Riemannian Metric and Affine Connections

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Preface

This note is written for the seminar Riemannian Geometry held by Prof. Bochen Liu in Southern University of Science and Technology 2025 Fall semester, and the main idea of this note follows the first two chapters of the book "Riemannian Geometry"[1] written by M.do Carmo, and I also use other resources to see concrete examples or motivation behind the definitions.

I assume the listeners are familiar with the basic structures and operations of a smooth manifold.

The main idea/flow of this lecture note is to demonstrate how mathematicians develop a general method to measure geometric properties, such as lengths and areas, and how this approach can be generalized to the smooth manifold structure.

The main theorem includes the existence of a Riemannian Metric and the uniqueness of a Riemannian connection.

1 Review

Before getting into the formal description of Riemannian metric and manifold, we quickly review the language of smooth manifold and some simple linear algebra.

1.1 Language of high-dimensional geometry

Definition 1.1 (Smooth Manifold)**.** A smooth manifold (M, Σ) contains a topological manifold M and a soft structure Σ on M.

The definition above has a few restrictions; for example, the curves can self-intersect

Definition 1.2 (Tangent Space)**.** Let (M, Σ) be the smooth manifold. Take a (differentiable) curve in M. Suppose that $\alpha(0) = p \in M$, and let $\mathcal{D}$ be the set of functions on M that are differentiable at p. The **tangent vector** to the curve α at $t = 0$ is a function $\alpha'(0) : \mathcal{D} \to \mathbb{R}$ given by:

$$\begin{aligned} \alpha'(0) : \mathcal{D} &\longrightarrow \mathbb{R} \\ f &\longrightarrow \frac{d(f \circ \alpha)}{dt}|_{t=0} \end{aligned}$$

The set of all tangent vectors to M at p will be indicated by T_pM.

Remark 1.3. Equivalently, a tangent vector at p can be defined as a derivation X_p : $C^\infty(M) \to \mathbb{R}$ satisfying the Leibniz rule $X_p(fg) = f(p)X_p(g) + g(p)X_p(f)$. This definition is coordinate-free and will be used later in defining connections.

Definition 1.4 (Tangent Bundle)**.**

$$TM = \ T_pM$$

Definition 1.5 (Vector field)**.** A **vector field** X on a differentiable manifold M corresponds to each point $p \in M$ a vector $X(p) \in T_pM$. Regarding mappings, X is a mapping from M into the tangent bundle TM.

We say the vector field is *differentiable* if the mapping $X : M \longrightarrow TM$ is *differentiable.*

Sometimes, we can also think of a vector field as a mapping $X : \mathcal{D} \longrightarrow \mathcal{F}$ by defining $X(f)(p) = \sum_i a_i(p)\frac{\partial f}{\partial x_i}(p)$

And we need to introduce an important technique of dealing with smooth manifolds.

Definition 1.6 (Smooth Partition of Unity)**.** Let M be a smooth manifold. A partition of unity is a collection $\{\lambda_\alpha \mid \alpha \in A\}$ of smooth functions $\lambda_\alpha : U_\alpha \to \mathbb{R}$ such that:

1. $0 \leq \lambda_\alpha \leq 1$ for all $x \in M$ and $\alpha \in A$.
2. The collection $\{supp(\lambda_\alpha) \mid \alpha \in A\}$ is locally finite, that is to say for any $x \in M$ there are most fintely many $\alpha \in A$ such that $x \in supp(\lambda_a)$
3. For all $x \in M$ one has:
$$\sum_{\alpha\in A} \lambda_\alpha(x) = 1$$

Theorem 1.7. *Let M be the smooth manifold and $\{U_a\}$ be an open cover; there always exists a smooth partition of unity belonging to $\{U_a\}$.*

It is customary to say that the *partition of unity is subordinate to the covering* $\{U_a\}$

By this theorem, we can prove the smooth version of the Urysohn lemma and the Tiestz expansion theorem, but we omit the proof of this theorem here; one can visit here[2] to get further details.

1.2 Basic linear algebra

Definition 1.8 (Bilinear Form)**.** Let V be an n-dimensional real vector space. A **bilinear form** is a map.

$$B : V \times V \longrightarrow \mathbb{R}$$

Such that B is linear in each argument.

Remark 1.9. Fix a basis $\{e_1, \dots, e_n\}$ of V. Then B is represented by a matrix

$$B = (b_{ij}), \quad b_{ij} = B(e_i, e_j).$$

For vectors $u = \sum u_i e_i$ and $v = \sum v_j e_j$ we have

$$B(u, v) = u^T Bv.$$

Under a change of basis by an invertible matrix P, the representing matrix transforms as

$$B \mapsto P^T BP.$$

Definition 1.10 (Inner Product)**.** An **inner product** on V is a bilinear form I that is

- symmetric: $I(u, v) = I(v, u)$,
- positive definite: $I(v, v) > 0$ for all $v \neq 0$.

Example 1.11. The standard Euclidean inner product on $\mathbb{R}^n$ is

$$\langle u, v\rangle = \sum_{i=1}^{n} u_i v_i,$$

with matrix representation $I = \delta_{ij}$ in the standard basis.

2 Riemannian Metric

2.1 Motivation

In classical differential geometry, a regular surface $S \subset \mathbb{R}^3$ inherits a way of measuring lengths and angles on its tangent spaces via the *first fundamental form*:

$$I_p(\cdot, \cdot) = \langle \cdot, \cdot\rangle_{\mathbb{R}^3}|_{T_pS}.$$

This allows one to compute the length of curves, the area of regions, and to define geometric notions such as angle or curvature purely in terms of the induced inner product.

The idea of a *Riemannian metric* is to generalize this construction to arbitrary smooth manifolds, not necessarily embedded in Euclidean space. Roughly speaking, it equips each tangent space T_pM with an inner product that varies smoothly with p.

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2.2 Definition

Definition 2.1 (Riemannian Metric)**.** A **Riemannian metric** on a smooth manifold M is a smooth assignment

$$g : M \longrightarrow \{\text{inner products on } T_pM\}, \quad p \longmapsto \langle \cdot, \cdot\rangle_p,$$

such that for any smooth vector fields $X, Y \in \mathcal{X}(M)$, the function

$$p \longmapsto \langle X_p, Y_p\rangle_p$$

is smooth on M.

Remark 2.2. There are many thinks need to be careful about the definition.

1. This definition is coordinate-free: it depends only on the tangent bundle structure, not on any particular coordinate chart.

2. The construction is directly motivated by the first fundamental form on a surface embedded in $\mathbb{R}^3$.
3. The Riemannian metric g is *not* a distance function itself — rather, it allows us to *define* one by measuring curve lengths.
4. The smoothness condition means precisely that for all smooth vector fields X, Y, the scalar function $p \mapsto g_p(X_p, Y_p)$ is C^∞.

2.3 Local Expression

In a coordinate chart $(U; x^1, \dots, x^n)$, let

$$\partial_i := \frac{\partial}{\partial x^i}$$

denote the local coordinate vector fields. Then the Riemannian metric is locally represented by a symmetric, positive-definite matrix of smooth functions:

$$g_{ij}(p) = \langle \partial_i(p), \partial_j(p) \rangle_p.$$

Hence, on U, the metric can be written in tensor form:

$$g = \sum_{i,j=1}^{n} g_{ij}\, dx^i \otimes dx^j.$$

2.4 Examples

Example 2.3 (Euclidean Space)**.** On $\mathbb{R}^n$, the standard Riemannian metric is

$$g_{ij} = \delta_{ij}, \qquad g = \sum_i (dx^i)^2,$$

which recovers the usual dot product and Euclidean length.

Example 2.4 (Induced Metric on Submanifolds)**.** Let $f : M^n \hookrightarrow N^{n+k}$ be an immersion of differentiable manifolds, and assume N has a Riemannian metric g_N. Then M inherits a natural metric g_M defined by

$$g_M(u, v)_p := g_N(df_p(u), df_p(v))_{f(p)}, \quad \forall\, u, v \in T_pM.$$

This is called the **induced metric**, and f is an **isometric immersion**.

Example 2.5 (Canonical Metric on the Sphere)**.** Consider $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = \|x\|^2 = \sum_i x_i^2$. Since 1 is a regular value, the level set

$$\mathbb{S}^{n-1} = f^{-1}(1)$$

is a smooth submanifold of $\mathbb{R}^n$. The metric induced from the Euclidean inner product on $\mathbb{R}^n$ is called the **canonical metric** on the unit sphere.

Example 2.6 (Product Metric)**.** If (M_1, g_1) and (M_2, g_2) are Riemannian manifolds, the product manifold $M_1 \times M_2$ carries a natural metric defined by

$$\langle u, v\rangle_{(p,q)} = \langle d\pi_1(u), d\pi_1(v)\rangle_p + \langle d\pi_2(u), d\pi_2(v)\rangle_q,$$

where π_1 and π_2 are the natural projections. The n-dimensional torus $T^n = (\mathbb{S}^1)^n$ endowed with this product metric is called the **flat torus**.

Example 2.7 (Isometries)**.** A diffeomorphism $f : (M, g_M) \to (N, g_N)$ is called an **isometry** if

$$g_M(u, v)_p = g_N(df_p(u), df_p(v))_{f(p)} \quad \forall\, p \in M,\ u, v \in T_pM.$$

If the condition holds locally near each point, f is a **local isometry**.

2.5 Length and Volume

The metric allows us to measure geometric quantities.

Definition 2.8 (Length of a Curve)**.** Let $c : [a, b] \to M$ be a smooth curve. Its **length** with respect to the Riemannian metric g is defined by

$$\mathcal{L}(c) = \int_a^b \sqrt{g_{c(t)}(\dot{c}(t), \dot{c}(t))}\, dt.$$

The definition is invariant under smooth reparametrizations of the curve.

Definition 2.9 (Riemannian Volume)**.** Let g_{ij} be the local representation of the metric on a coordinate chart $\mathcal{X} : U \to \mathbb{R}^n$. For a measurable region $R \subset U$, define its **Riemannian volume** by

$$\mathrm{vol}(R) = \int_{\mathcal{X}(R)} \sqrt{\det(g_{ij})}\, dx^1 \cdots dx^n.$$

Remark 2.10. Under a change of coordinates $x \mapsto \bar{x}$ with Jacobian matrix A, the matrices of the metric satisfy

$$G = A^T \bar{G} A,$$

and hence

$$\sqrt{\det(G)} = |\det(A)| \sqrt{\det(\bar{G})}.$$

This shows that the volume form

$$dV_g = \sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n$$

is globally well-defined and independent of the choice of coordinates.

2.6 Existence of Riemannian Metrics

Proposition 2.11 (Existence Theorem)**.** *Every smooth manifold admits at least one Riemannian metric.*

Proof. Let $\{U_\alpha\}$ be an open cover of M by coordinate charts, and let $\{f_\alpha\}$ be a smooth partition of unity subordinate to this cover. On each U_α, define a local metric g^α induced by the coordinate system. Then define, for $u, v \in T_pM$,

$$g_p(u,v) = \sum_\alpha f_\alpha(p)\, g_p^\alpha(u,v).$$

Since only finitely many $f_\alpha(p)$ are nonzero for each p, this sum is well-defined and smooth. Bilinearity, symmetry, and positive-definiteness are immediate from those of each g^α. Thus g defines a global Riemannian metric on M. □

Summary

- A Riemannian metric provides a smoothly varying inner product on tangent spaces.
- It allows for the **intrinsic** measurement of geometric quantities, such as length, area, and angle.
- Every smooth manifold admits a Riemannian metric via partitions of unity.

3 Riemannian Connections

3.1 Motivation

In Euclidean space $\mathbb{R}^n$, the directional derivative of a vector field Y along another vector field X is defined pointwise by the ordinary derivative:

$$D_X Y = \sum_i X^j \frac{\partial Y^i}{\partial x^j} \frac{\partial}{\partial x^i}.$$

However, this definition relies on the **global linear structure** of $\mathbb{R}^n$. On a general manifold M, the tangent spaces T_pM and T_qM at different points are distinct vector spaces, so we cannot directly subtract or compare vectors at different points.

To differentiate vector fields in a geometrically meaningful way, we introduce the notion of an **affine connection** (or **covariant derivative**), which allows us to "compare" vectors in nearby tangent spaces.

Definition 3.1 (Affine Connection)**.** An **affine connection** ∇ on a differentiable manifold M is a bilinear map

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{X}(M), \quad (X, Y) \longmapsto \nabla_X Y,$$

satisfying for all $X, Y, Z \in \mathcal{X}(M)$ and $f, g \in C^\infty(M)$:

1. $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$ (linearity in first argument)
2. $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$ (additivity in second)
3. $\nabla_X (fY) = X(f)Y + f\nabla_X Y$ (Leibniz rule)

Intuitively, $\nabla_X Y$ describes how the vector field Y "changes" along the direction of X. Note that $\nabla_X Y$ depends linearly on X but not $\mathbb{R}$-linearly on Y (it obeys Leibniz rule instead).

3.2 Covariant Derivative along a Curve

Let $c : I \subset \mathbb{R} \to M$ be a smooth curve. A **vector field along** c is a map $V : I \to TM$ such that $V(t) \in T_{c(t)}M$. We define the derivative of V along c in terms of the connection:

Definition 3.2 (Covariant Derivative along a Curve)**.** Given an affine connection ∇ on M, the **covariant derivative** of V along c is a new vector field $\frac{DV}{dt}$ along c, uniquely determined by:

1. $\dfrac{D}{dt}(V+W) = \dfrac{DV}{dt} + \dfrac{DW}{dt}$,
2. $\dfrac{D}{dt}(fV) = \dfrac{df}{dt}V + f\dfrac{DV}{dt}$ for $f \in C^\infty(I)$,
3. If $V(t) = Y(c(t))$ for some vector field Y on M, then

$$\frac{DV}{dt} = \nabla_{\dot{c}(t)}Y.$$

In local coordinates $(x^1, \dots, x^n)$ on M, if $V(t) = \sum_i V^i(t)\, \frac{\partial}{\partial x^i}$, then

$$\frac{DV^i}{dt} = \frac{dV^i}{dt} + \sum_{j,k} \Gamma^i_{jk} \frac{dx^j}{dt} V^k,$$

where Γ^i_{jk} are called the **Christoffel symbols** of the connection ∇.

3.3 Parallel Transport and Geodesics

Definition 3.3 (Parallel Vector Field)**.** Let $c : I \to M$ be a smooth curve. A vector field V along c is called **parallel** if

$$\frac{DV}{dt} = 0.$$

Parallel transport defines a way to move tangent vectors along a curve while keeping them "parallel" with respect to ∇. It gives a linear isomorphism between tangent spaces:

$$P_{a\to b} : T_{c(a)}M \longrightarrow T_{c(b)}M.$$

Definition 3.4 (Geodesic)**.** A smooth curve $c : I \to M$ is called a **geodesic** (with respect to ∇) if its velocity vector field is parallel along itself, i.e.

$$\frac{D\dot{c}}{dt} = 0.$$

In local coordinates, the geodesic equation reads:

$$\frac{d^2x^i}{dt^2} + \sum_{j,k} \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0.$$

If ∇ is the Levi-Civita connection associated to a Riemannian metric g, then geodesics are locally length-minimizing curves.

3.4 Riemannian Connections and Levi-Civita Connection

Let (M, g) be a Riemannian manifold. A connection ∇ is said to be **compatible with the metric** if it preserves the inner product under parallel transport:

Definition 3.5 (Metric Compatibility)**.** A connection ∇ satisfies

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \quad \forall X, Y, Z \in \mathcal{X}(M).$$

Definition 3.6 (Torsion Tensor)**.** The **torsion** of a connection ∇ is the $(1, 2)$-tensor

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

The connection is called **torsion-free** if $T(X, Y) = 0$ for all X, Y.

Remark 3.7. It is called symmetric in Do Carmo's book [1], but I think the definition of the torsion tensor and torsion-free better fit the later learning process.

Theorem 3.8 (Fundamental Theorem of Riemannian Geometry)**.** *On any Riemannian manifold (M, g), there exists a unique affine connection ∇ that is*

1. *metric compatible, and*
2. *torsion-free.*

*This unique connection is called the **Levi-Civita connection**.*

3.5 Local Expression: Christoffel Symbols

Let ∇ be the Levi-Civita connection associated with g. In a coordinate chart $(U, x^1, \ldots, x^n)$, we define:

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x^k}.$$

By metric compatibility and torsion-free conditions, one obtains the explicit formula:

$$\Gamma_{ij}^k = \frac{1}{2} g^{k\ell} \left(\frac{\partial g_{j\ell}}{\partial x^i} + \frac{\partial g_{i\ell}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^\ell} \right),$$

where $g^{k\ell}$ is the inverse matrix of $g_{k\ell}$.

Example 3.9 (Euclidean Space)**.** In $\mathbb{R}^n$ with the standard metric $g_{ij} = \delta_{ij}$, we have

$$\Gamma_{ij}^k = 0,$$

hence geodesics are straight lines.

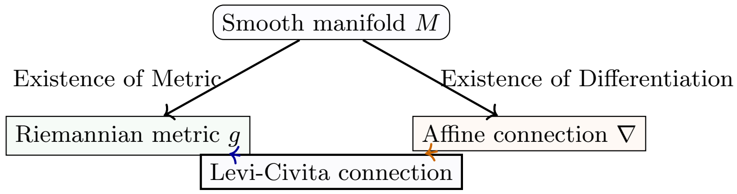
Example 3.10 (Unit Sphere $\mathbb{S}^2$)**.** On $\mathbb{S}^2 \subset \mathbb{R}^3$, the Levi-Civita connection is induced by orthogonal projection of the Euclidean derivative onto the tangent plane:

$$\nabla_X Y = D_X Y - \langle D_X Y, N\rangle N,$$

where N is the unit normal vector field. The geodesics are great circles.

Summary

- The affine connection ∇ generalizes directional differentiation to manifolds.
- The covariant derivative defines parallel transport and geodesics.
- The Levi-Civita connection is the unique torsion-free and metric-compatible connection associated to a Riemannian metric.



References

[1] Manfredo P. do Carmo. *Riemannian Geometry*. Birkhäuser, 1992. ISBN: 978-0-8176-3490-2.

[2] Zuoqin Wang. *Lecture Notes on Differential Geometry*. `http://staff.ustc.edu.cn/~wangzuoq/Courses/23F-Manifolds/index.html`. Accessed: 2025-09-14. 2023.