Lecture 4: Wrapping up Channel Modeling

Prof. Bobak Nazer 9/11/14

Linear, Time-Varying Systems

- We can model the wireless channel as a linear, time-varying system:

$$y(t) = \sum_i a_i(t)\, x\big(t - \tau_i(t)\big) + w(t)$$

$a_i(t) =$ attenuation of the signal traveling along the i$^{\text{th}}$ path

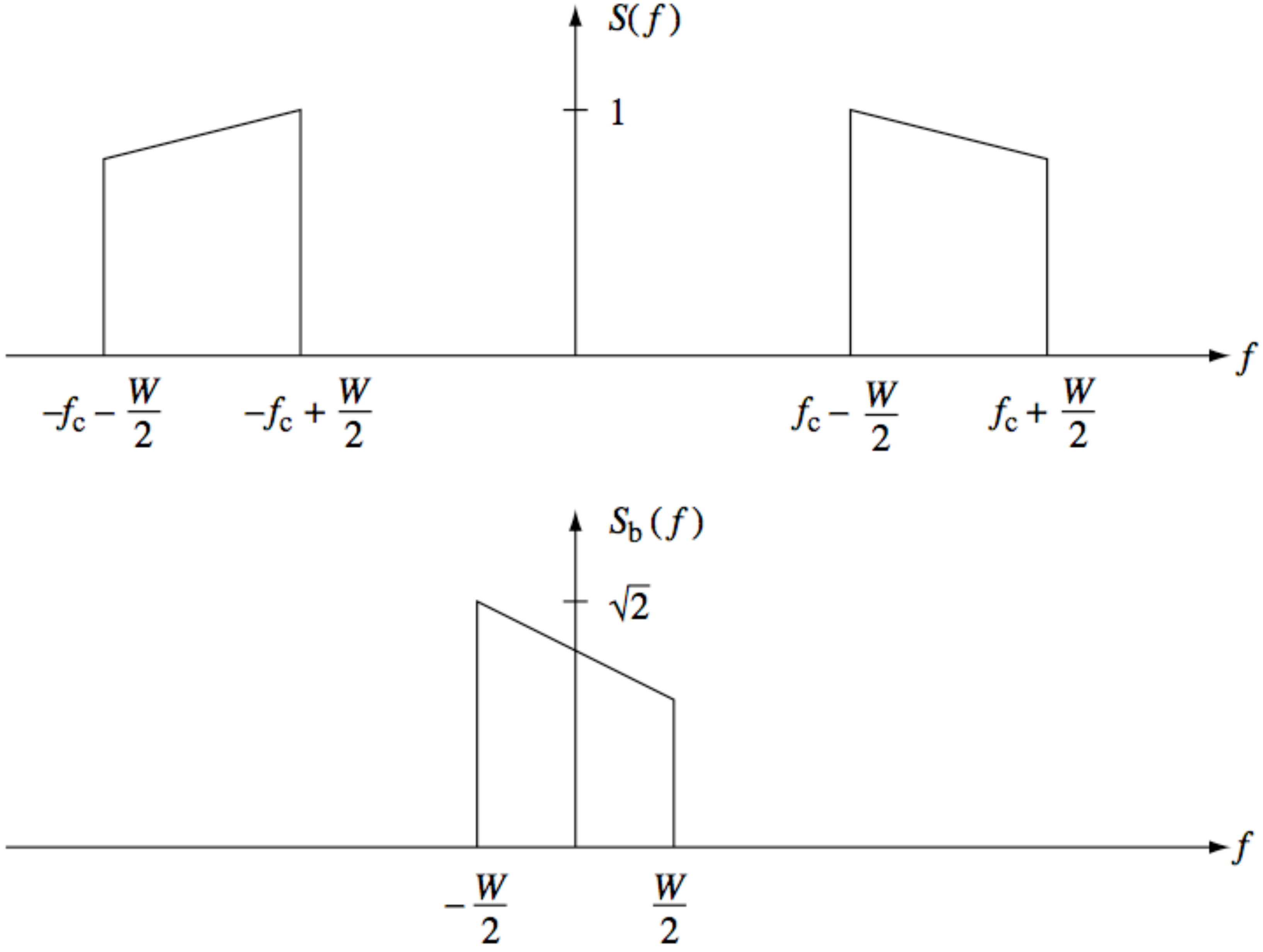
$\tau_i(t) =$ delay of the signal traveling along the i$^{\text{th}}$ path

- Equivalently, we can write the channel using a time-varying impulse response:

$$y(t) = \int_{-\infty}^{\infty} h(\tau, t) x(t - \tau) d\tau + w(t)$$

$$h(\tau, t) = \sum_i a_i(t) \delta\big(\tau - \tau_i(t)\big)$$ Channel impulse response

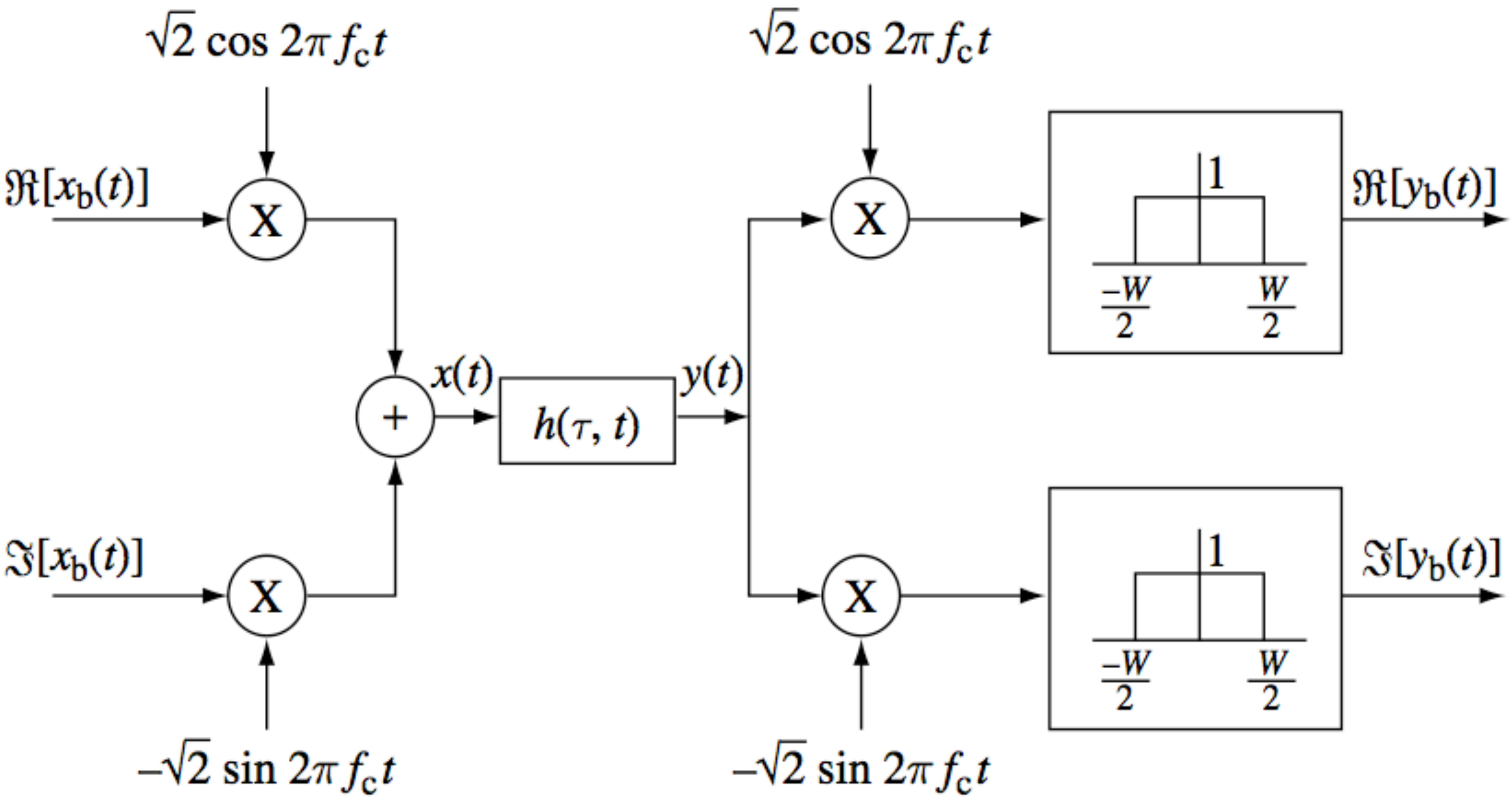
Complex Baseband Equivalent Signals

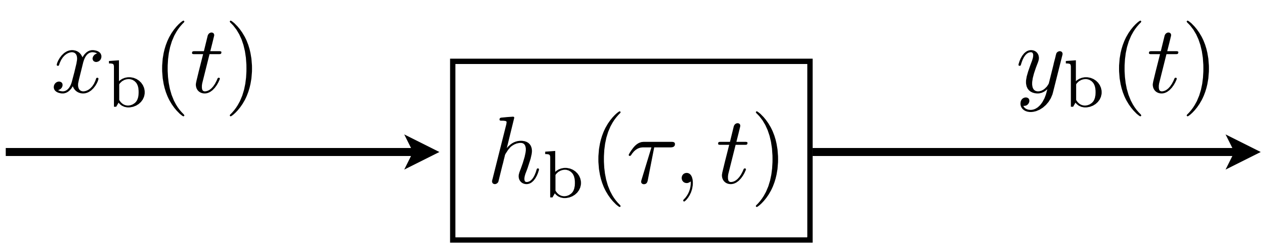


If $s(t)$ is a passband signal bandlimited to $\left[f_\mathrm{c} - \frac{W}{2}, f_\mathrm{c} + \frac{W}{2}\right]$ its complex baseband equivalent is the signal $s_\mathrm{b}(t)$ with Fourier transform

$$S_\mathrm{b}(f) = \begin{cases} \sqrt{2}S(f + f_c) & f + f_c > 0 \; , \\ 0 & f + f_c \leq 0 \; . \end{cases}$$

Quadrature Amplitude Modulation





Baseband Equivalent Channel

Complex Baseband Equivalent Channel Model

$$x_{\mathrm{b}}(t) \longrightarrow \boxed{h_{\mathrm{b}}(\tau, t)} \longrightarrow y_{\mathrm{b}}(t)$$

Baseband Equivalent Channel

- We can write the entire channel model in complex baseband:

$$y_{\mathrm{b}}(t) = \int_{-\infty}^{\infty} h_{\mathrm{b}}(\tau, t) x_{\mathrm{b}}(t - \tau) d\tau + w_{\mathrm{b}}(t)$$

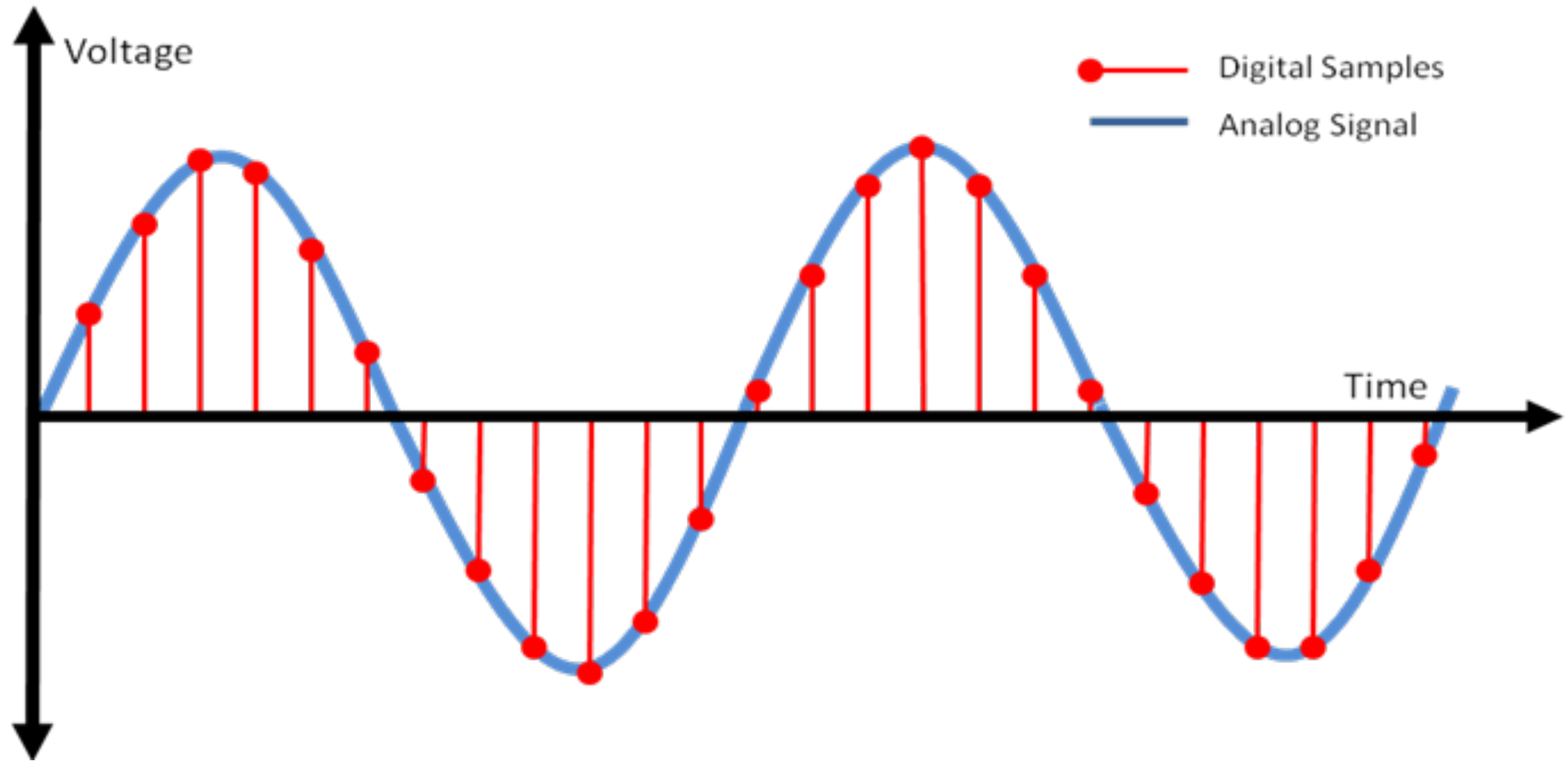
$$h_{\mathrm{b}}(\tau, t) = \sum_i a_i^{(\mathrm{b})}(t) \delta\big(\tau - \tau_i(t)\big) \qquad a_i^{(\mathrm{b})}(t) = a_i(t) e^{-j2\pi f_c \tau_i(t)}$$

Baseband Channel Impulse Response

- Magnitude $\left|a_i^{(\mathrm{b})}(t)\right| = a_i(t)$ changes slowly.

- Phase $2\pi f_{\mathrm{c}} \tau_i(t)$ changes quickly.

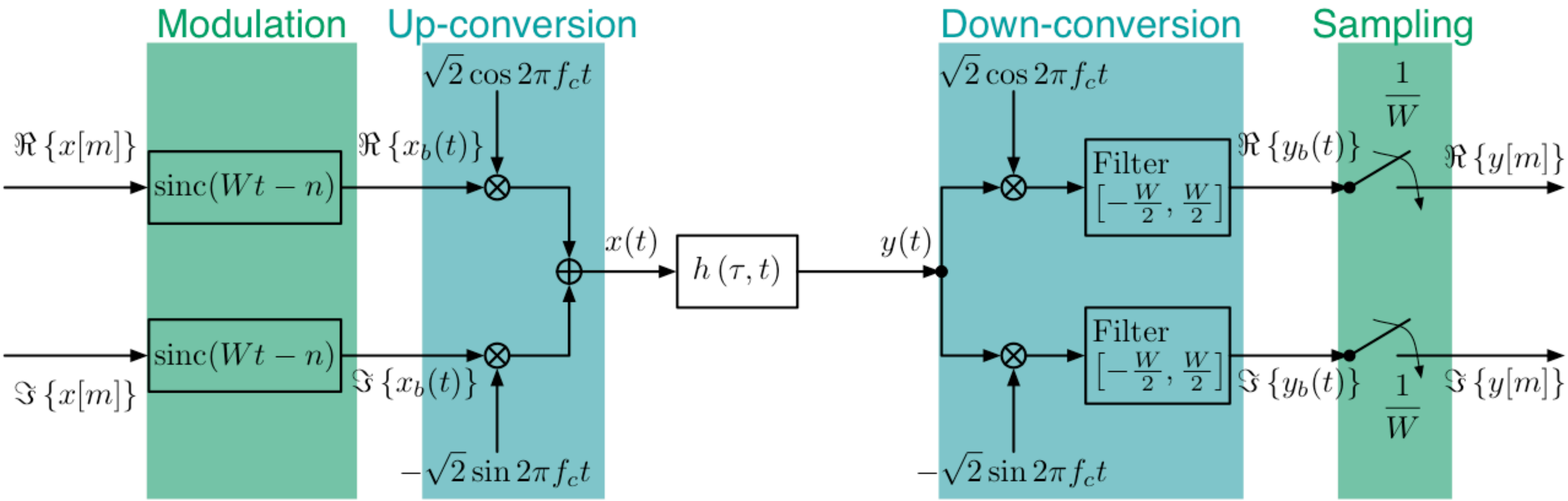
Shannon-Nyquist Sampling Theorem

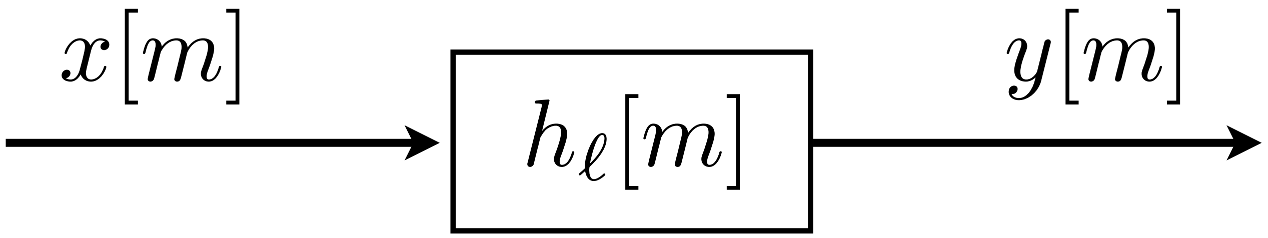


- Any signal $s_{\mathrm{b}}(t)$ that is bandlimited to $\left[-\frac{W}{2}, \frac{W}{2}\right]$ can be perfectly represented by its samples $s[n] = s_{\mathrm{b}}\big(n/W\big)$, $n \in \mathbb{Z}$
- Perfect reconstruction is possible by interpolating the samples with the sinc function:

$$s_{\mathrm{b}}(t) = \sum_n s[n] \operatorname{sinc}(Wt - n) \qquad \operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

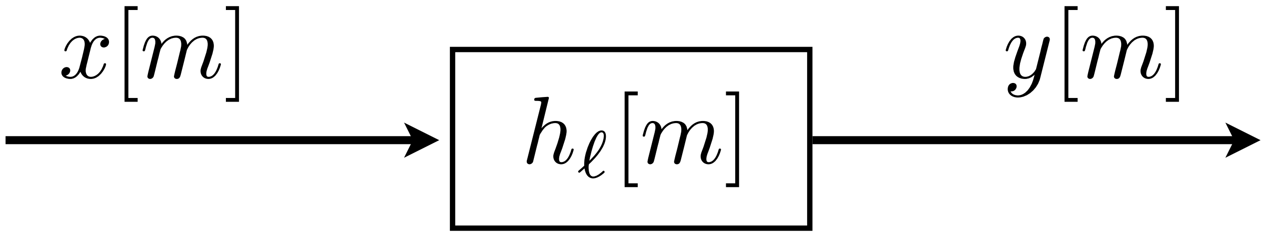
Sampling in a Communication System





Discrete-Time Baseband Equivalent Channel

Discrete-Time Baseband Equivalent Channel Model



Discrete-Time Baseband Equivalent Channel

$$y[m] = \sum_\ell h_\ell[m] x[m-\ell] + w[m]$$

$$y[m] = y_b\left(\frac{m}{W}\right)$$

$$x[m] = x_b\left(\frac{m}{W}\right)$$

$$w[m] = w_b\left(\frac{m}{W}\right)$$

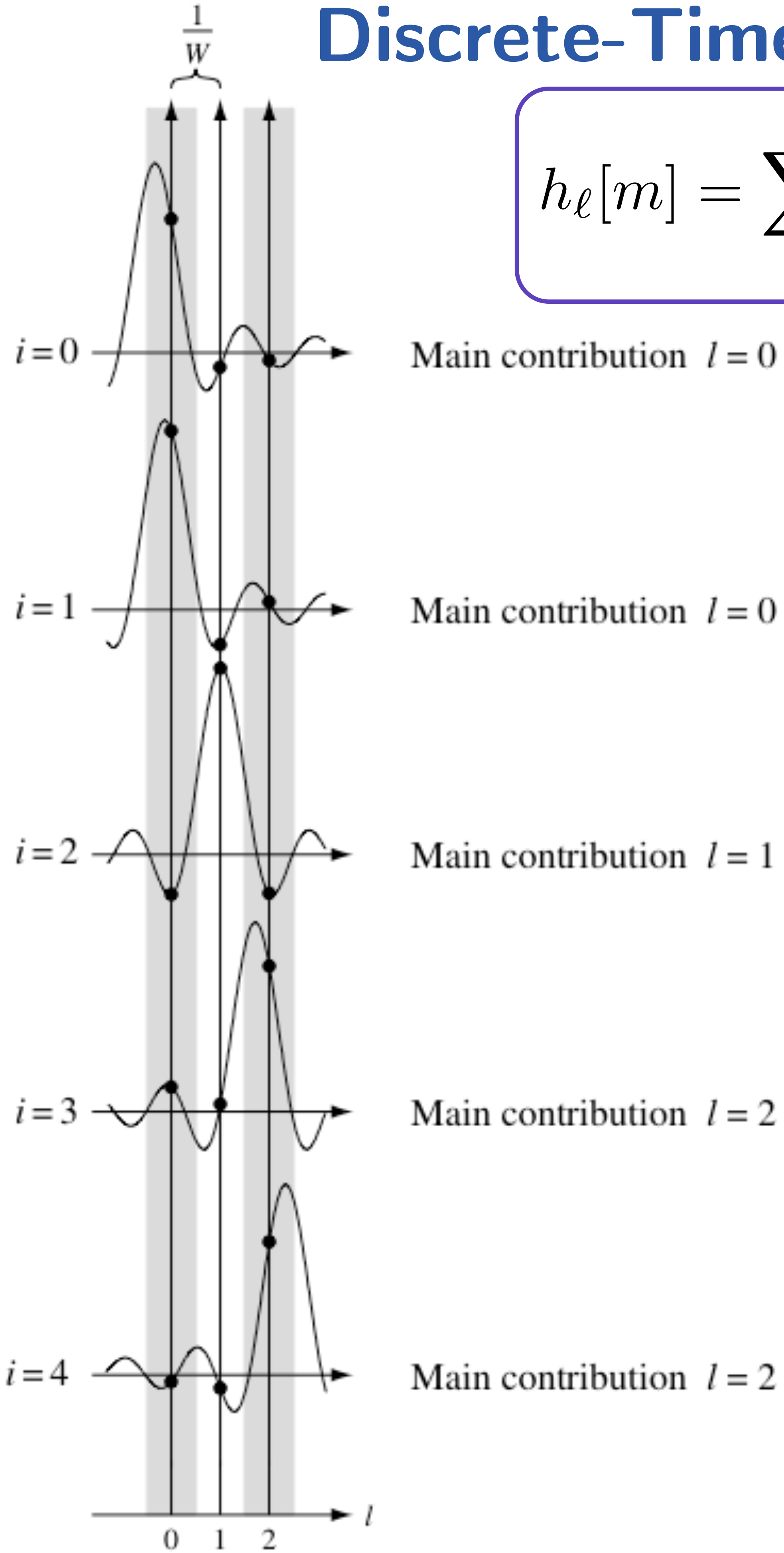
$$h_\ell[m] = \sum_i a_i\left(\frac{m}{W}\right) e^{-j2\pi f_c \tau_i\left(\frac{m}{W}\right)} \operatorname{sinc}\left(\ell - \tau_i\left(\frac{m}{W}\right) W\right)$$

Discrete-Time Baseband Channel Impulse Response

(This is a pretty complicated looking expression. Let's see if we can build up some intuition.)

Discrete-Time Baseband Impulse Response

$$\boxed{h_\ell[m] = \sum_i a_i\left(\frac{m}{W}\right)e^{-j2\pi f_c \tau_i\left(\frac{m}{W}\right)}\operatorname{sinc}\left(\ell - \tau_i\left(\frac{m}{W}\right)W\right)}$$



- sinc(t) is very small for t > 0.5

- The contribution of the i$^{\text{th}}$ path to the impulse response is a scaled, delayed sinc.

- To contribute significantly to the ℓ^{th} tap of the impulse response, the delay of the i$^{\text{th}}$ path must fall inside the interval $\left[\frac{\ell}{W} - \frac{1}{2W}, \frac{\ell}{W} + \frac{1}{2W}\right]$

Discrete-Time Baseband Impulse Response

$$x[m] \longrightarrow \boxed{h_\ell[m]} \longrightarrow y[m]$$

Discrete-Time Baseband Equivalent Channel

$$y[m] = \sum_{\ell} h_\ell[m] x[m-\ell] + w[m]$$

- As we discussed earlier, it is almost impossible to directly predict the impulse response from first principles. We will usually rely on measurements.

- However, we should try to answer **two key questions:**

1. How many taps will the impulse response have?

2. How quickly do these taps vary with time?

How many taps does the impulse response have?

- Remember that the continuous-time impulse response starts at the shortest path delay and ends at the longest path delay.

- The duration of the continuous-time impulse response is called the $\boxed{\text{delay spread } T_{\mathrm{d}} = \max_{i,k} \left| \tau_i(t) - \tau_k(t) \right|}$

- We will see only one tap if the sampling period 1/W exceeds the delay spread T_{d} and more than one tap otherwise.

Delay Spread and Coherence Bandwidth

- Delay for the i[th] path: $\tau_i(t)$
- Delay spread: $T_d = \max_{i,k} \left| \tau_i(t) - \tau_k(t) \right|$
- Coherence bandwidth: $W_c = \dfrac{1}{2T_d}$
- This is the range of frequencies over which the channel is relatively flat.
- Key question: How does the coherence bandwidth compare to the bandwidth used by our communication scheme?

Frequency-Selective Fading	Flat Fading
$W_c \ll W$	$W_c \gg W$
$\implies$ symbol duration $\frac{1}{W} \ll T_d$	$\implies$ symbol duration $\frac{1}{W} \gg T_d$
many taps: $h_0[m], h_1[m], h_2[m], \ldots$	one tap: $h_0[m]$

How quickly do the taps change with (discrete) time?

- Remember that the path attenuations change slowly with time but the phases $2\pi f_c \tau_i(t)$ change quickly.

- These phases determine the time scale over which the impulse response remains relatively constant.

- Each path has an associated Doppler shift. The maximum difference between these Doppler shifts is called the Doppler spread and it determines for roughly how long the channel remains static.

Doppler Spread and Coherence Time

- Doppler shift for the i[th] path: $D_i = f_c \tau_i'(t)$
- Doppler spread: $D_s = \max_{i,k} f_c \left| \tau_i'(t) - \tau_k'(t) \right|$
- Coherence time: $T_c = \dfrac{1}{4D_s}$
- This is the duration of time it takes for the channel to change significantly.
- Key question: How does the coherence time compare to the delay requirement of the application?

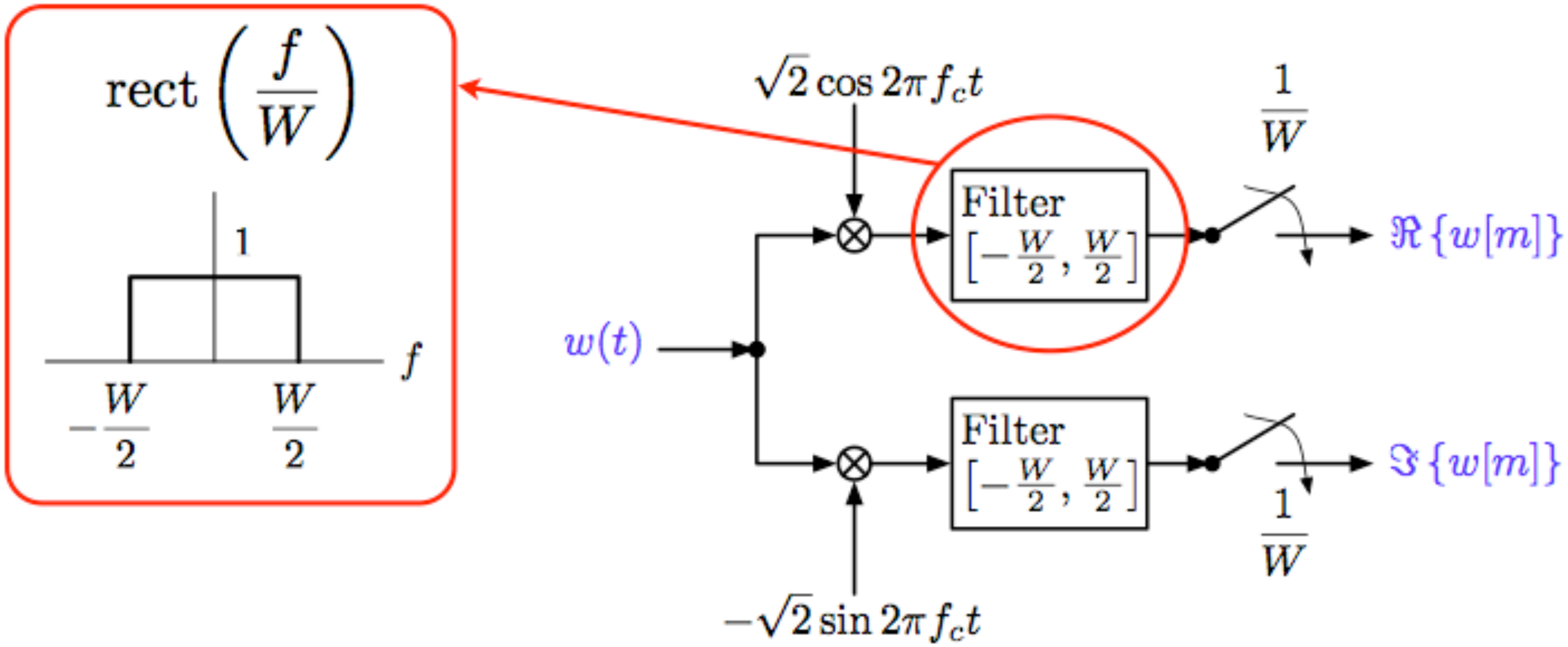
Fast Fading	Slow Fading
$T_c \ll$ delay requirement	$T_c \gg$ delay requirement
$\Longrightarrow$ time-varying channel	$\Longrightarrow$ static channel: $h_\ell[m] \approx h_\ell$

Typical Parameter Values

Key channel parameters and time-scales	Symbol	Representative values
Carrier frequency	f_c	1 GHz
Communication bandwidth	W	1 MHz
Distance between transmitter and receiver	d	1 km
Velocity of mobile	v	64 km/h
Doppler shift for a path	$D = f_c v/c$	50 Hz
Doppler spread of paths corresponding to a tap	D_s	100 Hz
Time-scale for change of path amplitude	d/v	1 minute
Time-scale for change of path phase	$1/(4D)$	5 ms
Time-scale for a path to move over a tap	$c/(vW)$	20 s
Coherence time	$T_c = 1/(4D_s)$	2.5 ms
Delay spread	T_d	1 μs
Coherence bandwidth	$W_c = 1/(2T_d)$	500 kHz

- Good to have a rough sense for these parameters to help with making approximations and back-of-the-envelope calculations.

Gaussian Noise



$$\Re\{w[m]\} = \int_{-\infty}^{\infty} w(t) \underbrace{\left[\sqrt{2}W\cos(2\pi f_c t)\,\mathrm{sinc}\,(Wt-m)\right]}_{\psi_{m,1}(t)} dt$$

$$= \boxed{\langle w(t), \psi_{m,1}(t)\rangle}$$

$$\Im\{w[m]\} = \int_{-\infty}^{\infty} w(t) \underbrace{\left[-\sqrt{2}W\sin(2\pi f_c t)\,\mathrm{sinc}\,(Wt-m)\right]}_{\psi_{m,2}(t)} dt$$

$$= \boxed{\langle w(t), \psi_{m,2}(t)\rangle}$$

Statistical Channel Models

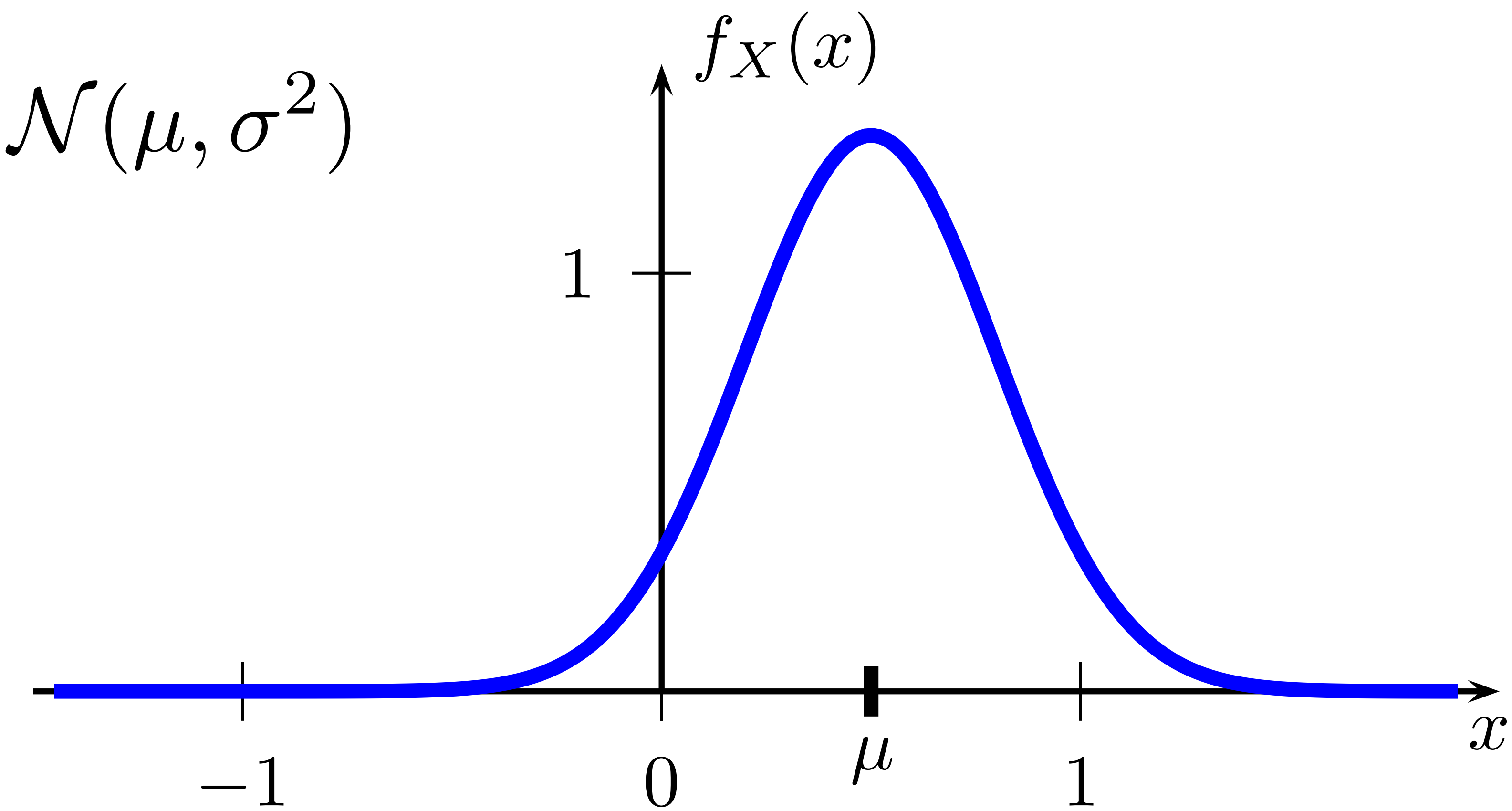
- By now, you should be convinced that it is very hard to model and predict the wireless channel exactly in real-time.
- When we need to know the channel, we will just measure it.
  - Simple example: Transmitter sends a brief pulse of energy (a Delta function). Receiver observes the channel impulse response (in noise).
- It is extremely useful to model the channel statistically.
  - Helps us come up with optimal channel measurement strategies.
  - More important, helps us design and analyze communication schemes.
- Channel characteristics discussed earlier (e.g., coherence time and bandwidth) determine what distribution to use.
- Different distributions for different environments (urban, rural).
- Prefer distributions that are analytically tractable.

Gaussian Distribution

- This is the most important distribution we will deal with in the class.
- Gaussian pdf: $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(- \frac{(x-\mu)^2}{2\sigma^2} \right)$
- Completely specified by two parameters:

  mean: $\mu = \mathbb{E}[X]$ variance: $\sigma^2 = \mathrm{Var}(X)$

- Often written as $X \sim \mathcal{N}(\mu, \sigma^2)$



Central Limit Theorem (CLT)

- Basic Intuition: The sum of many independent random variables will look approximately Gaussian.
- To be precise, consider the Lindeberg-Lévy CLT below:
- Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with mean μ and variance σ^2 . Then, the normalized sum of the random variables converges to a Gaussian random variable in distribution.

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right) \xrightarrow{d} \mathcal{N}(0, \sigma^2)$$

- More general versions of this result are known. (Including independent but not identically distributed, weakly correlated, random vectors, etc.)
- (Understanding the CLT is outside the scope of this class.)

Uniform Phase Assumption

- Recall that the discrete-time baseband impulse response is the result of the sum of the contributions from many paths.

$$h_\ell[m] = \sum_i a_i\left(\frac{m}{W}\right)e^{-j2\pi f_c \tau_i\left(\frac{m}{W}\right)}\operatorname{sinc}\left(\ell - \tau_i\left(\frac{m}{W}\right)W\right)$$

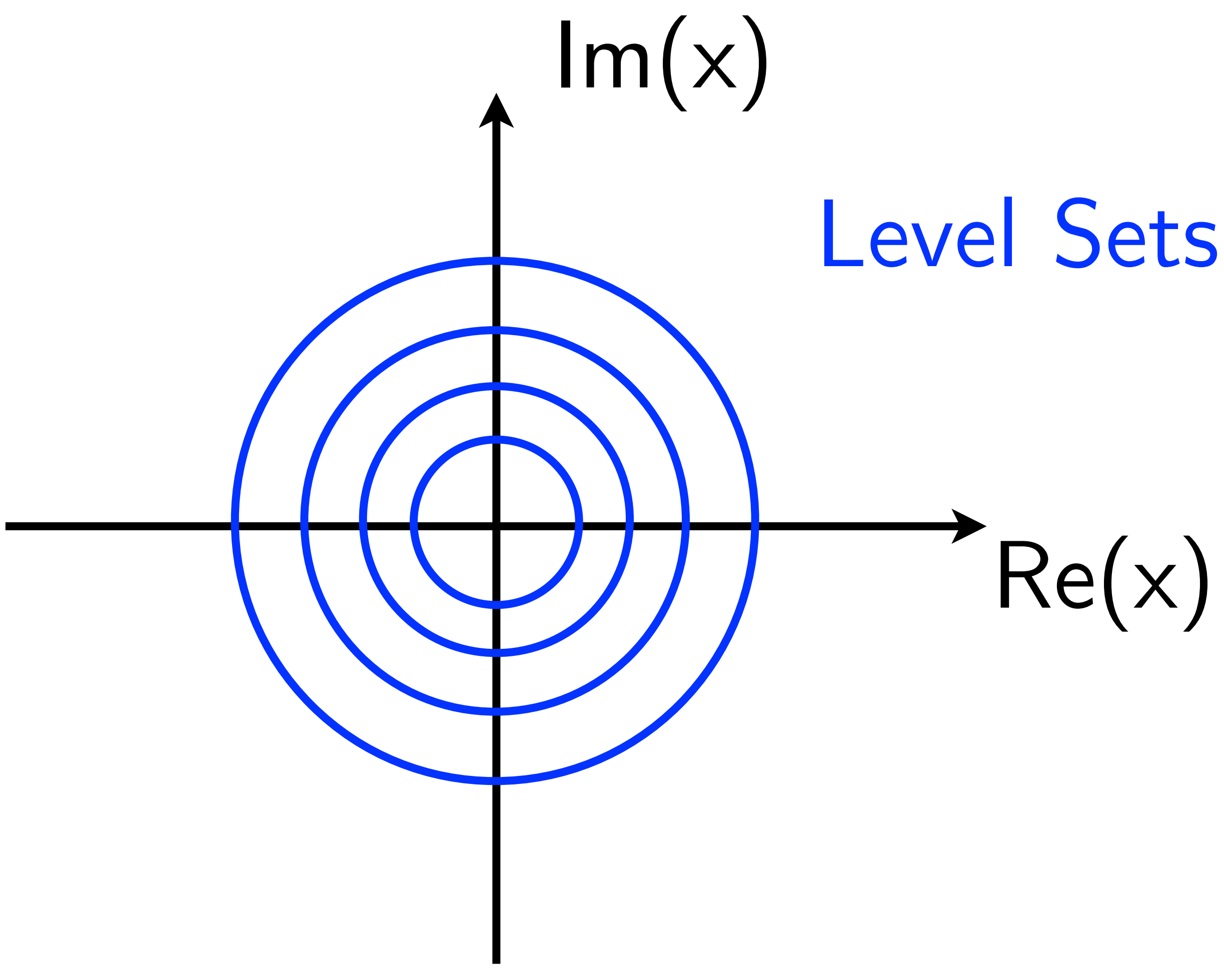
- The phase of the i$^{\text{th}}$ path is $2\pi f_c \tau_i \bmod 2\pi$.
- Note that $f_c \tau_i = \dfrac{d_i}{\lambda_c}$ where d_i is the length of the i$^{\text{th}}$ path.
- The path length is usually much larger than the carrier wavelength: $d_i \gg \lambda_c$ Example: $f_c = 900\text{MHz}$ $\lambda_c = 0.33m$

$$\Longrightarrow \frac{d_i}{\lambda_c} \gg 2\pi$$

- Quite reasonable to model phase as uniform over $[0, 2\pi)$ and independent across paths.

Uniform Phase Assumption

- This means that the contribution of the i^{th} path is a circularly symmetric random variable.
- Definition: X is a circularly symmetric random variable if it has the same distribution as $e^{j\theta}X$ for any phase θ.
- This implies that the mean must be equal to 0.



Rayleigh Fading

- Each channel tap $h_\ell[m]$ is the sum of many independent, circularly symmetric random variables.
- It follows that the real part $\mathrm{Re}\big(h_\ell[m]\big)$ is the sum of many independent, real-valued random variables.
- By the Central Limit Theorem, $\mathrm{Re}\big(h_\ell[m]\big)$ is Gaussian.
- By circular symmetry, $\mathrm{Re}\big(e^{j\theta}h_\ell[m]\big)$ is Gaussian for any θ.
- This implies that $h_\ell[m]$ is a circularly symmetric complex Gaussian random variable.
- A circularly symmetric complex Gaussian random variable w with variance $\sigma^2 = \mathrm{Var}(w) = \mathbb{E}\big[|w|^2\big]$ can be written as the sum of i.i.d. zero-mean real and imaginary Gaussian random variables with variance $\sigma^2/2$.

$$w = w_R + jw_I$$

$$w_R \sim \mathcal{N}(0, \sigma^2/2) \quad w_I \sim \mathcal{N}(0, \sigma^2/2) \qquad w \sim \mathcal{CN}(0, \sigma^2)$$

Rayleigh Fading

- If the $h_\ell[m]$ are circularly symmetric complex Gaussian random variables we say that the channel undergoes Rayleigh fading.
- This is because the magnitude $|w|$ of a circularly symmetric complex Gaussian random variable $w \sim \mathcal{CN}(0, \sigma^2)$ is Rayleigh distributed

$$f_{|w|}(u) = \begin{cases} \frac{u}{\sigma^2} \exp\left(- \frac{u^2}{2\sigma^2} \right) & u \geq 0 \ , \\ 0 & \text{otherwise.} \end{cases}$$

- Also useful to know that that the squared magnitude is exponentially distributed:

$$f_{|w|^2}(u) = \begin{cases} \frac{1}{\sigma^2} \exp\left(- \frac{u^2}{\sigma^2} \right) & u \geq 0 \ , \\ 0 & \text{otherwise.} \end{cases}$$

Tap Gain Auto-Correlation Function

- Key question: How does the channel change with time?
- For random channel models, this time variation can be partially captured using the tap gain auto-correlation function: $$R_\ell[n] = \mathbb{E}\Big[h_\ell^*[m]h_\ell[m+n]\Big]$$
- We assume that $R_\ell[n]$ does not depend on m and that the taps are independent of one another.
- Example: Fast Fading $R_\ell[n] = \sigma_\ell^2\delta[n]$
- Example: Slow Fading $R_\ell[n] = \sigma_\ell^2$

More Sophisticated Fading Distributions

- Rayleigh fading is a very nice distribution for analysis.
- However, we may sometimes want to capture some aspects of the propagation environment more carefully.
- Example: Rician fading includes a strong line-of-sight component. Can write as the sum of a constant term plus a Rayleigh fading term
$$h_\ell[m] = \sqrt{\frac{\kappa}{\kappa+1}}\sigma_\ell e^{j\theta} + \frac{1}{\kappa+1} g$$
$$\kappa = \text{energy ratio between line-of-sight and reflected paths}$$
$$\theta \sim \text{Unif}[0, 2\pi) \qquad g \sim \mathcal{CN}(0, \sigma_\ell^2)$$
- More generally, Nagakami fading has many parameters that can be used to capture various effects and includes Rayleigh and (almost) Rician fading as special cases.