UC Berkeley Math 228B, Spring 2020
Problem Set 2
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Suncica Canic

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1. Consider the PDE

$$u_t = \kappa u_{xx} - \gamma u, \tag{1}$$

which models a diffusion with decay provided $\kappa > 0$ and $\gamma > 0$. Consider methods of the form

$$U_j^{n+1} = U_j^n + \frac{k\kappa}{2h^2}[U_{j-1}^n - 2U_j^n + U_{j+1}^n + U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1}] - k\gamma[(1-\theta)U_j^n + \theta U_j^{n+1}] \tag{2}$$

where θ is a parameter. In particular, if $\theta = 1/2$ then the decay term is modeled with the same centered-in-time approach as the diffusion term and the method can be obtained by applying the Trapezoidal method to the MOL formulation of the PDE. If $\theta = 0$ then the decay term is handled explicitly. For more general reaction-diffusion equations it may be advantageous to handle the reaction terms explicitly since these terms are generally nonlinear, so making them implicit would require solving nonlinear systems in each time step (whereas handling the diffusion term implicitly only gives a linear system to solve in each time step).

(a) (In Problem Set 1)

(b) Using von Neumann analysis, show that this method is unconditionally stable if $\theta \geq 1/2$.

(c) Show that if $\theta = 0$ then the method is stable provided $k \leq 2/\gamma$, independent of h.

2. Consider Euler's equations of compressible gas dynamics in two space dimensions:

$$\boldsymbol{u}_t + \nabla \cdot \boldsymbol{F} = 0, \quad where \quad \boldsymbol{u} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{bmatrix} \quad and \quad \boldsymbol{F} = \begin{bmatrix} \rho u & \rho v \\ \rho u^2 + p & \rho uv \\ \rho uv & \rho v^2 + p \\ u(\rho E + p) & v(\rho E + p) \end{bmatrix} \tag{3}$$

Here, ρ is the fluid density, $u; v$ are the velocity components, and E is the total energy. For an ideal gas, the pressure p has the form $p = (\gamma-1)\rho(E-(u^2+v^2)/2)$, where γ is the adiabatic gas constant. We will solve these on a square domain with periodic boundary conditions, for $0 \le t \le T$. The spatial derivatives will be discretized with a fourth order compact Padé scheme, and the solution will be filtered using a sixth order compact filter. A standard $RK4$ scheme will be used for time integration.

(a) Write a function with the syntax

$$Frx, Fry, Frux, Fruy, Frvx, Frvy, FrEx, FrEy = euler\_fluxes(r, ru, rv, rE)$$

which returns the 8 flux functions in (3) for the 4 solution components. Assume $\gamma = 7/5$.

(b) Write a function with the syntax

$$divF = compact\_div(Fx, Fy, h)$$

which calculates the divergence of a grid function field $F = [Fx; Fy]$ using the 4th order compact Padé scheme with periodic boundary conditions and grid spacing h (see (10.79) in LeVeque, for more information):

$$f'_{i-1} + 4f'_i + f'_{i+1} = 3\frac{f_{i+1} - f_{i-1}}{h}$$

(c) Write a function with the syntax

$$u = compact\_filter(u, \alpha)$$

which filters the grid solution u using the 4th order compact filter with a given parameter α:

$$\alpha \hat{f}_{i-1} + \hat{f}_i + \alpha \hat{f}_{i+1} = af_i + \frac{c}{2}(f_{i+2} + f_{i-2}) + \frac{b}{2}(f_{i+1} + f_{i-1})$$

where a, b, and c depend on α:

$$a = 5/8 + 3\alpha/4, \quad b = \alpha + 1/2, \quad c = \alpha/4 - 1/8.$$

(d) Write a function with the syntax

$$fr, fru, frv, frE = euler\_rhs(r, ru, rv, rE, h)$$

which computes the right-hand side of the discretized $-\nabla \cdot \boldsymbol{F}$ (essentially just calling subroutines $euler\_fluxes$ and $compact\_div$).

(e) Write a function with the syntax

$$r, ru, rv, rE = euler\_rk4step(r, ru, rv, rE, h, k, \alpha)$$

which takes one $RK4$ step using $euler\_rhs()$ and filters each solution component using $compact\_filter()$.

(f) Validate your solver with the Isentropic Vortex Convection problem. (**Ref:** C.-W. Shu, "Essentially Non-oscillatory and Weighted Essentially Non-oscillatory Schemes for Hyperbolic Conservation Laws", ICASE Report 97-65, 1997.)

(f.1) Use a square domain $0 \le x; y \le 10$ with grid spacing $h = 10/N$ and $N = 32; 64; 128$. Use the time step $k \le 0.3 * h$, adjusted so the final time $T = 5\sqrt{2}$ is a multiple of k. Use the initial solution:

$$\rho_\infty = 1, \quad u_\infty = 0.1, \quad v_\infty = 0, \quad p_\infty = 1. \tag{4}$$

$$\rho = \left[1 - \frac{(\gamma - 1)b^2}{8\gamma\pi^2}e^{1-r^2}\right]^{\frac{1}{\gamma-1}}, \quad p = \rho^\gamma \tag{5}$$

$$u = u_\infty - \frac{b}{2\pi}e^{\frac{1}{2}(1-r^2)}(y - y_c), \quad v = v_\infty + \frac{b}{2\pi}e^{\frac{1}{2}(1-r^2)}(x - x_c), \tag{6}$$

where $b = 0.5$ is the vortex strength and $r = \sqrt{(x - x_c)^2 + (y - y_c)^2}$ is the distance from the inital vortex center $(x_c, y_c) = (5, 5)$.

(f.2) Compare with the exact final solution. We can easily obtain the exact solution since we know that the center of the vortex will be at $x_c + FinalTime * u_\infty, y_c$:

Exact solution:

$$x_{exact} = x_c + FinalTime * u_\infty, \tag{7}$$

$$d = (x - x_{exact})^2 + (y - y_c)^2, \tag{8}$$

$$\rho_{exact} = \left[1 - \frac{(\gamma - 1)b^2}{8\gamma\pi^2}e^{1-d^2}\right]^{\frac{1}{\gamma-1}}, \quad p = \rho_{exact}^\gamma \tag{9}$$

$$u_{exact} = u_\infty - \frac{b}{2\pi}e^{\frac{1}{2}(1-d^2)}(y - y_c), \quad v = v_\infty + \frac{b}{2\pi}e^{\frac{1}{2}(1-d^2)}(x - x_{exact}), \tag{10}$$

(f.3) Modify the filter parameters $\alpha = [0.48, 0.499]$ to see the numerical results.

(f.4) Calculate the errors in the infinity norm over all solution components. Plot the errors versus grid size h in a log-log plot, for the 3 different grid spacings h and the two filter coeficients $\alpha = 0.499$, $\alpha = 0.48$. Estimate the slopes of the two curves.

(g) Simulate a Kelvin-Helmholtz instability, using the unit square domain $0 \le x; y \le 1$ with grid spacing $h = 1/N$ and $N = 256$ grid points in each coordinate direction, $\alpha = 0.48$, time step $k \le 0.3 * h$, final time $T = 1$, and the initial condition:

$$\rho = \begin{cases} 2, & if |y - 0.5| < 0.15 + sin(2\pi x)/200 \\ 1, & otherwise. \end{cases} \tag{11}$$

$$u = \rho - 1, \quad v = 0, \quad p = 3. \tag{12}$$

Plot the final solution using a contour or color plot of the density ρ.

Code Submission: Submit a zip-file on bCourses which contains a single Matlab file named *euler_compact.m* where all requested functions are defined

(that is, $euler\_fluxes$, $compact\_div$, $compact\_filter$, $euler\_rhs$, and $euler\_rh4step$), and any other supporting documents, including a pdf file explaining what you did, and the results you obtained.