Assignment on Lambda Calculus and Types

1. This problem uses the untyped λ-calculus. Recall the Church numeral and boolean encoding:

$$
\begin{aligned}
\mathsf{true} &\stackrel{\text{def}}{=} \lambda x.\ \lambda y.\ x \\
\mathsf{false} &\stackrel{\text{def}}{=} \lambda x.\ \lambda y.\ y \\
\hat{0} &\stackrel{\text{def}}{=} \lambda f.\ \lambda y.\ y \\
\hat{1} &\stackrel{\text{def}}{=} \lambda f.\ \lambda y.\ f\ y \\
\hat{2} &\stackrel{\text{def}}{=} \lambda f.\ \lambda y.\ f\ (f\ y) \\
\mathsf{iszero} &\stackrel{\text{def}}{=} \lambda n.\ \lambda x.\ \lambda y.\ n\ (\lambda z.\ y)\ x \\
\mathsf{add} &\stackrel{\text{def}}{=} \lambda n.\ \lambda m.\ \lambda f.\ \lambda x.\ n\ f\ (m\ f\ x)
\end{aligned}
$$

Reduce the term $(\mathsf{iszero}\ (\mathsf{add}\ \hat{1}\ \hat{1}))$ to a normal form. You can find the reduction rules below, and you can choose any reduction strategy. Please do not skip steps.

Reduction rules:

$$
\frac{}{(\lambda x.\ M)\ N\ \rightarrow\ M[N/x]} \qquad \frac{M \rightarrow M'}{\lambda x.\ M\ \rightarrow\ \lambda x.\ M'}
$$

$$
\frac{M \rightarrow M'}{M\ N\ \rightarrow\ M'\ N} \qquad \frac{N \rightarrow N'}{M\ N\ \rightarrow\ M\ N'}
$$

Substitution:

$x[N/x] = N$

$y[N/x] = y$

$(M\,N)[N'/x] = (M[N'/x])\,(N[N'/x])$

$(\lambda x.\,M)[N/x] = \lambda x.\,M$

$(\lambda y.\,M)[N/x] = \lambda y.\,(M[N/x])$, where $y \notin fv(N)$

$(\lambda y.\,M)[N/x] = \lambda z.\,(M[z/y])[N/x]$, where $y \in fv(N)$ and z fresh

Free variables:

$fv(x) = \{x\} \qquad fv(M\,N) = fv(M) \cup fv(N) \qquad fv(\lambda x.\,M) = fv(M) - \{x\}$

Normal form: a term containing no redex.

2. (Modified from the course exam in Autumn 2018)

   In this problem we add the option types to the simply-typed λ-calculus. We can use None and Some to construct terms of the option type, just like `None` and `Some` in Coq. Intuitively, None represents a dummy element (i.e. there is no meaningful element), Some M means that there is a meaningful element M, and get M gives us the meaningful element contained in M of the option type.

   Syntax:

$$\begin{array}{rccl} (\mathsf{Types}) & \tau & ::= & \ldots \mid \mathsf{option}\ \tau \\ (\mathsf{Terms}) & M & ::= & \ldots \mid \mathsf{None} \mid \mathsf{Some}\ M \mid \mathsf{get}\ M \\ (\mathsf{Values}) & v & ::= & \ldots \mid \mathsf{None} \mid \mathsf{Some}\ v \end{array}$$

   Reduction rules:

$$\frac{M \to M'}{\mathsf{Some}\ M \to \mathsf{Some}\ M'}\ (\textsc{some}) \qquad \frac{M \to M'}{\mathsf{get}\ M \to \mathsf{get}\ M'}\ (\textsc{get-m})$$

$$\frac{}{\mathsf{get}\ (\mathsf{Some}\ M) \to M}\ (\textsc{get-some}) \qquad \frac{}{\mathsf{get}\ \mathsf{None} \to \mathsf{get}\ \mathsf{None}}\ (\textsc{get-none})$$

   (a) Give 3 appropriate new typing rules, one for each new form of term. Note that your rules should ensure the preservation and progress theorems.

   (b) Consider each of the following questions in isolation. Answer yes or no.

      i. Suppose we remove the above (GET-M) rule.
      Does the preservation theorem still hold?
      Does the progress theorem still hold?

      ii. Suppose we remove both the above (SOME) rule and the above (GET-M) rule.
      Does the preservation theorem still hold?
      Does the progress theorem still hold?

      iii. Suppose we add the following rule.

$$\frac{}{\mathsf{get}\ v \to \mathsf{get}\ v}\ (\textsc{get-v})$$

      Does the preservation theorem still hold?
      Does the progress theorem still hold?

      iv. Suppose we change the above (GET-SOME) rule to the following (GET-SOME') rule.

$$\frac{}{\mathsf{get}\ (\mathsf{Some}\ v) \to v}\ (\textsc{get-some'})$$

      Does the preservation theorem still hold?
      Does the progress theorem still hold?

v. Suppose we change the above (GET-SOME) rule to the following (GET-SOME") rule.

$$\frac{}{\mathsf{get}\ (\mathsf{Some}\ v) \to \mathsf{Some}\ (\mathsf{get}\ v)}\ (\text{GET-SOME"})$$

Does the preservation theorem still hold?
Does the progress theorem still hold?

vi. Suppose we change the above (GET-NONE) rule to the following (GET-NONE') rule.

$$\frac{}{\mathsf{get}\ \mathsf{None} \to \mathsf{None}}\ (\text{GET-NONE'})$$

Does the preservation theorem still hold?
Does the progress theorem still hold?