid \boldsymbol{\mu}^\top, \boldsymbol{\Sigma}, \boldsymbol{\gamma}^\top, \sigma^2, \boldsymbol{Y}^\top)$$

$$\propto \exp\left[-\frac{1}{2}(\boldsymbol{\beta}_i - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\mu}) + \frac{1}{2\sigma^2}\sum_{j=1}^{m_i}(Y_{i,j} - \mathbf{Z}_{i,j-1}^\top\boldsymbol{\beta}_i - \mathbf{X}_i^\top\boldsymbol{\gamma})^2\right]$$

$$\propto \exp\left[-\frac{1}{2}\left((\boldsymbol{\beta}_i - \boldsymbol{\mu})^\top\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\mu}) - \frac{1}{\sigma^2}\sum_{j=1}^{m_i}(-2\boldsymbol{\beta}_i^\top(Y_{i,j} - \mathbf{X}_i^\top\boldsymbol{\gamma})\mathbf{Z}_{i,j-1} + \boldsymbol{\beta}_i^\top(\mathbf{Z}_{i,j-1}\mathbf{Z}_{i,j-1}^\top)\boldsymbol{\beta}_i)\right)\right]$$

$$\propto \exp\left\{-\frac{1}{2}\left[\boldsymbol{\beta}_i^\top\left(\boldsymbol{\Sigma}^{-1} - \frac{1}{\sigma^2}\sum_{j=1}^{m_i}\mathbf{Z}_{i,j-1}\mathbf{Z}_{i,j-1}^\top\right)\boldsymbol{\beta}_i - 2\boldsymbol{\beta}_i^\top\left(\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \frac{1}{\sigma^2}\sum_{j=1}^{m_i}(Y_{i,j} - \mathbf{X}_i^\top\boldsymbol{\gamma})\mathbf{Z}_{i,j-1}\right)\right]\right\}.$$

Thus we have

$$\boldsymbol{\beta}_i \mid (\boldsymbol{\mu}^\top, \boldsymbol{\Sigma}, \boldsymbol{\gamma}^\top, \sigma^2, \boldsymbol{Y}^\top)$$

$$\sim N\left(\left(\boldsymbol{\Sigma}^{-1} - \frac{1}{\sigma^2}\sum_{j=1}^{m_i}\mathbf{Z}_{i,j-1}\mathbf{Z}_{i,j-1}^\top\right)^{-1}\left(\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \frac{1}{\sigma^2}\sum_{j=1}^{m_i}(Y_{i,j} - \mathbf{X}_i^\top\boldsymbol{\gamma})\mathbf{Z}_{i,j-1}\right), \left(\boldsymbol{\Sigma}^{-1} - \frac{1}{\sigma^2}\sum_{j=1}^{m_i}\mathbf{Z}_{i,j-1}\mathbf{Z}_{i,j-1}^\top\right)^{-1}\right).$$

2. **$\boldsymbol{\mu}$**:

$$\pi(\boldsymbol{\mu} \mid \mathbf{B}^\top, \boldsymbol{\Sigma}, \boldsymbol{\gamma}^\top, \sigma^2, \boldsymbol{Y}^\top)$$

$$\propto \exp\left[-\frac{1}{2}\left(\sum_{i=1}^n(\boldsymbol{\beta}_i - \boldsymbol{\mu})^\top\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\mu}) + \boldsymbol{\mu}^\top\boldsymbol{V}^{-1}\boldsymbol{\mu}\right)\right]$$

$$\propto \exp\left\{-\frac{1}{2}\left[\boldsymbol{\mu} - (n\boldsymbol{\Sigma}^{-1} + \boldsymbol{V}^{-1})^{-1}\boldsymbol{\Sigma}^{-1}\left(\sum_{i=1}^n\boldsymbol{\beta}_i\right)\right]^\top(n\boldsymbol{\Sigma}^{-1} + \boldsymbol{V}^{-1})\left[\boldsymbol{\mu} - (n\boldsymbol{\Sigma}^{-1} + \boldsymbol{V}^{-1})^{-1}\boldsymbol{\Sigma}^{-1}\left(\sum_{i=1}^n\boldsymbol{\beta}_i\right)\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\boldsymbol{\mu} - \boldsymbol{V}(n\boldsymbol{V} + \boldsymbol{\Sigma})^{-1}\left(\sum_{i=1}^n\boldsymbol{\beta}_i\right)\right]^\top(\boldsymbol{V}(n\boldsymbol{V} + \boldsymbol{\Sigma})^{-1})^{-1}\left[\boldsymbol{\mu} - \boldsymbol{V}(n\boldsymbol{V} + \boldsymbol{\Sigma})^{-1}\left(\sum_{i=1}^n\boldsymbol{\beta}_i\right)\right]\right\}.$$

Thus we have

$$\boldsymbol{\mu} \mid (\mathbf{B}^\top, \boldsymbol{\Sigma}, \boldsymbol{\gamma}^\top, \sigma^2, \boldsymbol{Y}^\top) \sim N\left(\boldsymbol{V}(n\boldsymbol{V} + \boldsymbol{\Sigma})^{-1}\left(\sum_{i=1}^n\boldsymbol{\beta}_i\right), \boldsymbol{V}(n\boldsymbol{V} + \boldsymbol{\Sigma})^{-1}\right).$$

3. $\boldsymbol{\Sigma}$:

$$\pi(\boldsymbol{\Sigma} \mid \mathbf{B}^\top, \boldsymbol{\mu}^\top, \boldsymbol{\gamma}^\top, \sigma^2, \boldsymbol{Y}^\top) \propto |\boldsymbol{\Sigma}|^{-(n+v+6)/2} \exp\left[-\frac{1}{2}\left(\sum_{i=1}^n (\boldsymbol{\beta}_i - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\mu}) + \operatorname{tr}(\boldsymbol{S}\boldsymbol{\Sigma}^{-1})\right)\right]$$

$$\propto |\boldsymbol{\Sigma}|^{-(n+v+6)/2} \exp\left[-\frac{1}{2}\operatorname{tr}\left(\left(\boldsymbol{S} + \sum_{i=1}^n (\boldsymbol{\beta}_i - \boldsymbol{\mu})(\boldsymbol{\beta}_i - \boldsymbol{\mu})^\top\right)\boldsymbol{\Sigma}^{-1}\right)\right].$$

Thus we have

$$\boldsymbol{\Sigma} \mid (\mathbf{B}^\top, \boldsymbol{\mu}^\top, \boldsymbol{\gamma}^\top, \sigma^2, \boldsymbol{Y}^\top) \sim \mathcal{W}^{-1}\left(\boldsymbol{S} + \sum_{i=1}^n (\boldsymbol{\beta}_i - \boldsymbol{\mu})(\boldsymbol{\beta}_i - \boldsymbol{\mu})^\top, n + \nu\right).$$

4. $\boldsymbol{\gamma}$:

$$\pi(\boldsymbol{\gamma} \mid \mathbf{B}^\top, \boldsymbol{\mu}^\top, \boldsymbol{\Sigma}, \sigma^2, \boldsymbol{Y}^\top)$$

$$\propto \exp\left(-200\|\boldsymbol{\gamma}\|_2^2 + \frac{1}{2\sigma^2}\sum_{i=1}^n \sum_{j=1}^{m_i}(Y_{i,j} - \mathbf{Z}_{i,j-1}^\top \boldsymbol{\beta}_i - \mathbf{X}_i^\top \boldsymbol{\gamma})^2\right)$$

$$\propto \exp\left\{-\frac{1}{2}\left[\boldsymbol{\gamma}^\top\left(400\boldsymbol{I} - \frac{1}{\sigma^2}\sum_{i=1}^n m_i \mathbf{X}_i \mathbf{X}_i^\top\right)\boldsymbol{\gamma} - 2\boldsymbol{\gamma}^\top \frac{1}{\sigma^2}\sum_{i=1}^n \sum_{j=1}^{m_i}(\mathbf{Z}_{i,j-1}^\top \boldsymbol{\beta}_i - Y_{i,j})\mathbf{X}_i\right]\right\}.$$

Thus we have

$$\boldsymbol{\gamma} \mid (\mathbf{B}^\top, \boldsymbol{\mu}^\top, \boldsymbol{\Sigma}, \sigma^2, \boldsymbol{Y}^\top)$$

$$\sim N\left(\left(400\boldsymbol{I} - \frac{1}{\sigma^2}\sum_{i=1}^n m_i \mathbf{X}_i \mathbf{X}_i^\top\right)^{-1}\left(\frac{1}{\sigma^2}\sum_{i=1}^n \sum_{j=1}^{m_i}(\mathbf{Z}_{i,j-1}^\top \boldsymbol{\beta}_i - Y_{i,j})\mathbf{X}_i\right), \left(400\boldsymbol{I} - \frac{1}{\sigma^2}\sum_{i=1}^n m_i \mathbf{X}_i \mathbf{X}_i^\top\right)^{-1}\right).$$

5. σ^2:

$$\pi(\sigma^2 \mid \mathbf{B}^\top, \boldsymbol{\mu}^\top, \boldsymbol{\Sigma}, \boldsymbol{\gamma}^\top, \boldsymbol{Y}^\top) \propto \frac{\sigma^{-\sum_{i=1}^n m_i}}{1 + (\sigma/10)^2} \exp\left(\frac{1}{2\sigma^2}\sum_{i=1}^n \sum_{j=1}^{m_i}(Y_{i,j} - \mathbf{Z}_{i,j-1}^\top \boldsymbol{\beta}_i - \mathbf{X}_i^\top \boldsymbol{\gamma})^2\right).$$