The Gaussian Free Field: Properties and Applications

Mark A. Olson

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The Discrete Gaussian Free Field

Let us have $G = (V, E)$ be a finite undirected graph. Let there be non-negative weights on the edges, $(w_e)_{e \in E}$. Let us also distinguish a set of vertices $\partial V \subset V$ and call this set of vertices "the boundary." Let $\hat{V} = V \setminus \partial V$ be "the interior" of V.

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Let us define a matrix, $Q = (q_{x,y})_{x,y \in V}$ such that

$$q_{x,y} = \begin{cases} w_{x,y}, & x \neq y \\ -\sum_{z \sim x} w_{x,z}, & x = y \end{cases}. \tag{1}$$

This is the **graph Laplacian**.

The Discrete Gaussian Free Field

Let us define the **Green's function**,

$$G(x,y) = \frac{1}{-q_{y,y}} \mathbb{E}_x \left[\sum_{n=0}^{\infty} \mathbf{1}_{\{Y_n = y; \tau > n\}} \right], \qquad (2)$$

where Y_n is a random walk on G, which jumps from z to w with probability proportional to $w_{z,w}$, and is killed at time $\tau = \min\{k : Y_k \in \partial V\}$.

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It is true that

$$G(x,y) = (-\hat{Q})^{-1}(x,y), \ x, y \in \hat{V} \tag{3}$$

where $\hat{Q}$ is the restriction of Q to $\hat{V} \times \hat{V}$.

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Let us define the **Dirichlet energy** of h as

$$\mathcal{E}[h] = \frac{1}{2} \sum_{x,y \in V} q_{x,y} (h(x) - h(y))^2. \tag{5}$$

Recall: given any boundary condition, the harmonic function satisfying the boundary constraints is also the minimizer of of $\mathcal{E}[h]$.

The Discrete Gaussian Free Field

Theorem

The law of the **discrete GFF** *is absolutely continuous with respect to Lebesgue measure on* $\mathbb{R}^{|V|}$*. The joint density with respect to Lebesgue measure is proportional to*

$$\exp\left(-\frac{1}{4}\sum_{x,y\in V} q_{x,y}(h(x)-h(y))^2\right) \tag{6}$$

$$= \exp\left(-\frac{1}{2}h(\hat{\mathbf{x}})^\intercal G^{-1} h(\hat{\mathbf{x}})\right), \tag{7}$$

with $h(x) = 0$ *for* $x \in \partial V$*.*

Notice, $h : \hat{V} \to \mathbb{R}$ is a centered Gaussian on the interior of V with a covariance structure given by the Green's function.

The 2-d GFF: Preliminaries

Facts about the Green's function in 2 dimensions.
Let $D \subset \mathbb{R}^2$.

1. $G_0^D(x, y) \to 0$ as $y \to y_0 \in \partial D$.

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4. As $y \to x$

$$G_0^D(x,y) = -\log(|x-y|) + \log(R(x;D)) + o(1), \tag{8}$$

where $R(x;D)$ is the conformal radius of $x \in D$.

The 2-d GFF: As a Stochastic Process

Let $D \subset \mathbb{R}^2$ be any domain on which the Green's function is finite. Let $\mathcal{M}_0^+$ be the set of non-negative measures with compact support such that

$$\int_{D^2} G_0^D \rho(dx)\rho(dy) < \infty.$$

Let $\mathcal{M}_0$ be the set of signed measures $\rho = \rho_- + \rho_+$.

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Let $\mathcal{M}_0$ be the set of signed measures $\rho = \rho_- + \rho_+$. For $\rho_1, \rho_2 \in \mathcal{M}_0$ let

$$\Gamma_0(\rho_1, \rho_2) := \int_{D^2} G_0^D(x, y)\rho_1(dx)\rho_2(dy). \tag{9}$$

Note that $\mathcal{M}_0$ includes the case that $\rho(x) = f(x)dx$ where $f(x)$ is a continuous function with compact support, but does not include any point masses due to logarithmic singularity of G_0^D on the diagonal.

The 2-d GFF: As a Stochastic Process

Theorem

There exists a unique stochastic process $(h_\rho)_{\rho \in \mathcal{M}_0}$, such that for every choice of $\rho_1, \ldots, \rho_n$, the vector $(h_{\rho_1}, \ldots, h_{\rho_n})$ is a centered Gaussian vector with covariances structure $\mathrm{Cov}(h_{\rho_i}, h_{\rho_j}) = \Gamma_0(\rho_i, \rho_j)$.

Recalling from last slide that,

$$\Gamma_0(\rho_1, \rho_2) := \int_{D^2} G_0^D(x, y)\rho_1(x)\rho_2(y). \tag{10}$$

The finite dimensional marginals are enough to uniquely characterize the infinite dimensional process by Kolmogorov's extension theorem.

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Reasoning then goes

- For $-\Delta f = 2\pi\rho$ with $\rho, f \in C_0^\infty(D)$ we have that $\|f\|_\nabla^2 = \Gamma_0(\rho,\rho)$.

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- Polarization would then suggest that $\mathrm{Cov}(\langle h, f\rangle_\nabla, \langle h, g\rangle_\nabla) = \langle f, g\rangle_\nabla$.
- So, f_n and f_m should be uncorrelated unit variance Gaussians, so h could be understood as a random series,

$$h_N = \sum_{n=1}^{N} X_n f_n,$$

with X_n i.i.d. $\mathcal{N}(0, 1)$.

What kind of convergence do we get as $N \to \infty$?

Does h_N converge in $H_0^1(D)$? No! But, does $\langle h_N, f \rangle_\nabla$ converge? Yes! It converges almost surely, and in $L^2(\mathbb{P})$.

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Theorem

Suppose D is bounded. If $(X_n)_{n\geq 1}$ are i.i.d standard Gaussian random variables and $(f_n)_{n\geq 1}$ is any orthonormal basis of $H_0^1(D)$, then the series $\sum_{n\geq 1} X_n f_n$ converges almost surely in $H_0^s(D)$, where

$$s = 1 - \frac{d}{2} - \epsilon,$$

for any $\epsilon > 0$. In particular, for $d = 2$, the series converges in $H_0^{-\epsilon}$ for any $\epsilon > 0$.

This also implies that h_N converges in the space of distributions, $(C_0^\infty)'(D)$.

The 2-d GFF: Conformal Invariance

Theorem

The Dirichlet inner product is conformaly invariant. That is, for $\phi : D \to D'$ a conformal map, and $f, g : D \to \mathbb{R}$ we have

$$\int_{D'} \nabla(f \circ \phi^{-1}) \cdot \nabla(g \circ \phi^{-1}) = \int_D \nabla f \cdot \nabla g.$$

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Theorem

If h is a random distribution on $(C_0^\infty)'(D)$ with the law of the Gaussian free field on D,, the the distribution of $h \circ \phi^{-1}$ defined by setting $(h \circ \phi^{-1}, f) = (h, |\phi'|^2(f \circ \phi))$ for $f \in C_0^\infty(D)$, has the law of a GFF on D'.

The 2-d GFF: Circle Averages

Let us fix $z \in D$. Let $\rho_{z,\epsilon}$ denote the uniform distribution on the circle of radius ϵ around z. Note, $\rho_{z,\epsilon} \in \mathcal{M}_0$. Set $h_\epsilon(z) = (h, \rho_{z,\epsilon})$.

Theorem

Let h be a GFF on D. Let $0 < \epsilon_0 < d(z, \partial D)$. For $t \geq t_0 = \log(1/\epsilon_0)$, set

$$B_t = h_{e^{-t}}(z).$$

Then $(B_t, t \geq t_0)$ has the law of a Brownian motion started from B_{t_0}.

The 2-d GFF: Thick Points

We let h be a GFF on $D \subset \mathbb{C}$ simply connected and let $\alpha > 0$. We way a point $z \in D$ is α-**thick** if

$$\liminf_{\epsilon \to 0} \frac{h_\epsilon(z)}{\log(1/\epsilon)} = \alpha.$$

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Theorem

Let $\mathcal{T}_\alpha$ denote the set of α-thick points. Then almost surely,

$$\dim(\mathcal{T}_\alpha) = (2 - \frac{\alpha^2}{2})_+$$

and $\mathcal{T}_\alpha$ is empty if $\alpha > 2$.

Liouville Quantum Gravity

It is our desire to construct the Liouville measure, formally,

$$\mu_\gamma(dz) = e^{\gamma h(z)} dz. \tag{13}$$

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Theorem

Suppose $0 \leq \gamma < 2$. If D is bounded, then the random measure μ_ϵ converges weakly amost surely to a random measure μ, the (bulk) Liouville measure, along the subsequence $\epsilon = 2^{-k}$. μ a.s. has no atoms, and for any $A \subset D$ open, we have $\mu(A) > 0$ almost surely. In fact, we have $\mathbb{E}(\mu(A)) = \int_A R(z, D)^{\gamma^2/2} dz$.

LQG: Thick points

Theorem

Suppose $D \subset \mathbb{C}$ is bounded. Let z be a point sampled according to the Liouville measure μ, normalized to be a probability measure. Then, a.s.,

$$\lim_{\epsilon \to 0} \frac{h_\epsilon(z)}{\log(1/\epsilon)} = \gamma.$$

That is, z is almost surely a γ-thick point.

So the γ-Liouville measure is supported on the set $\mathcal{T}_\gamma$. Thick points with $\alpha < \gamma$ don't carry enough weight, and there aren't enough thick points with $\alpha > \gamma$ to matter in the measure.

LQG: Conformal covariance

Theorem

Let $f : D \to D'$ be a conformal map, and let h be a GFF in D. Then $h' = f \circ f^{-1}$ is a GF in D', and

$$\mu_h \circ f^{-1} = e^{\gamma Q \log |(f^{-1})'|} \mu_{h'}$$

where $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$.

LQG: Convergence of planar maps to LQG

- Spanning tree decorated planar map converges to $\sqrt{2}$-LQG
- Uniform Random Triangulations converge to $\sqrt{8/3}$-LQG
- Conjectures on other maps such as the critical bond percolation decorated uniform map and critical Ising model decorated map.