Introduction to the minimal surface

Hongli Ye

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Preface

This brief lecture note aims to provide a basic understanding and intuition about what minimal surfaces are and how 3D printers can be used to create them.

The definitions and theorems mainly follow the book written by Do Carmo, [1]

1 Regular Surface and its curvature

1.1 Definitions

Definition 1.1 (Regular Surface)**.** A subset $\mathrm{S} \subset \mathrm{R}^n$ is a regular surface if, for each $\mathrm{p} \in \mathrm{S}$, there exists a neighborhood V in R^3 and a map $\mathbf{x} : \mathrm{U} \to \mathrm{V} \cap \mathrm{S}$ of an open set $\mathrm{U} \subset \mathrm{R}^2$ onto $\mathrm{V} \cap \mathrm{S} \subset \mathrm{R}^3$ such that:

1. $\mathbf{x}$ is differentiable. This means that if we write

$$\mathbf{x}(u, v) = (x_1(u, v), x_2(u, v), \cdots, x_n(u, v)), \quad (u, v) \in U,$$

For completeness, we will give the formal definition of the tangent space of the regular surface the functions $\mathrm{x}_j(\mathrm{u}, \mathrm{v})$ have continuous partial derivatives of all orders in U for all $j \in \{1, 2, \cdots, n\}$.

2. $\mathbf{x}$ is a homeomorphism. Since $\mathbf{x}$ is continuous by condition 1 , this means that $\mathbf{x}$ has an inverse $\mathbf{x}^{-1} : \mathrm{V} \cap \mathrm{S} \to \mathrm{U}$ which is continuous.

3. (The regularity condition.) For each $\mathrm{q} \in \mathrm{U}$, the differential $d\mathbf{x}_{\mathrm{q}} : \mathrm{R}^2 \to \mathrm{R}^n$ is one-to-one.

Remark 1.2. **In this lecture, we mainly discuss the case $n = .3$.**

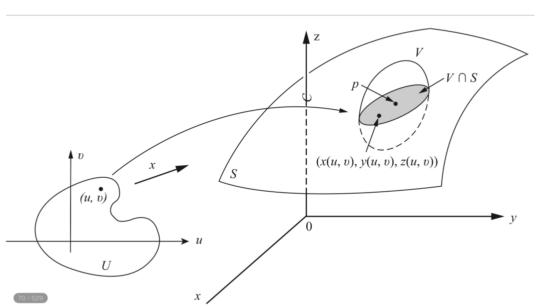

Figure 1: Regular surface when $n = 3$ [1]

Actually, condition 3 guarantees the existence of the tangent space everywhere on the surface.

Example 1.3 (Surface of Revolution)**.** Let $S \subset R^3$ be the set obtained by rotating a regular connected plane curve C about an axis in the plane which does not meet the curve; we shall take the xz plane as the plane of the curve and the z axis as the rotation axis. Let:

$$x = f(v), \quad z = g(v), \quad a < v < b, \quad f(v) > 0,$$

be a parametrization for C and denote by u the rotation angle about the z axis. Thus, we obtain a map

$$\mathbf{x}(u, v) = (f(v)\cos u, f(v)\sin u, g(v))$$

from the open set $U = \left\{(u, v) \in R^2; 0 < u < 2\pi, a < v < b\right\}$ into S (Fig. 2-18).

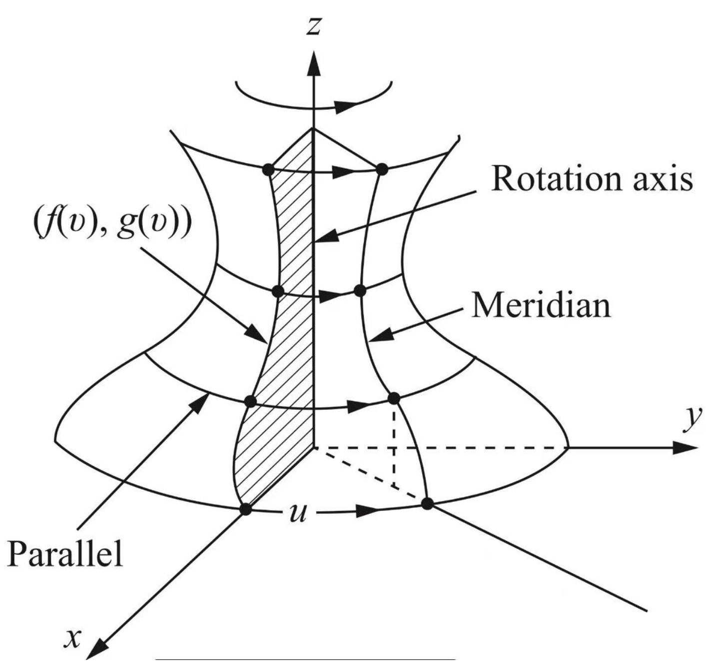

Figure 2: Surface of revolution[1]

For completeness, we will state the formal definitions here, but to have a basic intuition about what a tangent space T_pS at point $p \in S$ is, we can think of it as the best linear approximation of the plane to the surface S at p.

Definition 1.4 (Tangent Space, tangent vector)**.** A **tangent vector** to S, at a point $p \in S$, we mean the tangent vector $\alpha'(0)$ of a differentiable parametrized curve $\alpha : (-\epsilon, \epsilon) \to S$ with $\alpha(0) = p$.

The **tangent space** to S at p, denoted by T_pS, is the set of all tangent vectors at p.

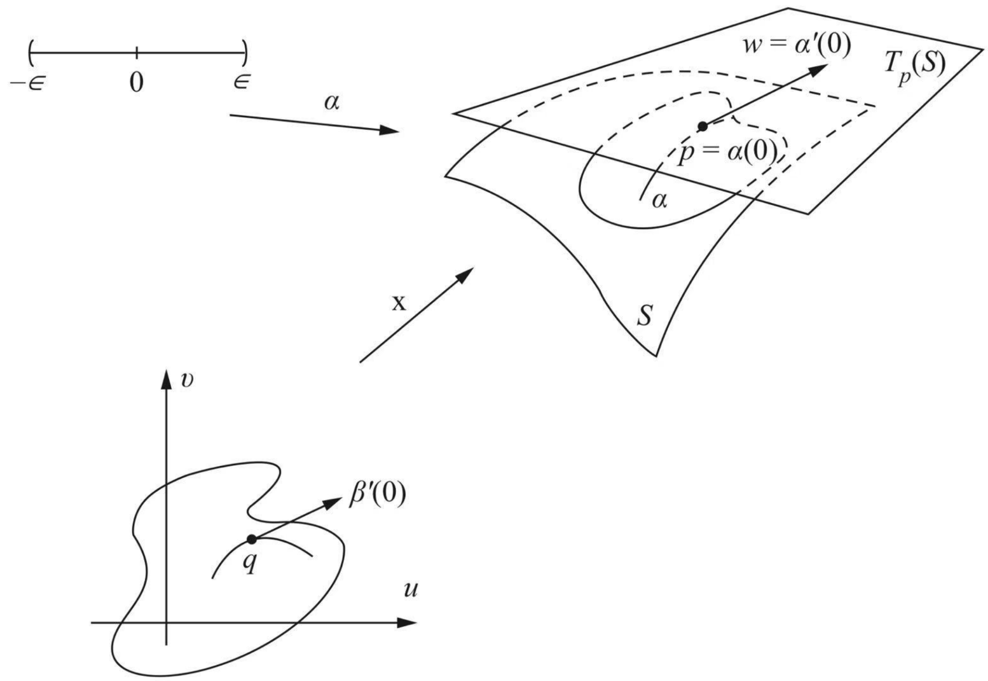

Figure 3: Tangent Vector[1]

Proposition 1.5. *Let $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ be a parametrization of a regular surface S and let $q \in U$. The vector subspace of dimension 2,*

$$dx_q(\mathbb{R}^2) \subset \mathbb{R}^3$$

coincides with the set of tangent vectors to S at $\mathbf{x}(q)$.

By this proposition, we can easily give an explicit expression of the T_pM if we know the local chart of the surface. i.e.

$$T_pM = dx_q(\mathbb{R}^2) = \text{span}\{X_u, X_v\} \text{ where } (u, v) \in \mathbb{R}^2$$

Definition 1.6 (First Fundamental Form)**.** [4] Let M be a regular surface and u, v tangent vectors in the tangential space T_pM at p. The inner product $I_p\langle u, v\rangle := \langle u, v\rangle$ is called the **first fundamental form**.

Here is just the Euclidean inner product in $\mathbb{R}^n$; in this way, we have a concrete way of measuring the **length** of the tangent vectors.

Definition 1.7 (Unit Normal Vector)**.** Let M be a regular surface, T_pM be the tangent space at p and (U, X) be the local chart, then the **unit normal vector** is defined as follows:

$$N(p) = \frac{X_u \times X_v}{\mid X_u \times X_v \mid}$$

It is easy to verify that $N(p)$ is well-defined, as the tangent space is well-defined, and we restrict the length of the vector.

Intuitively, the unit normal vector describes the 'direction' of the tangent space, so the idea of defining the **curvature** is to measure the rate of change of the $N(p)$ along p.

The following definition is to define how to measure the rate of change of the $N(p)$.

Definition 1.8 (Covariant Derivative)**.** Let Y be a smooth vector field on $p \in V \subset S$ and $v \in T_pS$, the covariant derivative is defined as follows:

$$D_yY := (\frac{d(Y \circ \alpha)}{dt} |_t = 0)^T = (dY_p(y))^T$$

where $\alpha : (-\epsilon, \epsilon) \to S$, $\alpha(0) = p, \alpha'(0) = v$

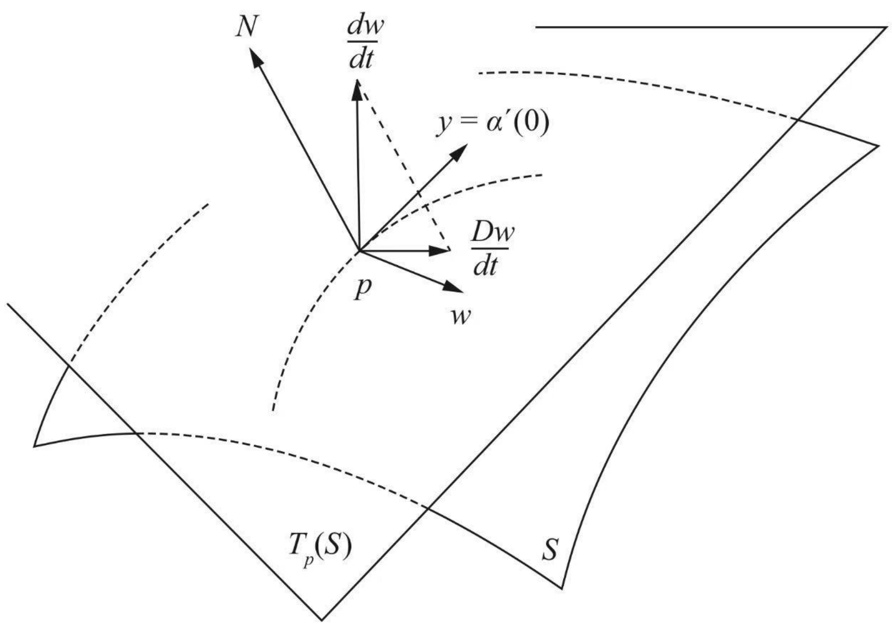

Figure 4: Covariant derivative[1]

Definition 1.9 (Shape Operator)**.** We define the shape operator S in the following way:

$$\begin{aligned} S(p) : T_pM &\longrightarrow T_pM \\ v &\longrightarrow D_v(N(p)) \end{aligned}$$

We can rewrite the definition:

$$S(p)(v) = (dN_p(v))^T = dN_p(v) \text{ since } N(p) \text{ is of constant length}$$

The shape operator is simply the differential of N at the point p.

We omit the formal definition of the general differential between two regular surfaces, and describe dN_p here:

The linear map $dN_p : T_pS \to T_pS$ operates as follows: For each tangent vector $v \in T_pS$, it has a parametrized curve $\alpha(t)$ in S with $\alpha(0) = p, \alpha'(0) = v$, we consider the parametrized curve $N \circ \alpha(t) = N(t)$ in the sphere $\mathbb{S}^2$, this amounts to restricting the normal vector N to the curve $\alpha(t)$. The tangent vector $N'(0) = dN_p(\alpha'(0)) = dN_p(v)$ is a vector in T_pS.

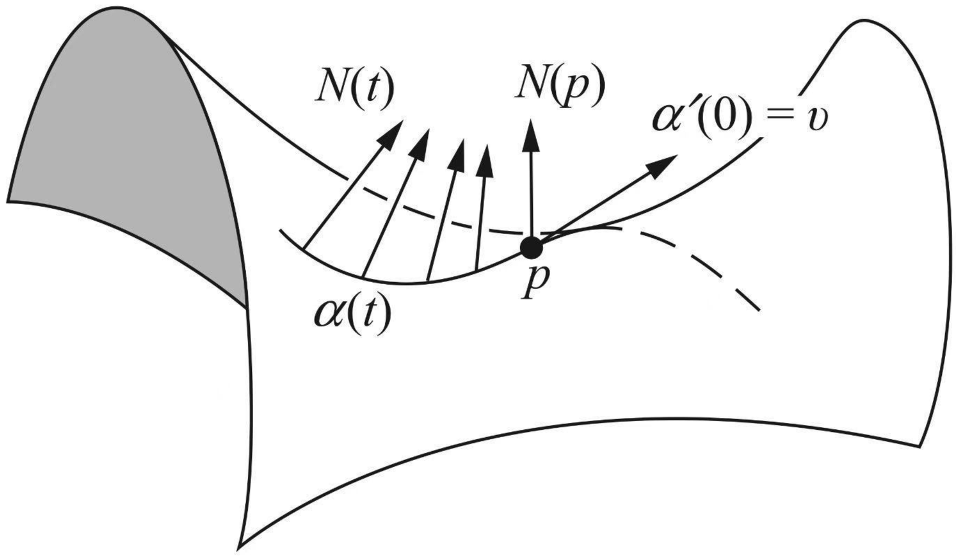

Figure 5: Differential of the map N[1]

Intuitively, it illustrates how the unit normal vector changes its direction as it aligns with the vector v.

Moreover, it is not hard to see that $S = dN_p$ is a **self-adjoint linear map**, so we can apply the techniques we learnt in linear algebra to it.

Definition 1.10 (Principal curvature[4]). The eigenvalues $\kappa_1(p), \kappa_2(p)$ of the shape operator $S : T_p\mathcal{S} \to T_p\mathcal{S}$ are called the **principal curvatures**.

Definition 1.11. The functions $H, K : \mathcal{M} \to \mathbb{R}$

$$H(p) = \frac{1}{m}\,\mathrm{tr}(S(p))$$
$$K(p) = \det(S(p))$$

denote the **mean** and **Gaussian** curvature, respectively

Choosing an orthonormal eigenbasis of the shape operator S yields a diagonal matrix with $\mathrm{diag}(S) = (\kappa_1, \kappa_2)$ and therefore, the following relationship between the mean, Gaussian, and principal curvature can be derived.

Proposition 1.12. *Let $H(p), K(p)$ denote the mean and Gaussian curvature, respectively. Then one has*

$$H(p) = \frac{1}{2}\,(\kappa_1 + \kappa_2)$$
$$K(p) = \kappa_1\kappa_2.$$

Actually, the sign of the mean curvature is related to the orientation, but since we are considering the case of a minimal surface, we can omit it.

1.2 Computations

In the first Calculus course, we have learned how to compute the length of a curve and the area of surfaces using the fundamental values E, F, G. Here, we provide their formal definitions.

Definition 1.13 (Coefficients of the first fundamental form)**.** Let $X : U \to S$ be the local patch of the regular surface S,we define:

$$\begin{cases} E(u, v) = \langle X_u, X_u \rangle \\ F(u, v) = \langle X_u, X_v \rangle \\ G(u, v) = \langle X_v, X_v \rangle \end{cases}$$

Definition 1.14 (Area)**.** Let S be the regular surface and $X : U \to S$ be the local patch, then we define the area of the region R:

$$Area(R) := \int_{X^{-1}(R)} |X_u \times X_v| \, dudv$$

By simple computations, we have:

$$Area(R) := \int_{X^{-1}(R)} \sqrt{EG - F^2} \, dudv$$

Definition 1.15 (Coefficients of the second fundamental form)**.** Let $X : U \to S$ be the local patch of the regular surface S,we define:

$$\begin{cases} e(u, v) = \langle N, X_{uu} \rangle \\ f(u, v) = \langle N, X_{uv} \rangle \\ g(u, v) = \langle N, X_{vv} \rangle \end{cases}$$

By these coefficients, we can give a formula for these two curvatures

$$H(p) = \frac{eG + Eg - 2fF}{2(EG - F^2)}$$

$$K(p) = \frac{eg - f^2}{EG - F^2}$$

2 Minimal Surface

Definition 2.1 (Minimal Surface[1])**.** A regular surface S is called a minimal surface if the mean curvature H vanishes for every point on S.

Definition 2.2 (Normal variation[4])**.** Let $x : \mathcal{U} \to \mathbb{R}^3$ be a regular local surface, $\mathcal{S} \subset \mathcal{U}$ be a bounded region in $\mathcal{U}$, and let $N(u, v)$ denote the unit normal vector to $x(u, v)$ for any

$(u, v) \in \mathcal{U}$. The **normal variation** of x under any differentiable mapping $h : S \to \mathbb{R}$ and $\epsilon > 0$ is defined as

$$\begin{aligned} X : (-\epsilon, \epsilon) \times \mathcal{S} &\longrightarrow \mathbb{R}^3 \\ (t, (u, v)) &\longmapsto x(u, v) + t \times h(u, v) \times N(u, v). \end{aligned}$$

In the following we use the abbreviation $X^t(u, v) := X(t, (u, v))$. The normal variation X^{t} describes for each t a slightly deformed (local) surface in normal direction and hence, to indicate the t different first fundamental forms, we use the notation.

$$E^t = \left\langle X_u^t, X_u^t \right\rangle, F^t = \left\langle X_u^t, X_v^t \right\rangle \text{ and } G^t = \left\langle X_v^t, X_v^t \right\rangle.$$

In preparation for the main result in this section, the relation of a vanishing mean curvature to a minimal surface area, the so-called first variation of a surface, will be derived.

Lemma 2.3. *Let $\mathcal{A}(t) = \int_S \|X_{\mathrm{u}}^t \times X_{\mathrm{v}}^t\| \, dudv$ denote the area of each normal variation $X^t(S)$. Then the first variation of A. i.e., $\left(\frac{dA}{dt}\right)\big|_{t=0}$, is*

$$\left(\frac{d\mathcal{A}}{dt}\right)\bigg|_{t=0} = -2 \int_S hH\sqrt{EG - F^2} d(u, v),$$

where H is the mean curvature of S.

Proof. . First, we describe the area operator:

$$\mathcal{A}(t) \left(X^t(S)\right) = \int_S \left\|X_{\mathrm{u}}^t \times X_v^t\right\| dudv = \int_S \sqrt{E^t G^t - \left(F^t\right)^2} dudv$$

Starting with E^t, G^t, and F^t, these will be subsequently calculated.

$$\begin{aligned} E^t &= \left\langle X_u^t, X_u^t \right\rangle \\ &= \langle x_u + th_u N + thN_u, x_u + th_u N + thN_u \rangle \\ &= \langle x_u + x_u \rangle + \langle x_u, th_u N \rangle + \langle x_u, thN_u \rangle + \langle th_u N, x_u \rangle + \langle thN_u, x_u \rangle + \mathcal{O}\left(t^2\right) \\ &= E + 2 \langle x_u, th_u N \rangle + 2 \langle x_u, thN_u \rangle + \mathcal{O}\left(t^2\right) \\ &= E + 2th_u \underbrace{\langle x_u, N \rangle}_{=0} - 2th \left(N, x_{uu}\right) + \mathcal{O}\left(t^2\right) \\ &= E - 2the + \mathcal{O}\left(t^2\right). \end{aligned}$$

analogously,

$$\begin{aligned} F^t &= F - 2thf + \mathcal{O}\left(t^2\right) \text{ and} \\ G^t &= G - 2thg + \mathcal{O}\left(t^2\right) \end{aligned}$$

Where e, f, and g are the coefficients of the second fundamental form.

$$
\begin{aligned}
E^tG^t - \left(F^t\right)^2 &= \left(E - 2the + \mathcal{O}\left(t^2\right)\right)\left(G - 2thg + \mathcal{O}\left(t^2\right)\right) - \left(F - 2thf + \mathcal{O}\left(t^2\right)\right)^2 \\
&= EG - F^2 - 2thgE - 2theG + 4thfF + \mathcal{O}\left(t^2\right) \\
&= EG - F^2 - 2th(eG - 2fF + gE) + \mathcal{O}\left(t^2\right) \\
&= EG - F^2 - 4thH\left(EG - F^2\right) + \mathcal{O}\left(t^2\right) \\
&= \left(EG - F^2\right)(1 - 4thH) + \mathcal{O}\left(t^2\right)
\end{aligned}
$$

hence,

$$
\begin{aligned}
\sqrt{E^tG^t - (F^t)^2} &= \sqrt{(EG - F^2)(1 - 4thH) + \mathcal{O}(t^2)} \\
&= \sqrt{(EG - F^2)}\sqrt{(1 - 4thH) + \mathcal{O}(t^2)} \\
&= \sqrt{(EG - F^2)}\sqrt{(1 - 2thH)^2 + \mathcal{O}(t^2)} \\
&= \sqrt{(EG - F^2)}(1 - 2thH) + \mathcal{O}\left(t^2\right)
\end{aligned}
$$

As a conclusion, the first variation is given as

$$
\begin{aligned}
\left.\left(\frac{d\mathcal{A}}{dt}\right)\right|_{t=0} &= \left.\left(\frac{d}{dt}\int_S \sqrt{(EG - F^2)}(1 - 2thH) + \mathcal{O}\left(t^2\right) d(u,v)\right)\right|_{t=0} \\
&= -2\int_{\mathcal{S}} hH\sqrt{(EG - F^2)}
\end{aligned}
$$

$\square$

This is just complicated computations.

Theorem 2.4. *Let $x : \mathcal{U} \to \mathbb{R}^3$ be a regular surface and $\mathcal{S} \subset \mathcal{U}$. Then x is a critical point of the area functional on $\mathcal{S}$ if and only if the mean curvature H vanishes.*

Proof. First, let the mean curvature be identically zero, i.e., $H = 0$, then by the result in the previous lemma, the first variation is zero for each h and therefore, $X^0 = x$ is a critical point of the area functional $\mathcal{A}$.

For the other direction, assume the contrary, i.e., $\left.\left(\frac{d\mathcal{A}}{dt}\right)\right|_{t=0} = 0$ for any differentiable $S \to \mathbb{R}$ and there is a point $p := (\bar{u}, \bar{v}) \in S$ such that the mean curvature $H(p) \neq 0$. nce h is arbitrary and smooth, choose h such that $h(p) = H(p)$ in a small neighborhood (p) around p and zero outside. Again, by (3.6) and the positive definiteness of the first fundamental form,

$$
\left.\left(\frac{d\mathcal{A}}{dt}\right)\right|_{t=0} = -2\int_s \underbrace{H^2\sqrt{EG - F^2}}_{\geq 0} d(u,v) < 0.
$$

This contradicts the assumption that the first variation equals zero. Hence, the mean value $H(p) = 0$ for any arbitrary p.

$\square$

Since x is only a critical point, it is uncertain whether the obtained surface is actually minimal. There are cases where x is not necessarily a minimum.

2.1 Equivalent definitions of the minimal surface

One can refer to Chapter 2.2 of the article [3] written by Meeks, William, and Pérez, Joaquín, for more equivalent definitions. To be explicit, eight equivalent definitions.

In this lecture note, we only use one.

Definition 2.5 (Isothermal Parametrization)**.** Let U be an open subset of S, a parametrization $X : U \to \mathbb{R}^3$ is called an isothermal parametrization if:

$$E = G \text{ and } F = 0$$

For the minimal surface, local isothermal parametrization always exists.

Proposition 2.6. *Let $\mathcal{X} = \mathcal{X}(u, v)$ be a (local) regular parametrized surface and assume that $\mathcal{X}$ is isothermal. Then*

$$\triangle \mathcal{X} = 2\lambda^2 H$$

where $\lambda^2 = \langle \mathcal{X}_u, \mathcal{X}_u \rangle = \langle \mathcal{X}_v, \mathcal{X}_v \rangle$

By these properties of isothermal parametrization, we can easily deduce the following corollary:

Corollary 2.7. *Let $\mathcal{X} = \mathcal{X}(u, v) = (x(u, v), y(u, v), z(u, v))$ be a (local) regular parametrized surface and assume that $\mathcal{X}$ is isothermal. Then*

$$\mathcal{X} \textit{ is minimal } \iff \textit{ coordinates function } x, y, z \textit{ are harmonic}$$

where $\lambda^2 = \langle \mathcal{X}_u, \mathcal{X}_u \rangle = \langle \mathcal{X}_v, \mathcal{X}_v \rangle$

2.2 Examples of the minimal surface

After all this discussion, we can finally provide concrete examples.

Therefore, it is very challenging to find a general form of the minimal surface. We will now present some specific types of minimal surfaces.

Definition 2.8 (Weierstrass–Enneper Parameterization[5])**.** where g is meromorphic and f is analytic, such that wherever g has a pole of order m and f has a zero of order $2m$, and letc_1, c_2, c_3 be constants.

Then the surface with coordinates (x_1, x_2, x_3) is minimal, where the x_k are defined using the real part of a complex integral, as follows:

$$\begin{aligned}
x_k(\zeta) &= \mathrm{Re}\left\{\int_0^\zeta \varphi_k(z)\,dz\right\} + c_k, \qquad k = 1, 2, 3 \\
\varphi_1 &= f(1 - g^2)/2 \\
\varphi_2 &= if(1 + g^2)/2 \\
\varphi_3 &= fg
\end{aligned}$$

here Re denotes the real part, $z \in \mathbb{C}$.

This representation guarantees that the resulting surface has zero mean curvature.

What is more important is that every nonplanar minimal surface defined over a connected domain can be given a parametrization of this type.

Definition 2.9 (Enneper's Surface of order n)**.** Based on the representation above, let $f(z) = 1$ and $g(z) = z^n$. Then we have:

$$\begin{cases} \phi_1 = \dfrac{1 - z^{2n}}{2} \\ \phi_2 = i\dfrac{1 + z^{2n}}{2} \\ \phi_3 = z^n \end{cases}$$

So the coordinates function (x_1, x_2, x_3) will be:

$$\begin{cases} x_1(\xi) = Re(\dfrac{\xi}{2} + \dfrac{\xi^{2n+1}}{2(2n+1)}) + c_1 \\ x_2(\xi) = Re(i(\dfrac{\xi}{2} + \dfrac{\xi^{2n+1}}{2(2n+1)})) + c_2 \\ x_3(\xi) = Re(\dfrac{\xi^{n+1}}{n+1}) + c_3 \end{cases}$$

Now we try to reparametrize it using the real numbers by $\xi = u + iv$ and do a binomial expansion, we have:

$$\begin{cases} x_1(u,v) = u - \dfrac{u^{2n+1} - \binom{2n+1}{2}u^{2n-1}v^2 + \binom{2n+1}{4}u^{2n-3}v^4 - \cdots + (-1)^n uv^{2n}}{2n+1} + c_1 \\ x_2(u,v) = -v + (-1)^{n+1}\dfrac{v^{2n+1} - \binom{2n+1}{2}v^{2n-1}u^2 + \binom{2n+1}{4}u^{2n-3}v^4 - \cdots + uv^{2n}}{2n+1} + c_2 \\ x_3(u,v) = \dfrac{u^{n+1} - \binom{n-1}{2}u^{n-1}v^2 + \cdots}{n+1} + c_3 \end{cases}$$

The surface with the above parametrization is called **Enneper's Surface of order** n, denoted as Enn_n[4].

Example 2.10 (Parametrization of Enn_1)**.** For convenience, we multiply some constants and let $\vec{c} = (c_1, c_2, c_3) = \vec{0}$ to make it more readable.

$$
\begin{cases}
x_1(u,v) = u - \dfrac{u^3}{3} - uv^2 \\
x_2(u,v) = -v + (-1)^n \dfrac{v^{2n+1} - \binom{2n+1}{2} v^{2n-1}}{} \\
x_3(u,v) = u^2 - v^2
\end{cases}
$$

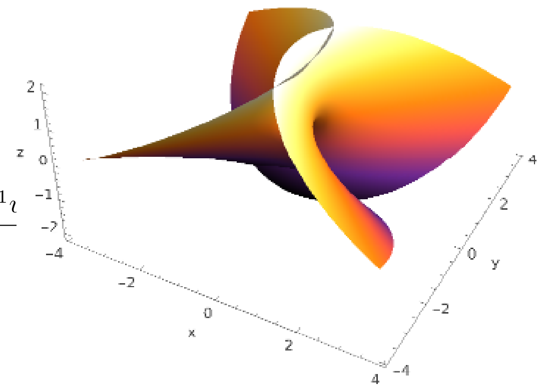

Figure 6: Enn_1 using mathematica

Example 2.11 (Parametrization of Enn_2)**.** For convenience, we multiply some constants and let $\vec{c} = (c_1, c_2, c_3) = \vec{0}$ to make it more readable.‘

$$
\begin{cases}
x_1(u,v) = u - \dfrac{u^5 - 10u^3v^2 + 5uv^4}{5} \\
x_2(u,v) = -v + \dfrac{v^5 - 10v^3u^2 + 5uv^4}{5} \\
x_3(u,v) = 2\dfrac{u^3 - 3uv^2}{3}
\end{cases}
$$

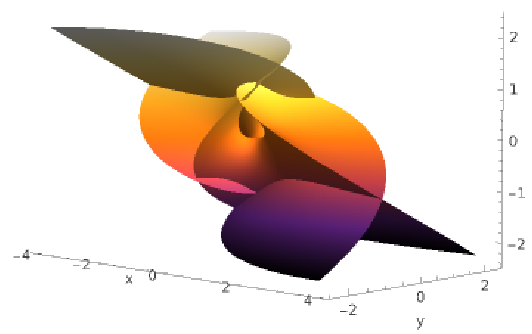

Figure 7: Enn_2 using mathematica

2.3 Some classification results on minimal surfaces[2]

Theorem 2.12 (Corollary of the Gauss-Bonnet Theorem)**.** *Let S be an orientable compact surface; then:*

$$\iint_S K d\sigma = 2\pi\chi(S)$$

*The left-hand side is also called the **total curvature***

We might want to give a general description of all minimal surfaces.

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a multivariable smooth function; thus, we can define a regular surface based on the **graph** of it.

$$F : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$
$$(x, y) \longrightarrow (x, y, f(x, y))$$

Suppose $F(R)$ is a regular surface where R is a region in $\mathbb{R}^2$, then we have the minimal surface partial differential equations:

$$(1 + f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1 + f_x^2)f_{yy} = 0$$

This is a very complicated 2^{nd} non-linear PDE with no general solutions discovered so far.

Mathematicians are still devoted to classifying minimal surfaces.

3 How to print them

A Codes used to generate the picture

```
enneper[n_Integer, urange_, vrange_] :=
 ParametricPlot3D[
   Re[
     {
       ( (u + I v)/2 - (u + I v)^(2 n + 1)/(2 (2 n + 1)) ),
       I ((u + I v)/2 + (u + I v)^(2 n + 1)/(2 (2 n + 1)) ),
       (u + I v)^(n + 1)/(n + 1)
     }
   ],
   {u, urange[[1]], urange[[2]]}, {v, vrange[[1]], vrange[[2]]},
   Mesh -> None, Boxed -> False,
   PlotRange -> All,
   AxesLabel -> {"x", "y", "z"},
   PlotPoints -> 100,
   Lighting -> "Neutral",
   ColorFunction -> Function[{x, y, z, u, v}, ColorData["SunsetColors"][z]]
 ];

(* Example: Enneper surface of order 1 *)
enneper[1, {-2, 2}, {-2, 2}
```

References

[1] M.P. do Carmo. *Differential Geometry of Curves and Surfaces: Revised and Updated Second Edition*. Dover Books on Mathematics. Dover Publications, 2016. ISBN: 9780486806990. URL: `https://books.google.com.sg/books?id=uXF6DQAAQBAJ`.

[2] Francisco J. López and Francisco Martín. "COMPLETE MINIMAL SURFACES IN ³". In: *Publicacions Matemàtiques* 43.2 (1999), pp. 341–449. ISSN: 02141493, 20144350. URL: `http://www.jstor.org/stable/43736665` (visited on 10/13/2025).

[3] William Meeks and Joaquin Pérez. "The classical theory of minimal surfaces". In: *Bulletin of the American Mathematical Society* 48 (Sept. 2011), pp. 325–325. DOI: `10.1090/S0273-0979-2011-01334-9`.

[4] Cornelia Michlits. "On the Generation of Minimal Surfaces". Accessed October 11, 2025. Bachelor's Thesis. TU Wien, 2023. URL: `https://repositum.tuwien.at/bitstream/20.500.12708/175789/1/Michlits%20Cornelia%20-%202023%20-%20On%20the%20generation%20of%20minimal%20surfaces.pdf`.

[5] Wikipedia contributors. *Weierstrass Enneper parameterization Wikipedia, The Free Encyclopedia*. Online; accessed 13-October-2025. 2025. URL: `https://en.wikipedia.org/w/index.php?title=Weierstrass%E2%80%93Enneper_parameterization&oldid=1310516764`.