System F

Reference: Chapter 23 in Pierce's *TAPL*

Recall the Simply-Typed Lambda-Calculus (STLC)

- Syntax

$$\begin{array}{rcll} \text{(Terms)} & M & ::= & x \mid \lambda x : \tau.\ M \mid M\,M \\ \text{(Types)} & \tau & ::= & \mathtt{T} \mid \tau \to \tau \\ \text{(Values)} & v & ::= & \lambda x : \tau.\ M \\ \text{(Contexts)} & \Gamma & ::= & \bullet \mid \Gamma, x : \tau \end{array}$$

- Reduction

$$\frac{}{(\lambda x : \tau.\ M_1)\ M_2 \longrightarrow M_1[M_2/x]}\ \text{(E-AppAbs)}$$

$$\frac{M_1 \longrightarrow M_1'}{M_1\ M_2 \longrightarrow M_1'\ M_2}\ \text{(E-App1)} \qquad \frac{M_2 \longrightarrow M_2'}{M_1\ M_2 \longrightarrow M_1\ M_2'}\ \text{(E-App2)}$$

$$\frac{M \longrightarrow M'}{\lambda x : \tau.\ M \longrightarrow \lambda x : \tau.\ M'}\ \text{(E-Abs)}$$

Recall the Simply-Typed Lambda-Calculus (STLC)

- Typing

$$\frac{}{\Gamma, x : \tau \vdash x : \tau}\ (\text{T-Var}) \qquad \frac{\Gamma, x : \tau_1 \vdash M : \tau_2}{\Gamma \vdash (\lambda x : \tau_1.\ M) : \tau_1 \to \tau_2}\ (\text{T-Abs})$$

$$\frac{\Gamma \vdash M_1 : \tau \to \tau' \qquad \Gamma \vdash M_2 : \tau}{\Gamma \vdash M_1\ M_2 : \tau'}\ (\text{T-App})$$

- Soundness

Theorem (Preservation)

For all M, M' and τ, if $\bullet \vdash M : \tau$ and $M \longrightarrow M'$, then $\bullet \vdash M' : \tau$.

Theorem (Progress)

For all M and τ, if $\bullet \vdash M : \tau$, then either $M \in \text{Values}$ or $\exists M'.\ M \longrightarrow M'$.

Recall the Simply-Typed Lambda-Calculus (STLC)

We can write an infinite number of “doubling” functions in STLC:

$$\begin{aligned}
\mathsf{doubleNat} &\overset{\text{def}}{=} \lambda f : \mathsf{Nat} \to \mathsf{Nat}.\ \lambda x : \mathsf{Nat}.\ f\,(f\,x) \\
\mathsf{doubleBool} &\overset{\text{def}}{=} \lambda f : \mathsf{Bool} \to \mathsf{Bool}.\ \lambda x : \mathsf{Bool}.\ f\,(f\,x) \\
\mathsf{doubleFun} &\overset{\text{def}}{=} \lambda f : (\mathsf{Nat} \to \mathsf{Nat}) \to (\mathsf{Nat} \to \mathsf{Nat}).\ \lambda x : \mathsf{Nat} \to \mathsf{Nat}.\ f\,(f\,x)
\end{aligned}$$

Different types of arguments, but the same function body.

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Different types of arguments, but the same function body.

Can we abstract out the types?

Polymorphism

poly = many, *morph* = form
Allow a single piece of code to be used with multiple types.

Our focus: *parametric polymorphism*.

- Code is typed "generically", using variables in place of actual types, and then instantiated with particular types as needed.
- *Uniform*: all of their instances behave the same.
- By contrast, *ad-hoc polymorphism* (e.g. overloading) allows the code to exhibit different behaviors at different types.

System F

System F was first discovered by Jean-Yves Girard (1972), in the context of proof theory in logic.

John Reynolds (1974) independently developed a type system with the same power, called *the polymorphic lambda-calculus*.

It is also sometimes called *the second-order lambda-calculus*, because it corresponds, via the Curry-Howard correspondence, to second-order intuitionistic logic, which allows quantification not only over individuals [terms], but also over predicates [types].

Syntax

$$\begin{array}{rrcl} \text{(Terms)} & M & ::= & x \mid \lambda x : \tau.\, M \mid M\, M \mid \Lambda\alpha.\, M \mid M\,\langle\tau\rangle \\ \text{(Types)} & \tau & ::= & \alpha \mid \mathtt{T} \mid \tau \to \tau \mid \forall\alpha.\, \tau \\ \text{(Values)} & v & ::= & \lambda x : \tau.\, M \mid \Lambda\alpha.\, M \end{array}$$

- Type variable α
- Type abstraction $\Lambda\alpha.\, M$
- Type application $M\,\langle\tau\rangle$
- Universal type $\forall\alpha.\, \tau$

Reduction

$$\frac{}{(\lambda x : \tau.\ M_1)\ M_2 \longrightarrow M_1[M_2/x]}\ (\textsc{E-AppAbs})$$

$$\frac{M_1 \longrightarrow M_1'}{M_1\ M_2 \longrightarrow M_1'\ M_2}\ (\textsc{E-App1}) \qquad \frac{M_2 \longrightarrow M_2'}{M_1\ M_2 \longrightarrow M_1\ M_2'}\ (\textsc{E-App2})$$

$$\frac{M \longrightarrow M'}{\lambda x : \tau.\ M \longrightarrow \lambda x : \tau.\ M'}\ (\textsc{E-Abs})$$

$$\frac{}{(\Lambda\alpha.\ M_1)\ \langle\tau_2\rangle \longrightarrow M_1[\tau_2/\alpha]}\ (\textsc{E-TAppTAbs})$$

$$\frac{M_1 \longrightarrow M_1'}{M_1\ \langle\tau_2\rangle \longrightarrow M_1'\ \langle\tau_2\rangle}\ (\textsc{E-TApp}) \qquad \frac{M \longrightarrow M'}{\Lambda\alpha.\ M \longrightarrow \Lambda\alpha.\ M'}\ (\textsc{E-TAbs})$$

Statics

$$
\begin{array}{rrcl}
\text{(Terms)} & M & ::= & x \mid \lambda x : \tau.\, M \mid M\, M \mid \Lambda\alpha.\, M \mid M\langle\tau\rangle \\
\text{(Types)} & \tau & ::= & \alpha \mid \mathtt{T} \mid \tau \to \tau \mid \forall\alpha.\, \tau \\
\text{(Values)} & v & ::= & \lambda x : \tau.\, M \mid \Lambda\alpha.\, M \\
\\
\text{(Contexts)} & \Gamma & ::= & \bullet \mid \Gamma, x : \tau \\
\text{(TypeVarContexts)} & \Delta & ::= & \bullet \mid \Delta, \alpha
\end{array}
$$

Type well-formedness: $\Delta \vdash \tau$

Typing judgment: $\Delta; \Gamma \vdash M : \tau$

Type Well-Formedness

$$\frac{}{\Delta, \alpha \vdash \alpha} \qquad \frac{}{\Delta \vdash \mathtt{T}} \qquad \frac{\Delta \vdash \tau_1 \qquad \Delta \vdash \tau_2}{\Delta \vdash \tau_1 \to \tau_2} \qquad \frac{\Delta, \alpha \vdash \tau}{\Delta \vdash \forall \alpha.\, \tau}$$

An alternative formulation :

$$\frac{fv(\tau) \subseteq \Delta}{\Delta \vdash \tau}$$

$$fv(\alpha) \overset{\text{def}}{=} \{\alpha\} \qquad fv(\mathtt{T}) \overset{\text{def}}{=} \emptyset \qquad fv(\tau_1 \to \tau_2) \overset{\text{def}}{=} fv(\tau_1) \cup fv(\tau_2)$$
$$fv(\forall \alpha.\, \tau) \overset{\text{def}}{=} fv(\tau) - \{\alpha\}$$

Typing

$$\frac{}{\Delta; \Gamma, x : \tau \vdash x : \tau} \text{ (T-VAR)}$$

$$\frac{\Delta \vdash \tau_1 \qquad \Delta; \Gamma, x : \tau_1 \vdash M : \tau_2}{\Delta; \Gamma \vdash (\lambda x : \tau_1.\, M) : \tau_1 \to \tau_2} \text{ (T-ABS)}$$

$$\frac{\Delta; \Gamma \vdash M_1 : \tau \to \tau' \qquad \Delta; \Gamma \vdash M_2 : \tau}{\Delta; \Gamma \vdash M_1\, M_2 : \tau'} \text{ (T-APP)}$$

$$\frac{\Delta, \alpha; \Gamma \vdash M : \tau}{\Delta; \Gamma \vdash (\Lambda\alpha.\, M) : \forall\alpha.\tau} \text{ (T-TABS)}$$

$$\frac{\Delta; \Gamma \vdash M_1 : \forall\alpha.\tau \qquad \Delta \vdash \tau_2}{\Delta; \Gamma \vdash M_1\, \langle\tau_2\rangle : \tau[\tau_2/\alpha]} \text{ (T-TAPP)}$$

Examples

- $\mathsf{id} \overset{\text{def}}{=} \Lambda\alpha.\, \lambda x : \alpha.\, x$
  - $\mathsf{id} : \forall\alpha.\, \alpha \to \alpha$
  - $\mathsf{id}\,\langle\mathsf{Nat}\rangle : \mathsf{Nat} \to \mathsf{Nat}$
  - $\mathsf{id}\,\langle\mathsf{Nat} \to \mathsf{Nat}\rangle : (\mathsf{Nat} \to \mathsf{Nat}) \to (\mathsf{Nat} \to \mathsf{Nat})$

Examples

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  - $\mathsf{id}\,\langle\mathsf{Nat}\rangle : \mathsf{Nat} \to \mathsf{Nat}$
  - $\mathsf{id}\,\langle\mathsf{Nat} \to \mathsf{Nat}\rangle : (\mathsf{Nat} \to \mathsf{Nat}) \to (\mathsf{Nat} \to \mathsf{Nat})$
- $\mathsf{double} \overset{\text{def}}{=} \Lambda\alpha.\, \lambda f : \alpha \to \alpha.\, \lambda x : \alpha.\, f\,(f\,x)$
  - $\mathsf{double} : \forall\alpha.\, (\alpha \to \alpha) \to \alpha \to \alpha$
  - $\mathsf{double}\,\langle\mathsf{Nat}\rangle : (\mathsf{Nat} \to \mathsf{Nat}) \to \mathsf{Nat} \to \mathsf{Nat}$

Examples

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  - $\mathsf{double} : \forall\alpha.\ (\alpha \to \alpha) \to \alpha \to \alpha$
  - $\mathsf{double}\,\langle \mathsf{Nat} \rangle : (\mathsf{Nat} \to \mathsf{Nat}) \to \mathsf{Nat} \to \mathsf{Nat}$
- $\mathsf{quadruple} \overset{\text{def}}{=} \Lambda\alpha.\ \mathsf{double}\,\langle \alpha \to \alpha \rangle\,(\mathsf{double}\,\langle \alpha \rangle)$
  - $\mathsf{quadruple} : \forall\alpha.\ (\alpha \to \alpha) \to \alpha \to \alpha$

Examples

- id $\overset{\text{def}}{=}$ $\Lambda\alpha.\ \lambda x : \alpha.\ x$
  - id : $\forall\alpha.\ \alpha \to \alpha$
  - id $\langle\text{Nat}\rangle$: Nat → Nat
  - id $\langle\text{Nat} \to \text{Nat}\rangle$: (Nat → Nat) → (Nat → Nat)
- double $\overset{\text{def}}{=}$ $\Lambda\alpha.\ \lambda f : \alpha \to \alpha.\ \lambda x : \alpha.\ f\,(f\,x)$
  - double : $\forall\alpha.\ (\alpha \to \alpha) \to \alpha \to \alpha$
  - double $\langle\text{Nat}\rangle$: (Nat → Nat) → Nat → Nat
- quadruple $\overset{\text{def}}{=}$ $\Lambda\alpha.$ double $\langle\alpha \to \alpha\rangle$ (double $\langle\alpha\rangle$)
  - quadruple : $\forall\alpha.\ (\alpha \to \alpha) \to \alpha \to \alpha$
- selfApp $\overset{\text{def}}{=}$ $\lambda x : (\forall\alpha.\ \alpha \to \alpha).\ \ x\,\langle\forall\alpha.\ \alpha \to \alpha\rangle\,x$
  - selfApp : $(\forall\alpha.\ \alpha \to \alpha) \to (\forall\alpha.\ \alpha \to \alpha)$
  - Recall in STLC there's no way to type $\lambda x.\ x\,x$.

Properties

Theorem (Preservation)

For all M, M' and τ, if $\bullet; \bullet \vdash M : \tau$ and $M \longrightarrow M'$, then $\bullet; \bullet \vdash M' : \tau$.

Theorem (Progress)

For all M and τ, if $\bullet; \bullet \vdash M : \tau$, then either $M \in \text{Values}$ or $\exists M'.\ M \longrightarrow M'$.

Strong normalization: Every reduction path starting from a well-typed System F term is guaranteed to terminate.

Church Encodings

Recall in the untyped λ-calculus, we can encode boolean values:

$$\begin{aligned} \text{True} &\overset{\text{def}}{=} \lambda x.\, \lambda y.\, x \\ \text{False} &\overset{\text{def}}{=} \lambda x.\, \lambda y.\, y \end{aligned}$$

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In System F:

$$\begin{aligned} \text{True} &\overset{\text{def}}{=} \Lambda\alpha.\, \lambda x : \alpha.\, \lambda y : \alpha.\, x \\ \text{False} &\overset{\text{def}}{=} \Lambda\alpha.\, \lambda x : \alpha.\, \lambda y : \alpha.\, y \end{aligned}$$

Their type: Bool $\overset{\text{def}}{=} \forall\alpha.\, \alpha \to \alpha \to \alpha$.

$$\text{not} \overset{\text{def}}{=} \lambda b : \text{Bool}.\, \Lambda\alpha.\, \lambda x : \alpha.\, \lambda y : \alpha.\, b\,\langle\alpha\rangle\, y\, x$$

Its type: Bool $\to$ Bool.

Church Encodings

Recall the untyped Church numerals:

$$\begin{aligned}
\underline{0} &\overset{\text{def}}{=} \lambda f.\, \lambda x.\, x \\
\underline{1} &\overset{\text{def}}{=} \lambda f.\, \lambda x.\, f\, x \\
\underline{2} &\overset{\text{def}}{=} \lambda f.\, \lambda x.\, f\, (f\, x)
\end{aligned}$$

In System F:

$$\begin{aligned}
\underline{0} &\overset{\text{def}}{=} \Lambda\alpha.\, \lambda f : \alpha \to \alpha.\, \lambda x : \alpha.\, x \\
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\end{aligned}$$

Read *TAPL* for the encodings of many other data and operators.

Incompleteness?

Recall that the type system for STLC is not complete: it may reject terms that do not go wrong. For instance,

$$(\lambda x.\, (x\,(\lambda y.\, y))\,(x\,3))\,(\lambda z.\, z)$$

In System F, this term can be typed:

$$(\lambda x : \forall\alpha.\, \alpha \to \alpha.\, (x\,\langle \mathrm{Nat} \to \mathrm{Nat}\rangle\,(\lambda y : \mathrm{Nat}.\, y))\,(x\,\langle \mathrm{Nat}\rangle\,3))\,(\Lambda\alpha.\, \lambda z : \alpha.\, z)$$

Incompleteness

Non-terminating functions *cannot* be typed in System F.

While $(\lambda x.\ x\,x)$ can be typed in System F:

$$\mathsf{selfApp} \overset{\text{def}}{=} \lambda x : (\forall\alpha.\,\alpha \to \alpha).\ \ x\,\langle\forall\alpha.\,\alpha \to \alpha\rangle\,x$$

the non-terminating term $(\lambda x.\ x\,x)\,(\lambda x.\ x\,x)$ cannot be typed.

Parametricity

Parametricity: polymorphic terms behave uniformly on their type variables.

- Given a parametrically polymorphic type, we know quite a bit about the behavior of any term of that type.

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Example

Write down all the functions that have type $\forall\alpha.\ \alpha \to \alpha$.

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Example

Write down all the functions that have type $\forall \alpha.\ \alpha \to \alpha$.

Every term you write behaves identically to $\Lambda\alpha.\ \lambda x : \alpha.\ x$.

Intuition: Because the term with type $\forall \alpha.\ \alpha \to \alpha$ is polymorphic in α, whatever it wants to do needs to work for every possible type α, and the lambda calculus is so *simple* that the only such thing it can do is to *return the argument*.

Parametricity

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- Given a parametrically polymorphic type, we know quite a bit about the behavior of any term of that type.

Example

Consider the type Bool $\overset{\text{def}}{=} \forall\alpha.\,\alpha \to \alpha \to \alpha$.

Only two terms: $\Lambda\alpha.\,\lambda x:\alpha.\,\lambda y:\alpha.\,x$ *and* $\Lambda\alpha.\,\lambda x:\alpha.\,\lambda y:\alpha.\,y$.

They are exactly the terms True and False.

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They are exactly the terms True and False.

Read the paper *Theorems for free!* written by Phil Wadler in 1989. It's a fun paper and a famous application of parametricity.

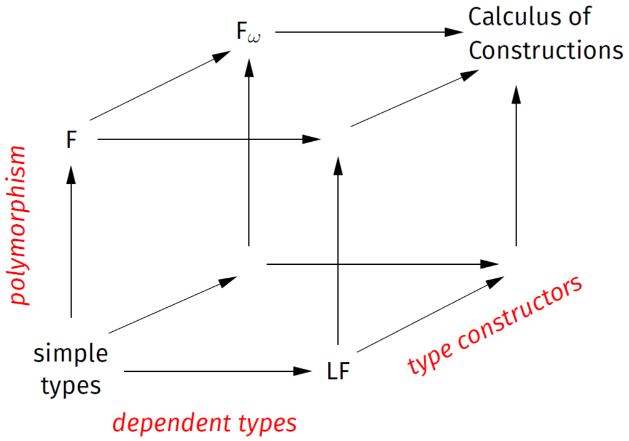
Impredicativity

The polymorphism of System F is often called *impredicative*.

In general, a definition (of a set, a type, etc.) is called *impredicative* if it involves a quantifier whose domain includes the very thing being defined.

For example, in System F, the type variable α in the type $\tau = \forall\alpha.\, \alpha \to \alpha$ ranges over all types, including τ itself (so that, for example, we can instantiate a term of type τ at type τ, yielding a function from τ to τ).

Lambda Cube



Proposed by Henk Barendregt in 1991.
The theoretical basis of Coq: Calculus of Inductive Constructions (CC + inductive definitions).