Summer 2012 Research Grant Proposal: The Logarithmic Sobolev Inequality

1 Introduction

I took a graduate level course in mathematical analysis (Math 410-1) with Professor Elton Hsu this fall. I found the material in this class fascinating, and began doing independent work under Professor Hsu's guidance during the quarter. I would like to extend this work into a full-time project this summer. The topic of my project is the logarithmic Sobolev inequality, an inequality for a certain type of integral which has numerous applications in both pure and applied mathematics—including partial differential equations, mathematical statistics, and quantum mechanics. In the course of my research, I will attempt to develop a new proof of this inequality, based on a special integration by parts formula. Such a proof will yield a hitherto unknown approach to the study of the logarithmic Sobolev inequality, as well as that of a family of related inequalities. It may thereby lead to advances in the areas of math and science where these inequalities are applied. Moreover, my project will give me additional experience doing research in analysis, a field in which I am considering specializing in my future career.

2 Background and Literature Review

A *Borel measure* μ on the real numbers $\mathbb{R}$ is a means of assigning a size $\mu(E)$ to subsets E of $\mathbb{R}$. The most familiar measure is Lebesgue measure, which assigns to each interval $[a, b]$ a size equal to its length, $b - a$. In probability theory and mathematical statisics, the most important measure is the standard *Gaussian Measure*,

$$\mu(E) = \frac{1}{\sqrt{2\pi}} \int_E e^{-x^2/2} dx$$

This is a probability measure, in the sense that $\mu(\mathbb{R}) = 1$[1]. So, we can think of $\mu(E)$ as representing the probability that a random variable lies in the set E.

We can also compute the integral of a real-valued function f with respect to a measure μ. We denote such an integral by replacing dx by $d\mu(x)$. If μ is a probability measure, the integral of a function with respect to μ can be interpreted as its expected value.

In my research, I will study the *logarithmic Sobolev inequality*: if μ is the standard Gaussian measure, f is a real valued function, $|f|^2$ is integrable with respect to μ, and the generalized derivative f' of f exists, then

$$\int_{\mathbb{R}} |f(x)| \log |f(x)| d\mu(x) \leq \frac{1}{2} \int_{\mathbb{R}} |f'(x)|^2 d\mu(x) + \int_{\mathbb{R}} |f(x)|^2 d\mu(x) \log \left(\int_{\mathbb{R}} |f(x)|^2 d\mu(x) \right)^{1/2}.$$

This inequality is important in many areas of mathematics and physics such as quantum mechanics, statistical mechanics, infinite dimensional analysis, probability theory, mathematical statistics, and partial differential equations. Indeed, the logarithmic Sobolev inequality can be viewed as a sharpened form of Heisenberg's uncertainty principle [2]. It is also used to obtain bounds for the solutions of partial differential equations and to characterize the behavior of stochastic processes like Brownian motion on manifolds [7]. The inequality has several known proofs, for example those in [1] and [3].

A related but simpler inequality is the *Poincaré inequality*: under the same hypotheses as the logarithmic Sobolev inequality,

$$\int_{\mathbb{R}} |f|^2 d\mu(x) - \left(\int_{\mathbb{R}} f d\mu(x) \right)^2 \leq \int_{\mathbb{R}} |f'|^2 d\mu(x).$$

In probabilistic terms, the variance of f is no greater than the expectation of the square of f'. The Poincaré inequality is weaker than the logarithmic inequality stated above. However, in [2], the author exhibits a family of inequalities which

[1] In fact, it is the probability measure corresponding to the normal distribution.

interpolate smoothly between the Poincaré inequality and the logarithmic Sobolev inequality. That is, the inequalities depend smoothly on a parameter p, and reduce to the Poincaré inequality and the logarithmic Sobolev inequality in the cases where $p = 1$ and as $p \to 2$, respectively.

The standard Gaussian measure satisfies a special integration by parts formula:

$$\int_{\mathbb{R}} f'(x) d\mu(x) = \int_{\mathbb{R}} x f(x) d\mu(x)$$

for every differentiable function f on $\mathbb{R}$. In fact, this formula uniquely characterizes Gaussian measure, in the sense that no other probability measure satisfies it. So, it is natural to ask whether other properties of Gaussian measure can be deduced directly from this formula. Recently, my advisor, Professor Hsu, discovered a proof of the Poincaré inequality based on the integration by parts formula. We believe that it is possible to prove the intermediate inequalities in [2], and thereby the logarithmic Sobolev inequality, directly from the formula as well.

3 Research and Methodology

The goal of my project is to devise a new proof of the inequalities in [2], including the logarithmic Sobolev inequality. My intended method of proof is motivated by Professor Hsu's proof of the Poincaré inequality; that is, I will try to deduce the inequalities from the integration by parts formula for Gaussian measure.

If this can be done, then Professor Hsu and I will have found a new approach to the study of this important class of inequalities. This research will yield a fuller understanding of the logarithmic Sobolev inequality and the inequalities in [2], which may, in turn, further the development of the areas of mathematics and science—such as statistics and quantum mechanics—to which they have been applied. Moreover, it is likely that the techniques of my intended proof can be generalized to prove similar inequalities for related classes of functions, for example in higher dimensions or for other measures.

I will officially commence work on my project at the beginning of the summer. However, Professor Hsu will be out of the country for the earlier portion of the summer, so we will collaborate by email during this period. I plan to carry out the bulk of my research during the final eight weeks of the summer, beginning in late July. During this period I will work full time on the project: thinking of strategies to approach the inequality, testing out mathematical ideas, discussing results with other mathematicians, and formulating proofs. By the end of the summer, I hope to have a paper ready for submission to a research journal. If I do not succeed in my attempts at a proof during the summer, I will still be able to write a summary paper on the theory and history behind the logarithmic Sobolev inequality, along with any new insights I gain during my work. I may also continue my research next year as part of a senior thesis. The research grant will pay for room and board during the summer, and perhaps for travel to a research conference to present my results.

4 Preparation

I completed two mathematical research projects of similar scope at a Research Experience for Undergraduates program (REU) at Indiana University in the summer of 2011. Both resulted in publications ([5] and [6]). The REU acquainted me with the general process of conducting and publishing mathematical research. I enjoyed my time there immensely, and it solidified my desire to pursue further research and an eventual career in mathematics.

By the end of the school year, I will already have most of the background knowledge necessary for my proposed project. I am taking a year long graduate level course in analysis (the Math 410 sequence) which covers measure theory and analysis on function spaces, both of which will be essential for my research. I am also studying some topics directly relevant to the project independently, including advanced material about function spaces and Gaussian measure, proofs of related inequalities, and known proofs of the logarithmic Sobolev inequality. This preparation will enable me to do a substantial amount of work on my own during the initial part of the summer, and to commence immediately with full time research when Professor Hsu returns.

5 Conclusion

I intend to complete an honors thesis in mathematics next year. This thesis will probably relate to my proposed project: I may extend my research, or investigate a mathematical question which I encounter during my investigations this summer. After I graduate, I plan to enroll in graduate school to pursue a Ph.D in mathematics, and then go on to a research and teaching career. I am considering specializing in analysis, so this project will be particularly beneficial in preparing me for my future work. While studying the logarithmic Sobolev inequality, I will learn more about topics in analysis such as advanced measure theory and operators on function spaces, and familiarize myself with an area of active research which I hope to explore further as a professional mathematician.

References

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