UC Berkeley Math 228B, Spring 2020
Problem Set 5
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1. Let K be a triangle with vertices a^i, $i = 1, 2, 3$, and let a^{ij}, $i < j$, denote the midpoints of the sides of K. Prove that $v \in \mathbb{P}^2(K)$ is uniquely determined by the following degrees of freedom:

$$v(a^i), i = 1, 2, 3,$$
$$v(a^{ij}), i, j = 1, 2, 3, i < j.$$

Also show that the functions in the corresponding finite element space V_h are continuous.

2. Consider the Neumann problem:

$$\left\{ \begin{array}{rcll} \triangle u &=& f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} &=& g, & \text{on } \Gamma, \end{array} \right. \tag{1}$$

(a) Is it true that if u is a solution of (1) then so is $u + c$ for any constant c?

(b) Will the following condition guarantee uniqueness:

$$\int_\Omega u \, dx = 0?$$

Prove your statement.

(c) Give a variational formulation of

$$\left\{ \begin{array}{rcll} \triangle u &=& f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} &=& g, & \text{on } \Gamma, \\ \displaystyle\int_\Omega u \, dx &=& 0, & \end{array} \right.$$

using the space

$$V = \{v \in H^1(\Omega) : \int_\Omega u \; dx = 0\},$$

and prove that the classical elliptic problem conditions are satisfied:

1. $a(\cdot, \cdot)$ is symmetric,
2. $a(\cdot, \cdot)$ is continuous,
3. $a(\cdot, \cdot)$ is V-elliptic,
4. L is continuous.

3. Let V_h be a finite element space of a triangulation T_h of the domain $\Omega \subset \mathbb{R}^d$, where T_h satisfies:

$$\|u - \pi_h u\|_{L^2(\Omega)} \leq Ch^{r+1}|u|_{H^{r+1}(\Omega)},$$

Here $\pi_h u$ is a polynomial interpolant of u of degree $r \geq 1$, and $|u|_{H^{r+1}(\Omega)}$ is the seminorm

$$|u|_{H^{r+1}(\Omega)} = \left(\sum_{|\alpha|=r+1} \int_\Omega |D^\alpha u|^2 dx \right)^{1/2}.$$

Given $u \in L^2(\Omega)$ let $u_h \in V_h$ be the $L^2(\Omega)$-projection of u onto V_h, i.e.,

$$(u_h, v) = (u, v), \forall v \in V_h,$$

where $(\cdot, \cdot)$ is the scalar product in $L^2(\Omega)$. Prove the following two statements:

1. $\|u - u_h\|_{L^2(\Omega)} \leq \inf_{v \in V_h} \|u - u_h\|_{L^2(\Omega)} \leq Ch^{r+1}|u|_{H^{r+1}(\Omega)}$;
2. $\|u_h\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)}$.

Project Submission: Submit your pdf file with the solutions on bCourses.