Finding large inscribed ellipsoid

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1 Problem statement

Given many points $v_1, \ldots, v_m \in \mathbb{R}^n$, we want to find a large ellipsoid in the convex hull $ConvexHull(v_1, \ldots, v_m)$, and the ellipsoid doesn't contain any of the point $v_1, \ldots, v_m$.

2 Approach

We consider to parameterize the ellipsoid as

$$\mathcal{E} = \{x | x^T P x + 2q^T x < r\} \tag{1}$$

The unknowns are P, q, r.

The condition that the ellipsoid doesn't contain any of the point $v_1, \ldots, v_m$ is the following linear constraint on P, q, r

$$v_i \notin \mathcal{E} \Leftrightarrow v_i^T P v_i + 2q^T v_i \geq r \tag{2}$$

If we write the convex hull of the points $v_1, \ldots, v_m$ as the polytope

$$ConvexHull(v_1, \ldots, v_m) = \{x | Cx \leq d\} \tag{3}$$

Then using s-lemma [1], we know that the ellipsoid is within this convex hull if and only if

$$\mathcal{E} \subset ConvexHull(v_1, \ldots, v_m) \tag{4}$$

$$\Leftrightarrow \exists \lambda_i \geq 0, \text{s.t} \begin{bmatrix} P & q - \frac{1}{2}\lambda_i c_i \\ (q - \frac{1}{2}\lambda_i c_i)^T & \lambda_i d_i - r \end{bmatrix} \succeq 0 \tag{5}$$

where c_i, d_i is the i'th row of C, d respectively. $\succeq 0$ means the matrix on the left hand side is positive semidefinite.

To guarantee that the quadratic function $x^T P x + 2q^T x \leq r$ actually describes an ellipsoid (not other shapes like a hyperbola), we require the following two conditions

$$P \succeq 0 \tag{6a}$$

$$\exists z, s.t\ z^T P z + 2q^T z \leq r \tag{6b}$$

Namely (6b) demands that there exists a point z such that $z \in \mathcal{E}$.

The volume of the ellipsoid is proportional to this quantity

$$\text{volume}(\mathcal{E}) \propto \frac{(r + q^T P^{-1} q)^{n/2}}{\sqrt{\det(P)}} \tag{7}$$

where n is the dimensionality of the geometries.

To maximize the ellipsoid volume, we can maximize the logarithm of this function

$$\max\ n\log(r+q^TP^{-1}q)-\log\det(P) \tag{8}$$

So finding the largest inscribed ellipsoid can be formulated as the following optimization problem

$$\max_{P,q,r,\lambda}\ n\log(r+q^TP^{-1}q)-\log det(P) \tag{9a}$$

$$\text{s.t } v_i^TPv_i+2q^Tv_i\geq r,\ \forall i=1,...,m \tag{9b}$$

$$\lambda\geq 0 \tag{9c}$$

$$\begin{bmatrix} P & q-\frac{1}{2}\lambda_ic_i \\ (q-\frac{1}{2}\lambda_ic_i)^T & \lambda_id_i-r\end{bmatrix}\succeq 0 \tag{9d}$$

$$P\succeq 0 \tag{9e}$$

$$z^TPz+2q^Tz\leq r \tag{9f}$$

we will specify and fix the point z in this program. As you can see, all of the constraints are convex constraints. The only nonconvexity is in the objective function.

2.1 maximizing volume through approximation

To solve the optimization problem in (9) through convex optimization, we have to convert the objective function (9a) to a concave function (maximizing a concave function can be handled in convex optimization). First we notice that the term $r+q^TP^{-1}q$ is not a linear function of the decision variable P,q,r, but it can be regarded the *Schur complement* of another matrix, namely

$$\det(-P)\det(r+q^TP^{-1}q)=\det\left(\begin{bmatrix} r & q^T \\ q & -P\end{bmatrix}\right) \tag{10}$$

where we use the property of the *Schur complement*. Hence the objective function can be re-written as

$$\max n\log\det\left(\begin{bmatrix} r & q^T \\ q & -P\end{bmatrix}\right)-(n+1)\log\det(P)\text{ if } n \text{ is even} \tag{11a}$$

$$\max n\log-\det\left(\begin{bmatrix} r & q^T \\ q & -P\end{bmatrix}\right)-(n+1)\log\det(P)\text{ if } n \text{ is odd} \tag{11b}$$

Now in this new objective function (11) all the terms inside determinant functions are linear w.r.t P,q,r. But still the objective function is not a concave function of P,q,r because the log-determinant function.

To resolve this, we consider to take the linear approximation of the nonlinear non-concave objective by using the gradient of the log-determinant function. We know that

$$\frac{\partial\log\det(X)}{\partial X}=X^{-1}\text{ if }\det(X)>0,X=X^T \tag{12}$$

$$\frac{\partial\log-\det(X)}{\partial X}=X^{-1}\text{ if }\det(X)<0,X=X^T \tag{13}$$

So our general idea is to solve the non-convex optimization problem (9) iteratively. In each iteration we linearize the objective (9a) and solve the convex optimization problem within a trust region. In the next section, I will explain this idea to iteratively maximize the ellipsoid volume through trust region method.

2.2 Sequential convex optimization

First let's assume that at iteration k, we have the value $P^{(k)},q^{(k)},r^{(k)}$, we linearize the objective function (9a) as

$$\max_{P^{(k+1)},q^{(k+1)},r^{(k+1)}} n\text{Tr}\left(\begin{bmatrix} r^{(k)} & (q^{(k)})^T \\ q^{(k)} & -P^{(k)}\end{bmatrix}^{-1}\begin{bmatrix} r^{(k+1)} & (q^{(k+1)})^T \\ q^{(k+1)} & -P^{(k+1)}\end{bmatrix}\right)-(n+1)\text{Tr}\left((P^{(k)})^{-1}P^{(k+1)}\right) \tag{14}$$

Now that we have this linearized objective function (14), we can consider to solve a sequence of convex optimization programs. At each iteration we linearize the objective function at certain point (for example the solution in the previous iteration) and update the ellipsoid based on the solution to the convex program.

At the k'th iteration, we consider to solve the following program within a trust region

$$\max_{P,q,r} n\text{Tr}\left(\begin{bmatrix} r^{(k-1)} & (q^{(k-1)})^T \\ q^{(k-1)} & -P^{(k-1)} \end{bmatrix}^{-1} \begin{bmatrix} r & q^T \\ q & -P \end{bmatrix}\right) - (n+1)\text{Tr}\left((P^{(k-1)})^{-1}P\right) \tag{15a}$$

$$\text{s.t } v_i^T P v_i + 2q^T v_i \geq r,\ \forall i = 1, ..., m \tag{15b}$$

$$\lambda \geq 0 \tag{15c}$$

$$\begin{bmatrix} P & q - \frac{1}{2}\lambda_i c_i \\ (q - \frac{1}{2}\lambda_i c_i)^T & \lambda_i d_i - r \end{bmatrix} \succeq 0 \tag{15d}$$

$$P \succeq 0 \tag{15e}$$

$$\left(z^{(k)}\right)^T P z^{(k)} + 2q^T z^{(k)} \leq r \tag{15f}$$

$$|P - P^{(k-1)}|^2 + |q - q^{(k-1)}|^2 + (r - r^{(k-1)})^2 \leq \Delta \tag{15g}$$

where (15g) is the trust-region constraint that the change on P, q, r should be within a neighbourhood of the previous iterations $P^{(k-1)}, q^{(k-1)}, r^{(k-1)}$. In (15f), the point $z^{(k)}$ is a given point computed as the center of the ellipsoid in the previous iteration $\mathcal{E}^{(k-1)} = \{x | x^T P^{(k-1)} x + 2x^T q^{(k-1)} \leq r^{(k-1)}\}$, namely $z^{(k)} = -\left(P^{(k-1)}\right)^{-1/2} q^{(k-1)}$, i.e., the new ellipsoid will contain the center of the previous ellipsoid. I denote the solution to the convex optimization program (15) as $\bar{P}^{(k)}, \bar{q}^{(k)}, \bar{r}^{(k)}$. This solution give a search direction $\bar{P}^{(k)} - P^{(k-1)}, \bar{q}^{(k)} - q^{(k-1)}, \bar{r}^{(k)} - r^{(k-1)}$. We then do a line search to find the appropriate step size, and set $P^{(k)}, q^{(k)}, r^{(k)}$ based on both the search direction and the step size.

Here is the complete algorithm

Algorithm 1 Given points $v_1, \ldots, v_m \in \mathbb{R}^n$, together with the convex hull of these points as $\{x | Cx \leq d\}$, the goal is to find a large ellipsoid $\{x | x^T P x + 2q^T x \leq r\}$ contained within the convex hull but not containing any point $v_i, i = 1, \ldots, m$. Start with $P^{(0)}, q^{(0)}, r^{(0)}$

1: $i \leftarrow 0$.
2: **while** not converged **do**
3: $\quad z^{(k)} = -\left(P^{(k-1)}\right)^{-1/2} q^{(k-1)}$
4: $\quad$ Solve (15) with solution $\bar{P}^{(k)}, \bar{q}^{(k)}, \bar{r}^{(k)}$.
5: $\quad$ Do line search to find the step size α along the direction $[\bar{P}^{(k)}, \bar{q}^{(k)}, \bar{r}^{(k)}] - [P^{(k-1)}, q^{(k-1)}, r^{(k-1)}]$.
6: $\quad P^{(k)} \leftarrow P^{(k-1)} + \alpha\left(\bar{P}^{(k)} - P^{(k-1)}\right)$
7: $\quad q^{(k)} \leftarrow q^{(k-1)} + \alpha\left(\bar{q}^{(k)} - q^{(k-1)}\right)$
8: $\quad r^{(k)} \leftarrow r^{(k-1)} + \alpha\left(\bar{r}^{(k)} - r^{(k-1)}\right)$
9: $\quad i \leftarrow i + 1$.
10: **end while**

At line 4 of Algorithm 1, we can use different line search algorithm and conditions, the idea is to find the step length such that the original nonlinear cost $n \log(r + q^T P^{-1} q) - \log\det(P)$ is increased.

References

[1] Imre Pólik and Tamás Terlaky. A survey of the s-lemma. *SIAM review*, 49(3):371–418, 2007.