This note describes the construction of an equal priority auction.

The Model. There are n potential buyers of a single homogenous good. Each buyer has a privately known valuation that is independently drawn from the interval $[0,1]$. Assume all valuations are distributed according to some distribution F with strictly positive density f. Buyers' payoff when they buy at price p is given by $v-p$. The seller's cost is zero, so the profit from selling at price p is just p.

In what follows, we make the usual assumption that virtual valuation is increasing.

Definition 1. The virtual valuation function is given by $\phi(v) \equiv v-(1-F(v))/f(v)$. It is assumed throughout that it is strictly increasing in v.

Notice that this function has to be negative where $v=0$ and has to be positive when $v=1$.

Definition. The profit function is given by

$$\pi(p) \equiv (1-F(p))p$$

The function $\pi(p)$ describes the expected revenue of a seller who makes a single take it or leave it offer to a buyer whose valuation is unknown but distributed according to the distribution F. The function π attains its maximum where the derivative is zero, that is

$$(1-F(p)) = pf(p)$$

which implies that

$$p-\frac{1-F(p)}{p} = \phi(p) = 0.$$

We add another definition.

Definition. Define r^* as the solution to $\phi(r^*) = 0$.

Since it is well known that this is the optimal reserve price in a standard independent private value auction, we refer to r^* as the optimal reserve price.

What makes the argument here different is that we are going to assume that some buyers misunderstand the rules of the auction. These buyers don't understand that the seller is using an auction. They simply wait to receive an offer without being able to communicate any information to sellers and without understanding that they can bid.

The seller can make them an offer, but if he does, he can't make an offer to someone who bid. This is trade-off is the central problem we analyze in the paper.

Definition. Whether or not each particiant in informed about the auction is private information. The variable $\alpha \in (0,1)$ is the probability with which each participant is informed.

Just a remark before starting. If a bidder is uninformed, they can do nothing about it, they just wait for an offer. If a bidder is informed, however, they can act as if they are uninformed if they want. This has implications for the payoff functions that will be discussed below. The seller can identify informed bidders in what follows simply by seeing that they submit a bid. Uninformed buyers don't know how to submit bids. If the seller sees a buyer who appears to be uninformed in the sense that they don't submit a bid, then the seller can't be sure if they are really uniformed, or if they are informed bidders who are pretending to be uninformed.

Equal Priority Auction. An equal priority auction is fully characterized by four numbers, a 'reserve price' r, a price offer t, and the upper and lower bound v_+ and v_- of an interval of buyer types. We'll assume throughout that $0 \leq r \leq r^* \leq t \leq v_- \leq v_{+-}$

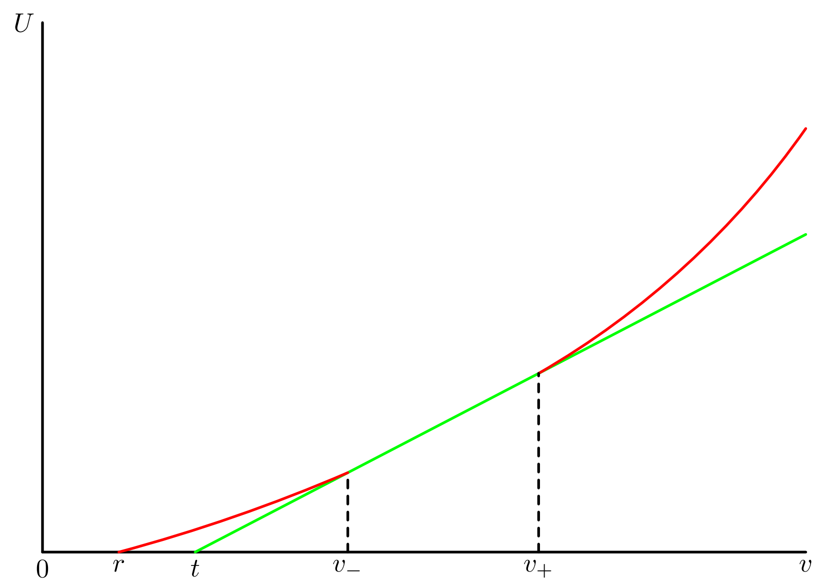
The allocation in an equal priority auction is determined in the following way:

- If all buyers are informed and have values less than or equal to v_-, then the seller holds an auction with reserve price r;
- If the highest bid received from an informed bidders lies in the interval $[v_-, v_+]$, then the seller makes an offer either to one of the uniformed bidders, or to one of the informed bidders who has a bid in the interval $[v_-, v_+]$. The seller chooses each uninformed buyer, and each informed buyer whose bid lies in the interval $[v_-, v_+]$ with equal probability to be the recipient of the offer.
- Finally if there are one or more informed bidders who submit a bid above v_+, then the seller sells to the one who made the highest bid. If there is more than one bidder who bids above v_+, then the price offer is equal to the second highest bid.

The variable t is the offer made to an uninformed bidder conditional on receiving an offer. It is also the expected offer made to an informed bidder who value lies in the interval $[v_-, v_+]$ conditional on receiving the offer.

When an informed bidder has a valuation above v_+ then conditional on no other bidders having a valuation above v_+ his expected offer is is a convex combination of v_+ and t. This is analogous to a reserve price.

The following figure describes the payoffs to bidders with various values in an incentive compatible equal priority auction.



The straight green line gives the payoff to a uninformed bidder. The curved red line segments represent the payoffs to informed bidders. The payoffs of informed bidders whose values lie on the line segment $[v_-, v_+]$ coincide with the payoffs of uninformed bidders.

Trading Probability for the Informed. In an equal priority auction, every offer that an informed bidder receives is acceptable. So the probability that an informed bidder trades is equal to the probability with which the bidder gets an offer.

Definition. The probability with which an informed bidder who value is w trades is $\mathcal{F}^{\epsilon}(w)$.

This is the formula for $\mathcal{F}^{\epsilon}(w)$ is an equal priority auction:

$$\mathcal{F}^{\epsilon}(w) = \tag{0.1}$$

$$\begin{cases} 0 & \text{if} \quad w < r \\ (1-\alpha)^{n-1} F^{n-1}(w) & \text{if} \quad w \in [r, v_-) \\ \chi(v_-, v_+) & \text{if} \quad w \in [v_-, v_+] \\ \sum_{m=0}^{n-1} B(m; n-1, \alpha) F^{n-1-m}(w) & \text{if} \quad w > v_+, \end{cases}$$

where

$$B_k^{n-1-m}(v_-, v_+) = \binom{n-1-m}{k} ((F(v_+) - F(v_-))^k F^{n-1-m-k}(v_-)$$

is the probability that $k \leq n-1-m$ informed buyers have valuations on the pooling interval $[v_-, v_+]$ and $n-1-m-k$ informed buyers have valuations lower than v_-; and the function

$$\chi(v_-, v_+) = \sum_{m=0}^{n-1} B(m; n-1, \alpha) \sum_{k=0}^{n-1-m} B_k^{n-1-m}(v_-, v_+) \frac{1}{m+k+1}.$$

The payoff to an informed bidder. Now we adopt a well known theorem from mechanism design with independent private values.

Definition. The payoff to an informed bidder of type v is

$$U(v, \epsilon) = \int_r^v \mathcal{F}^{\epsilon}(w)\, dw \tag{0.2}$$

The payoff to an uninformed bidder.

Definition. The payoff to an unformed bidder is

$$U(v, \mu) = \chi(v_-, v_+) \max\{(v - t), 0\}$$

All uninformed bidders are treated the same way. They all receive an offer with the same probability. Since informed bidders can pretend to be uninformed we must have the probability with which they receive an offer equal to $\chi(v_-, v_+)$.

Revenues for the seller. Using standard arguments in mechanism design, the expected revenue of the seller is

$$n(1-\alpha) \int_r^1 \mathcal{F}^{\epsilon}(v) \phi(v) f(v) dv + n\alpha\chi(v_-, v_+)\pi(t).$$

This has to be maximized with respect to r, t, v_- and v_+subject to the constraint that

$$(1-\alpha)^{n-1} \int_r^{v_-} F^{n-1}(s)\, ds = \chi(v_-, v_+)(v_- - t).$$

The Solution. The necessary conditions for maximization are given by

$$\alpha(\pi(t) - \phi(v_+)) =$$

$$(1-\alpha)\Big((v_- - t)(\phi(v_+) - \phi(v_-))f(v_-) + (F(v_+) - F(v_-))\phi(v_+) - (\pi(v_-) - \pi(v_+))\tag{0.3}$$

$$-\alpha\pi'(t) = (1-\alpha)(\phi(v_+) - \phi(v_-))f(v_-)\tag{0.4}$$

$$-\phi(r)f(r) = (\phi(v_+) - \phi(v_-))f(v_-)\tag{0.5}$$

$$(1-\alpha)^{n-1}\int_r^{v_-} F^{n-1}(s)\,ds = \chi(v_-, v_+)(v_- - t)\tag{0.6}$$

If there is a corner solution to r at $r = 0$, then we should have

$$-\phi(0)f(0) < (\phi(v_+) - \phi(v_-))f(v_-)$$

Objective. The object is to find solutions for equations (0.3), (0.4), (0.5), and (0.6) and see how they depend on the distribution function F and the probability of being uninformed α.

Computational Hints. The expression $\chi(v_-, v_+)$ can be rewritten as

$$\chi(v_-, v_+) = \frac{((1-\alpha)F(v_+) + \alpha)^n - ((1-\alpha)F(v_-))^n}{n((1-\alpha)(F(v_+) - F(v_-)) + \alpha)}$$