Chapter 1
The Model

Small damped oscillations in the absence of gyroscopic forces are described by the vector differential equation

$$M\ddot{x} + C\dot{x} + Kx = f(t). \tag{1.1}$$

Here $x = x(t)$ is an $\mathbb{R}^n$-valued function of time $t \in \mathbb{R}$; M, C, K are real symmetric matrices of order n. Typically M, K are positive definite whereas C is positive semidefinite. $f(t)$ is a given vector function. The physical meaning of these objects is

$x_j(t)$	position or displacement
M	mass
C	damping
K	stiffness
$f(t)$	external force

while the dots mean time derivatives. So, any triple M, C, K will be called a *damped system*, whereas the solution $x = x(t)$ is called *the motion* or also *the response* of the linear system to the external force $f(t)$.

In this chapter we will describe common physical processes, governed by these equations and give an outline of basic mechanical principles which lead to them. It is hoped that this introduction is self-contained enough to accomodate readers with no background in Physics.

Example 1.1 As a model example consider the spring-mass system like the one in Fig. 1.1. Here $x = \begin{bmatrix} x_1 \cdots x_n \end{bmatrix}^T$ where $x_i = x_i(t)$ is the horizontal displacement of the ith mass point from its equilibrium position and

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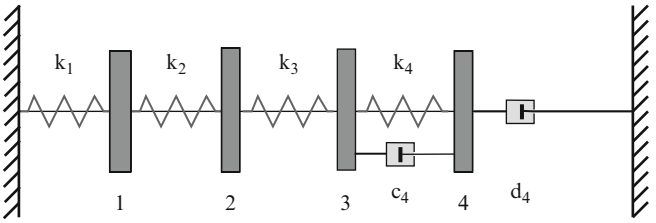

Fig. 1.1 Oscillator ladder

$$M = \operatorname{diag}(m_1, \ldots, m_n), \quad m_1, \ldots, m_n > 0 \tag{1.2}$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & & & \\ -k_2 & k_2 + k_3 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -k_n \\ & & & -k_n & k_n + k_{n+1} \end{bmatrix}, \quad k_1, \ldots k_{n+1} > 0 \tag{1.3}$$

(all void positions are zeros) and

$$C = C_{in} + C_{out}, \tag{1.4}$$

$$C_{in} = \begin{bmatrix} c_1 + c_2 & -c_2 & & & \\ -c_2 & c_2 + c_3 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -c_n \\ & & & -c_n & c_n + c_{n+1} \end{bmatrix}, \quad c_1, \ldots c_{n+1} \geq 0 \tag{1.5}$$

$$C_{out} = \operatorname{diag}(d_1, \ldots, d_n), \quad d_1, \ldots, d_n \geq 0 \tag{1.6}$$

In the case of Fig. 1.1 we have $n = 4$ and

$$k_5 = 0, \quad c_2 = c_3 = 0,$$

$$d_1 = d_2 = d_3 = 0, \quad d_4 > 0.$$

Various dynamical quantities such as 'force', 'work', 'energy' will play an important role in the course of these Notes. We shall next give a short compact overview of the physical background of (1.1).

1.1 Newton's Law

We will derive (1.1) from Newton's law for our model example. The coordinate x_i of the ith mass point is measured from its static equilibrium position, that is, from the point where this mass is at rest and $f = 0$. Newton's second law of motion for the ith particle reads

$$m_i \ddot{x}_i = f_i^{tot},$$

where f_i^{tot} is the sum of all forces acting upon that mass point. These forces are:

- The external force $f_i(t)$.
- The elastic force from the neighbouring springs, negative proportional to the relative displacement.

$$k_i(x_{i-1} - x_i) + k_{i+1}(x_{i+1} - x_i), \quad i = 1, \ldots, n, \tag{1.7}$$

  where k_j is the jth *spring stiffness*.
- The inner damping force from the neighbouring dampers, negative proportional to the relative velocity

$$c_i(\dot{x}_{i-1} - \dot{x}_i) + c_{i+1}(\dot{x}_{i+1} - \dot{x}_i), \quad i = 1, \ldots, n. \tag{1.8}$$

- The external damping force, negative proportional to the velocity $-d_i\dot{x}_i$, where d_j, c_j are the respective *inner and external damper viscosities*.

Here to simplify the notation we have set $x_0 = x_{n+1} = 0, \dot{x}_0 = \dot{x}_{n+1} = 0$, these are the fixed end points. Altogether we obtain (1.1) with M, C, K, from (1.2)–(1.4). All these matrices are obviously real and symmetric. By

$$x^T M x = \sum_{j=1}^{n} m_j x_j^2 \tag{1.9}$$

and

$$x^T K x = k_1 x_1^2 + \sum_{j=2}^{n} k_j (x_j - x_{j-1})^2 + k_{n+1} x_n^2 \tag{1.10}$$

both M and K are positive definite. By the same argument C_{in} and C_{out} are positive semidefinite.

Exercise 1.2 *How many coefficients k_i in (1.3) may vanish while keeping K positive definite? Interpret the response physically!*

Example 1.3 A typical external force stems from a given movement of the frame in which the vibrating structure is anchored (this is the so-called

inertial force). On the model example in Fig. 1.1 the system is anchored on two points. The force caused by the horizontal movement of these two points is taken into account by replacing the zero values of x_0, x_{n+1} in (1.7) and (1.8) by given functions $x_0(t), x_{n+1}(t)$, respectively. This does not change the matrices M, C, K whereas $f(t)$ reads

$$f(t) = \begin{bmatrix} k_1 x_0(t) + c_1 \dot{x}_0(t) \\ 0 \\ \vdots \\ 0 \\ k_{n+1} x_{n+1}(t) + c_{n+1} \dot{x}_{n+1}(t) \end{bmatrix}.$$

Exercise 1.4 *Find the equilibrium configuration (i.e. the displacement x at rest) of our model example if the external force f (i) vanishes or (ii) is a constant vector. Hint: the matrix K can be explicitly inverted.*

The model in Example 1.1 is important for other reasons too. It describes a possible discretisation of a continuous damped system (vibrating string). There the parameters c_i come from the inner friction within the material whereas d_i describe the external damping caused by the medium in which the system is moving (air, water) or just artificial devices (dashpots) purposely built in to calm down dangerous vibrations. In the latter case there usually will be few such dashpots resulting in the low rank matrix C_{out}.

It should be mentioned that determining the inner damping matrix for complex vibrating systems in real life may be quite a difficult task, it involves special mathematical methods as well as experimental work.

From the derivations above we see that determining the equilibrium *precedes* any work on oscillations. The equilibrium is the first approximation to the true behaviour of the system, it is found by solving a linear or nonlinear system of equations. The next approximation, giving more detailed information are the linear oscillations around the equilibrium, that is, their movement is governed by a system of linear differential equations. Their linearity will be seen to be due to the assumption that the oscillations have small amplitudes.

1.2 Work and Energy

We will now introduce some further relevant physical notions based on Example 1.1. *The work* performed by any force $\phi_j(t)$ on the jth point mass in the time interval $t_1 \le t \le t_2$ is given by

$$\int_{t_1}^{t_2} \phi_j(t) \dot{x}_j dt,$$

this corresponds to the rule 'work equals force times distance'. So is the work of the external force:

$$\int_{t_1}^{t_2} f_j(t)\dot{x}_j dt, \tag{1.11}$$

of the damping force:

$$\int_{t_1}^{t_2} [c_j(\dot{x}_{j-1} - \dot{x}_j) + c_{j+1}(\dot{x}_{j+1} - \dot{x}_j)]\dot{x}_j dt - \int_{t_1}^{t_2} d_j \dot{x}_j^2 dt,$$

of the elastic force:

$$\int_{t_1}^{t_2} [k_j(x_{j-1} - x_j) + k_{j+1}(x_{j+1} - x_j)]\dot{x}_j dt.$$

To this we add the work of the so-called 'inertial force' which is given by

$$\int_{t_1}^{t_2} m_j \ddot{x}_j \dot{x}_j dt \tag{1.12}$$

The total work is the sum over all mass points, so from (1.11) and (1.12) we obtain

$$\int_{t_1}^{t_2} \dot{x}^T f(t) dt, \tag{1.13}$$

$$-\int_{t_1}^{t_2} \dot{x}^T C\dot{x} dt,$$

$$-\int_{t_1}^{t_2} \dot{x}^T Kx dt = 2(E_p(x(t_1)) - E_p(x(t_2))),$$

$$\int_{t_1}^{t_2} \dot{x}^T M\ddot{x} dt = 2(E_k(x(t_2)) - E_k(x(t_1))), \tag{1.14}$$

as the total work of the external forces, damping forces, elastic forces and inertial forces, respectively; here

$$E_p(x) = \frac{1}{2} x^T Kx, \tag{1.15}$$

$$E_k(\dot{x}) = \frac{1}{2} \dot{x}^T M\dot{x} \tag{1.16}$$

are *the potential* and *the kinetic energy* and

$$E(x, \dot{x}) = E_p(x) + E_k(\dot{x}) \tag{1.17}$$

the total energy. Note the difference: in the first two cases the work depends on the whole motion $x(t)$, $t_1 \le t \le t_2$ whereas in the second two cases it depends just on the values of E_p, E_k, respectively, taken at the end points of the motion. In the formulae (1.9), (1.10), (1.13) and (1.14) we observe a property of both potential and kinetic energy: they are 'additive' magnitudes, that is, the kinetic energy is a sum of the kinetic energies of single point masses whereas the potential energy is a sum of the potential energies of single springs.

The relations (1.13)–(1.17) make sense and will be used for general damped systems. By premultiplying (1.1) by $\dot{x}^T$ we obtain *the differential energy balance*

$$\frac{d}{dt}E(x,\dot{x}) + \dot{x}^T C\dot{x} = \dot{x}^T f(t). \tag{1.18}$$

This is obviously equivalent to *the integral energy balance*

$$E(x,\dot{x})\Big|_{t_1}^{t_2} = -\int_{t_1}^{t_2} \dot{x}^T C\dot{x}dt + \int_{t_1}^{t_2} \dot{x}^T f(t)dt. \tag{1.19}$$

for any $t_1 \le t \le t_2$. We see that the work of the damping forces is always non-positive. This effect is called *the energy dissipation*. Thus, (1.19) displays the amount of energy which is transformed into thermal energy. If $f \equiv 0$ then this energy loss is measured by the decrease of the total energy.

If both $f \equiv 0$ and $C = 0$ then the total energy is preserved in time; such systems are called *conservative*.

1.3 The Formalism of Lagrange

The next (and last) step in our short presentation of the dynamics principles is to derive the Lagrangian formalism which is a powerful and simple tool in modelling mechanical systems. The position (also called *configuration*) of a mechanical system is described by the generalised coordinates $q_1, \ldots, q_s$ as components of the vector q from some region of $\mathbb{R}^s$ which is called *the configuration space*. The term 'generalised' just means that q_i need not be a Euclidian coordinate as in (1.1) but possibly some other measure of position (angle and the like). To this we add one more copy of $\mathbb{R}^s$ whose elements are the *generalised velocities* $\dot{q}$. The dynamical properties are described by the following four functions, defined on $\mathbb{R}^{2s}$:

- The kinetic energy $T = T(q,\dot{q})$, having a minimum equal to zero at $(q,0)$ for any q and with $T''_{\dot{q}}(q,\dot{q})$ positive semidefinite for any q.
- The potential energy $V = V(q)$ with $V'(q_0) = 0$ and $V''(q)$ everywhere positive definite (that is, V is assumed to be uniformly convex).

- The dissipation function $Q = Q(q, \dot{q})$, having a minimum equal to zero at $(q_0, 0)$, with $Q''_{\dot{q}}(q, \dot{q})$ positive semidefinite for any q as well as.
- The time dependent *generalised external force* $f = f(t) \in \mathbb{R}^s$.

Here the first three functions are assumed to be smooth enough (at least twice continuously differentiable) and the symbols $'$ and $''$ denote the vector of first derivatives (gradient) and the matrix of second derivatives (Hessian), respectively. The subscript (if needed) denotes the set of variables with respect to which the derivative is taken. From these functions we construct *the Lagrangian*

$$L = T - V + f^T q.$$

The time development of this system is described by the *Lagrange equations*

$$\frac{d}{dt} L'_{\dot{q}} - L'_q + Q'_{\dot{q}} = 0. \tag{1.20}$$

This is a system of differential equations of at most second order in time. Its solution, under our, rather general, conditions is not easily described. If we suppose that the solution $q = q(t)$ does not depend on time, i.e. the system is at rest then $\dot{q} \equiv 0$ in (1.20) yields

$$-V'(q) = f$$

with the unique solution $q = \hat{q}_0$ (note that here f must also be constant in t and the uniqueness is a consequence of the uniform convexity of V). The point $\hat{q}_0$ is called *the point of equilibrium* of the system. Typical such situations are those in which f vanishes; then $\hat{q}_0 = q_0$ which is the minimum point of the potential energy. The energy balance in this general case is completely analogous to the one in (1.18) and (1.19) and its derivation is left to the reader.

Remark 1.5 If f is constant in time then we can modify the potential energy function into

$$\hat{V}(q) = V(q) - q^T f$$

thus obtaining a system with the vanishing generalised external force. If, in addition, $Q = 0$ then the system is conservative. This shows that there is some arbitrariness in the choice of the potential energy function.

Exercise 1.6 *Using the expressions (1.16), (1.15) for the kinetic and the potential energy, respectively, choose the dissipation function and the generalised external force such that, starting from (1.20) the system (1.1) is obtained.*

Solution. As the dissipation function take

$$Q = \frac{1}{2} \dot{x}^T C \dot{x};$$

then

$$L = \frac{1}{2}\dot{x}^T M\dot{x} - \frac{1}{2}x^T Kx + f^T x,$$

$$L'_{\dot{x}} = M, \quad L'_x = -Kx + f, \quad Q'_{\dot{x}} = C$$

and (1.20) immediately yields (1.1).

Now consider small oscillations around q_0, that is, the vectors $q - q_0$ and $\dot{q}$ will be considered small for all times. This will be expected, if the distance from the equilibrium together with the velocity, taken at some initial time as well as the external force $f(t)$ at all times are small enough. This assumed, we will approximate the functions T, V, Q by their Taylor polynomials of second order around the point $(q_0, 0)$:

$$V(q) \approx V(q_0) + (q - q_0)^T V'(q_0) + \frac{1}{2}(q - q_0)^T V''(q_0)(q - q_0),$$

$$T(q, \dot{q}) \approx T(q_0, 0) + \dot{q}^T T'_{\dot{q}}(q_0, 0) + \frac{1}{2}\dot{q}^T T''_{\dot{q}}(q_0, 0)\dot{q}$$

and similarly for Q. Note that, due to the conditions on T the matrix $T''(q_0, 0)$ looks like

$$\begin{bmatrix} 0 & 0 \\ 0 & T''_{\dot{q}}(q_0, 0) \end{bmatrix}$$

(the same for $Q''(q_0, 0)$). Using the properties listed above we have

$$T(q_0, 0) = 0, \quad T'_{\dot{q}}(q_0, 0) = 0,$$

$$Q(q_0, 0) = 0, \quad Q'_{\dot{q}}(q_0, 0) = 0,$$

$$V'(q_0) = 0.$$

With this approximation the Lagrange equations (1.20) yield (1.1) with $x = q$ and

$$M = T''_{\dot{q}}(q_0, 0), \quad C = Q''_{\dot{q}}(q_0, 0), \quad K = V''(q_0). \tag{1.21}$$

This approximation is called *linearisation*,[1] because non-linear equations are approximated by linear ones via Taylor expansion.

Under our conditions all three matrices in (1.21) are real and symmetric. In addition, K is positive definite while the other two matrices are positive semidefinite. If we additionally assume that T is also uniformly convex then M will be positive definite as well. We have purposely allowed M to be only positive semidefinite as we will analyse some cases of this kind later.

Example 1.7 Consider a hydraulic model which describes the oscillation of an incompressible homogeneous heavy fluid (e.g. water) in an open vessel.

[1] The term 'linearisation' will be used later in a very different sense.

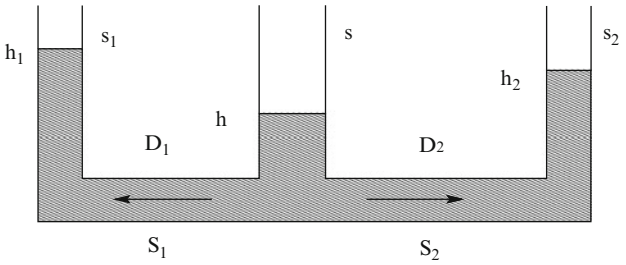

Fig. 1.2 Open vessel

The vessel consists of coupled thin tubes which, for simplicity, are assumed as cylindrical (see Fig. 1.2).

The parameters describing the vessel and the fluid are

- h_1, h, h_2: The fluid heights in the vertical tubes,
- s_1, s, s_2: The cross sections of the vertical tubes,
  - D_1, D_2: The lengths,
  - S_1, S_2: The cross sections,
  - v_1, v_2: The velocities (in the sense of the arrows),

  in the horizontal tubes.
- ρ: The mass density of the fluid

While the values h_1, h, h_2, v_1, v_2 are functions of time t all others are constant. Since the fluid is incompressible the volume conservation laws give

$$s_1\dot{h}_1 = S_1 v_1 \tag{1.22}$$

$$S_1 v_1 + s\dot{h} + S_2 v_2 = 0 \tag{1.23}$$

$$s_2\dot{h}_2 = S_2 v_2 \tag{1.24}$$

These equalities imply

$$s_1\dot{h}_1 + s\dot{h} + s_2\dot{h}_2 = 0 \tag{1.25}$$

which, integrated, gives

$$s_1 h_1 + sh + s_2 h_2 = \Gamma, \tag{1.26}$$

where Γ is a constant (in fact, $\Gamma + D_1 S_1 + D_2 S_2$ is the fixed total volume of the fluid). Thus, the movement of the system is described by $h_1 = h_1(t)$, $h_2 = h_2(t)$.

To obtain the Lagrange function we must find the expressions for the kinetic and the potential energy as well as for a dissipation function – as functions of $h_1, h_2, \dot{h}_1, \dot{h}_2$.

In each vertical tube the potential energy equals the mass times half the height (this is the center of gravity) times g, the gravity acceleration. The total potential energy V is given by

$$\frac{V}{\rho g} = \frac{V(h_1, h_2)}{\rho g} = s_1 \frac{h_1^2}{2} + s\frac{h^2}{2} + s_2\frac{h_2^2}{2} \tag{1.27}$$

$$= s_1 \frac{h_1^2}{2} + \frac{(\Gamma - s_1 h_1 - s_2 h_2)^2}{2s} + s_2\frac{h_2^2}{2} \tag{1.28}$$

(the contributions from the horizontal tubes are ignored since they do not depend on $h_1, h_2, \dot{h}_1, \dot{h}_2$ and hence do not enter the Lagrange equations). It is readily seen that V takes its minimum at

$$h_1 = h = h_2 = \hat{h} := \frac{\Gamma}{s_1 + s + s_2}. \tag{1.29}$$

The kinetic energy in each tube equals half the mass times the square of the velocity. The total kinetic energy T is given by

$$T/\rho = T(h_1, h_2, \dot{h}_1, \dot{h}_2)/\rho$$

$$= h_1 s_1 \frac{\dot{h}_1^2}{2} + D_1 S_1 \frac{v_1^2}{2} + hs\frac{\dot{h}^2}{2} + D_2 S_2 \frac{v_2^2}{2} + h_2 s_2 \frac{\dot{h}_2^2}{2} \tag{1.30}$$

$$= \left(h_1 s_1 + \frac{D_1 s_1^2}{S_1}\right)\frac{\dot{h}_1^2}{2} + (\Gamma - s_1 h_1 - s_2 h_2)\frac{(s_1\dot{h}_1 + s_2\dot{h}_2)^2}{2s^2} + \left(h_2 s_2 + \frac{D_2 s_2^2}{S_2}\right)\frac{\dot{h}_2^2}{2}, \tag{1.31}$$

where $\dot{h}$ was eliminated by means of (1.25). While T and V result almost canonically from first principles and the geometry of the system the dissipation function allows more freedom. We will assume that the frictional force in each tube is proportional to the velocity of the fluid and to the fluid-filled length of the tube. This leads to the dissipation function

$$Q = Q(h_1, h_2, \dot{h}_1, \dot{h}_2)$$

$$= h_1\theta_1\frac{\dot{h}_1^2}{2} + D_1\Theta_1\frac{v_1^2}{2} + h\theta\frac{\dot{h}^2}{2} + D_2\Theta_2\frac{v_2^2}{2} + h_2\theta_2\frac{\dot{h}_2^2}{2} \tag{1.32}$$

$$= \left(h_1\theta_1 + \frac{D_1\Theta_1 s_1^2}{S_1^2}\right)\frac{\dot{h}_1^2}{2} + (\Gamma - s_1h_1 - s_2h_2)\frac{(s_1\dot{h}_1 + s_2\dot{h}_2)^2}{2s^3}\Theta \tag{1.33}$$

$$+ \left(h_2\theta_2 + \frac{D_2\Theta_2 s_2^2}{S_2^2}\right)\frac{\dot{h}_2^2}{2}, \tag{1.34}$$

where $\theta_1, \Theta_1, \theta, \Theta_2, \theta_2$ are positive constants characterising the friction per unit length in the respective tubes. By inserting the obtained T, V, Q into (1.20) and by taking q as $\begin{bmatrix} h_1 & h_2 \end{bmatrix}^T$ we obtain two non-linear differential equations of second order for the unknown functions h_1, h_2.

Now comes the linearisation at the equilibrium position which is given by (1.29). The Taylor expansion of second order for T, V, Q around the point $h_1 = h_2 = \hat{h},\ \dot{h}_1 = \dot{h}_2 = 0$ takes into account that all three gradients at that point vanish thus giving

$$V \approx V(\hat{h}, \hat{h}) + \rho g\left(s_1\frac{\chi_1^2}{2} + \frac{(s_1\chi_1 + s_2\chi_2)^2}{2s} + s_2\frac{\chi_2^2}{2}\right)$$

$$= V(\hat{h}, \hat{h}) + \frac{1}{2}\chi^T K\chi \tag{1.35}$$

with $\chi = \begin{bmatrix} h_1 - \hat{h} & h_2 - \hat{h} \end{bmatrix}^T$ and

$$T \approx \rho\left(\left(\hat{h}s_1 + \frac{D_1 s_1^2}{S_1}\right)\frac{\dot{\chi}_1^2}{2} + \hat{h}\frac{(s_1\dot{\chi}_1 + s_2\dot{\chi}_2)^2}{2s} + \left(\hat{h}s_2 + \frac{D_2 s_2^2}{S_2}\right)\frac{\dot{\chi}_2^2}{2}\right) \tag{1.36}$$

$$= \frac{1}{2}\dot{\chi}^T M\dot{\chi} \tag{1.37}$$

$$Q \approx \left(\hat{h}\theta_1 + \frac{D_1\Theta_1 s_1^2}{S_1^2}\right)\frac{\dot{\chi}_1^2}{2} + \hat{h}\frac{(s_1\dot{\chi}_1 + sa_2\dot{\chi}_2)^2}{2s^2}\Theta + \left(\hat{h}\theta_2 + \frac{D_2\Theta_2 s_2^2}{S_2^2}\right)\frac{\dot{\chi}_2^2}{2} \tag{1.38}$$

$$= \frac{1}{2}\dot{\chi}^T C\dot{\chi}. \tag{1.39}$$

The symmetric matrices M, C, K are immediately recoverable from (1.35)–(1.38) and are positive definite by their very definition. By inserting these T, V, Q into (1.20) the equations (1.1) result with $f = 0$.

Any linearised model can be assessed in two ways:

- By estimating the error between the true model and the linearised one and
- By investigating the *well-posedness* of the linear model itself: do small forces $f(t)$ and small initial data produce small solutions?

While the first task is completely out of the scope of the present text, the second one will be given our attention in the course of these Notes.

1.4 Oscillating Electrical Circuits

Equation (1.1) governs some other physical systems, most relevant among them is the electrical circuit system which we will present in the following example.

Example 1.8 n simple wire loops (circuits) are placed in a chain. The jth circuit is characterised by the current I_j, the impressed electromagnetic force $E_j(t)$, the capacity $C_j > 0$, the resistance $R_j \geq 0$, and the left and the right inductance $L_j > 0$, $M_j > 0$, respectively. In addition, the neighbouring jth and $j+1$th inductances are accompanied by the mutual inductance $N_j \geq 0$. The circuits are shown in Fig. 1.3 with $n = 3$. The inductances satisfy the inequality

$$N_j < \sqrt{M_j L_{j+1}}.$$

The equation governing the time behaviour of this system is derived from the fundamental electromagnetic laws and it reads in vector form

$$\mathcal{L}\ddot{I} + \mathcal{R}\dot{I} + \mathcal{K}I = \dot{E}(t).$$

Here

$$\mathcal{L} = \begin{bmatrix} L_1 + M_1 & N_1 & & & \\ N_1 & L_2 + M_2 & N_2 & \ddots & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & N_{n-1} \\ & & & N_{n-1} & L_n + M_n \end{bmatrix}$$

is *the inductance matrix*,

$$\mathcal{R} = \operatorname{diag}(R_1, \ldots, R_n)$$

is *the resistance matrix* and

$$\mathcal{K} = \operatorname{diag}(1/C_1, \ldots, 1/C_n)$$

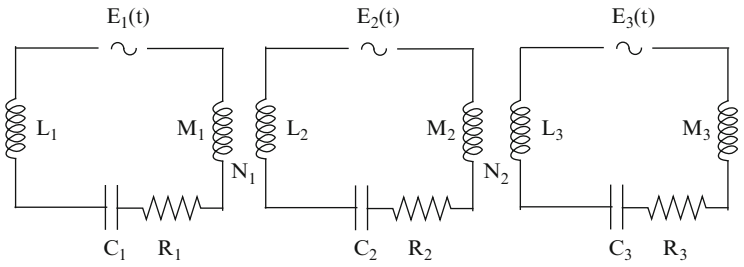

Fig. 1.3 Circuits

is *the inverse capacitance matrix*. The latter two matrices are obviously positive definite whereas the positive definiteness of the first one is readily deduced from its structure and the inequality (1.40).

Exercise 1.9 *Show the positive definiteness of the inductance matrix* $\mathcal{L}$. *Hint: use the fact that each of the matrices*

$$\begin{bmatrix} M_j & N_j \\ N_j & L_{j+1} \end{bmatrix}$$

is positive definite.