Minimum outer ellipsoid

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1 Problem statement

Given a basic semi-algebraic set in $\mathbb{R}^n$.

$$\mathcal{K} = \{x | p_i(x) \le 0, i = 1, \ldots, N\}, \tag{1}$$

where $p_i(x)$ is a polynomial of x, we want to find the smallest ellipsoid $\mathcal{E}$ (measured by the ellipsoid volume) that covers this basic semialgebraic set $\mathcal{K}$, namely $\mathcal{E} \supset \mathcal{K}$.

2 Approach

We parameterize the ellipsoid as

$$\mathcal{E} = \{x | x^T S x + b^T x + c \le 0\}, \tag{2}$$

where $S \succeq 0, b, c$ are parameters of the ellipsoid. Notice that these parameters show up linearly in the ellipsoid (2).

2.1 Containment constraint

To impose the containment constraint $\mathcal{E} \supset \mathcal{K}$, we use the *Positivestellasatz* (p-satz), a common technique for proving containment between basic semi-algebraic sets:

$$-(1 + \lambda_0(x))(x^T S x + b^T x + c) + \sum_{i=1}^{N} \lambda_i(x) p_i(x) \text{ is sos} \tag{3a}$$

$$\lambda_0(x) \text{ is sos}, \lambda_i(x) \text{ is sos}, i = 1, \ldots, N \tag{3b}$$

where "is sos" means the polynomial is a sum-of-squares (sos) polynomial. A polynomial being sos is a convex constraint on the polynomial coefficients. In (3), the polynomial $\lambda_0(x)$ is given, and we search for $S, b, c, \lambda_i(x), i = 1, \ldots, N$.

2.2 Minimize volume

Our goal is to minimize the volume of the ellipsoid $\mathcal{E}$. We know that

$$\text{vol}(\mathcal{E}) \propto \left(\frac{b^T S^{-1} b/4 - c}{\det(S)^{1/n}} \right)^{\frac{n}{2}} \tag{4}$$

where n is the dimensionality of x. So our goal is to minimize this volume

$$\min_{S,b,c} \frac{b^T S^{-1} b/4 - c}{\det(S)^{1/n}} \tag{5}$$

2.2.1 Attempt 1

How can we minimize this term in (4) through convex optimization? We know that $-\log\det(S)$ is a convex function of S, so how about taking the logarithm of (4) as

$$\log(b^T S^{-1} b/4 - c) - \frac{1}{n}\log\det(S) \tag{6}$$

. The second term $-\frac{1}{n}\log\det(S)$ is a convex function. Moreover, the term $b^T S^{-1} b/4 - c$ is convex. Indeed, we can minimize this term through convex optimization

$$\min_{S,b,c,t} t \tag{7a}$$

$$\text{s.t} \begin{bmatrix} t+c & b^T/2 \\ b/2 & S \end{bmatrix} \succeq 0. \tag{7b}$$

We can prove that (7) is equivalent to $\min b^T S^{-1} b/4 - c$ from the Schur complement: $t \geq b^T S^{-1} b/4 - c \Leftrightarrow \begin{bmatrix} t+c & b^T/2 \\ b/2 & S \end{bmatrix} \succeq 0$. As a result, the objective (6) can be rewritten as $\min \log t - \frac{1}{n}\log\det(S)$. Unfortunately its first term $\log t$ is not a convex function of t (it is a concave function). Hence I cannot minimize this objective (6) through convex optimization.

2.2.2 Attempt 2

As we see in the previous subsection, we can minimize the numerator $b^T S^{-1} b/4 - c$ of (4) through convex optimization, but not the logarithm of this numerator. To resolve this, let's look at the volume in (4) again. Notice that the volume is homogeneous w.r.t (S, b, c), namely if we scale it by a factor of k to (kS, kb, kc), the volume is still the same. Hence without loss of generality, we can assume that the denominator is lower bounded by 1, and only minimize the numerator

$$\min_{S,b,c} b^T S^{-1} b/4 - c \tag{8a}$$

$$\text{s.t } \det(S)^{\frac{1}{n}} \geq 1. \tag{8b}$$

The optimization problem (8) is equivalent to the original problem (5) which minimizes the ellipsoid volume. The constraint (8b) is non-convex, but we can convert it to a convex constraint by taking its logarithm

$$\min_{S,b,c} b^T S^{-1} b/4 - c \tag{9a}$$

$$\text{s.t } \log\det(S) \geq 0. \tag{9b}$$

Both the objective and the constraint in (9) are convex, hence we can minimize the volume of this ellipsoid through a convex optimization problem.

Some solvers would prefer formulating the convex problem as a conic optimization problem, with linear objective function and linear or conic constraints. To do so, we use (7) to convert the objective in (9) to the linear objective with conic constraints. Also we bring in the containment constraints in (3) to the optimization problem

$$\min_{\substack{S,b,c,t \\ \phi_i(x), i=1,\ldots,N}} \quad t \tag{10a}$$

$$\text{s.t} \begin{bmatrix} c+t & b^T/2 \\ b/2 & S \end{bmatrix} \succeq 0 \tag{10b}$$

$$\log\det(S) \geq 0 \tag{10c}$$

$$-(1+\lambda_0(x))(x^T S x + b^T x + c) + \sum_{i=1}^N \lambda_i(x) p_i(x) \text{ is sos} \tag{10d}$$

$$\lambda_i(x) \text{ is sos}, i = 1, \ldots, N, \tag{10e}$$

where we assume $\lambda_0(x)$ is a given sos polynomial (and often set to $\lambda_0(x) = 0$). Note that the constraint $\log\det(S) \geq 0$ can also be written in the conic form (with positive semidefinite constraints and exponential cone constraints), as described in Mosek doc.

3 Appendix

3.1 Ellipsoid volume

In this sub-section, we derive the ellipsoid volume equation (4). We can rewrite the ellipsoid as

$$\|A(x + S^{-1}b/2)\|_2 \leq \sqrt{b^T S^{-1} b/4 - c} \tag{11}$$

where $A^T A = S$. If we denote $y = A(x + S^{-1}b/2)$, then we know the volume of the hypersphere $\mathcal{O} = \{y | \|y\|_2 \leq \sqrt{b^T S^{-1} b/4 - c}\}$ is proportional to $(b^T S^{-1} b/4 - c)^{\frac{n}{2}}$. Since $y = A(x + S^{-1}b/2)$ is an affine transformation of x, and the volume of a set after an affine transformation is scaled by the determinant of the transformation matrix, we know that the volume of the ellipsoid $\mathcal{E}$ can be computed from the volume of the hypersphere $\mathcal{O}$ as

$$\mathrm{vol}(\mathcal{E}) = \frac{\mathrm{vol}(\mathcal{O})}{\det(A)} \propto \left(\frac{b^T S^{-1} b/4 - c}{\det(S)^{1/n}}\right)^{\frac{n}{2}}, \tag{12}$$

where we use the fact that $\det(A) = \det(S)^{\frac{1}{2}}$.