11

Nonlinear Scalar Conservation Laws

We begin our study of nonlinear conservation laws by considering scalar conservation laws of the form

$$q_t + f(q)_x = 0. \tag{11.1}$$

When the flux function $f(q)$ is linear, $f(q) = \bar{u}q$, then this is simply the advection equation with a very simple solution. The equation becomes much more interesting if $f(q)$ is a nonlinear function of q. The solution no longer simply translates uniformly. Instead, it deforms as it evolves, and in particular *shock waves* can form, across which the solution is discontinuous. At such points the differential equation (11.1) cannot hold in the classical sense. We must remember that this equation is typically derived from a more fundamental integral conservation law that does still make sense even if q is discontinuous. In this chapter we develop the mathematical theory of nonlinear scalar equations, and then in the next chapter we will see how to apply high-resolution finite volume methods to these equations. In Chapter 13 we turn to the even more interesting case of nonlinear hyperbolic systems of conservation laws.

In this chapter we assume that the flux function $f(q)$ has the property that $f''(q)$ does not change sign, i.e., that $f(q)$ is a convex or concave function. This is often called a *convex flux* in either case. The nonconvex case is somewhat trickier and is discussed in Section 16.1.

For the remainder of the book we consider only conservation laws. In the linear case, hyperbolic problems that are not in conservation form can be solved using techniques similar to those developed for conservation laws, and several examples were explored in Chapter 9. In the nonlinear case it is possible to apply similar methods to a nonconservative quasilinear hyperbolic problem $q_t + A(q)q_x = 0$, but equations of this form must be approached with caution for reasons discussed in Section 16.5. The vast majority of physically relevant nonlinear hyperbolic problems arise from integral conservation laws, and it is generally best to keep the equation in conservation form.

11.1 Traffic Flow

As a specific motivating example, we consider the flow of cars on a one-lane highway, as introduced in Section 9.4.2. This is frequently used as an example of nonlinear conservation laws, and other introductory discussions can be found in [175], [457], [486], for example.

This model provides a good introduction to the effects of nonlinearity in a familiar context where the results will probably match readers' expectations. Moreover, a good understanding of this example provides at least some physical intuition for the types of solutions seen in gas dynamics, an important application area for this theory. A gas is a dilute collection of molecules that are quite far apart and hence can be compressed or expanded. The density (measured in mass or molecules per unit volume) can vary by many orders of magnitude depending on how much space is available, and so a gas is a *compressible* fluid.

Cars on the highway can also be thought of as a compressible fluid. The density (measured in cars per car length, as introduced in Section 9.4.2), can vary from 0 on an empty highway to 1 in bumper-to-bumper traffic. (We ignore the compressibility of individual cars in a major collision)

Since the flux of cars is given by uq, we obtain the conservation law

$$q_t + (uq)_x = 0. \tag{11.2}$$

In Section 9.4.2 we considered the variable-coefficient linear problem in which cars always travel at the speed limit $u(x)$. This is a reasonable assumption only for very light traffic. As traffic gets heavier, it is more reasonable to assume that the speed also depends on the density. At this point we will suppose the speed limit and road conditions are the same everywhere, so that u depends only on q and not explicitly on x. Suppose that the velocity is given by some specific known function $u = U(q)$ for $0 \le q \le 1$. Then (11.2) becomes

$$q_t + (q\,U(q))_x = 0, \tag{11.3}$$

which in general is now a *nonlinear conservation law* since the flux function

$$f(q) = qU(q) \tag{11.4}$$

will be nonlinear in q. This is often called the LWR model for traffic flow, after Lighthill & Whitham [299] and Richards [368]. (See Section 17.17 for a brief discussion of other models.)

Various forms of $U(q)$ might be hypothesized. One simple model is to assume that $U(q)$ varies linearly with q,

$$U(q) = u_{\max}(1-q) \quad \text{for } 0 \le q \le 1. \tag{11.5}$$

At zero density (empty road) the speed is $u_{\max}$, but decreases to zero as q approaches 1. We then have the flux function

$$f(q) = u_{\max} q(1-q). \tag{11.6}$$

This is a quadratic function. Note that the flux of cars is greatest when $q = 1/2$. As q decreases, the flux decreases because there are few cars on the road, and as q increases, the flux decreases because traffic slows down.

Before attempting to solve the nonlinear conservation law (11.3), we can develop some intuition for how solutions behave by doing simulations of traffic flow in which we track the motion of individual vehicles $X_k(t)$, as described in Section 9.4.2. If we define the density

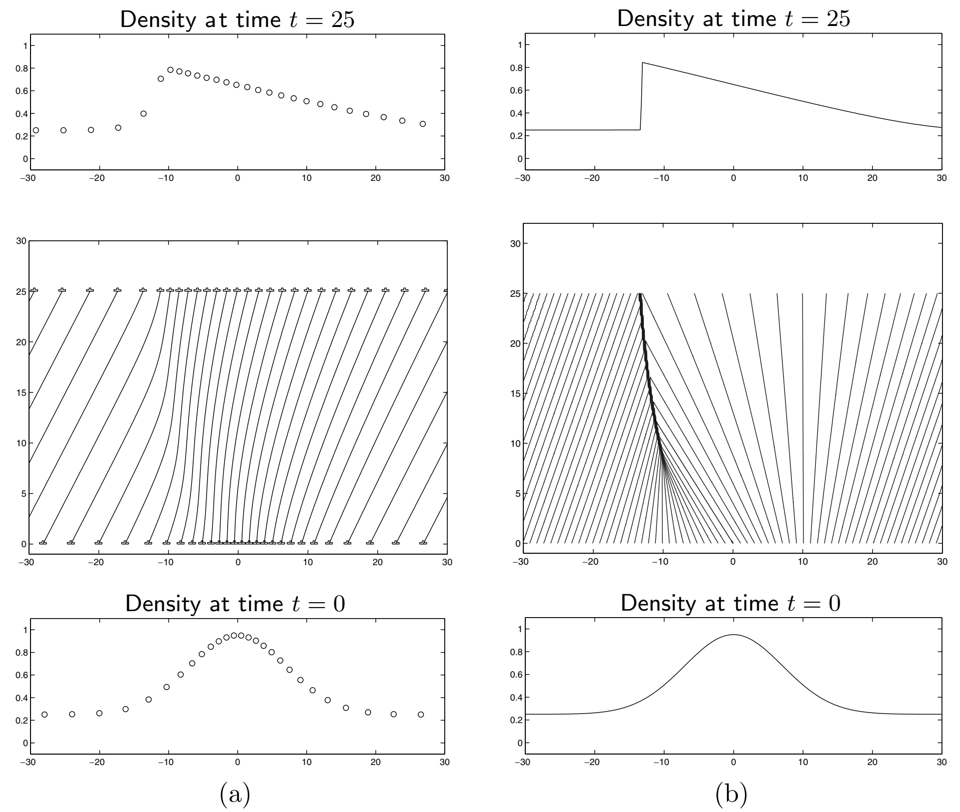


Fig. 11.1. Solution to the traffic flow model starting with a bulge in the density. (a) Trajectories of individual vehicles and the density observed by each driver. (b) The characteristic structure is shown along with the density as computed by CLAWPACK. `[claw/book/chap11/congestion]`

seen by the kth driver as $q_k(t)$, as in (9.32), then the velocity of the kth car at time t will be $U(q_k(t))$. This is a reasonable model of driver behavior: the driver chooses her speed based only on the distance to the car she is following. This viewpoint can be used to justify the assumption that U depends on q. The ordinary differential equations for the motion of the cars now become a coupled set of nonlinear equations:

$$X_k'(t) = U(q_k(t)) = U([X_{k+1}(t) - X_k(t)]^{-1}) \tag{11.7}$$

for $k = 1, 2, \ldots, m$, where m is the number of cars.

Figure 11.1(a) shows an example of one such simulation. The speed limit is $u_{\max} = 1$, and the cars are initially distributed so there is a Gaussian increase in the density near $x = 0$. Note the following:

- The disturbance in density does not move with the individual cars, the way it would for the linear advection equation. Individual cars move through the congested region, slowing down and later speeding up.
- The hump in density changes shape with time. By the end of the simulation a *shock wave* is visible, across which the density increases and the velocity decreases very quickly (drivers who were zipping along in light traffic must suddenly slam on their brakes).

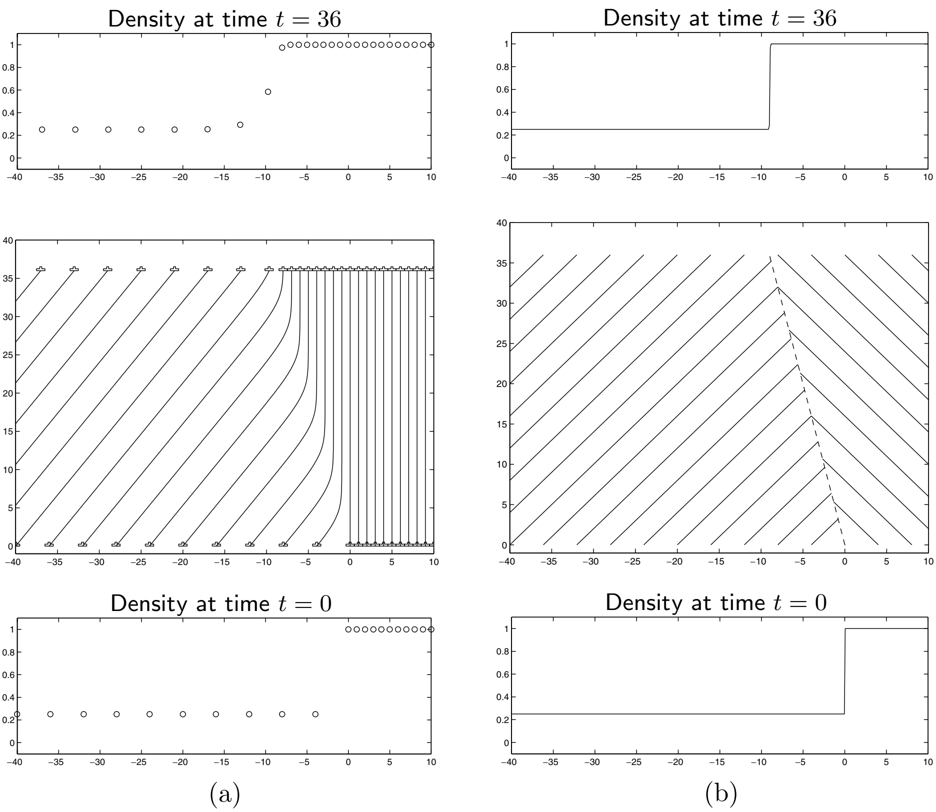


Fig. 11.2. Riemann problem for the traffic flow model at a red light. (a) Trajectories of individual vehicles and the density observed by each driver. (b) The characteristic structure is shown along with the density as computed by CLAWPACK. [claw/book/chap11/redlight]

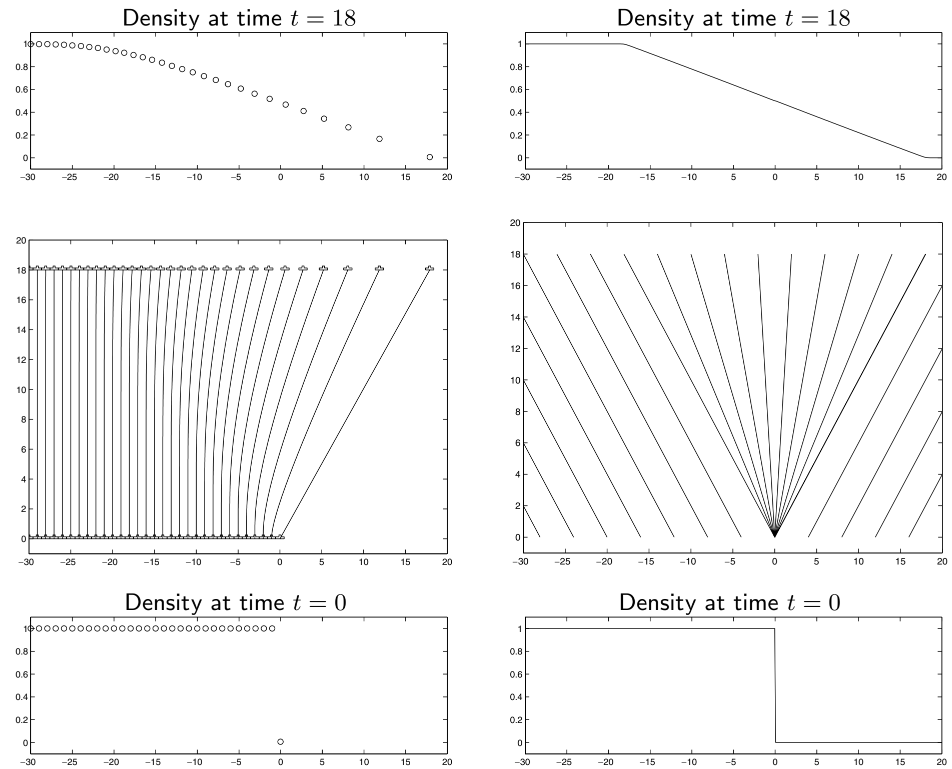
- As cars move out of the congested region, they accelerate smoothly and the density decreases smoothly. This is called a *rarefaction wave*, since the fluid is becoming more rarefied as the density decreases.

Figures 11.2 and 11.3 show two more simulations for special cases corresponding to *Riemann problems* in which the initial data is piecewise constant. Figure 11.2 can be interpreted as cars approaching a traffic jam, or a line of cars waiting for a light to change; Figure 11.3 shows how cars accelerate once the light turns green. Note that the solution to the Riemann problem may consist of either a shock wave as in Figure 11.2 or a rarefaction wave as in Figure 11.3, depending on the data.

11.2 Quasilinear Form and Characteristics

By differentiating the flux function $f(q)$ we obtain the quasilinear form of the conservation law (11.1),

$$q_t + f'(q)q_x = 0. \tag{11.8}$$



(a) (b)

Fig. 11.3. Riemann problem for the traffic flow model at a green light. (a) Trajectories of individual vehicles and the density observed by each driver. (b) The characteristic structure is shown, along with the density as computed by CLAWPACK. `[claw/book/chap11/greenlight]`

It is normally preferable to work directly with the conservative form, but it is the quasilinear form that determines the characteristics of the equation. If we assume that $q(x, t)$ is smooth, then along any curve $X(t)$ satisfying the ODE

$$X'(t) = f'(q(X(t), t)), \tag{11.9}$$

we have

$$\begin{aligned} \frac{d}{dt} q(X(t), t) &= X'(t) q_x + q_t \\ &= 0 \end{aligned} \tag{11.10}$$

by (11.8) and (11.9). Hence q is constant along the curve $X(t)$, and consequently $X'(t)$ is also constant along the curve, and so the characteristic curve must be a straight line.

We thus see that *for a scalar conservation law, q is constant on characteristics, which are straight lines, as long as the solution remains smooth*. The structure of the characteristics depends on the initial data $q(x, 0)$. Figure 11.1(b) shows the characteristics for the traffic flow problem of Figure 11.1(a). Actually, Figure 11.1(b) shows a contour plot (in x and t) of the density q as computed using CLAWPACK on a very fine grid. Since q is constant on characteristics, this essentially shows the characteristic structure, as long as the solution remains smooth. What we observe, however, is that characteristics may collide. This is an essential feature of nonlinear hyperbolic problems, giving rise to shock formation. For a linear advection equation, even one with variable coefficients, the characteristics will never cross.

Also note that the characteristic curves of Figure 11.1(b) are quite different from the vehicle trajectories plotted in Figure 11.1(a). Only in the linear case $f(q) = \bar{u}q$ does the characteristic speed agree with the speed at which particles are moving, since in this case $f'(q) = \bar{u}$.

We will explore shock waves starting in Section 11.8. For now suppose the initial data $\mathring{q}(x) = q(x, 0)$ is smooth and the solution remains smooth over some time period $0 \le t \le T$ of interest. Constant values of q propagate along characteristic curves. Using the fact that these curves are straight lines, it is possible to determine the solution $q(x, t)$ as

$$q(x, t) = \mathring{q}(\xi), \tag{11.11}$$

where ξ solves the equation

$$x = \xi + f'(\mathring{q}(\xi))t. \tag{11.12}$$

This is generally a nonlinear equation for ξ, but will have a unique solution provided $0 \le t \le T$ is within the period that characteristics do not cross.

11.3 Burgers' Equation

The traffic flow model gives a scalar conservation law with a quadratic flux function. An even simpler scalar equation of this form is the famous *Burgers equation*

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0. \tag{11.13}$$

This should more properly be called the *inviscid Burgers equation*, since Burgers [54] actually studied the viscous equation

$$u_t + \left(\frac{1}{2}u^2\right)_x = \epsilon u_{xx}. \tag{11.14}$$

Rather than modeling a particular physical problem, this equation was introduced as the simplest model equation that captures some key features of gas dynamics: the nonlinear hyperbolic term and viscosity. In the literature of hyperbolic equations, the inviscid problem (11.13) has been widely used for developing both theory and numerical methods.

Around 1950, Hopf, and independently Cole, showed that the *exact* solution of the nonlinear equation (11.14) could be found using what is now called the *Cole–Hopf transformation*. This reduces (11.14) to a linear heat equation. See Chapter 4 of Whitham [486] for details and Exercise 11.2 for one particular solution.

Solutions to the inviscid Burgers' equation have the same basic structure as solutions to the traffic flow problem considered earlier. The quasilinear form of Burgers' equation,

$$u_t + uu_x = 0, \tag{11.15}$$

shows that in smooth portions of the solution the value of u is constant along characteristics traveling at speed u. Note that (11.15) looks like an advection equation in which the value of u is being carried at velocity u. This is the essential nonlinearity that appears in the conservation-of-momentum equation of fluid dynamics: the velocity or momentum is carried in the fluid at the fluid velocity.

Detailed discussions of Burgers' equation can be found in many sources, e.g., [281], [486]. All of the issues illustrated below for the traffic flow equation can also be illustrated with Burgers' equation, and the reader is encouraged to explore this equation in the process of working through the remainder of this chapter.

11.4 Rarefaction Waves

Suppose the initial data for the traffic flow model satisfies $q_x(x, 0) < 0$, so the density falls with increasing x. In this case the characteristic speed

$$f'(q) = U(q) + q\,U'(q) = u_{\max}(1 - 2q) \tag{11.16}$$

is increasing with x. Hence the characteristics are spreading out and will never cross. The density observed by each car will decrease with time, and the flow is being *rarefied.*

A special case is the Riemann problem shown in Figure 11.3. In this case the initial data is discontinuous, but if we think of smoothing this out very slightly as in Figure 11.4(a), then each value of q between 0 and 1 is taken on in the initial data, and each value propagates with its characteristic speed $f'(q)$, as is also illustrated in Figure 11.4(a). Figure 11.3(b) shows the characteristics in the x–t plane. This is called a *centered rarefaction wave*, or rarefaction fan, emanating from the point (0, 0).

For Burgers' equation (11.13), the characteristics are all spreading out if $u_x(x, 0) > 0$, since the characteristic speed is $f'(u) = u$ for this equation.

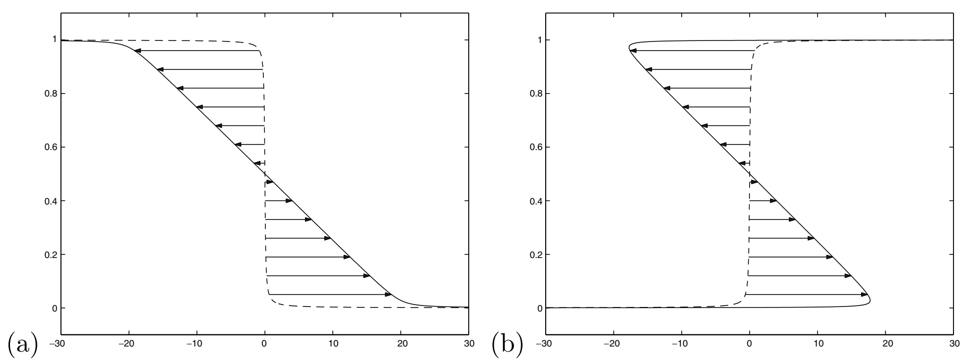


Fig. 11.4. Tracing each value of q at its characteristic speed from the initial data (shown by the dashed line) to the solution at time $t = 20$. (a) When q is decreasing, we obtain a rarefaction wave. (b) When q is increasing, we obtain an unphysical triple-valued solution.

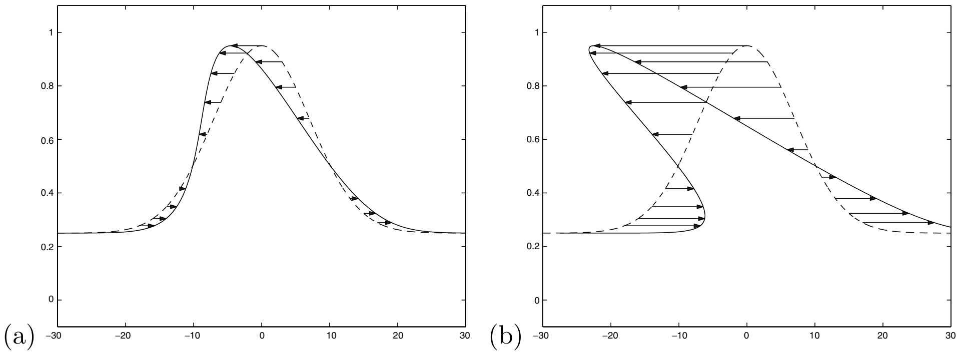


Fig. 11.5. Tracing each value of q at its characteristic speed from the initial data from Figure 11.1 (shown by the dashed line) to the solution at two different times. (a) At $t = 5$ we see a compression wave and a rarefaction wave. (b) At $t = 25$ we obtain an unphysical triple-valued solution.

11.5 Compression Waves

Figure 11.5(a) shows a picture similar to Figure 11.4(a) for initial data consisting of a hump of density as in the example of Figure 11.1. In this case the right half of the hump, where the density falls, is behaving as a rarefaction wave. The left part of the hump, where the density is rising with increasing x, gives rise to a *compression wave*. A driver passing through this region will experience increasing density with time (this can be observed in Figure 11.1).

If we try to draw a similar picture for larger times then we obtain Figure 11.5(b). As in the case of Figure 11.4(b), this does not make physical sense. At some points x the density appears to be triple-valued. At such points there are three characteristics reaching (x, t) from the initial data, and the equation (11.12) would have three solutions.

An indication of what should instead happen physically is seen in Figure 11.1(a). Recall that this was obtained by simulating the traffic flow directly rather than by solving a conservation law. The compression wave should steepen up into a discontinuous shock wave. At some time T_b the slope $q_x(x, t)$ will become infinite somewhere. This is called the *breaking time* (by analogy with waves breaking on a beach; see Section 13.1). Beyond time T_b characteristics may cross and a shock wave appears. It is possible to determine T_b from the initial data $q(x, 0)$; see Exercise 11.1.

11.6 Vanishing Viscosity

As a differential equation, the hyperbolic conservation law (11.1) breaks down once shocks form. This is not surprising, since it was derived from the more fundamental integral equation by manipulations that are only valid when the solution is smooth. When the solution is not smooth, a different formulation must be used, such as the integral form or the *weak form* introduced in Section 11.11.

Another approach is to modify the differential equation slightly by adding a bit of viscosity, or diffusion, obtaining

$$q_t + f(q)_x = \epsilon q_{xx}, \tag{11.17}$$

where $\epsilon > 0$ is a small parameter. The term "viscosity" is motivated by fluid dynamics

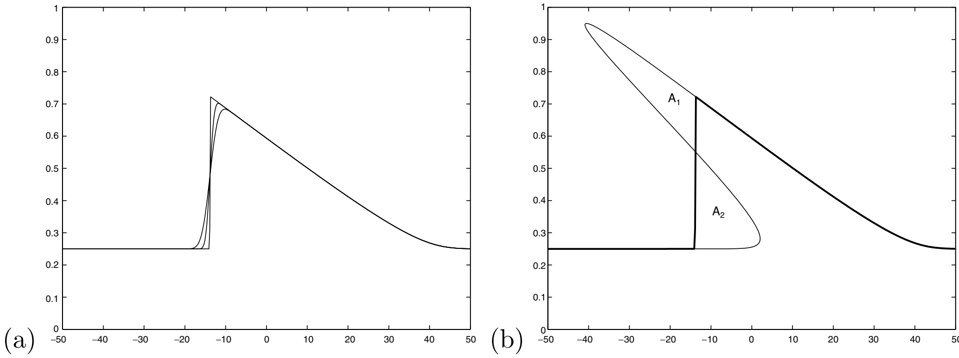


Fig. 11.6. (a) Solutions to the viscous traffic flow equation (11.17) for two values of $\epsilon > 0$, along with the limiting shock-wave solution when $\epsilon = 0$. (b) The shock wave (heavy line) can be determined by applying the equal-area rule to the triple-valued solution of Figure 11.5(b) so that the areas A_1 and A_2 are equal.

equations, as discussed below. If ϵ is extremely small, then we might expect solutions to (11.17) to be very close to solutions of (11.1), which has $\epsilon = 0$. However, the equation (11.17) is *parabolic* rather than hyperbolic, and it can be proved that this equation has a unique solution for all time $t > 0$, for any set of initial conditions, provided only that $\epsilon > 0$. Away from shock waves, q_{xx} is bounded and this new term is negligible. If a shock begins to form, however, the derivatives of q begin to blow up and the ϵq_{xx} term becomes important. Figure 11.6 shows solutions to the traffic flow model from Figure 11.1 with this new term added, for various values of ϵ. As $\epsilon \to 0$ we approach a limiting solution that has a discontinuity corresponding to the shock wave seen in Figure 11.1.

The idea of introducing the small parameter ϵ and looking at the limit $\epsilon \to 0$ is called the *vanishing-viscosity* approach to defining a sensible solution to the hyperbolic equation. To motivate this, remember that in general any mathematical equation is only a model of reality, in which certain effects may have been modeled well but others are necessarily neglected. In gas dynamics, for example, the hyperbolic equations (2.38) only hold if we ignore thermal diffusion and the effects of viscosity in the gas, which is the frictional force of molecules colliding and converting kinetic energy into internal energy. For a dilute gas this is reasonable most of the time, since these forces are normally small. Including the (parabolic) viscous terms in the equations would only complicate them without changing the solution very much. Near a shock wave, however, the viscous terms are important. In the real world shock waves are not sharp discontinuities but rather smooth transitions over very narrow regions. By using the hyperbolic equation we hope to capture the big picture without modeling exactly how the solution behaves within this thin region.

11.7 Equal-Area Rule

From Figure 11.1 it is clear that after a shock forms, the solution contains a discontinuity, but that away from the shock the solution remains smooth and still has the property that it is constant along characteristics. This solution can be constructed by taking the unphysical solution of Figure 11.4(b) and eliminating the triple-valued portion by inserting a discontinuity at some point as indicated in Figure 11.6(b). It can be shown that the correct place to insert the shock is determined by the *equal-area rule*: The shock is located so that the two regions cut off by the shock have equal areas. This is a consequence of conservation – the area under the discontinuous single-valued solution (the integral of the density) must be the same as the area "under" the multivalued solution, since it can be shown that the rate of change of this area is the difference between the flux at the far boundaries, even after it becomes multivalued.

11.8 Shock Speed

As the solution evolves the shock will propagate with some speed $s(t)$ that may change with time. We can use the integral form of the conservation law to determine the shock speed at any time in terms of the states $q_l(t)$ and $q_r(t)$ immediately to the left and right of the shock. Suppose the shock is moving as shown in Figure 11.7, where we have zoomed in on a very short time increment from t_1 to $t_1 + \Delta t$ over which the shock speed is essentially a constant value s. Then the rectangle $[x_1,\ x_1 + \Delta x] \times [t_1,\ t_1 + \Delta t]$ shown in Figure 11.7 is split by the shock into two triangles and the value of q is roughly constant in each. If we apply the integral form of the conservation law (2.6) to this region, we obtain

$$\begin{aligned} &\int_{x_1}^{x_1+\Delta x} q(x, t_1 + \Delta t)\, dx - \int_{x_1}^{x_1+\Delta x} q(x, t_1)\, dx \\ &\quad = \int_{t_1}^{t_1+\Delta t} f(q(x_1, t))\, dt - \int_{t_1}^{t_1+\Delta t} f(q(x_1 + \Delta x, t))\, dt. \end{aligned} \tag{11.18}$$

Since q is essentially constant along each edge, this becomes

$$\Delta x\, q_r - \Delta x\, q_l = \Delta t f(q_l) - \Delta t f(q_r) + \mathcal{O}(\Delta t^2), \tag{11.19}$$

where the $\mathcal{O}(\Delta t^2)$ term accounts for the variation in q. If the shock speed is s, then $\Delta x = -s\, \Delta t$ (for the case $s < 0$ shown in the figure). Using this in (11.19), dividing by $-\Delta t$, and

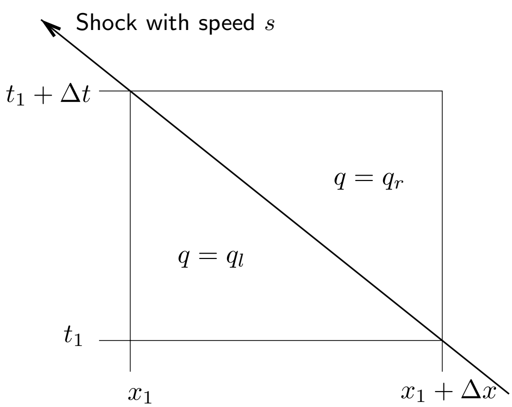


Fig. 11.7. The Rankine–Hugoniot jump conditions are determined by integrating over an infinitesimal rectangular region in the x–t plane.

taking the limit as $\Delta t \to 0$ gives

$$s(q_r - q_l) = f(q_r) - f(q_l). \tag{11.20}$$

This is called the *Rankine–Hugoniot jump condition*. This is sometimes written simply as $s[\![q]\!] = [\![f]\!]$, where $[\![\cdot]\!]$ represents the jump across the shock.

For a scalar conservation law we can divide by $q_r - q_l$ and obtain the *shock speed*:

$$s = \frac{f(q_r) - f(q_l)}{q_r - q_l}. \tag{11.21}$$

In general $q_l(t)$ and $q_r(t)$, the states just to the left and the right of the shock, vary with time and the shock speed also varies. For the special case of a Riemann problem with piecewise constant data q_l and q_r, the resulting shock moves with constant speed, given by (11.21).

Note that if $q_l \approx q_r$, then the expression (11.21) approximates $f'(q_l)$. A weak shock, which is essentially an acoustic wave, thus propagates at the characteristic velocity, as we expect from linearization.

For the traffic flow flux (11.6) we find that

$$s = u_{\max}[1 - (q_l + q_r)] = \frac{1}{2}[f'(q_l) + f'(q_r)], \tag{11.22}$$

since $f'(q) = u_{\max}(1 - 2q)$. In this case, and for any quadratic flux function, the shock speed is simply the average of the characteristic speeds on the two sides. Another quadratic example is Burgers' equation (11.13), for which we find

$$s = \frac{1}{2}(u_l + u_r). \tag{11.23}$$

11.9 The Rankine–Hugoniot Conditions for Systems

The derivation of (11.20) is valid for systems of conservation laws as well as for scalar equations. However, for a system of m equations, $q_r - q_l$ and $f(q_r) - f(q_l)$ will both be m-vectors and we will not be able to simply divide as in (11.21) to obtain the shock speed. In fact, for arbitrary states q_l and q_r there will be no scalar value s for which (11.20) is satisfied. Special relations must exist between the two states in order for them to be connected by a shock: the vector $f(q_l) - f(q_r)$ must be a scalar multiple of the vector $q_r - q_l$. We have already seen this condition in the case of a linear system, where $f(q) = Aq$. In this case the Rankine–Hugoniot condition (11.20) becomes

$$A(q_r - q_l) = s(q_r - q_l),$$

which means that $q_r - q_l$ must be an *eigenvector* of the matrix A. The propagation speed s of the discontinuity is then the corresponding eigenvalue λ. In Chapter 3 we used this condition to solve the linear Riemann problem, and in Chapter 13 we will see how this theory can be extended to nonlinear systems.

11.10 Similarity Solutions and Centered Rarefactions

For the special case of a Riemann problem with data

$$q(x,0) = \begin{cases} q_l & \text{if } x < 0, \\ q_r & \text{if } x > 0, \end{cases} \tag{11.24}$$

the solution to a conservation law is a *similarity solution*, a function of x/t alone that is self-similar at different times. The solution

$$q(x,t) = \tilde{q}(x/t) \tag{11.25}$$

is constant along any ray $x/t =$ constant through the origin, just as in the case of a linear hyperbolic system (see Chapter 3). From (11.25) we compute

$$q_t(x,t) = -\frac{x}{t^2}\,\tilde{q}'(x/t) \quad \text{and} \quad f(q)_x = \frac{1}{t}\,f'(\tilde{q}(x/t))\,\tilde{q}'(x/t).$$

Inserting these in the quasilinear equation $q_t + f'(q)q_x = 0$ shows that

$$f'(\tilde{q}(x/t))\,\tilde{q}'(x/t) = \frac{x}{t}\tilde{q}'(x/t). \tag{11.26}$$

For the scalar equation we find that either $\tilde{q}'(x/t) = 0$ ($\tilde{q}$ is constant) or that

$$f'(\tilde{q}(x/t)) = x/t. \tag{11.27}$$

This allows us to determine the solution through a centered rarefaction wave explicitly. (For a system of equations we cannot simply cancel $\tilde{q}'(x/t)$ from equation (11.26). See Section 13.8.3 for the construction of a rarefaction wave in a nonlinear system.)

Consider the traffic flow model, for example, with f given by (11.6). The solution to a Riemann problem with $q_l > q_r$ consists of a rarefaction fan. The left and right edges of this fan propagate with the characteristic speeds $f'(q_l)$ and $f'(q_r)$ respectively (see Figure 11.3), and so we have

$$\tilde{q}(x/t) = \begin{cases} q_l & \text{for } x/t \le f'(q_l), \\ q_r & \text{for } x/t \ge f'(q_r). \end{cases} \tag{11.28}$$

In between, $\tilde{q}$ varies and so (11.27) must hold, which gives

$$u_{\max}[1 - 2\tilde{q}(x/t)] = x/t$$

and hence

$$\tilde{q}(x/t) = \frac{1}{2}\left(1 - \frac{x}{u_{\max}t}\right) \quad \text{for } f'(q_l) \le x/t \le f'(q_r). \tag{11.29}$$

Note that at any fixed time t the solution $q(x,t)$ is linear in x as observed in Figure 11.3(b). This is a consequence of the fact that $f(q)$ is quadratic and so $f'(q)$ is linear. A different flux function could give rarefaction waves with more interesting structure (see Exercise 11.8).

11.11 Weak Solutions

We have observed that the differential equation (11.1) is not valid in the classical sense for solutions containing shocks (though it still holds in all regions where the solution is smooth). The integral form of the conservation law (2.6) does hold, however, even when q is discontinuous, and it was this form that we used to determine the Rankine–Hugoniot condition (11.20) that must hold across shocks. More generally we can say that a function $q(x, t)$ is a solution of the conservation law if (2.2) holds for all t and any choice of x_1, x_2.

This formulation can be difficult to work with mathematically. In this section we look at a somewhat different integral formulation that is useful in proving results about solutions. In particular, in Section 12.10 we will investigate the convergence of finite volume methods as the grid is refined and will need this formulation to handle discontinuous solutions.

To motivate this *weak form*, first suppose that $q(x, t)$ is smooth. In Chapter 2 we derived the differential equation (11.1) by rewriting (2.7) as (2.9). Integrating this latter equation in time between t_1 and t_2 gives

$$\int_{t_1}^{t_2} \int_{x_1}^{x_2} [q_t + f(q)_x]\, dx\, dt = 0. \tag{11.30}$$

Rather than considering this integral for arbitrary choices of x_1, x_2, t_1, and t_2, we could instead consider

$$\int_0^\infty \int_{-\infty}^\infty [q_t + f(q)_x]\phi(x, t)\, dx\, dt \tag{11.31}$$

for a certain set of functions $\phi(x, t)$. In particular, if $\phi(x, t)$ is chosen to be

$$\phi(x, t) = \begin{cases} 1 & \text{if } (x, t) \in [x_1, x_2] \times [t_1, t_2], \\ 0 & \text{otherwise,} \end{cases} \tag{11.32}$$

then this integral reduces to the one in (11.30). We can generalize this notion by letting $\phi(x, t)$ be any function that has *compact support*, meaning it is identically zero outside of some bounded region of the x–t plane. If we now also assume that ϕ is a smooth function (unlike (11.32)), then we can integrate by parts in (11.31) to obtain

$$\int_0^\infty \int_{-\infty}^\infty [q\phi_t + f(q)\phi_x]\, dx\, dt = -\int_0^\infty q(x, 0)\phi(x, 0)\, dx. \tag{11.33}$$

Only one boundary term along $t = 0$ appears in this process, since we assume ϕ vanishes at infinity.

A nice feature of (11.33) is that the derivatives are on ϕ, and no longer of q and $f(q)$. So (11.33) continues to make sense even if q is discontinuous. This motivates the following definition.

Definition 11.1. *The function $q(x, t)$ is a* weak solution *of the conservation law (11.1) with given initial data $q(x, 0)$ if (11.33) holds for all functions ϕ in C_0^1.*

The function space C_0^1 denotes the set of all functions that are C^1 (continuously differentiable) and have compact support. By assuming ϕ is smooth we rule out the special choice

(11.32) that gave (11.30), but we can approximate this function arbitrarily well by a slightly smoothed-out version. It can be shown that any weak solution also satisfies the integral conservation laws and vice versa.

11.12 Manipulating Conservation Laws

One danger to observe in dealing with conservation laws is that transforming the differential form into what appears to be an equivalent differential equation may not give an equivalent equation in the context of weak solutions.

Example 11.1. If we write Burgers' equation

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0 \tag{11.34}$$

in the quasilinear form $u_t + uu_x = 0$ and multiply by $2u$, we obtain $2uu_t + 2u^2u_x = 0$, which can be rewritten as

$$(u^2)_t + \left(\frac{2}{3}u^3\right)_x = 0. \tag{11.35}$$

This is again a conservation law, now for u^2 rather than u itself, with flux function $f(u^2) = \frac{2}{3}(u^2)^{3/2}$. The differential equations (11.34) and (11.35) have precisely the same smooth solutions. However, they have different weak solutions, as we can see by considering the Riemann problem with $u_l > u_r$. The unique weak solution of (11.34) is a shock traveling at speed

$$s_1 = \frac{[\![\frac{1}{2}u^2]\!]}{[\![u]\!]} = \frac{1}{2}(u_l + u_r), \tag{11.36}$$

whereas the unique weak solution to (11.35) is a shock traveling at speed

$$s_2 = \frac{[\![\frac{2}{3}u^3]\!]}{[\![u^2]\!]} = \frac{2}{3}\left(\frac{u_r^3 - u_l^3}{u_r^2 - u_l^2}\right). \tag{11.37}$$

It is easy to check that

$$s_2 - s_1 = \frac{1}{6}\frac{(u_l - u_r)^2}{u_l + u_r}, \tag{11.38}$$

and so $s_2 \neq s_1$ when $u_l \neq u_r$, and the two equations have different weak solutions. The derivation of (11.35) from (11.34) requires manipulating derivatives in a manner that is valid only when u is smooth.

11.13 Nonuniqueness, Admissibility, and Entropy Conditions

The Riemann problem shown in Figure 11.3 has a solution consisting of a rarefaction wave, as determined in Section 11.10. However, this is not the only possible weak solution to the equation with this data. Another solution is

$$q(x,t) = \begin{cases} q_l & \text{if } x/t < s, \\ q_r & \text{if } x/t > s, \end{cases} \tag{11.39}$$

where the speed s is determined by the Rankine–Hugoniot condition (11.21). This function consists of the discontinuity propagating at speed s, the *shock speed*. We don't expect a shock wave in this case, since characteristics are spreading out rather than colliding, but we did not use any information about the characteristic structure in deriving the Rankine–Hugoniot condition. The function (11.39) is a weak solution of the conservation law regardless of whether $q_l < q_r$ or $q_l > q_r$. The discontinuity in the solution (11.39) is sometimes called an *expansion shock* in this case.

We see that the weak solution to a conservation law is not necessarily unique. This is presumably another failing of our mathematical formulation, since physically we expect only one thing to happen for given initial data, and hence a unique solution. Again this results from the fact that the hyperbolic equation is an imperfect model of reality. A better model might include "viscous terms" as in Section 11.6, for example, and the resulting parabolic equation would have a unique solution for any set of data. As in the case of shock waves, what we hope to capture with the hyperbolic equation is the correct limiting solution as the viscosity ϵ vanishes. As Figure 11.4(a) suggests, if the discontinuous data is smoothed only slightly (as would happen immediately if the equation were parabolic), then there is a unique solution determined by the characteristics. This solution clearly converges to the rarefaction wave as $\epsilon \to 0$. So the expansion shock solution (11.39) is an artifact of our formulation and is not physically meaningful.

The existence of these spurious solution is not merely a mathematical curiosity. Under some circumstances nonphysical solutions of this type are all too easily computed numerically, in spite of the fact that numerical methods typically contain some "numerical viscosity." See Section 12.3 for a discussion of these numerical difficulties.

In order to effectively use the hyperbolic equations we must impose some additional condition along with the differential equation in order to insure that the problem has a unique weak solution that is physically correct. Often the condition we want to impose is simply that the weak solution must be the vanishing-viscosity solution to the proper viscous equation. However, this condition is hard to work with directly in the context of the hyperbolic equation. Instead, a variety of other conditions have been developed that can be applied directly to weak solutions of hyperbolic equations to check if they are physically admissible. Such conditions are sometimes called *admissibility conditions*, or more often *entropy conditions*. This name again comes from gas dynamics, where the second law of thermodynamics demands that the entropy of a system must be nondecreasing with time (see Section 14.5). Across a physically admissible shock the entropy of the gas increases. Across an expansion shock, however, the entropy of the gas would decrease, which is not allowed. The entropy at each point can be computed as a simple function of the pressure and density, (2.35), and the behavior of this function can be used to test a weak solution for admissibility. For other conservation laws it is sometimes possible to define a function $\eta(q)$, called an *entropy function*, which has similar diagnostic properties. This approach to developing entropy conditions is discussed in Section 11.14. Expansion shocks are often called *entropy-violating shocks*, since they are weak solutions that fail to satisfy the appropriate entropy condition.

First we discuss some other admissibility conditions that relate more directly to the characteristic structure. We discuss only a few possibilities here and concentrate on the scalar case. For some systems of equations the development of appropriate admissibility conditions remains a challenging research question. In some cases it is not even well understood what the appropriate viscous regularization of the conservation law is, and it may be that different choices of the viscous term lead to different vanishing-viscosity solutions.

For scalar equations there is an obvious condition suggested by Figures 11.2(b) and 11.3(b). A shock should have characteristics going *into* the shock, as time advances. A propagating discontinuity with characteristics coming *out of* it would be unstable to perturbations. Either smearing out the initial profile a little, or adding some viscosity to the system, will cause this to be replaced by a rarefaction fan of characteristics, as in Figure 11.3(b). This gives our first version of the entropy condition:

Entropy Condition 11.1 (Lax). For a convex scalar conservation law, a discontinuity propagating with speed s given by (11.21) satisfies the Lax entropy condition if

$$f'(q_l) > s > f'(q_r). \tag{11.40}$$

Note that $f'(q)$ is the characteristic speed. For convex or concave f, the Rankine–Hugoniot speed s from (11.21) must lie between $f'(q_l)$ and $f'(q_r)$, so (11.40) reduces to simply the requirement that $f'(q_l) > f'(q_r)$. For the traffic flow flux (11.4), this then implies that we need $q_l < q_r$ in order for the solution to be an admissible shock since $f''(q) < 0$. If $q_l > q_r$ then the correct solution would be a rarefaction wave. For Burgers' equation with $f(u) = u^2/2$, on the other hand, the Lax Entropy Condition 11.1 requires $u_l > u_r$ for an admissible shock, since $f''(u)$ is everywhere positive rather than negative.

A more general form of (11.40), due to Oleinik, also applies to nonconvex scalar flux functions and is given in Section 16.1.2. The generalization to systems of equations is discussed in Section 13.7.2.

Another form of the entropy condition is based on the spreading of characteristics in a rarefaction fan. We state this for the convex case with $f''(q) > 0$ (such as Burgers' equation), since this is the form usually seen in the literature. If $q(x,t)$ is an increasing function of x in some region, then characteristics spread out in this case. The rate of spreading can be quantified, and gives the following condition, also due to Oleinik [346].

Entropy Condition 11.2 (Oleinik). $q(x,t)$ is the entropy solution to a scalar conservation law $q_t + f(q)_x = 0$ with $f''(q) > 0$ if there is a constant $E > 0$ such that for all $a > 0$, $t > 0$, and $x \in \mathbb{R}$,

$$\frac{q(x+a,t) - q(x,t)}{a} < \frac{E}{t}. \tag{11.41}$$

Note that for a discontinuity propagating with constant left and right states q_l and q_r, this can be satisfied only if $q_r - q_l \le 0$, so this agrees with (11.40). The form of (11.41) also gives information on the rate of spreading of rarefaction waves as well as on the form of allowable jump discontinuities, and is easier to apply in some contexts. In particular, this formulation has advantages in studying numerical methods, and is related to one-sided Lipschitz conditions and related stability concepts [339], [340], [437].

11.14 Entropy Functions

Another approach to the entropy condition is to define an entropy function $\eta(q)$, motivated by thermodynamic considerations in gas dynamics as described in Chapter 14. This approach often applies to systems of equations, as in the case of gas dynamics, and is also used in studying numerical methods; see Section 12.11.1.

In general an entropy function should be a function that is conserved (i.e., satisfies some conservation law) whenever the function $q(x,t)$ is smooth, but which has a source or a sink at discontinuities in q. The entropy of a gas has this property: entropy is produced in an admissible shock but would be reduced across an expansion shock. The second law of thermodynamics requires that the total entropy must be nondecreasing with time. This entropy condition rules out expansion shocks.

We now restate this in mathematical form. Along with an entropy function $\eta(q)$, we need an *entropy flux* $\psi(q)$ with the property that whenever q is smooth an integral conservation law holds,

$$\int_{x_1}^{x_2} \eta(q(x,t_2))\,dx = \int_{x_1}^{x_2} \eta(q(x,t_1))\,dx + \int_{t_1}^{t_2} \psi(q(x_1,t))\,dt - \int_{t_1}^{t_2} \psi(q(x_2,t))\,dt. \tag{11.42}$$

The entropy function and flux must also be chosen in such a way that if q is discontinuous in $[x_1, x_2] \times [t_1, t_2]$, then the equation (11.42) does *not* hold with equality, so that the total entropy in $[x_1, x_2]$ at time t_2 is either less or greater than what would be predicted by the fluxes at x_1 and x_2. Requiring that an inequality of the form

$$\int_{x_1}^{x_2} \eta(q(x,t_2))\,dx \le \int_{x_1}^{x_2} \eta(q(x,t_1))\,dx + \int_{t_1}^{t_2} \psi(q(x_1,t))\,dt - \int_{t_1}^{t_2} \psi(q(x_2,t))\,dt \tag{11.43}$$

always holds gives the entropy condition. (For the physical entropy in gas dynamics one would require an inequality of this form with $\ge$ in place of $\le$, but in the mathematical literature $\eta(q)$ is usually chosen to be a convex function with $\eta''(q) > 0$, leading to the inequality (11.43).)

If $q(x,t)$ is smooth, then the conservation law (11.42) can be manipulated as in Chapter 2 to derive the differential form

$$\eta(q)_t + \psi(q)_x = 0. \tag{11.44}$$

Moreover, if η and ψ are smooth function of q, we can differentiate these to rewrite (11.44) as

$$\eta'(q)q_t + \psi'(q)q_x = 0. \tag{11.45}$$

On the other hand, the smooth function q satisfies

$$q_t + f'(q)q_x = 0. \tag{11.46}$$

Multiplying (11.46) by $\eta'(q)$ and comparing with (11.45) yields

$$\psi'(q) = \eta'(q)f'(q) \tag{11.47}$$

as a relation that should hold between the entropy function and the entropy flux. For a scalar conservation law this equation admits many solutions $\eta(q)$, $\psi(q)$. One trivial choice of η and ψ satisfying (11.47) would be $\eta(q) = q$ and $\psi(q) = f(q)$, but then η would be conserved even across discontinuities and this would not help in defining an admissibility criterion. Instead we also require that $\eta(q)$ be a *convex function* of q with $\eta''(q) > 0$ for all q. This will give an entropy function for which the inequality (11.43) should hold.

For a system of equations η and ψ are still *scalar* functions, but now $\eta'(q)$ and $\psi'(q)$ must be interpreted as the row-vector gradients of η and ψ with respect to q, e.g.,

$$\eta'(q) = \left[\frac{\partial \eta}{\partial q^1}, \frac{\partial \eta}{\partial q^2}, \dots, \frac{\partial \eta}{\partial q^m}\right], \tag{11.48}$$

and $f'(q)$ is the $m \times m$ Jacobian matrix. In general (11.47) is a system of m equations for the two variables η and ψ. Moreover, we also require that $\eta(q)$ be convex, which for a system requires that the Hessian matrix $\eta''(q)$ be positive definite. For $m > 2$ this may have no solutions. Many physical systems do have entropy functions, however, including of course the Euler equations of gas dynamics, where the negative of physical entropy can be used. See Exercise 13.6 for another example. For symmetric systems there is always an entropy function $\eta(q) = q^T q$, as observed by Godunov [158]. Conversely, if a system has a convex entropy, then its Hessian matrix $\eta''(q)$ symmetrizes the system [143]; see also [433].

In order to see that the physically admissible weak solution should satisfy (11.43), we go back to the more fundamental condition that the admissible $q(x, t)$ should be the vanishing-viscosity solution. Consider the related viscous equation

$$q_t^\epsilon + f(q^\epsilon)_x = \epsilon q_{xx}^\epsilon \tag{11.49}$$

for $\epsilon > 0$. We will investigate how $\eta(q^\epsilon)$ behaves for solutions $q^\epsilon(x, t)$ to (11.49) and take the limit as $\epsilon \to 0$. Since solutions to the parabolic equation (11.49) are always smooth, we can derive the corresponding evolution equation for the entropy following the same manipulations we used for smooth solutions of the inviscid equation, multiplying (11.49) by $\eta'(q^\epsilon)$ to obtain

$$\eta(q^\epsilon)_t + \psi(q^\epsilon)_x = \epsilon\eta'(q^\epsilon)q_{xx}^\epsilon.$$

We can now rewrite the right-hand side to obtain

$$\eta(q^\epsilon)_t + \psi(q^\epsilon)_x = \epsilon\left(\eta'(q^\epsilon)q_x^\epsilon\right)_x - \epsilon\eta''(q^\epsilon)\left(q_x^\epsilon\right)^2.$$

Integrating this equation over the rectangle $[x_1, x_2] \times [t_1, t_2]$ gives

$$\begin{aligned}\int_{x_1}^{x_2} \eta(q^\epsilon(x, t_2))\,dx = &\int_{x_1}^{x_2} \eta(q^\epsilon(x, t_1))\,dx \\ &- \left(\int_{t_1}^{t_2} \psi(q^\epsilon(x_2, t))\,dt - \int_{t_1}^{t_2} \psi(q^\epsilon(x_1, t))\,dt\right) \\ &+ \epsilon \int_{t_1}^{t_2} \left[\eta'(q^\epsilon(x_2, t))\, q_x^\epsilon(x_2, t) - \eta'(q^\epsilon(x_1, t))\, q_x^\epsilon(x_1, t)\right] dt \\ &- \epsilon \int_{t_1}^{t_2} \int_{x_1}^{x_2} \eta''(q^\epsilon)\left(q_x^\epsilon\right)^2 dx\,dt. \end{aligned} \tag{11.50}$$

In addition to the flux differences, the total entropy is modified by two terms involving ϵ. As $\epsilon \to 0$, the first of these terms vanishes. (This is clearly true if the limiting function $q(x, t)$ is smooth at x_1 and x_2, and can be shown more generally.) The other term, however, involves integrating $(q_x^\epsilon)^2$ over the rectangle $[x_1, x_2] \times [t_1, t_2]$. If the limiting weak solution is discontinuous along a curve in this rectangle, then this term will not vanish in the limit. However, since $\epsilon > 0$, $(q_x^\epsilon)^2 > 0$, and $\eta'' > 0$ (by our convexity assumption), we can conclude that this term is nonpositive in the limit and hence the vanishing-viscosity weak solution satisfies (11.43).

Just as for the conservation law, an alternative weak form of the entropy condition can be formulated by integrating against smooth test functions ϕ, now required to be nonnegative, since the entropy condition involves an inequality. A weak solution q satisfies the *weak form of the entropy inequality* if

$$\int_0^\infty \int_{-\infty}^\infty [\phi_t \eta(q) + \phi_x \psi(q)]\,dx\,dt + \int_{-\infty}^\infty \phi(x, 0)\eta(q(x, 0))\,dx \ge 0 \tag{11.51}$$

for all $\phi \in C_0^1(\mathbb{R} \times \mathbb{R})$ with $\phi(x, t) \ge 0$ for all x, t. In Section 12.11.1 we will see that this form of the entropy condition is convenient to work with in proving that certain numerical methods converge to entropy-satisfying weak solutions.

The entropy inequalities (11.43) and (11.51) are often written informally as

$$\eta(q)_t + \psi(q)_x \le 0, \tag{11.52}$$

with the understanding that where q is smooth (11.44) is in fact satisfied and near discontinuities the inequality (11.52) must be interpreted in the integral form (11.43) or the weak form (11.51).

Another form that is often convenient is obtained by applying the integral inequality (11.43) to an infinitesimal rectangle near a shock as illustrated in Figure 11.7. When the integral form of the original conservation law was applied over this rectangle, we obtained the Rankine–Hugoniot jump condition (11.20),

$$s(q_r - q_l) = f(q_r) - f(q_l).$$

Going through the same steps using the inequality (11.43) leads to the inequality

$$s(\eta(q_r) - \eta(q_l)) \ge \psi(q_r) - \psi(q_l). \tag{11.53}$$

We thus see that a discontinuity propagating with speed s satisfies the entropy condition if and only if the inequality (11.53) is satisfied.

Example 11.2. For Burgers' equation (11.34), the discussion of Section 11.12 shows that the convex function $\eta(u) = u^2$ can be used as an entropy function, with entropy flux $\psi(u) = \frac{2}{3}u^3$. Note that these satisfy (11.47). Consider a jump from u_l to u_r propagating with speed $s = (u_l + u_r)/2$, the shock speed for Burgers' equation. The entropy condition (11.53) requires

$$\frac{1}{2}(u_l + u_r)\big(u_r^2 - u_l^2\big) \geq \frac{2}{3}\big(u_r^3 - u_l^3\big).$$

This can be rearranged to yield

$$\frac{1}{6}(u_l - u_r)^3 \geq 0,$$

and so the entropy condition is satisfied only if $u_l > u_r$, as we already found from the Lax Entropy Condition 11.1.

11.14.1 The Kružkov Entropies

In general an entropy function should be strictly convex, with a second derivative that is strictly positive at every point. This is crucial in the above analysis of (11.50), since the $(q_x^\epsilon)^2$ term that gives rise to the inequality in (11.51) is multiplied by $\eta''(q^\epsilon)$.

Rather than considering a single strictly convex function that can be used to investigate every value of q^ϵ, a different approach was adopted by Kružkov [248], who introduced the idea of entropy inequalities. He used a whole family of entropy functions and corresponding entropy fluxes,

$$\eta_k(q) = |q - k|, \qquad \psi_k(q) = \operatorname{sgn}(q - k)\,[f(q) - f(k)], \tag{11.54}$$

where k is any real number. Each function $\eta_k(q)$ is a piecewise linear function of q with a kink at $q = k$, and hence $\eta''(q) = \delta(q - k)$ is a delta function with support at $q = k$. It is a weakly convex function whose nonlinearity is concentrated at a single point, and hence it is useful only for investigating the behavior of weak solutions near the value $q = k$. However, it is sometimes easier to obtain entropy results by studying the simple piecewise linear function $\eta_k(q)$ than to work with an arbitrary entropy function. If it can be shown that if a weak solution satisfies the entropy inequality (11.51) for an arbitrary choice of $\eta_k(q)$ (i.e., that the entropy condition is satisfied for *all* the Kružkov entropies), then (11.51) also holds more generally for any strictly convex entropy $\eta(q)$.

11.15 Long-Time Behavior and N-Wave Decay

Figure 11.8 shows the solution to Burgers' equation with some smooth initial data having compact support. As time evolves, portions of the solution where $u_x < 0$ steepen into shock waves while portions where $u_x > 0$ spread out as rarefaction waves. Over time these shocks

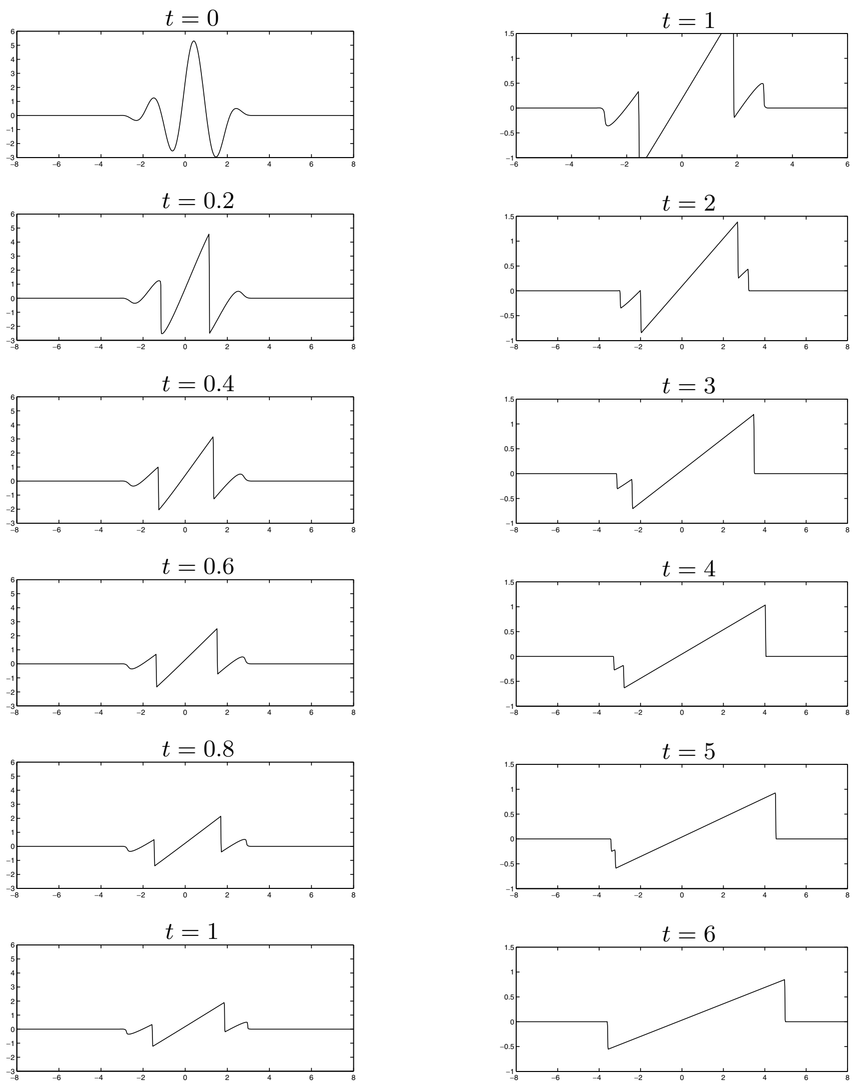


Fig. 11.8. Decay of general initial data to an N-wave with Burgers' equation. The left column shows the initial behavior from time $t = 0$ to $t = 1$ (going down). The right column shows later times $t = 1$ to $t = 6$ on a different scale. `[claw/book/chap11/burgers]`

and rarefactions interact. The shocks travel at different speeds, and those going in the same direction merge into stronger shocks. Meanwhile, the rarefaction waves weaken the shocks. The most striking fact is that in the long run all the structure of the initial data completely disappears and the solution behaves like an *N-wave*: one shock propagates to the left, another to the right, and in between the rarefaction wave is essentially linear. Similar long-time behavior would be seen starting from any other initial data with compact support. The position and strength of the two shocks does depend on the data, but this same N-wave shape will always arise. The same is true for other nonlinear conservation laws, provided the flux function is convex, although the shape of the rarefaction wave between the two shocks will depend on this flux. Only for a quadratic flux (such as Burgers' equation) will it be linear. For more details about the long-time behavior of solutions, see for example [98], [308], [420].

Note that this loss of structure in the solution implies that in general a nonlinear conservation law models irreversible phenomena. With a linear equation, such as the advection equation or the acoustics equations, the solution is reversible. We can start with arbitrary data, solve the equation $q_t + Aq_x = 0$ for some time T, and then solve the equation $q_t - Aq_x = 0$ for another T time units, and the original data will be recovered. (This is equivalent to letting t run backwards in the original equation.) For a nonlinear equation $q_t + f(q)_x = 0$, we can do this only over times T for which the solution remains smooth. Over such time periods the solution is constant along the characteristics, which do not cross, and the process is reversible. Once shocks form, however, characteristics disappear into the shock and information is lost. Many different sets of data can give rise to exactly the same shock wave. If the equation is now solved backwards in time (or, equivalently, we solve $q_t - f(q)_x = 0$) then compression at the shock becomes expansion. There are infinitely many different weak solutions in this case. Spreading back out into the original data is one of these, but the solution at the final time contains no information about which of the infinitely many possibilities is "correct."

The fact that information is irretrievably lost in a shock wave is directly related to the notion that the physical entropy increases across a shock wave. Entropy is a measure of the amount of disorder in the system (see Section 14.5) and a loss of structure corresponds to an increase in entropy. Since entropy can only increase once shocks have formed, it is not possible to recover the initial data.

Exercises

11.1. Show that in solving the scalar conservation law $q_t + f(q)_x = 0$ with smooth initial data $q(x, 0)$, the time at which the solution "breaks" is given by

$$T_b = \frac{-1}{\min_x[f''(q(x,0))q_x(x,0)]} \tag{11.55}$$

if this is positive. If this is negative, then characteristics never cross. *Hint:* Use $q(x,t) = q(\xi(x,t), 0)$ from (11.11), differentiate this with respect to x, and determine where q_x becomes infinite. To compute ξ_x, differentiate the equation (11.12) with respect to x.

11.2. Show that the viscous Burgers equation $u_t + uu_x = \epsilon u_{xx}$ has a *traveling-wave solution* of the form $u^\epsilon(x,t) = w^\epsilon(x - st)$, by deriving an ODE for w and verifying that this ODE has solutions of the form

$$w(\xi) = u_r + \frac{1}{2}(u_l - u_r)\left[1 - \tanh\left(\frac{(u_l - u_r)\xi}{4\epsilon}\right)\right], \tag{11.56}$$

when $u_l > u_r$, with the propagation speed s agreeing with the shock speed (11.23). Note that $w(\xi) \to u_l$ as $\xi \to -\infty$, and $w(\xi) \to u_r$ as $\xi \to +\infty$. Sketch this solution and indicate how it varies as $\epsilon \to 0$. What happens to this solution if $u_l < u_r$, and why is there no traveling-wave solution with limiting values of this form?

11.3. For a general smooth scalar flux functions $f(q)$, show by Taylor expansion of (11.21) that the shock speed is approximately the average of the characteristic speed on each side,

$$s = \frac{1}{2}[f'(q_l) + f'(q_r)] + \mathcal{O}(|q_r - q_l|^2).$$

The exercises below require determining the exact solution to a scalar conservation law with particular initial data. In each case you should sketch the solution at several instants in time as well as the characteristic structure and shock-wave locations in the x–t plane.

You may wish to solve the problem numerically by modifying the CLAWPACK codes for this chapter in order to gain intuition for how the solution behaves and to check your formulas.

11.4. Determine the exact solution to Burgers' equation $u_t + (\frac{1}{2}u^2)_x = 0$ for all $t > 0$ when each of the following sets of initial data is used:

(a)

$$\overset{\circ}{u}(x) = \begin{cases} 1 & \text{if } x < -1, \\ 0 & \text{if } -1 < x < 1, \\ -1 & \text{if } x > 1. \end{cases}$$

(b)

$$\overset{\circ}{u}(x) = \begin{cases} -1 & \text{if } x < -1, \\ 0 & \text{if } -1 < x < 1, \\ 1 & \text{if } x > 1. \end{cases}$$

11.5. Determine the exact solution to Burgers' equation for $t > 0$ with initial data

$$\overset{\circ}{u}(x) = \begin{cases} 2 & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the rarefaction wave catches up to the shock at some time T_c. For $t > T_c$ determine the location of the shock by two different approaches:

(a) Let $x_s(t)$ represent the shock location at time t. Determine and solve an ODE for $x_s(t)$ by using the Rankine–Hugoniot jump condition (11.21), which must hold across the shock at each time.

(b) For $t > T_c$ the exact solution is triangular-shaped. Use conservation to determine $x_s(t)$ based on the area of this triangle. Sketch the corresponding "overturned" solution, and illustrate the equal-area rule (as in Figure 11.6).

11.6. Repeat Exercise 11.5 with the data

$$\overset{\circ}{u}(x) = \begin{cases} 2 & \text{if } 0 < x < 1, \\ 4 & \text{otherwise.} \end{cases}$$

Note that in this case the shock catches up with the rarefaction wave.

11.7. Determine the exact solution to Burgers' equation for $t > 0$ with the data

$$\mathring{u}(x) = \begin{cases} 12 & \text{if } 0 < x, \\ 8 & \text{if } 0 < x < 14, \\ 4 & \text{if } 14 < x < 17, \\ 2 & \text{if } 17 < x. \end{cases}$$

Note that the three shocks eventually merge into one shock.

11.8. Consider the scalar conservation law $u_t + (e^u)_x = 0$. Determine the exact solution with the following sets of initial data:

(a)

$$\mathring{u}(x) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x > 0. \end{cases}$$

(b)

$$\mathring{u}(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x > 0. \end{cases}$$

(c)

$$\mathring{u}(x) = \begin{cases} 2 & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hint: Use the approach outlined in Exercise 11.5(a).

11.9. Determine an entropy function and entropy flux for the traffic flow equation with flux (11.4). Use this to show that $q_l < q_r$ is required for a shock to be admissible.