

# Bayesian Modeling of Hurricane Trajectories

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# Introduction

## **Background:**

- ▶ A hurricane is a large and powerful tropical cyclone that typically forms over warm ocean waters and can cause significant damage and destruction to coastal areas.

## **Motivation:**

- ▶ Researchers are interested in modeling hurricane trajectories to forecast wind speed to predict the severity or to develop protective measures.

# Introduction

## Data Source:

- ▶ “hurricane703.csv” collected the track data (every 6 hours) of 702 hurricanes in the North Atlantic area since 1950.

## Variables:

- ▶ ID: ID of the hurricanes
- ▶ Season: In which the hurricane occurred
- ▶ Month: In which the hurricane occurred
- ▶ Nature: Nature of the hurricane
  - ET: Extra Tropical
  - DS: Disturbance
  - NR: Not Rated
  - SS: Sub Tropical
  - TS: Tropical Storm
- ▶ time: dates and time of the record
- ▶ Latitude and Longitude: The location of a hurricane check point
- ▶ Wind.kt: Maximum wind speed (in Knot) at each check point

## **Data pre-processing:**

- ▶ We only kept observations that occurred on 6-hour intervals.
- ▶ We found that some hurricanes had the same ID but were actually different ones.
- ▶ We excluded hurricanes that had fewer than 3 observations.
- ▶ We defined August, September, and October as active season, the rest as inactive season.
- ▶ After the process, there are 21691 observations across 704 unique hurricanes.

Atlantic named Windstorm Trajectories by Month (1950 – 2013)

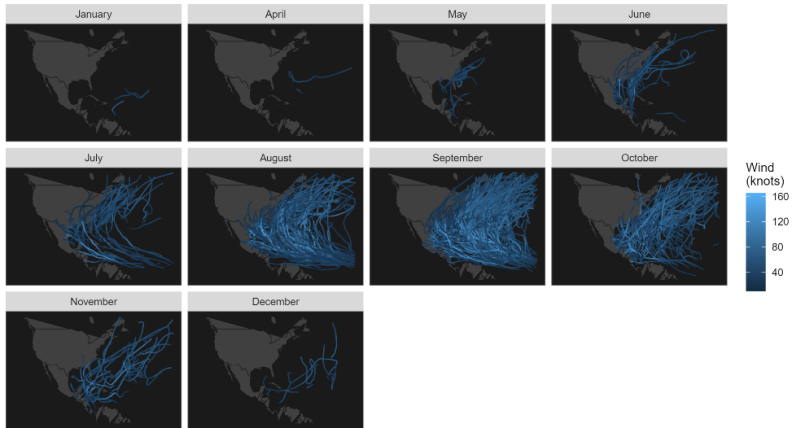


Figure 1: Trajectories of hurricanes by month

## Bayesian Model

Let  $Y_i(t)$  denote the wind speed of the  $i$ th hurricane at time  $t$  (in hours) since the hurricane began. The following Bayesian model was suggested to model the wind speed of the  $i$ th hurricane 6 hours later:

$$Y_i(t+6) = \beta_{0,i} + \beta_{1,i}Y_i(t) + \beta_{2,i}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t) \\ + \mathbf{X}_i^\top \boldsymbol{\gamma} + \epsilon_i(t),$$

where

- ▶  $\Delta_{i,1}(t)$ ,  $\Delta_{i,2}(t)$  and  $\Delta_{i,3}(t)$ : changes of latitude, longitude and wind speed between  $t - 6$  and  $t$ , with random coefficients  $\boldsymbol{\beta}_i = (\beta_{0,i}, \beta_{1,i}, \dots, \beta_{4,i})^\top$
- ▶  $\mathbf{X}_i = (x_{i,1}, \dots, x_{i,6})^\top$ : covariates with fixed effects  $\boldsymbol{\gamma}$ , where
  - ▶  $x_{i,1}$ : the calendar year of the  $i$ -th hurricane
  - ▶  $x_{i,2}$ : indicator variable of the month in active season (August-October) when the  $i$ -th hurricane started
  - ▶  $x_{i,3}, \dots, x_{i,6}$ : indicator variables of the type (ES, NR, SS, TS) of the  $i$ -th hurricane
- ▶  $\epsilon_{i,t} \sim N(0, \sigma^2)$ , independent across  $t$

# Task 1 - Prior Distributions

**Objective:** Let  $\mathbf{B} = (\beta_1^\top, \dots, \beta_n^\top)^\top$ , derive the posterior distribution of the parameters  $\Theta = (\mathbf{B}^\top, \boldsymbol{\mu}^\top, \boldsymbol{\Sigma}, \boldsymbol{\gamma}^\top, \sigma)$ .

We assume that

- ▶  $\beta_i \stackrel{i.i.d.}{\sim} N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$
- ▶  $\boldsymbol{\mu} \sim N(\mathbf{0}, \mathbf{V})$
- ▶  $\boldsymbol{\Sigma}$ : an inverse-Wishart distribution with d.f.  $\nu$  and scale matrix  $\mathbf{S}$
- ▶  $\boldsymbol{\gamma} \sim N(\mathbf{0}, 0.05^2 \mathbf{I}_6)$
- ▶  $\sigma$ : a half-Cauchy distribution with scale parameter 10

We set  $\mathbf{V} = \mathbf{S} = \mathbf{I}_5$ , and  $\nu = 5$ .



## Task 1 - Joint Prior Distribution of Parameters

Let  $n$  denote the number of hurricanes in the dataset. The prior distribution of  $\Theta = (\mathbf{B}^\top, \boldsymbol{\mu}^\top, \boldsymbol{\Sigma}, \boldsymbol{\gamma}^\top, \sigma)$  is given by

$$\begin{aligned}\pi(\Theta) &= \pi(\mathbf{B}^\top, \boldsymbol{\mu}^\top, \boldsymbol{\Sigma}, \boldsymbol{\gamma}^\top, \sigma) \\ &= \pi(\mathbf{B}^\top \mid \boldsymbol{\mu}^\top, \boldsymbol{\Sigma}) \pi(\boldsymbol{\mu}^\top, \boldsymbol{\Sigma}) \pi(\boldsymbol{\gamma}) \pi(\sigma) \\ &= \left( \prod_{i=1}^n \pi(\beta_i^\top \mid \boldsymbol{\mu}^\top, \boldsymbol{\Sigma}) \right) \pi(\boldsymbol{\mu}) \pi(\boldsymbol{\Sigma}) \pi(\boldsymbol{\gamma}) \pi(\sigma) \\ &\propto |\boldsymbol{\Sigma}|^{-n/2} \exp \left( -\frac{1}{2} \sum_{i=1}^n (\beta_i - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\beta_i - \boldsymbol{\mu}) \right) \\ &\quad \times \exp \left( -\frac{1}{2} \boldsymbol{\mu}^\top \mathbf{V}^{-1} \boldsymbol{\mu} \right) \\ &\quad \times |\boldsymbol{\Sigma}|^{-(\nu+6)/2} \exp \left( -\frac{1}{2} \text{tr}(\mathbf{S} \boldsymbol{\Sigma}^{-1}) \right) \\ &\quad \times \exp \left( -\frac{1}{2} \cdot 400 \boldsymbol{\gamma}^\top \boldsymbol{\gamma} \right) \times \frac{I(\sigma > 0)}{1 + (\sigma/10)^2}.\end{aligned}$$

## Task 1 - Likelihood

Let  $m_i$  denote the number of observations and

$\mathbf{Y}_i = (Y_{i,1}, \dots, Y_{i,m_i})^\top$  denote the wind speed data of the  $i$ -th hurricane (excluding the first and second observations), where  $Y_{i,k} = Y_i(6k + 6)$ . Denote  $\mathbf{Y} = (\mathbf{Y}_1^\top, \mathbf{Y}_2^\top, \dots, \mathbf{Y}_n^\top)^\top$ , and  $\mathbf{Z}_{i,k} = (1, Y_{i,k}, \Delta_{i,1}(6k + 6), \Delta_{i,2}(6k + 6), \Delta_{i,3}(6k + 6))^\top$ .

Given that

$$Y_{i,j} \mid (\beta_i^\top, \gamma^\top, \sigma) \sim N(\mathbf{Z}_{i,j-1}^\top \beta_i + \mathbf{x}_i^\top \gamma, \sigma^2),$$

we have

$$\begin{aligned} L(\Theta \mid \mathbf{Y}^\top) &= \prod_{i=1}^n \prod_{j=1}^{m_i} L(\Theta \mid Y_{i,j}) \\ &= \prod_{i=1}^n \prod_{j=1}^{m_i} \left[ \frac{1}{\sqrt{2\pi}\sigma} \exp \left( -\frac{(Y_{i,j} - \mathbf{Z}_{i,j-1}^\top \beta_i - \mathbf{x}_i^\top \gamma)^2}{2\sigma^2} \right) \right]. \end{aligned}$$

## Task 1 - Joint Posterior Distribution of Parameters

$$\begin{aligned}\pi(\Theta \mid \mathbf{Y}^\top) &\propto L(\Theta \mid \mathbf{Y}^\top)\pi(\Theta) \\ &\propto \frac{I(\sigma > 0)\sigma^{-\sum_{i=1}^n m_i}}{1 + (\sigma/10)^2} |\boldsymbol{\Sigma}|^{-(n+\nu+6)/2} \\ &\quad \times \exp \left[ -\frac{1}{2} \left( \sum_{i=1}^n (\boldsymbol{\beta}_i - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\boldsymbol{\beta}_i - \boldsymbol{\mu}) + \boldsymbol{\mu}^\top \mathbf{V}^{-1} \boldsymbol{\mu} \right. \right. \\ &\quad \left. \left. + \text{tr}(\mathbf{S}\boldsymbol{\Sigma}^{-1}) + 400\|\boldsymbol{\gamma}\|_2^2 \right) \right] \\ &\quad \times \exp \left( -\frac{1}{2\sigma^2} \sum_{i=1}^n \sum_{j=1}^{m_i} (Y_{i,j} - \mathbf{z}_{i,j-1}^\top \boldsymbol{\beta}_i - \mathbf{x}_i^\top \boldsymbol{\gamma})^2 \right).\end{aligned}$$

## Task 2

### **Objective:**

Design and implement a custom MCMC algorithm for the outlined Bayesian hierarchical model. Monitor the convergence of the MCMC chains, using diagnostic plots and summary statistics to check for any issues.

# Task 2 - MCMC Algorithm

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## Algorithm 1 MCMC Algorithm (Part 1)

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Require:  $\mathbf{Y}; \beta_0, \mu_0, \Sigma_0, \sigma_0, \gamma_0$

Ensure:  $\hat{\beta}, \hat{\mu}, \hat{\Sigma}, \hat{\sigma}, \hat{\gamma} \approx \beta, \mu, \Sigma, \sigma, \gamma$

$i \leftarrow 0$ , where  $i$  is the current number of iterations

**while** iteration times is not met **do**

$i \leftarrow i + 1$

Gibbs sampling for  $\beta$

$$\beta^{(k)} \sim \mathcal{N}(\beta | \mathbf{Y}, \mu^{(k-1)}, \Sigma^{(k-1)}, \sigma^{(k-1)}, \gamma^{(k-1)})$$

Gibbs sampling for  $\mu$

$$\mu^{(k)} \sim \mathcal{N}(\mu | \mathbf{Y}, \beta^{(k)}, \Sigma^{(k-1)}, \sigma^{(k-1)}, \gamma^{(k-1)})$$

Gibbs sampling for  $\Sigma$

$$\Sigma^{(k)} \sim \mathcal{W}(\Sigma | \nabla \sqcup (\Sigma | \mathbf{Y}, \beta^{(k)}, \mu^{(k)}, \sigma^{(k-1)}, \gamma^{(k-1)}))$$

Gibbs sampling for  $\gamma$   $\gamma^{(k)} \sim p(\Sigma | \mathbf{Y}, \beta^{(k)}, \mu^{(k)}, \Sigma^{(k)}, \sigma^{(k-1)})$

Metropolis-Hastings for  $\sigma$

Propose a new value  $\sigma^*$  from a normal distribution with mean  $\sigma^{(k-1)}$  and a small variance.

Compute the acceptance ratio

$$\gamma = \frac{p(\gamma^{(k)} | \mathbf{Y}, \beta^{(k)}, \mu^{(k)}, \Sigma^{(k)}, \sigma^{(k)}) q(\gamma^{(k-1)} | \gamma^{(k)})}{p(\gamma^{(k-1)} | \mathbf{Y}, \beta^{(k)}, \mu^{(k)}, \Sigma^{(k)}, \sigma^{(k)}) q(\gamma^{(k)} | \gamma^{(k-1)})}$$

Generate a random number  $u$  from a uniform distribution between 0 and 1.

If  $u \leq \gamma$ , set  $\sigma^{(k)} = \sigma^*$ , otherwise set  $\sigma^{(k)} = \sigma^{(k-1)}$ .

**end while**

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# Task 2 - Starting Values

## Final Initial Value Selection and Core Information of MH Algorithm

- ▶  $\beta_i$ : This can be obtained through the random effects term in the lmm model. The random effects term can be added to the fixed effects term to obtain  $\beta_i^{(0)}$ .
- ▶  $\mu$ : This can be obtained through the fixed effects of windpre, latdif, longdif, winddif term.
- ▶  $\gamma$ : This can be obtained through the fixed effects of Season, Active Month, and Nature term.
- ▶  $\sigma^2$ : This can be obtained through the model residual sigma0.
- ▶  $\Sigma^{-1}$ : This can be obtained through the VarCorr(lmm) function which returns the covariance matrix of the random effects in the model. The inverse of this matrix can be taken to obtain  $\Sigma^{-1(0)}$ .

Table 1: Initial Value Setting

Parameter	Value
$\mu$	(24.25, 0.94, -0.02, -0.24, 0.47)
$\gamma$	(-0.01, 0.35, 0.28, 0.37, 0.12, 0.08)
$\Sigma$	$\begin{pmatrix} 0.36 & -0.01 & 0.04 & 0.12 & 0.03 \\ -0.01 & 0.00 & -0.00 & -0.00 & 0.00 \\ 0.04 & -0.00 & 0.04 & 0.03 & -0.02 \\ 0.12 & -0.00 & 0.03 & 0.07 & 0.00 \\ 0.03 & 0.00 & -0.02 & 0.00 & 0.02 \end{pmatrix}$
$\sigma^2$	5.27

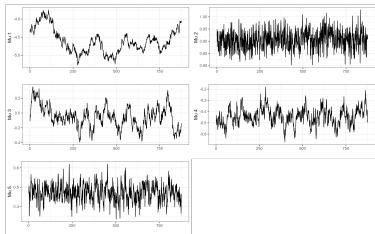
## Task 2 - MCMC Algorithm R code

```
# Gimble sampling algorithm
B_sample <- function(mu, Sigma, gamma, sigma) {
  Sigma.inv <- solve(Sigma)
  B_mean_cov <- function(i) {
    cov <- solve(Sigma.inv + 1/sigma^2 * t(Z[[i]]) %*% Z[[i]])
    mean <- cov %*% (Sigma.inv %*% mu + 1/sigma^2 * colSums((Y[[i]] - (X[i,] %*% gamma)[,]) * Z[[i]]))
    list(mean = mean, cov = cov)
  }
  mean_cov_list <- lapply(1:n, B_mean_cov)
  B <- sapply(mean_cov_list, function(x) {mvnrm(mu = x$mean, Sigma = x$cov)})
  return(B)
}

# MH algorithm (random walk)
sigma_sample <- function(sigma, B, gamma, a) {
  sigma_new <- sigma + (runif(1) - 0.5) * 2 * a # candidate sigma
  if (sigma_new <= 0) {
    return(sigma)
  }
  RSS <- sum(sapply(1:n, function(i) sum((Y[[i]] - Z[[i]] %*% B[,i] - (X[i,] %*% gamma)[,])^2)))
  log_kernal_ratio <- -sum(m) * log(sigma_new/sigma) +
    log(1 + (sigma/10)^2) - log(1 + (sigma_new/10)^2) -
    0.5 * (1/sigma_new^2 - 1/sigma^2) * RSS
  log_prob <- min(0, log_kernal_ratio)
  sigma <- ifelse(log_prob > log(runif(1)), sigma_new, sigma)
  return(sigma)
}
```

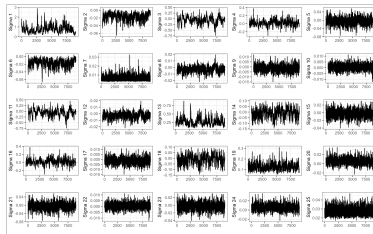
# Task 2 - Results Presentation(Time Series Plot for Each Parameters(Burn-In 8000))

Time Series Plot of Mu



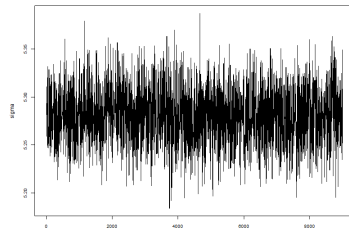
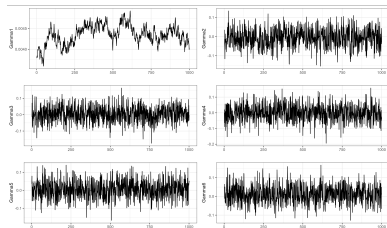
(a) Mu

Time Series Plot of Sigma



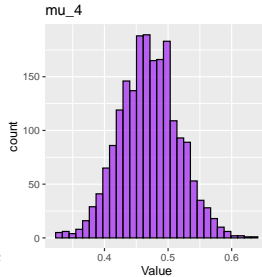
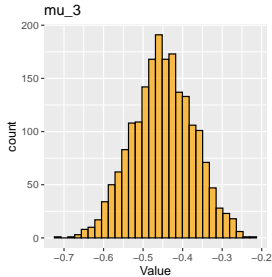
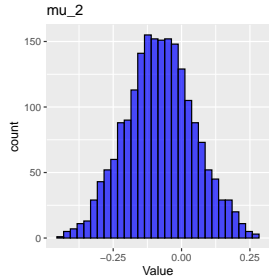
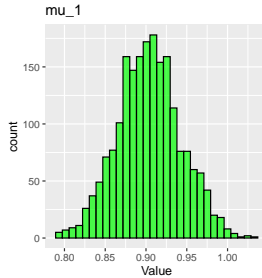
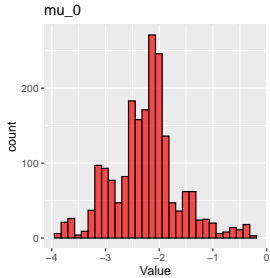
(b) Sigma

Time Series Plot of Gamma





# Task 2 - Results Presentation(Histogram Plot for $\mu$ )



## Task 2 - Results Presentation(Values of $\mu$ and $\Sigma$ )

$$\hat{\mu}$$

$$\begin{pmatrix} \hat{\mu}_0 & 4.6170057 \\ \hat{\mu}_1 & 0.9035517 \\ \hat{\mu}_2 & -0.0425631 \\ \hat{\mu}_3 & -0.4558512 \\ \hat{\mu}_4 & 0.4706897 \end{pmatrix}$$

$$\hat{\Sigma}$$

$$\begin{pmatrix} 0.7521 & -0.0155 & -0.0860 & 0.0136 & -0.0060 \\ -0.0155 & 0.0050 & -0.0023 & -0.0013 & 0.0006 \\ -0.0860 & -0.0023 & 0.2705 & -0.0085 & -0.0024 \\ 0.0136 & -0.0013 & -0.0085 & 0.1287 & 0.0057 \\ -0.0060 & 0.0006 & -0.0024 & 0.0057 & 0.0268 \end{pmatrix}$$

$$\hat{\rho}$$

$$\begin{pmatrix} 1.0000 & -0.2529 & -0.1907 & 0.0436 & -0.0423 \\ -0.2529 & 1.0000 & -0.0635 & -0.0521 & 0.0490 \\ -0.1907 & -0.0635 & 1.0000 & -0.0457 & -0.0277 \\ 0.0436 & -0.0521 & -0.0457 & 1.0000 & 0.0962 \\ -0.0423 & 0.0490 & -0.0277 & 0.0962 & 1.0000 \end{pmatrix}$$

## Task 2 - Results Presentation(CI of $\mu$ and $Beta_{mean}$ )

variable	mean	lower_CI	upper_CI
beta_mean_0	-3.6630742	-4.9777068	-1.1445976
beta_mean_1	0.9025330	0.8896760	0.9156107
beta_mean_2	-0.0585258	-0.2852565	0.1540752
beta_mean_3	-0.4508275	-0.5631008	-0.3268690
beta_mean_4	0.4722175	0.4325184	0.5126448
mu_0	4.6170057	-4.6014745	22.2632268
mu_1	0.9035517	0.8399851	0.9661758
mu_2	-0.0425631	-0.2496227	0.1581210
mu_3	-0.4558512	-0.5891481	-0.3209113
mu_4	0.4706897	0.3979018	0.5432620

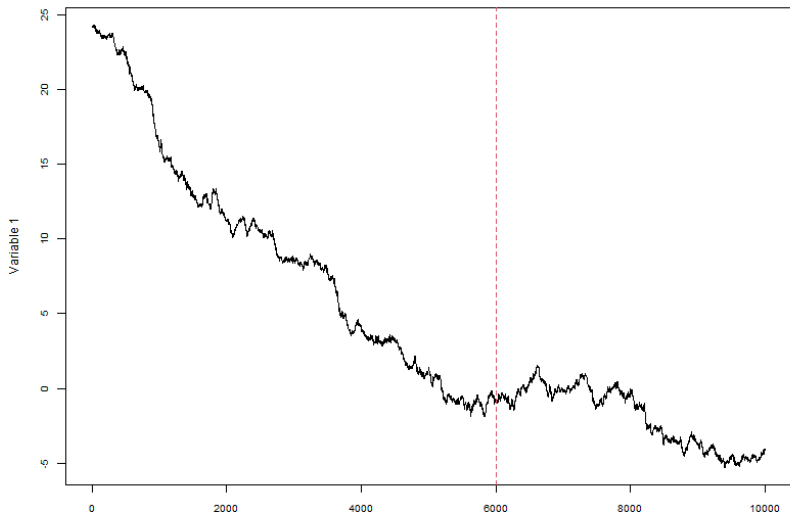
## Task 3

### **Objective:**

Compute posterior summaries and 95% credible intervals of  $\gamma$ , the fixed effects associated with the covariates in the model. Using the estimated Bayesian model, answer the following questions:

- (1) Are there seasonal differences in hurricane wind speeds?
- (2) Is there evidence to support the claim that hurricane wind speeds have been increasing over the years?

## Task 3 - Parameters Convergence



## Task 3 - 95% Credible Intervals

	gamma_year	gamma_active
2.5%	0.0025498	-0.1047691
97.5%	0.0038603	0.0880498

## Task 4 - Objective

### **Objective:**

With the estimated model parameters and covariate values, you can calculate the predicted wind speed for each time point using the model equation. This way, you can track the hurricane and compare the predicted wind speeds with the actual wind speeds recorded during the hurricane. Please evaluate how well the estimated Bayesian model can track individual hurricanes.

## Task 4

**Prediction:** Using the parameters after burn-in, we can obtain the predicted value for each hurricane.

$$\hat{Y}_i(t+6) = \hat{\beta}_{0,i} + \hat{\beta}_{1,i} Y_i(t) + \hat{\beta}_{2,i} \Delta_{i,1}(t) + \hat{\beta}_{3,i} \Delta_{i,2}(t) + \hat{\beta}_{4,i} \Delta_{i,3}(t) + \mathbf{X}_i^\top \hat{\gamma}$$

**Performance evaluation:** For each hurricane, we can evaluate the estimated Bayesian model performance by calculating

$$RMSE = \sqrt{\frac{\sum_{i=1}^n (y_i - \hat{y}_i)^2}{n}}$$

$$R^2 = 1 - \frac{\sum_{i=1}^n (y_i - \hat{y}_i)^2}{\sum_{i=1}^n (y_i - \bar{y})^2}$$



## Task 4

Table 3: Summary of RMSE and R-squared for selected hurricanes

ID	Year	RMSE	R-squared
ABBY.1960	1960	8.8804	0.7700
ABBY.1964	1964	9.6430	0.3033
ABBY.1968	1968	3.5043	0.9360
ABLE.1950	1950	3.6755	0.9813
ABLE.1951	1951	3.4802	0.9767
ABLE.1952	1952	4.5183	0.9583
AGNES.1972	1972	5.2483	0.8881
ALBERTO.1982	1982	8.0473	0.7499
ALBERTO.1988	1988	2.6121	0.7420
ALBERTO.1994	1994	4.3941	0.8807
ALBERTO.2000	2000	3.7896	0.9625
ALBERTO.2006	2006	4.3591	0.7882
ALBERTO.2012	2012	3.2193	0.8036
ALEX.1998	1998	2.9351	0.7289
ALEX.2004	2004	5.4552	0.9539

## Task 4

Prediction performance on random chosen example hurricanes.

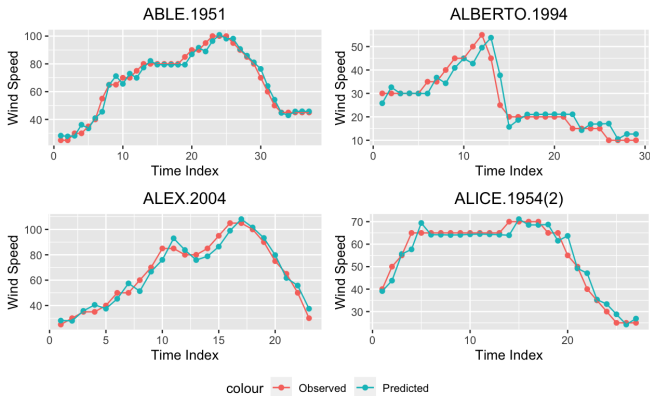


Figure 2: Time series prediction plot

## Task 4

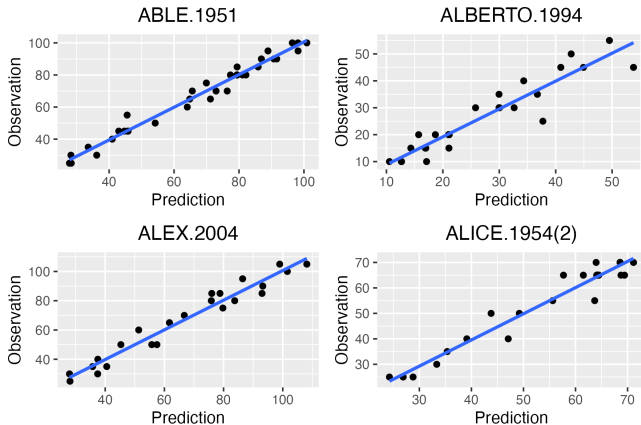


Figure 3: Prediction vs. observation

## Task 4

**Performance evaluation:** We plot the  $RMSE$  and  $R^2$  distribution for all the hurricanes

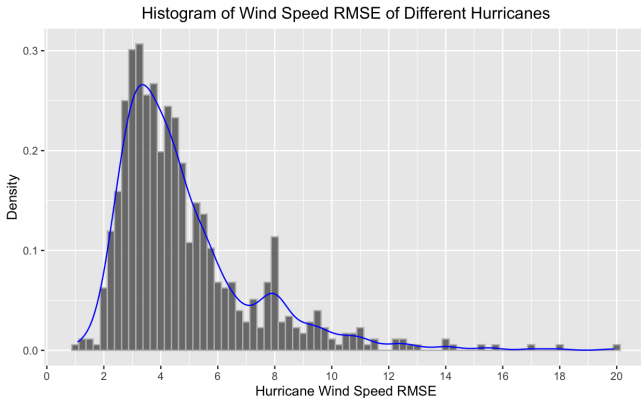


Figure 4: RMSE distribution

## Task 4

### Performance evaluation:

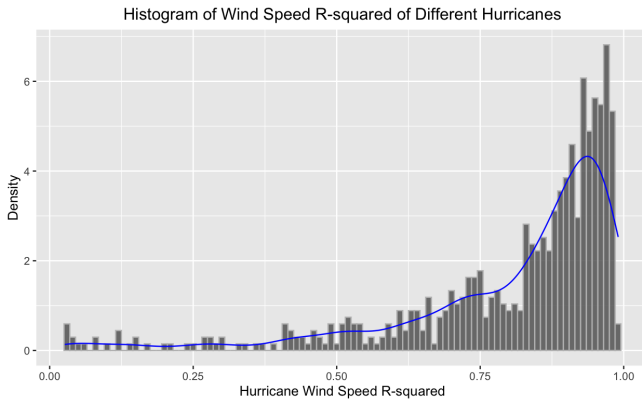


Figure 5:  $R^2$  distribution

# Discussions

- ▶ Parameters Convergence Problem

# Reference

Reference

## Q&A

- ▶ Thanks for listening!