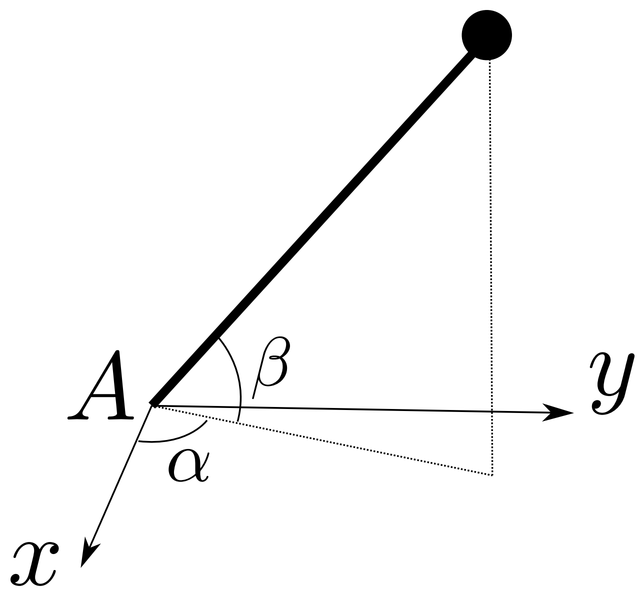

Consider a massless pole with a sphere attached to one end, with sphere mass being m_s. The other end of the pole (point A) is attached to the end-effector with mass m_e at point A. The state of the pole is the location/velocity of A, and the azimuth/altitude angle and their velocity, as shown in this plot

The position of the mass is

$$\begin{bmatrix} x_A + l\cos\beta\cos\alpha \\ y_A + l\cos\beta\sin\alpha \\ z_A + l\sin\beta \end{bmatrix} \tag{1}$$

The velocity of the mass is

$$\begin{bmatrix} \dot{x}_A - l\dot{\alpha}\cos\beta\sin\alpha - l\dot{\beta}\sin\beta\cos\alpha \\ \dot{y}_A + l\dot{\alpha}\cos\beta\cos\alpha - l\dot{\beta}\sin\beta\sin\alpha \\ \dot{z}_A + l\dot{\beta}\cos\beta \end{bmatrix} \tag{2}$$

The total kinetic energy of the system is

$$\begin{aligned} T = {} & 0.5m_e(\dot{x}_A^2 + \dot{y}_A^2 + \dot{z}_A^2) + 0.5m_s(\dot{x}_A^2 + \dot{y}_A^2 + \dot{z}_A^2 + l^2\dot{\alpha}^2\cos^2\beta + \\ & l^2\dot{\beta}^2 - 2\dot{x}_A l\dot{\alpha}\cos\beta\sin\alpha + 2\dot{y}_A l\dot{\alpha}\cos\beta\cos\alpha - 2\dot{x}_A l\dot{\beta}\sin\beta\cos\alpha - 2\dot{y}_A l\dot{\beta}\sin\beta\sin\alpha + 2\dot{z}_A l\dot{\beta}\cos\beta) \end{aligned} \tag{3}$$

The total potential energy is

$$V = m_e g z_A + m_s g(z_A + l\sin\beta) \tag{4}$$

Using Lagrangian $L = T - V$ and $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}$, we have

$$(m_s + m_e)\ddot{x}_A - m_s l(\ddot{\alpha}\cos\beta\sin\alpha - \ddot{\beta}\sin\beta\cos\alpha - \cos\beta\cos\alpha(\dot{\alpha}^2 + \dot{\beta}^2) + 2\dot{\alpha}\dot{\beta}\sin\beta\sin\alpha) = u_x \tag{5}$$

$$(m_s + m_e)\ddot{y}_A + m_s l(\ddot{\alpha}\cos\beta\cos\alpha - \ddot{\beta}\sin\beta\sin\alpha - 2\dot{\alpha}\dot{\beta}\sin\beta\cos\alpha - (\dot{\alpha}^2 + \dot{\beta}^2)\cos\beta\sin\alpha) = u_y \tag{6}$$

$$(m_s + m_e)\ddot{z}_A + m_s l(\ddot{\beta}\cos\beta - \dot{\beta}^2\sin\beta) - (m_s + m_e)g = u_z \tag{7}$$

$$\begin{aligned} & l\ddot{\alpha}\cos^2\beta - l\dot{\alpha}\dot{\beta}\sin 2\beta - \ddot{x}_A\cos\beta\sin\alpha + \dot{x}_A\dot{\beta}\sin\beta\sin\alpha - \dot{x}_A\dot{\alpha}\cos\beta\cos\alpha \\ & \qquad + \ddot{y}_A\cos\beta\cos\alpha - \dot{y}_A\dot{\beta}\sin\beta\cos\alpha - \dot{y}_A\dot{\alpha}\cos\beta\sin\alpha = 0 \end{aligned} \tag{8}$$

$$
\begin{aligned}
l\ddot{\beta} - \ddot{x}_A \sin\beta\cos\alpha - \dot{x}_A\dot{\beta}\cos\beta\cos\alpha + \dot{x}_A\dot{\alpha}\sin\beta\sin\alpha - \ddot{y}_A\sin\beta\sin\alpha - \\
\dot{y}_A\dot{\beta}\cos\beta\sin\alpha - \dot{y}_A\dot{\alpha}\sin\beta\cos\alpha + \ddot{z}_A\cos\beta - \dot{z}_A\dot{\beta}\sin\beta - g\cos\beta = 0 \quad (9)
\end{aligned}
$$