Lecture 7: Detection under Fading

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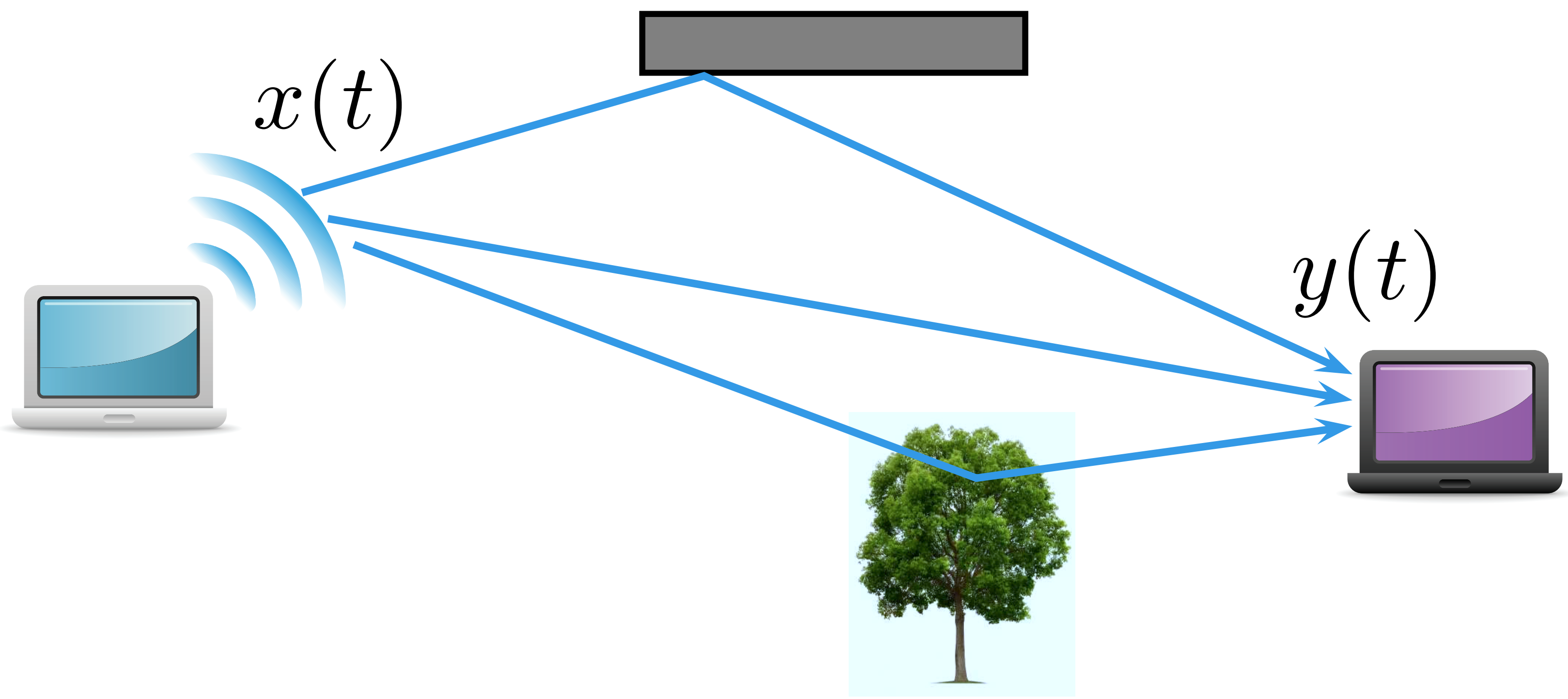
- Recall the continuous-time channel model:

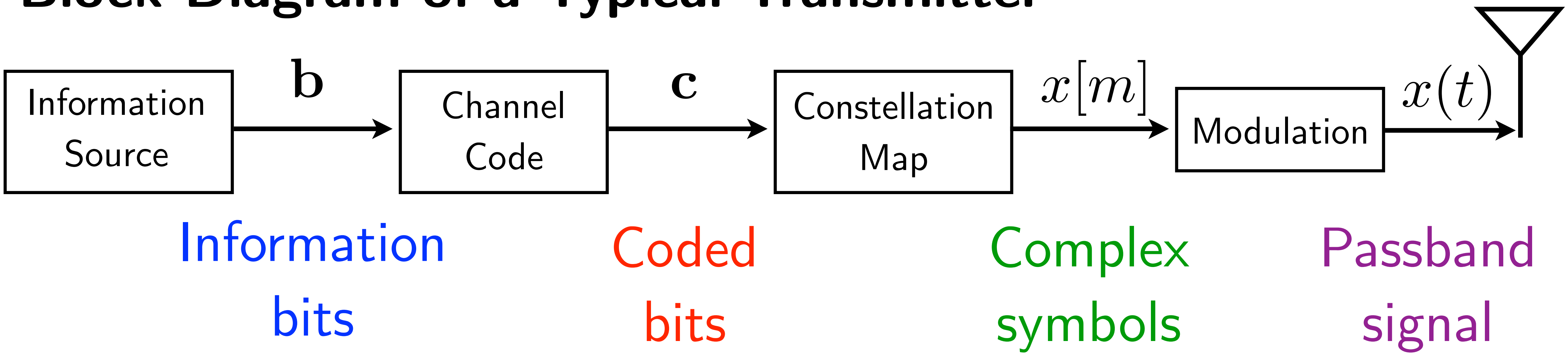
$$y(t) = \int_{-\infty}^{\infty} h(\tau, t) x(t-\tau) d\tau + w(t)$$

$$h(\tau, t) = \sum_i a_i(t) \delta\big(\tau - \tau_i(t)\big) \quad \text{Channel impulse response}$$

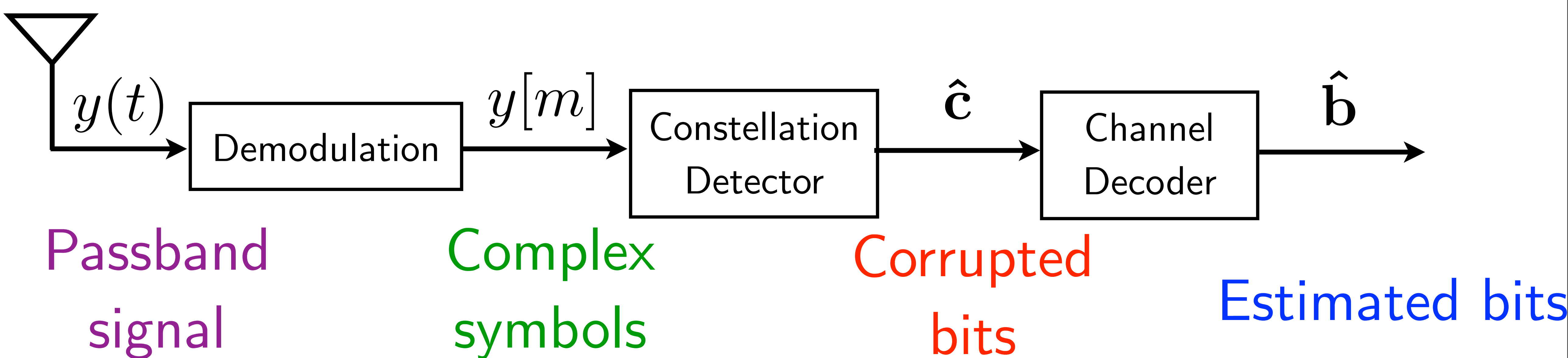


Typical Transmitter and Receiver

Block Diagram of a Typical Transmitter



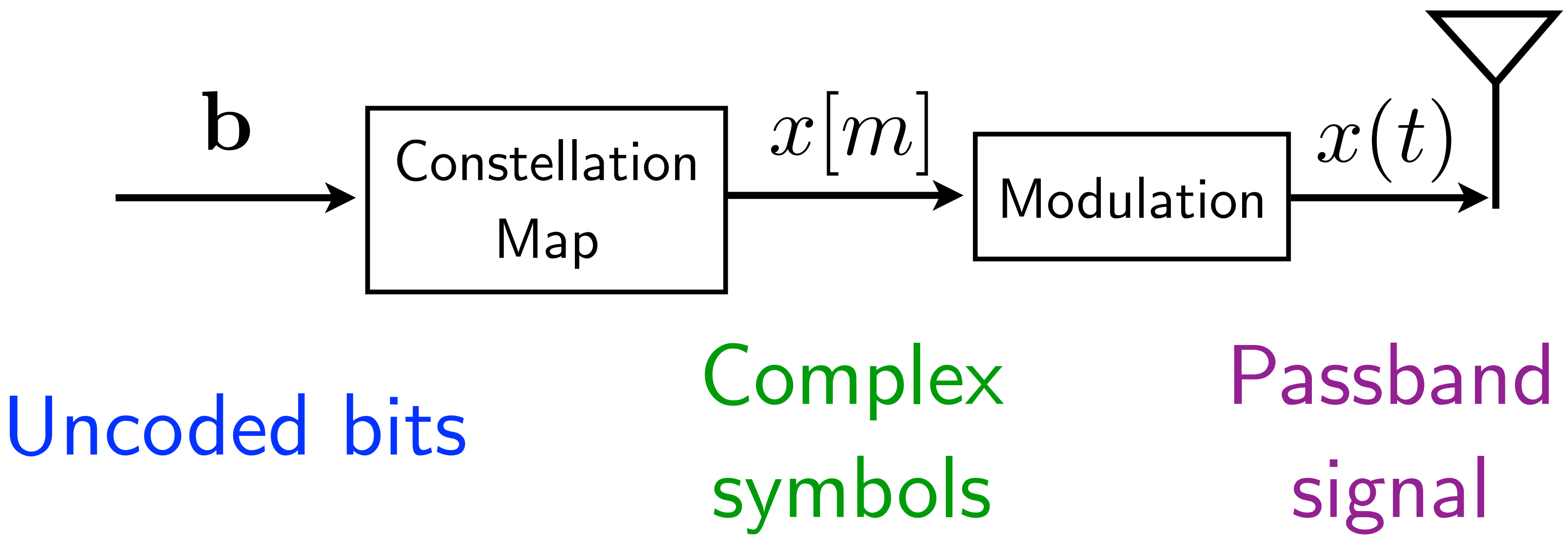
Block Diagram of a (Hard Decision) Receiver



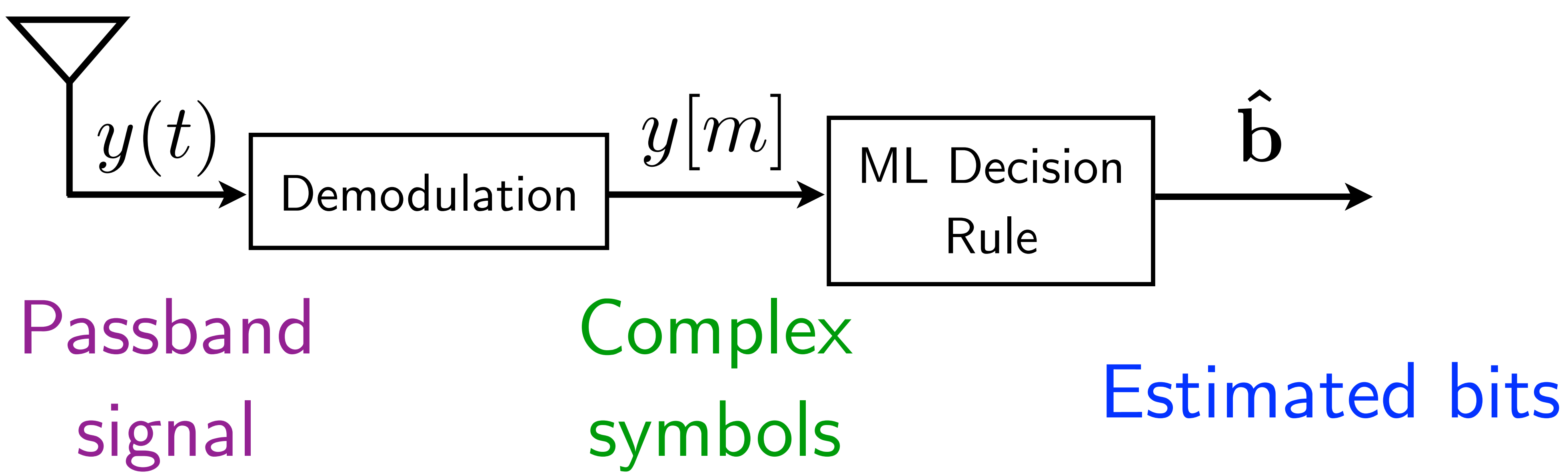
Uncoded Bits and Optimal Detection

- For now, we will ignore the possibility of channel coding:

Transmitter Block Diagram (without channel coding)



Receiver Block Diagram (without channel coding)

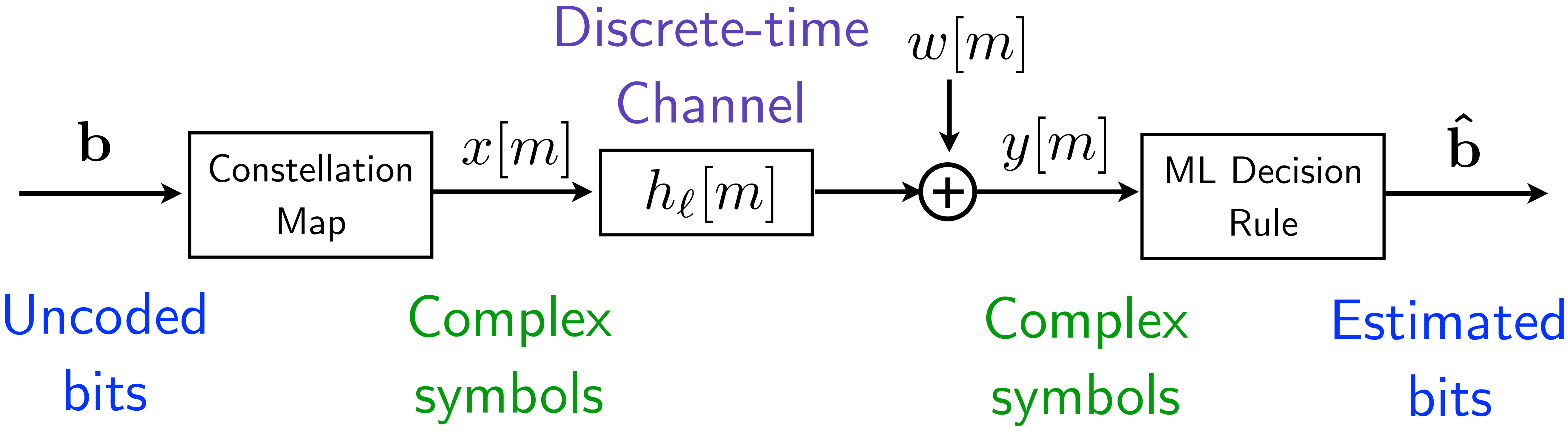


Discrete-Time Equivalent Block Diagram

- Easier to evaluate the performance in discrete time.
- Recall that the discrete-time equivalent channel is:

$$y[m] = \sum_{\ell} h_\ell[m] x[m-\ell] + w[m]$$

Discrete-Time Equivalent Transmitter and Receiver



Flat Fading

- To keep things simple, we will start by considering flat fading channels:
$$y[m] = h[m]x[m] + w[m]$$

- Note that we have dropped the tap index ℓ to simplify the notation.

- We will assume that the noise is i.i.d. across time and is distributed as $w[m] \sim \mathcal{CN}(0, N_0)$

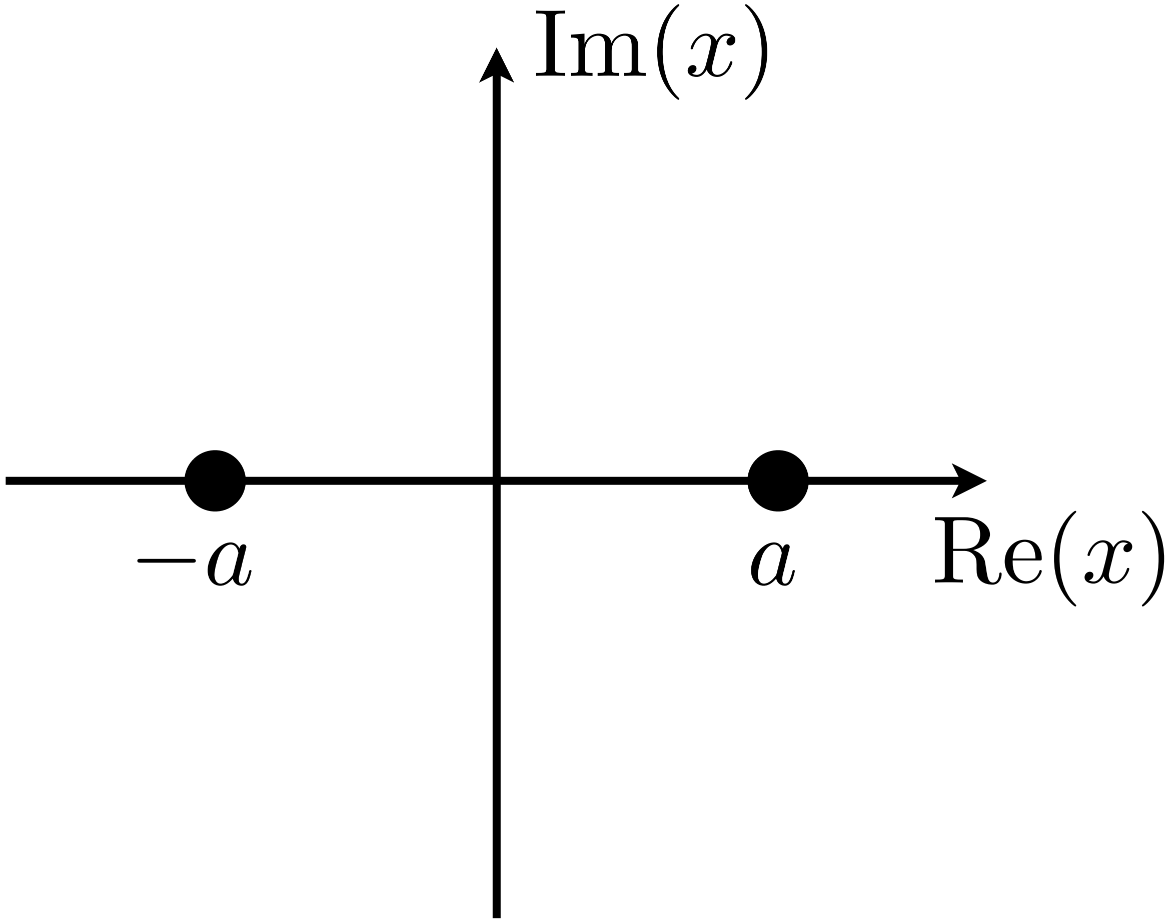
- Let us start with the simplest possible setting: We have only one bit to communicate from the transmitter to the receiver.

AWGN Channels and BPSK

- For the sake of comparison, let us consider the case where there is no fading: $y[m] = x[m] + w[m]$

- To send one bit, we can use a BPSK constellation:

$$x[0] \in \{+a, -a\}$$



- ML Decision Rule:

$$\hat{x}[0] = \begin{cases} a & y[0] > 0 \ , \\ -a & y[0] \leq 0 \ . \end{cases}$$

AWGN Channels and BPSK

- The probability of error is $\mathbb{P}(\hat{x}[0] \neq x[0]) = Q\left(\frac{a}{\sqrt{N_0/2}}\right)$
- It is often useful to express the probability of error (and other quantities) in terms of the signal-to-noise ratio (SNR).
- SNR $= \dfrac{\text{Average Received Signal Energy (per Complex Sample)}}{\text{Average Noise Energy (per Complex Sample)}}$
- For this simple BPSK scenario,

$$\text{Average Received Signal Energy} = \mathbb{E}\left[\left|x[0]\right|^2\right] = a^2$$

$$\text{Average Noise Energy} = \mathbb{E}\left[\left|w[0]\right|^2\right] = N_0$$

$$\boxed{\text{SNR} = \frac{a^2}{N_0}}$$

- The probability of error is $\mathbb{P}(\hat{x}[0] \neq x[0]) = Q\left(\sqrt{2\text{SNR}}\right)$

AWGN Channels and BPSK

- The probability of error is $\mathbb{P}(\hat{x}[0] \neq x[0]) = Q\left(\sqrt{2\mathsf{SNR}}\right)$
- Let's develop some intuition about how fast the probability of error decays with SNR.
- Upper and lower bounds on Q-function:

$$Q(v) < \exp\left(-\frac{v^2}{2}\right) \qquad \text{for } v > 0$$

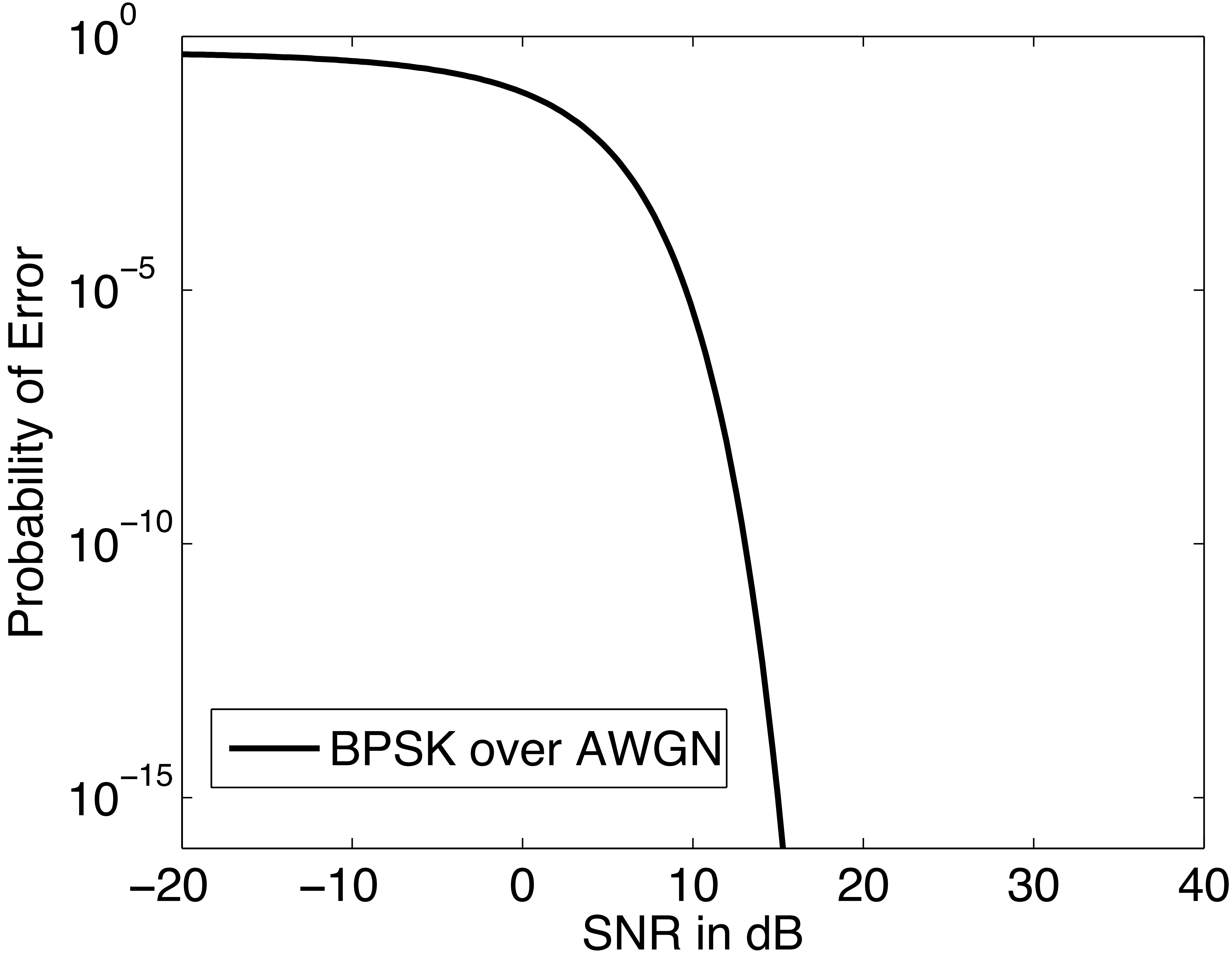
$$Q(v) > \frac{1}{v\sqrt{2\pi}}\left(1 - \frac{1}{v^2}\right)\exp\left(-\frac{v^2}{2}\right) \qquad \text{for } v > 1$$

- Leads to upper bound on the probability of error:

$$\mathbb{P}(\hat{x}[0] \neq x[0]) = Q\left(\sqrt{2\mathsf{SNR}}\right) < \exp\left(-\mathsf{SNR}\right)$$

- This is great! Probability of error falls exponentially fast with the SNR.

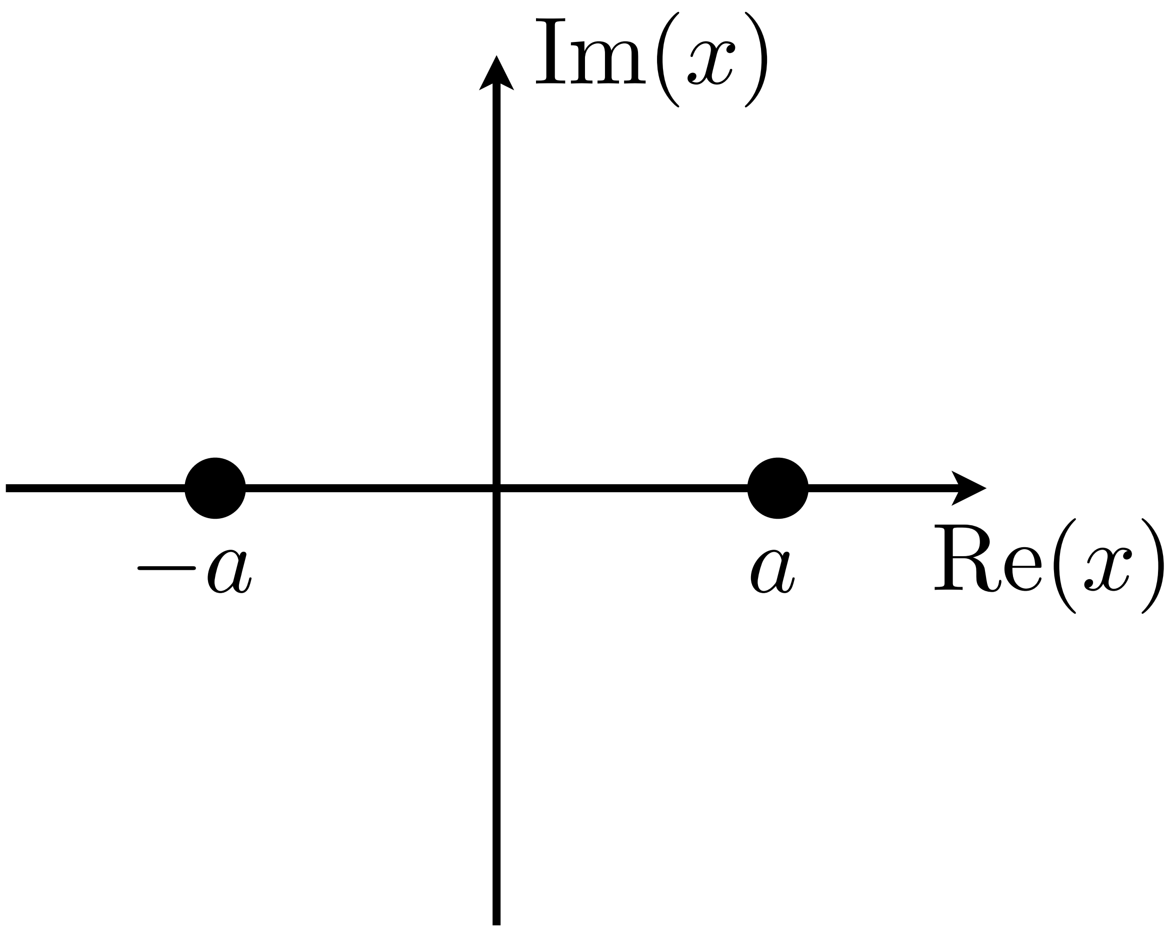
AWGN Channels and BPSK



Noncoherent Rayleigh Fading Channels and BPSK

- Now that we have a solid understanding of how BPSK performs without fading, let's take a careful look at what happens under Rayleigh fading, $y[m] = h[m]x[m] + w[m]$

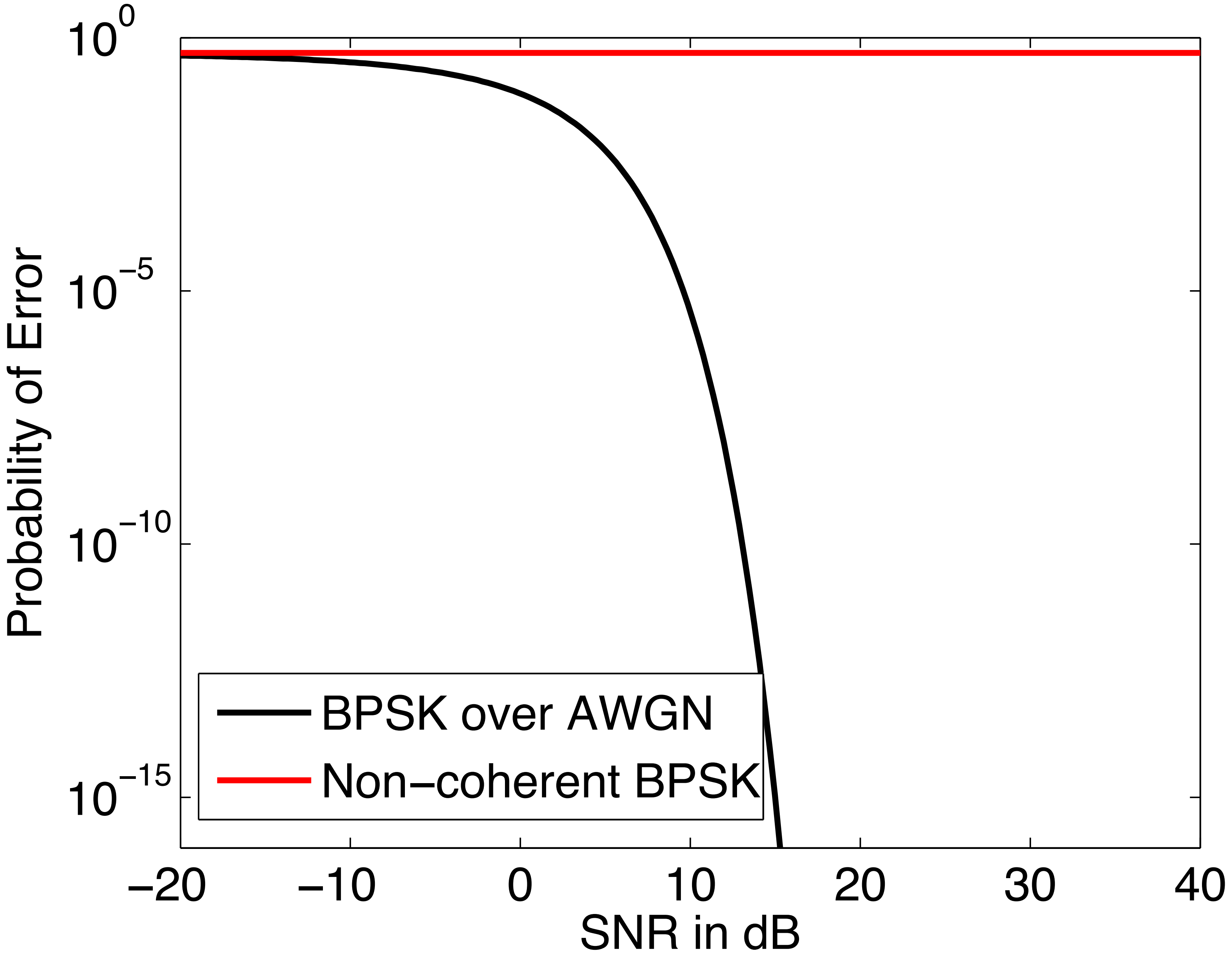
- By Rayleigh fading, we mean that $h[m]$ is i.i.d. across time and distributed as $h[m] \sim \mathcal{CN}(0,1)$



- Let's assume that the channel is non-coherent, meaning that neither the transmitter nor the receiver know the realization of $h[m]$ (but they do know its statistics).

Noncoherent Rayleigh Fading Channels and BPSK

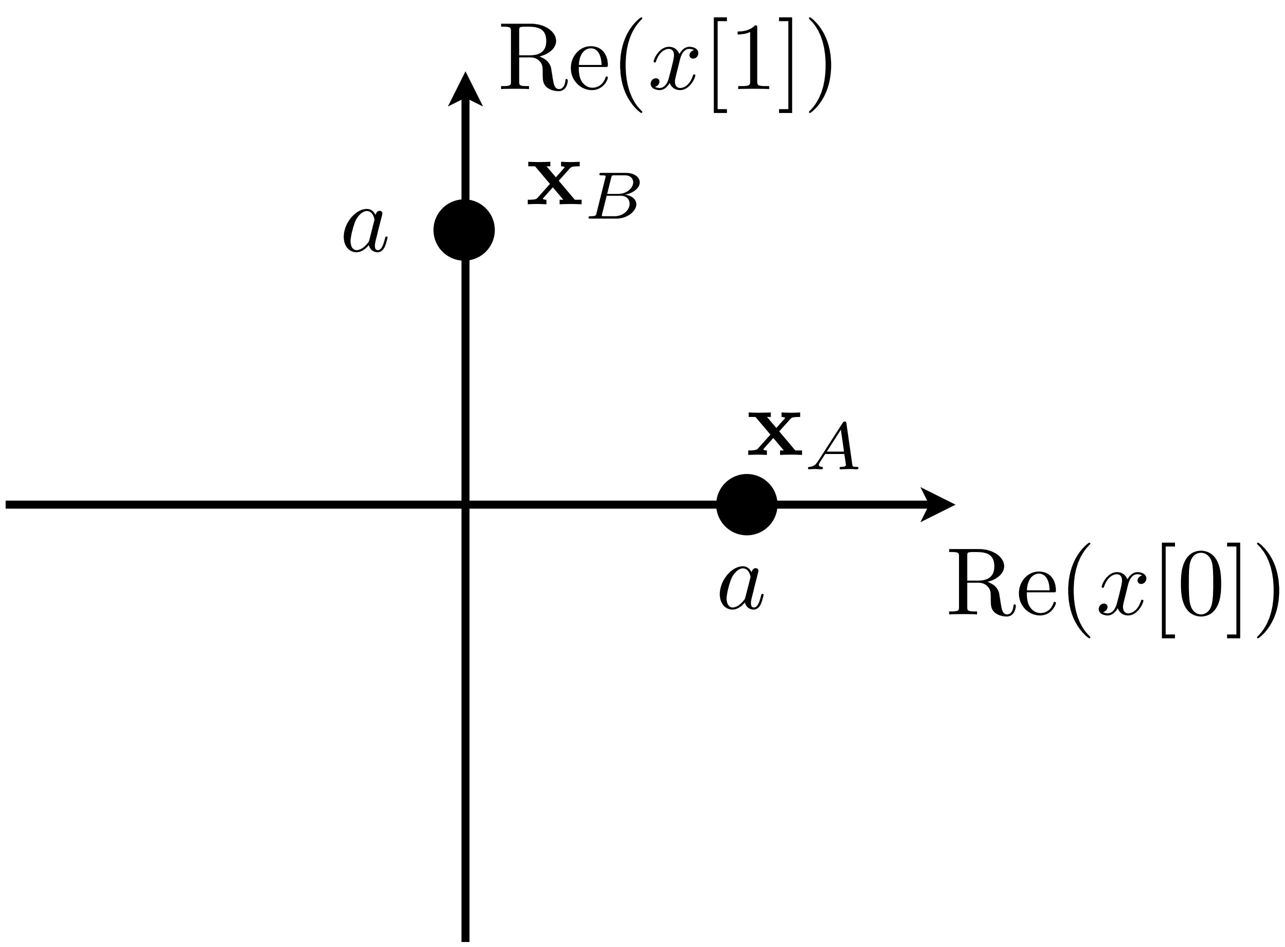
- ML Decision Rule: $\hat{x}[0] = a$
- Probability of Error: $\mathbb{P}\big(\hat{x}[0] \neq x[0]\big) = \dfrac{1}{2}$



Noncoherent Rayleigh Fading and Orthogonal Signaling

- Since the channel is non-coherent, we cannot encode information in the phase.
- Consider the following orthogonal signaling scheme over two time slots.

$\mathbf{x} = \begin{pmatrix} x[0] \\ x[1] \end{pmatrix}$ is equally likely to be $\mathbf{x}_A = \begin{pmatrix} a \\ 0 \end{pmatrix}$ or $\mathbf{x}_B = \begin{pmatrix} 0 \\ a \end{pmatrix}$



Noncoherent Rayleigh Fading and Orthogonal Signaling

- ML Decision Rule: $\hat{\mathbf{x}} = \begin{cases} \mathbf{x}_A & \left|y[0]\right|^2 \geq \left|y[1]\right|^2 , \\ \mathbf{x}_B & \left|y[0]\right|^2 < \left|y[1]\right|^2 . \end{cases}$

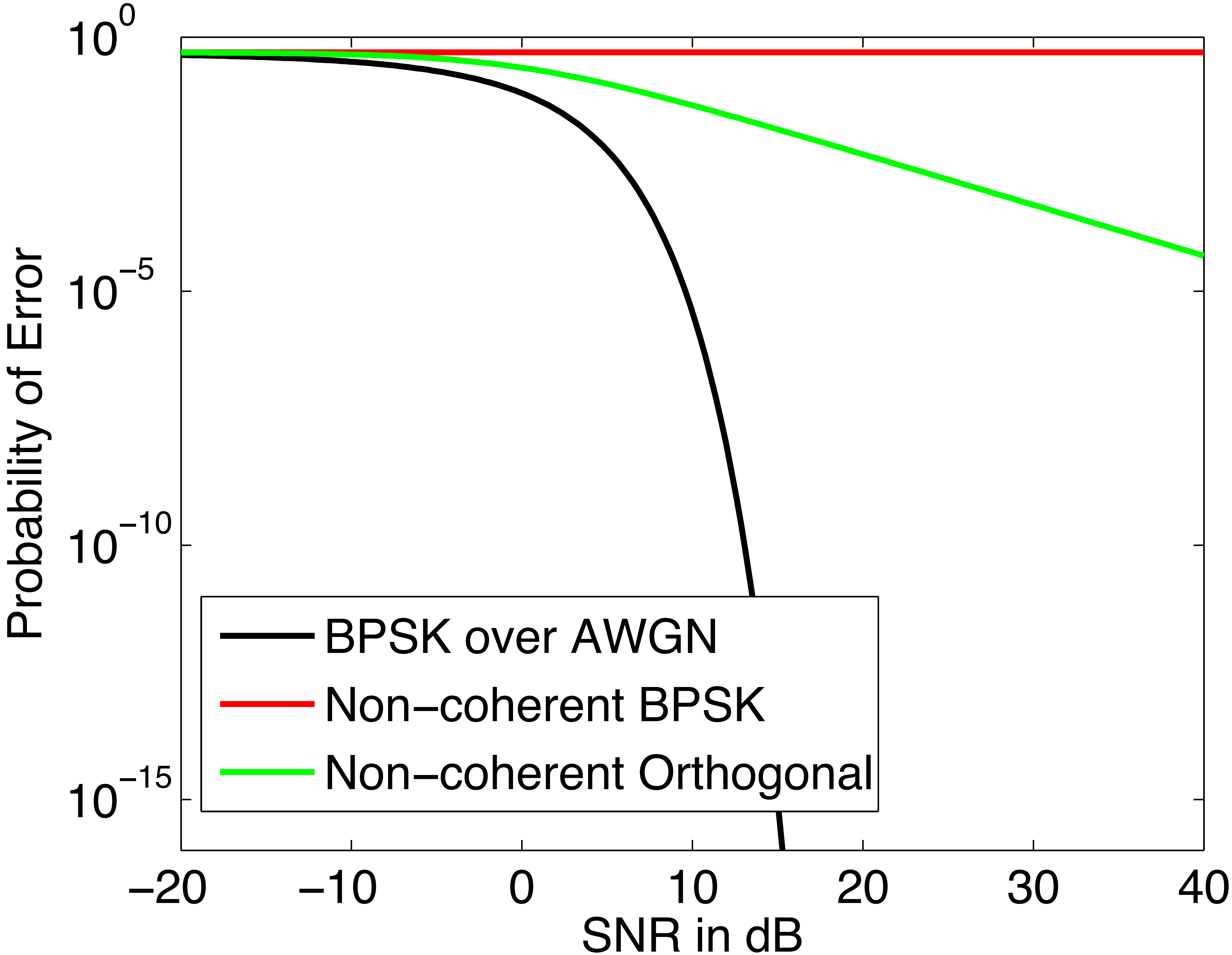
Average Received Signal Energy $= \mathbb{E}\left[\left|x[0]\right|^2\right] = \dfrac{a^2}{2}$

Average Noise Energy $= \mathbb{E}\left[\left|w[0]\right|^2\right] = N_0$

$$\boxed{\text{SNR} = \frac{a^2}{2N_0}}$$

- The probability of error is $\mathbb{P}\left(\hat{\mathbf{x}} \neq \mathbf{x}\right) = \dfrac{1}{2 + \frac{a^2}{N_0}} = \dfrac{1}{2(1 + \text{SNR})}$
- Overall, the error decays like 1/SNR rather than exp(-SNR)!
- Maybe this is because we have assumed the channel is non-coherent...

Noncoherent Rayleigh Fading and Orthogonal Signaling

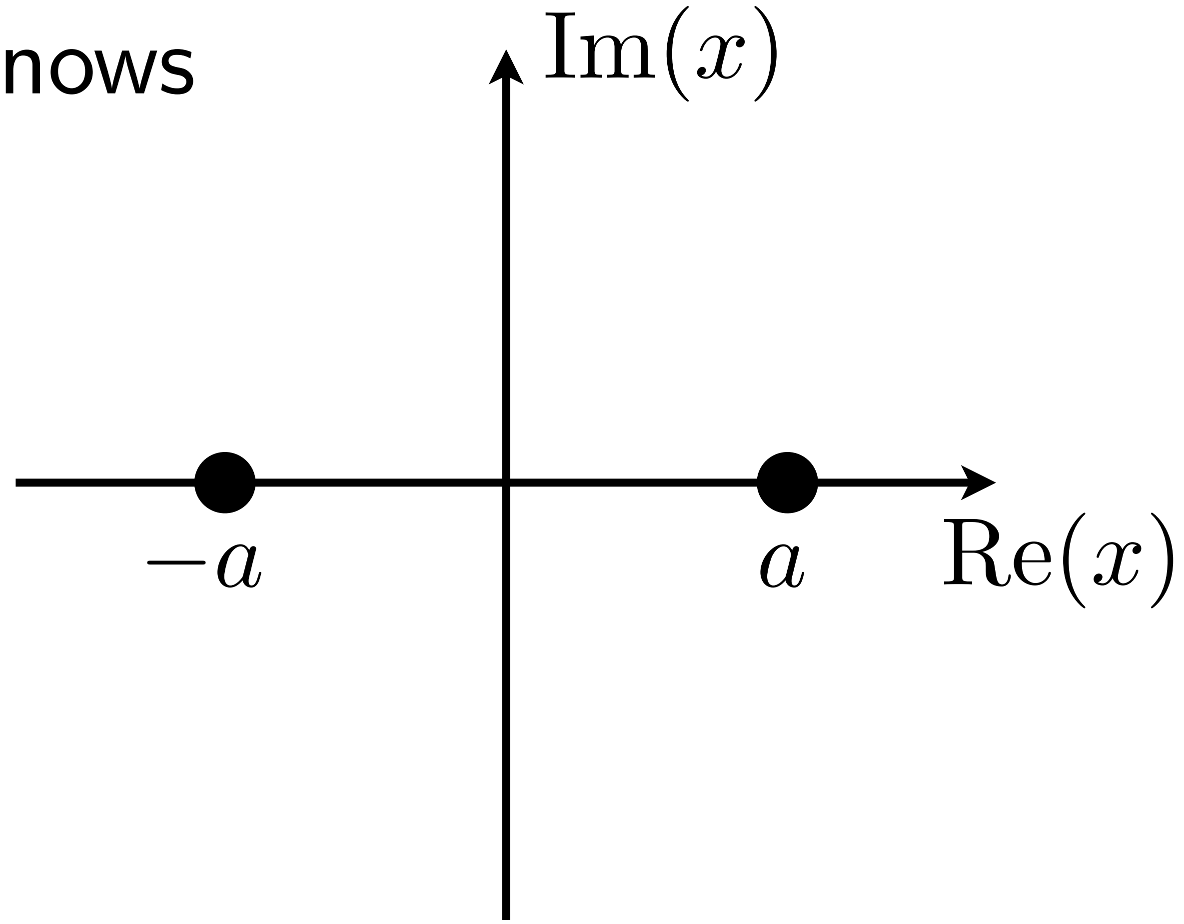


Coherent Rayleigh Fading Channels and BPSK

- Let's go back to BPSK and now assume that we have a coherent Rayleigh fading channel

$$y[m] = h[m]x[m] + w[m]$$

- By coherent, we mean that the receiver knows $h[m]$ perfectly while the transmitter does not (but it still knows the channel statistics).



Coherent Rayleigh Fading Channels and BPSK

- ML Decision Rule: $\hat{x}[0] = \begin{cases} a & \frac{h^*[0]}{|h[0]|} y[0] \geq 0 \\ -a & \frac{h^*[0]}{|h[0]|} y[0] < 0 \end{cases}$

Average Received Signal Energy $= \mathbb{E}\left[|x[0]|^2\right] = a^2$

Average Noise Energy $= \mathbb{E}\left[|w[0]|^2\right] = N_0$

$$\boxed{\text{SNR} = \frac{a^2}{N_0}}$$

- The probability of error is

$$\mathbb{P}\left(\hat{x}[0] \neq x[0]\right) = \frac{1}{2}\left(1 - \sqrt{\frac{\text{SNR}}{1 + \text{SNR}}}\right) \approx \frac{1}{4\text{SNR}}$$

- The error still decays like 1/SNR rather than exp(-SNR)!

- So whether or not we know the channel at the receiver, the error scaling is fundamentally different with fading.

Coherent Rayleigh Fading Channels and BPSK