14

Gas Dynamics and the Euler Equations

A brief introduction to gas dynamics was given in Section 2.6, where the equations for conservation of mass and momentum were stated. In this chapter we will consider the energy equation and the equation of state in more detail, along with a few other quantities of physical (and mathematical) significance, such as the entropy. We will also look at some special cases, including isentropic and isothermal flows, where systems of two equations are obtained. These provide simplified examples that can be used to illustrate the nonlinear theory.

The derivations here will be very brief, with an emphasis on the main ideas without a detailed description of the physics. More thorough introductions may be found in several books on hyperbolic equations, such as [92], [156], [420], [486], or on gas dynamics, such as [58], [70], [297], [405], [474].

Recall that ρ is the density, u the velocity, E the total energy, and p the pressure of the gas. In Section 2.6 the continuity equation

$$\rho_t + (\rho u)_x = 0 \tag{14.1}$$

was derived from the more fundamental integral form, obtained by integrating the density over a test section $[x_1, x_2]$ and using the mass flux ρu at each end. More generally, for any quantity z that is advected with the flow there will be a contribution to the flux for z of the form zu. Thus, the momentum equation has a contribution of the form $(\rho u)u = \rho u^2$, and the energy equation has a flux contribution Eu.

14.1 Pressure

The velocity $u(x, t)$ used in gas dynamics is a macroscopic quantity that represents an average over a huge number of gas molecules in the neighborhood of the point x. The advective momentum flux ρu^2 mentioned above is a macroscopic flux, the same as what would be obtained if all nearby gas molecules were moving at this same speed. In reality they are not, however, and this microscopic variation leads to an additional microscopic contribution to the momentum flux, which is given by the *pressure*. To understand this, consider a gas that is "at rest", with macroscopic velocity $u = 0$. The individual molecules are still moving, however, at least if the temperature is above absolute zero. One way to calculate the pressure at a point x_1 is to think of inserting an imaginary wall in our

one-dimensional tube of gas at this point and calculate the force (per unit area) exerted on each side of this wall by the gas. These forces will normally be of equal magnitude and opposite in sign, and arise from the molecules on each side colliding with the wall and bouncing off. Of course, if the wall is not really there, then the molecules don't bounce off it. Instead, molecules that approach x_1 from $x < x_1$ with positive momentum move into the test section $[x_1, x_2]$ and by doing so will increase the momentum in this section, and hence make a positive contribution to the flux of momentum past x_1. Likewise molecules that approach x_1 from $x > x_1$ must have negative momentum, so as they pass the point they are *removing negative momentum* from the test section, which also makes a positive contribution to the momentum flux past this point. Note that the two contributions to momentum flux due to molecules moving rightwards and leftwards do not cancel out, but rather add together to give a net positive flux of momentum. The momentum in the section $[x_1, x_2]$ is thus always increasing due to the flux at the left boundary, resulting simply from the random movement of molecules near x_1. This seems rather paradoxical, but note that if the pressure is constant over the test section then there will be an equal flux of momentum at x_2, out of the section, so that the total momentum in the section remains constant (and equal to zero, since $u = 0$). Note that if the pressure differs between x_1 and x_2, then there will be a net nonzero flux of momentum into this section, and hence an apparent macroscopic acceleration of the gas. (Individual molecules are not actually accelerated, however – it's just that the distribution of velocities observed in the test section is changing.)

In general the pressure gives the microscopic momentum flux that must be added to the advective flux ρu^2 to obtain the total momentum flux,

$$\text{momentum flux} = \rho u^2 + p, \tag{14.2}$$

leading to the integral conservation law

$$\frac{d}{dt} \int_{x_1}^{x_2} \rho(x, t) u(x, t)\, dx = -[\rho u^2 + p]_{x_1}^{x_2}. \tag{14.3}$$

The differential form of the momentum equation is

$$(\rho u)_t + (\rho u^2 + p)_x = 0. \tag{14.4}$$

There may also be external forces acting on the gas, such as gravity, that do cause the acceleration of individual molecules. In this case the external force must be integrated over $[x_1, x_2]$ and this contribution added to (14.3). This leads to the addition of a source term on the right-hand side of (14.4); see Section 2.5 and Chapter 17.

14.2 Energy

The total energy E is often decomposed as

$$E = \rho e + \frac{1}{2}\rho u^2. \tag{14.5}$$

The term $\frac{1}{2}\rho u^2$ is the kinetic energy, while ρe is the internal energy. The variable e, internal energy per unit mass, is called the *specific internal energy*. (In general, "specific" means

"per unit mass"). Internal energy includes translational, rotational, and vibrational energy and possibly other forms of energy in more complicated situations. In the Euler equations we assume that the gas is in local chemical and thermodynamic equilibrium and that the internal energy is a known function of pressure and density:

$$e = e(p, \rho). \tag{14.6}$$

This is the *equation of state* for the gas, which depends on the particular gas under study.

The total energy advects with the flow, leading to the macroscopic energy flux term Eu. In addition, the microscopic momentum flux measured by p leads to a flux in kinetic energy that is given by pu. In the absence of outside forces, the conservation law for total energy thus takes the differential form

$$E_t + [(E + p)u]_x = 0. \tag{14.7}$$

If outside forces act on the gas, then a source term must be included, since the total energy will be modified by work done on the gas.

14.3 The Euler Equations

Putting these equations together gives the system of *Euler equations*

$$\begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}_t + \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (E + p)u \end{bmatrix}_x = 0. \tag{14.8}$$

These are a simplification of the more realistic *Navier–Stokes* equations, which also include effects of fluid viscosity and heat conduction. The terms dropped involve second-order derivatives that would make the system parabolic rather than hyperbolic, and cause them to have smooth solutions for all time. However, when the viscosity and heat conductivity are very small, the vanishing-viscosity hyperbolic equations are a good approximation. The resulting discontinuous shock waves are good approximations to what is observed in reality – very thin regions over which the solution is rapidly varying. (In some cases viscous effects may be nonnegligible. For example, viscous boundary layers may have a substantial effect on the overall solution; see Section 21.8.4.)

To obtain a closed system of equations, we still need to specify the equation of state relating the internal energy to pressure and density.

14.4 Polytropic Ideal Gas

For an ideal gas, internal energy is a function of temperature alone, $e = e(T)$. The *temperature T* is related to p and ρ by the *ideal gas law*,

$$p = R\rho T \tag{14.9}$$

where R is a constant obtained by dividing the universal gas constant $\mathcal{R}$ by the molecular weight of the gas. To good approximation, the internal energy is simply proportional to the

temperature,

$$e = c_v T, \tag{14.10}$$

where c_v is a constant known as the *specific heat at constant volume*. Such gases are called *polytropic*. If energy is added to a fixed volume of a polytropic gas, then the change in energy and change in temperature are related via

$$de = c_v dT. \tag{14.11}$$

On the other hand, if the gas is allowed to expand at constant pressure, not all of the energy goes into increasing the internal energy. The work done in expanding the volume $1/\rho$ by $d(1/\rho)$ is $p\, d(1/\rho)$, and we obtain another relation

$$de + p\, d(1/\rho) = c_p\, dT \tag{14.12}$$

or

$$d(e + p/\rho) = c_p\, dT, \tag{14.13}$$

where c_p is the *specific heat at constant pressure*. The quantity

$$h = e + p/\rho \tag{14.14}$$

is called the (*specific*) *enthalpy* of the gas. For a polytropic gas, c_p is constant, so that (14.13) yields

$$h = c_p T, \tag{14.15}$$

and the enthalpy is simply proportional to the temperature. Note that by the ideal gas law,

$$c_p - c_v = R. \tag{14.16}$$

The equation of state for a polytropic gas turns out to depend only on the *ratio of specific heats*, usually denoted by

$$\gamma = c_p/c_v. \tag{14.17}$$

This parameter is also often called the *adiabatic exponent*.

Internal energy in a molecule is typically split up between various degrees of freedom (translational, rotational, vibrational, etc.). How many degrees of freedom exist depends on the nature of the gas. The general *principle of equipartition of energy* says that the average energy in each of these is the same. Each degree of freedom contributes an average energy of $\frac{1}{2}kT$ per molecule, where k is *Boltzmann's constant*. This gives a total contribution of $\frac{\alpha}{2}kT$ per molecule if there are α degrees of freedom. Multiplying this by n, the number of molecules per unit mass (which depends on the gas), gives

$$e = \frac{\alpha}{2} nkT. \tag{14.18}$$

The product nk is precisely the gas constant R, so comparing this with (14.10) gives

$$c_v = \frac{\alpha}{2} R. \tag{14.19}$$

From (14.16) we obtain

$$c_p = \left(1 + \frac{\alpha}{2}\right) R, \tag{14.20}$$

and so

$$\gamma = c_p/c_v = \frac{\alpha + 2}{\alpha}. \tag{14.21}$$

Note that $T = p/R\rho$, so that

$$e = c_v T = \left(\frac{c_v}{R}\right) \frac{p}{\rho} = \frac{p}{(\gamma - 1)\rho} \tag{14.22}$$

by (14.16) and (14.17). Using this in (14.5) gives the common form of the *equation of state for an ideal polytropic gas*:

$$E = \frac{p}{\gamma - 1} + \frac{1}{2}\rho u^2. \tag{14.23}$$

An ideal gas with this equation of state is also sometimes called a *gamma-law gas*.

For a monatomic gas the only degrees of freedom are the three translational degrees, so $\alpha = 3$ and $\gamma = 5/3$. For a diatomic gas there are also two rotational degrees of freedom and $\alpha = 5$, so that $\gamma = 7/5 = 1.4$. Under ordinary circumstances air is composed primarily of N_2 and O_2, and so $\gamma \approx 1.4$.

14.5 Entropy

The fundamental thermodynamic quantity is the entropy. Roughly speaking, this measures the disorder in the system, and indicates the degree to which the internal energy is available for doing useful work. The greater the entropy, the less available the energy.

The specific entropy s (entropy per unit mass) is given by

$$s = c_v \log(p/\rho^\gamma) + \text{constant}. \tag{14.24}$$

This can be solved for p to give

$$p = \kappa e^{s/c_v} \rho^\gamma, \tag{14.25}$$

where κ is a constant.

We can manipulate the Euler equations to derive the relation

$$s_t + u s_x = 0, \tag{14.26}$$

which says that entropy is constant along particle paths in regions of smooth flow. In fact, (14.26) can be derived from fundamental principles, and this equation, together with the

conservation of mass and momentum equations, gives an alternative formulation of the Euler equations for smooth flows (though not in conservation form):

$$\rho_t + (\rho u)_x = 0,$$
$$(\rho u)_t + (\rho u^2 + p)_x = 0,$$
$$s_t + u s_x = 0.$$
(14.27)

The equation of state in these variables gives p as a function of ρ and s, e.g. (14.25) for a polytropic gas.

From our standpoint the most important property of entropy is that in smooth flow it remains constant on each particle path, whereas if a particle crosses a shock, then the entropy may jump, but only to a *higher* value. This results from the fact that the viscous processes (molecular collisions) in the thin physical shock profile cause the entropy to increase. This gives the physical *entropy condition* for shocks. (The term "fluid particle" is used to mean an infinitesimal volume of fluid that nonetheless contains a huge number of molecules.)

Note that along a particle path in smooth flow, since s is constant, we find by (14.25) that

$$p = \hat{\kappa} \rho^\gamma,$$
(14.28)

where $\hat{\kappa} = \kappa e^{s/c_v}$ is a constant that depends only on the initial entropy of the particle. This explicit relation between density and pressure along particle paths is sometimes useful. Of course, if the initial entropy varies in space, then $\hat{\kappa}$ will be different along different particle paths.

Note that it appears we can combine the first and third equations of (14.27) to obtain a conservation law for the entropy per unit volume, $S = \rho s$,

$$S_t + (uS)_x = 0.$$
(14.29)

This equation does hold for smooth solutions, but it does not follow from an integral conservation law that holds more generally, and in fact entropy is *not* conserved across shocks. Hence the apparent system of conservation laws obtained by replacing the third equation of (14.27) by (14.29) is not equivalent to the conservative Euler equations (14.8) for weak solutions, and would result in physically incorrect shock speeds.

14.5.1 Isentropic Flow

If we consider very small smooth perturbations around some background state (as in acoustics), then no shocks will form over reasonable time periods, and so we can use these nonconservative equations (14.27) and obtain the same results as with the conservative Euler equations. Moreover, since s simply advects with the flow, if s is initially uniform throughout the gas, then s will remain constant and we do not need to bother solving the third equation in (14.27). This justifies the use of the *isentropic* equations for small disturbances, which were presented in Section 2.3. Taking $s = $ constant in (14.25) gives the equation of

state (14.28), as was introduced earlier in (2.35). The isentropic equations again are

$$\begin{bmatrix} \rho \\ \rho u \end{bmatrix}_t + \begin{bmatrix} \rho u \\ \rho u^2 + \hat{\kappa} \rho^\gamma \end{bmatrix}_x = 0. \tag{14.30}$$

Recall that the sound speed c is given by

$$c = \sqrt{\gamma p / \rho}. \tag{14.31}$$

This was derived from the linearized equations using the ideal gas equation of state presented above. With a more general equation of state we can still study acoustics by linearizing for small perturbations. Assuming the entropy is constant then results in the more general expression for the sound speed,

$$c = \sqrt{\left.\frac{\partial p}{\partial \rho}\right|_{s=\text{const}}}. \tag{14.32}$$

The sound speed is computed in general from the equation of state $p = p(\rho, s)$ by taking the partial derivative with respect to ρ. This corresponds to the fact that within an acoustic wave the density and pressure vary, but the entropy does not, and the "stiffness" of the gas (i.e., its response to compression in the form of increased pressure) dictates the velocity with which an acoustic wave propagates.

Note that the isentropic equations are still nonlinear, and so in general we expect shocks to form if we take arbitrary data. In particular, if we look at a Riemann problem with discontinuous data, then we may have shocks in the solution immediately. What is the physical meaning of these shocks, in view of the fact that across a real gas-dynamic shock we know the entropy cannot remain constant?

In reducing the Euler equations to the isentropic equations, we have dropped the conservation-of-energy equation. If we study only flows for which the entropy is truly constant (no shocks), then this equation will be automatically satisfied. However, if we use the isentropic equations for a problem with shocks, then conservation of energy will not hold across the shock. Mathematically such weak solutions of the isentropic equations make perfectly good sense, but they no longer model reality properly, since they do not model conservation of energy. (However, if the shocks are weak enough, then very little entropy is produced physically and the "isentropic" shock may be a good approximation to reality.)

Another way to view this is by the following thought experiment. We could, in principle, create a physical shock wave across which there is no increase in entropy if we could find a way to reduce the entropy of each gas particle just after it passes through the shock. In principle we could accomplish this by doing some work on each fluid particle to eliminate the "disorder" created by the trauma of passing through the shock. Doing so would require an input of energy right at the shock. Hence across this hypothetical isentropic shock there must be a jump in the energy of the gas, reflecting the outside work that has been done on it.

Moreover, across an isentropic shock we see that the energy must jump to a *higher* value. This can be used as an admissibility criterion for shocks in the weak formulation of the isentropic equations. A shock is the correct vanishing-viscosity solution to the isentropic equations only if the energy *increases* across the shock. The *energy* can thus be used as

an "entropy function" for the isentropic system of equations, in the sense introduced in Chapter 11 (though that terminology is particularly confusing here).

14.6 Isothermal Flow

Taking $\gamma = 1$ in the isentropic equations (14.30) gives a particularly simple set of equations. By (14.21), the case $\gamma = 1$ is not physically realizable but can be viewed as a limiting case as $\alpha \to \infty$, i.e., for very complex molecules with many degrees of freedom. Note that such gases also have large heat capacities c_v and c_p, meaning that it requires a substantial input of energy to change the temperature very much. In the limit $\gamma \to 1$ the gas becomes *isothermal*, with constant temperature.

One can also obtain the isothermal flow equations by considering an ordinary gas in a tube that is immersed in a bath at a constant temperature $\bar{T}$. If we assume that this bath maintains a constant temperature within the gas, then we again obtain isothermal flow within the gas.

In isothermal flow, the ideal gas law (14.9) reduces to

$$p = a^2 \rho, \tag{14.33}$$

where $a^2 \equiv R\bar{T}$ is a constant and a is the sound speed (which is constant in isothermal flow). Note that maintaining this constant temperature requires heat flux through the wall of the tube (to take away heat generated at a shock or supply heat to a rarefaction), and so energy is no longer conserved in the tube. But mass and momentum are still conserved, and these equations, together with the equation of state (14.33), lead to the *isothermal equations*,

$$\begin{bmatrix} \rho \\ \rho u \end{bmatrix}_t + \begin{bmatrix} \rho u \\ \rho u^2 + a^2 \rho \end{bmatrix}_x = 0. \tag{14.34}$$

Isothermal flow is also an appropriate model for some astrophysical problems, particularly when modeling shock waves traveling through low-density interstellar space. In many cases the temperature increase caused by a shock wave leads to a radiative loss of energy via electromagnetic waves (at the speed of light) and very little of this energy is reabsorbed by the gas nearby.

In practice the temperature of a gas will never stay exactly constant, but it may *relax* towards a constant temperature very quickly as energy flows in or out of the gas via radiation or other mechanisms. A better physical model can be obtained by considering the full Euler equations with a source term that models the flow of heat into or out of the gas. A discussion of this *relaxation system* can be found in Section 17.17.3.

The isothermal equations are a system of two conservation laws for which the Hugoniot loci and integral curves are easy to compute, similarly to what was done in Chapter 13 for the shallow water equations. These are worked out for isothermal flow in [281].

14.7 The Euler Equations in Primitive Variables

For smooth solutions it is possible to rewrite the Euler equations in various nonconservative forms that are sometimes easier to work with or more revealing than the conservative form

(14.8). One example is the system (14.27), which shows that entropy is constant along streamlines for smooth solutions.

Another form that is more comprehensible physically is obtained by working in the *primitive variables* ρ, u, and p instead of the conserved variables, since the density, velocity and pressure are more intuitively meaningful. (Indeed, when we plot solutions to the the the Euler equations it is generally these variables that are plotted, even if the calculation was done in terms of the conserved variables.)

From the mass and momentum conservation equations it is easy to derive the equations

$$\rho_t + u\rho_x + \rho u_x = 0 \tag{14.35}$$

for the density and

$$u_t + uu_x + (1/\rho)p_x = 0 \tag{14.36}$$

for the velocity. With more manipulations one can also derive the equation

$$p_t + \gamma p u_x + u p_x = 0 \tag{14.37}$$

for a polytropic gas. These three equations yield the quasilinear hyperbolic system

$$\begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_t + \begin{bmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \gamma p & u \end{bmatrix} \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_x = 0. \tag{14.38}$$

This matrix has a considerably simpler form than the Jacobian matrix $f'(q)$ obtained from the conservative equations (14.8), which is given in (14.43) below. The eigenvalues and eigenvectors of the coefficient matrix in (14.38) are easily computed to be

$$\lambda^1 = u - c, \qquad \lambda^2 = u, \qquad \lambda^3 = u + c,$$
$$r^1 = \begin{bmatrix} -\rho/c \\ 1 \\ -\rho c \end{bmatrix}, \quad r^2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad r^3 = \begin{bmatrix} \rho/c \\ 1 \\ \rho c \end{bmatrix}, \tag{14.39}$$

where c is the sound speed of the polytropic gas,

$$c = \sqrt{\frac{\gamma p}{\rho}}. \tag{14.40}$$

We see a familiar pattern in these eigenvalues: information can advect at the fluid velocity or move as acoustic waves at speeds $\pm c$ relative to the gas. The ratio

$$M = |u|/c \tag{14.41}$$

is called the *Mach number*. The flow is *transonic* at any point where M passes through 1.

Note that linearizing these equations about a state (ρ_0, u_0, p_0) gives a result that is easily related to the acoustic equations derived in Section 2.7, and the eigenvectors (14.39) have been normalized in a manner analogous to that chosen in (2.58). (Other normalizations can

be used instead. In particular, multiplying r^1 and r^3 by c and r^2 by ρ would give a form where the physical units agree with those of the vector (ρ, u, p).)

From the above expressions we see that the Euler equations are hyperbolic provided ρ and p are positive. Moreover we can compute the gradients of the eigenvalues and find that the first and third characteristic field are genuinely nonlinear while the second field is linearly degenerate:

$$\nabla \lambda^1 = \begin{bmatrix} -\partial c/\partial \rho \\ 1 \\ -\partial c/\partial p \end{bmatrix} = \begin{bmatrix} c/2\rho \\ 1 \\ -c/2p \end{bmatrix} \implies \nabla \lambda^1 \cdot r^1 = \tfrac{1}{2}(\gamma + 1),$$

$$\nabla \lambda^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \implies \nabla \lambda^2 \cdot r^2 = 0, \qquad (14.42)$$

$$\nabla \lambda^3 = \begin{bmatrix} \partial c/\partial \rho \\ 1 \\ \partial c/\partial p \end{bmatrix} = \begin{bmatrix} -c/2\rho \\ 1 \\ c/2p \end{bmatrix} \implies \nabla \lambda^3 \cdot r^3 = \tfrac{1}{2}(\gamma + 1).$$

Simple waves in the second characteristic field consist of variations in density that are advecting with constant speed u, since u and p must be constant in such a wave. Such waves are often called *entropy waves* because the entropy satisfies the advection equation (14.26) and varies along with the density if p is constant. For the Riemann problem, the second field corresponds to *contact discontinuities* as described in Section 13.12, across which the two initial gases are in contact.

Simple waves in the first or third family will deform, since λ^1 and λ^3 vary along the integral curves of these families, sharpening into shocks or spreading out as rarefactions.

14.8 The Riemann Problem for the Euler Equations

To discuss the Riemann problem, we must return to the conservative form of the equations, which are valid across shock waves. If we compute the Jacobian matrix $f'(q)$ from (14.8), with the polytropic equation of state (14.23), we obtain

$$f'(q) = \begin{bmatrix} 0 & 1 & 0 \\ \tfrac{1}{2}(\gamma - 3)u^2 & (3 - \gamma)u & \gamma - 1 \\ \tfrac{1}{2}(\gamma - 1)u^3 - uH & H - (\gamma - 1)u^2 & \gamma u \end{bmatrix}, \qquad (14.43)$$

where

$$H = \frac{E + p}{\rho} = h + \frac{1}{2}u^2 \qquad (14.44)$$

is the *total specific enthalpy*. The eigenvalues are again

$$\lambda^1 = u - c, \qquad \lambda^2 = u, \qquad \lambda^3 = u + c, \qquad (14.45)$$

as for the coefficient matrix resulting from the primitive equations. They agree because the two forms are equivalent and should yield the same characteristic speeds. The eigenvectors will appear different, of course, in these different variables. We have

$$r^1 = \begin{bmatrix} 1 \\ u - c \\ H - uc \end{bmatrix}, \qquad r^2 = \begin{bmatrix} 1 \\ u \\ \frac{1}{2}u^2 \end{bmatrix}, \qquad r^3 = \begin{bmatrix} 1 \\ u + c \\ H + uc \end{bmatrix}. \tag{14.46}$$

Note that in these variables

$$\nabla \lambda^2(q) = \begin{bmatrix} -u/\rho \\ 1/\rho \\ 0 \end{bmatrix}, \tag{14.47}$$

and so we again find that $\nabla \lambda^2 \cdot r^2 \equiv 0$ and the second field is linearly degenerate.

14.9 Contact Discontinuities

We can have neither rarefaction waves nor shocks in the 2-characteristic field. Instead we have *contact discontinuities*, which are linear discontinuities that propagate with speed equal to the characteristic speed λ^2 on each side.

Note that because $\lambda^2 = u$ is constant on the integral curves of r^2, and because r^2 depends only on u, the vector r^2 is itself constant on these curves, and hence the integral curves are straight lines in phase space. Moreover, these integral curves also form the Hugoniot loci for contact discontinuities. Along these curves u and p are constant; only ρ varies.

It may seem strange that this discontinuity can sustain a jump in density – it seems that the denser gas should try to expand into the thinner gas. But that's because our intuition tends to equate higher density with higher pressure. It is only a *pressure* difference that can provide the force for expansion, and here the pressures are equal.

We can achieve two different densities at the same pressure by taking gases at two different temperatures. In fact, from (14.9) it is clear that there must be a jump in temperature if there is a jump in density but not in pressure. There must also be a jump in entropy by (14.24). This explains why contact discontinuities do not appear in solutions to the isothermal or isentropic equations considered previously. In the reduction of the Euler equations to one of these systems of only two equations, it is this linearly degenerate characteristic field that disappears.

Also, in the shallow water equations, where the pressure is related to the depth h by (13.3), it is not possible to have a jump in depth without also having a jump in pressure. Hence we see contact discontinuities only if we introduce another passive tracer advected with the fluid, as we did in Section 13.12.

Note that in all these hyperbolic systems we are ignoring diffusive effects, such as molecular diffusion of a tracer or diffusion of heat in a gas. These effects would smear out contact discontinuities in reality. We are assuming the diffusion coefficients are sufficiently small that these effects are negligible over the time scales of interest.

14.10 Riemann Invariants

Recall that for each family, the Riemann invariants are functions of q that are constant along any integral curve of this family and hence are constant through any simple wave in this family. Knowing these functions is very helpful in constructing solutions to the Riemann problem.

Since u and p are both constant across a contact discontinuity, these functions of q are both Riemann invariants of the 2-family.

The entropy s satisfies the advection equation (14.26) and hence is constant along particle paths. It follows that s is constant through any rarefaction wave or other simple wave in the 1-family or 3-family, and hence entropy is a Riemann invariant for these families. There's also a second set of Riemann invariants for each of these families. All of the Riemann invariants for a polytropic ideal gas are summarized below:

$$
\begin{aligned}
\text{1-Riemann invariants:} \quad & s, \quad u + \frac{2c}{\gamma - 1}, \\
\text{2-Riemann invariants:} \quad & u, \quad p, \\
\text{3-Riemann invariants:} \quad & s, \quad u - \frac{2c}{\gamma - 1}.
\end{aligned}
\tag{14.48}
$$

14.11 Solution to the Riemann Problem

The solution to a Riemann problem typically has a contact discontinuity and two nonlinear waves, each of which may be either a shock or a rarefaction wave, depending on q_l and q_r. The structure of a typical Riemann solution is shown in Figure 14.1 (see also the examples in Section 14.13). The first and third characteristic fields for the Euler equations are genuinely nonlinear and have behavior similar to the two characteristic fields in the isothermal or isentropic equations, and also similar to what we have seen for the two fields in the shallow water equations in Chapter 13. The contact discontinuity is also sometimes called the *entropy wave*, since it carries a jump in entropy. The first and third wave families are called *acoustic waves*, since in the small-disturbance limit these reduce to acoustics equations.
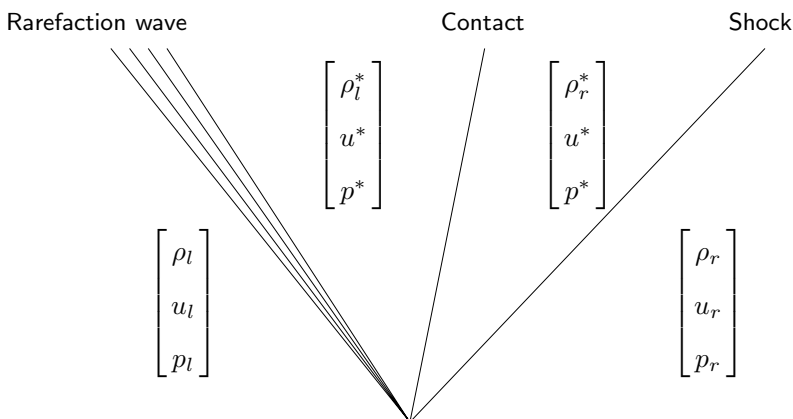

Fig. 14.1. Typical solution to the Riemann problem for the Euler equations.

Because u and p are constant across the contact discontinuity, it is often easier to work in the primitive variables (ρ, u, p) rather than $(\rho, \rho u, E)$, although of course the jump conditions must be determined using the conserved variables. The resulting Hugoniot locus and integral curves can be transformed into (ρ, u, p) space.

If the Riemann data is (ρ_l, u_l, p_l) and (ρ_r, u_r, p_r), then the two new constant states that appear in the Riemann solution will be denoted by $q_l^* = (\rho_l^*, u^*, p^*)$ and $q_r^* = (\rho_r^*, u^*, p^*)$. (See Figure 14.1.) Note that across the 2-wave we know there is a jump only in density.

Solution of the Riemann problem proceeds in principle just as in the previous chapters. Given the states q_l and q_r in the phase space, we need to determine the two intermediate states in such a way that q_l and q_l^* are connected by a 1-wave, q_l^* and q_r^* are connected by a 2-wave, and finally q_r^* and q_r are connected by a 3-wave. We need to consider three families of curves in the three-dimensional state space and find the appropriate intersections.

This seems difficult, but we can take advantage of the fact that we know the 2-wave will be a contact discontinuity across which u and p are constant to make the problem much simpler. Instead of considering the full three-dimensional (ρ, u, p) phase space, consider the p–u plane, and project the integral curves and Hugoniot loci for the 1-waves and 3-waves onto this plane. In particular, project the locus of all states that can be connected to q_l by a 1-wave (entropy satisfying shocks or rarefactions) onto this plane, and also the locus of all states that can be connected to q_r by a 3-wave. This gives Figure 14.2.

We see in this example that we can go from q_l (or actually, the projection of q_l) to q^* by a 1-rarefaction and from q^* to q_r by a 3-shock. The problem with this construction, of course, is that these curves are really curves in three-space, and the mere fact that their projections intersect does not mean the original curves intersect. However, the curve $R_1(q_l)$ must go through some state $q_l^* = (\rho_l^*, u^*, p^*)$ for some ρ_l^* (so that its projection onto the u–p plane is (u^*, p^*)). Similarly, the curve $\mathcal{S}_3(q_r)$ must pass through some state $q_r^* = (\rho_r^*, u^*, p^*)$. But these two states differ only in ρ, and hence can be connected by a 2-wave (contact discontinuity). We have thus achieved our objective. Note that this technique depends on the fact that *any* jump in ρ is allowed across the contact discontinuity.

Based on the given state q_l, we can find a function $u = \phi_l(p)$ that describes how u varies as we adjust p in a state (p, u) that can be connected to q_l by a 1-shock or 1-rarefaction. For
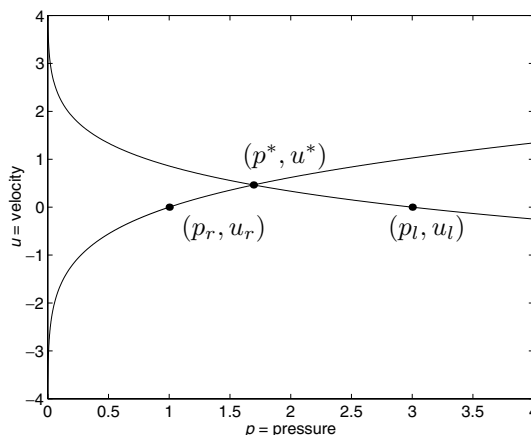

Fig. 14.2. Projection of shock and rarefaction curves onto the two-dimensional p–u plane, and determination of (p^*, u^*), for Sod's Riemann problem discussed in Section 14.13.

$p < p_l$ this function is defined by the integral curve of r^1 (projected to the p–u plane), since such states can be connected to q_l by a 1-rarefaction. For $p > p_l$ this function is defined by the Hugoniot locus, since these states can be connected by a shock. Similarly, for a given state q_r we can find a function $u = \phi_r(p)$ that describes how u varies as we adjust p in a state (p, u) that can be connected to q_r by a 3-shock or 3-rarefaction.

To solve the Riemann problem we need only solve for a pressure p^* for which $\phi_l(p^*) = \phi_r(p^*)$, which is a scalar nonlinear equation for p^*. In general this must be solved by an iterative method. Once p^* is known, u^*, ρ_l^*, and ρ_r^* are easily determined. Godunov first proposed a numerical method based on the solution of Riemann problems and presented one such iterative method in his paper [157] (also described in §12.15 of [369]). Chorin [67] describes an improvement of this method, and many other variants have been developed more recently.

We now briefly summarize how the functions $\phi_l(p)$ and $\phi_r(p)$ are determined. First consider the case of 1-rarefactions through q_l. Since the entropy s is a 1-Riemann invariant, we know that p/ρ^γ is constant through any rarefaction wave. This allows us to determine how ρ varies with p through a 1-rarefaction:

$$\rho = (p/p_l)^{1/\gamma} \rho_l. \tag{14.49}$$

We also know that the other 1-Riemann invariant from (14.48) is constant through any rarefaction wave, so

$$u + \frac{2}{\gamma - 1}\sqrt{\frac{\gamma p}{\rho}} = u_l + \frac{2}{\gamma - 1}\sqrt{\frac{\gamma p_l}{\rho_l}}. \tag{14.50}$$

We can use (14.49) to eliminate ρ on the left hand side and obtain

$$u + \frac{2}{\gamma - 1}\sqrt{\frac{\gamma p}{\rho_l}\left(\frac{p_l}{p}\right)^{1/\gamma}} = u_l + \frac{2}{\gamma - 1}\sqrt{\frac{\gamma p_l}{\rho_l}}$$

or

$$u + \frac{2}{\gamma - 1}\sqrt{\frac{\gamma p_l}{\rho_l}\left(\frac{p}{p_l}\right)^{(\gamma-1)/\gamma}} = u_l + \frac{2}{\gamma - 1}\sqrt{\frac{\gamma p_l}{\rho_l}}.$$

Rearranging this and using $c_l = \sqrt{\gamma p_l/\rho_l}$ to denote the speed of sound in the left state, we can solve this for u and obtain

$$u = u_l + \frac{2 c_l}{\gamma - 1}\left[1 - (p/p_l)^{(\gamma-1)/(2\gamma)}\right] \equiv \phi_l(p) \quad \text{for } p \le p_l. \tag{14.51}$$

This defines the function $\phi_l(p)$ for $p \le p_l$.

To determine this function for $p > p_l$ we need to use the Rankine–Hugoniot conditions instead of the Riemann invariants. We omit the detailed manipulations (see [420], for example) and just present the final formula, in which $\beta = (\gamma + 1)/(\gamma - 1)$:

$$u = u_l + \frac{2 c_l}{\sqrt{2\gamma(\gamma - 1)}}\left(\frac{1 - p/p_l}{\sqrt{1 + \beta p/p_l}}\right) \equiv \phi_l(p) \quad \text{for } p \ge p_l. \tag{14.52}$$

For each point (p, u) on the Hugoniot locus, there is a unique density ρ associated with it, given by

$$\rho = \left(\frac{1 + \beta p/p_l}{p/p_l + \beta} \right) \rho_l. \tag{14.53}$$

Similarly, the function $\phi_r(p)$ can be determined for a given state q_r by the same procedure. We obtain

$$u = u_r - \left(\frac{2 c_r}{\gamma - 1} \right) \left(1 - (p/p_r)^{(\gamma-1)/(2\gamma)} \right) \equiv \phi_r(p) \quad \text{for } p \leq p_r, \tag{14.54}$$

and

$$u = u_r - \frac{2 c_r}{\sqrt{2\gamma(\gamma - 1)}} \left(\frac{1 - p/p_r}{\sqrt{1 + \beta p/p_r}} \right) \equiv \phi_r(p) \quad \text{for } p \geq p_r. \tag{14.55}$$

The corresponding density for points on this 3-locus is given by

$$\rho = \left(\frac{1 + \beta p/p_r}{p/p_r + \beta} \right) \rho_r. \tag{14.56}$$

It is these functions $\phi_l(p)$ and $\phi_r(p)$ that are plotted in Figure 14.2 (for the particular case of the Sod's Riemann problem described in the Section 14.13). An iterative procedure can be used to determine the intersection (p^*, u^*).

Note that the formulas (14.55) and (14.56) are useful in setting initial data for numerical tests if we wish to specify data that corresponds to a single shock wave (a 3-shock in this case). We can choose the right state and the pressure p_l as we please, and then determine u_l and ρ_l using these formulas.

14.12 The Structure of Rarefaction Waves

In Section 14.11 we saw how to determine the Riemann solution for the Euler equations (with the polytropic equation of state) in the sense of finding the intermediate states q_l^* and q_r^*. To fully specify the Riemann solution we must also determine the structure of any rarefaction waves, i.e., determine ρ, u, and p as functions of $\xi = x/t$.

Suppose there is a 1-rarefaction connecting q_l to a state q_l^*. Then at each point in the rarefaction, $\xi = \lambda^1(q) = u - c$, and hence the sound speed c is given by

$$c = u - \xi. \tag{14.57}$$

We can use this expression to eliminate $c = \sqrt{\gamma p/\rho}$ from the equality (14.50) to obtain

$$u + \frac{2}{\gamma - 1} (u - \xi) = u_l + \frac{2}{\gamma - 1} c_l. \tag{14.58}$$

We can solve this for u as a function of ξ:

$$u(\xi) = \frac{(\gamma - 1)u_l + 2(c_l + \xi)}{\gamma + 1}. \tag{14.59}$$

From (14.57) and (14.59) we now know how c varies with ξ:

$$c(\xi) = u(\xi) - \xi. \tag{14.60}$$

Next we can use the fact that p/ρ^γ is constant to obtain the relation between c and ρ in the rarefaction:

$$
\begin{aligned}
c^2 &= \gamma p/\rho \\
&= \gamma (p/\rho^\gamma)\rho^{\gamma-1} \\
&= \gamma \left(p_l/\rho_l^\gamma\right)\rho^{\gamma-1}.
\end{aligned} \tag{14.61}
$$

Using (14.60), this allows us to determine how ρ varies with ξ:

$$\rho(\xi) = \left(\frac{\rho_l^\gamma [u(\xi) - \xi]^2}{\gamma p_l} \right)^{1/(\gamma-1)}. \tag{14.62}$$

Finally, again using the constancy of p/ρ^γ, we obtain

$$p(\xi) = \left(p_l/\rho_l^\gamma\right)[\rho(\xi)]^\gamma. \tag{14.63}$$

The same procedure can be used for 3-rarefactions, obtaining

$$
\begin{aligned}
u(\xi) &= \frac{(\gamma - 1)u_r - 2(c_r - \xi)}{\gamma + 1}, \\
\rho(\xi) &= \left(\frac{\rho_r^\gamma [\xi - u(\xi)]^2}{\gamma p_r} \right)^{1/(\gamma-1)}, \\
p(\xi) &= \left(p_r/\rho_r^\gamma\right)[\rho(\xi)]^\gamma.
\end{aligned} \tag{14.64}
$$

Having developed the formulas for the exact Riemann solution, we should note that in practical computations all the details of this structure are generally not needed. In practice methods based on Riemann solutions often use approximate Riemann solvers, as discussed in Section 15.3.

14.13 Shock Tubes and Riemann Problems

An experimental shock tube is filled with two different gases (or the same gas at different pressures and perhaps densities), separated by a membrane at $x = 0$. Initially the gas is at rest, so $u = 0$ everywhere. At time $t = 0$ the membrane is ruptured. The problem is a special case of the Riemann problem (special in that $u_l = u_r = 0$), and in this case it can be shown that the solution consists of a shock moving into the gas at lower pressure and a rarefaction wave that expands into the gas at higher pressure. The interface between the two gases moves at speed u^*, and so this interface is exactly the contact discontinuity. This is very similar to the dam-break Riemann problem for the shallow water equations described in Section 13.2.

Figure 14.3 shows particle paths in the x–t plane for one example with $\rho_l = p_l = 3$ and $\rho_r = p_r = 1$. This particular problem is called the *Sod problem* because Sod [421] used it
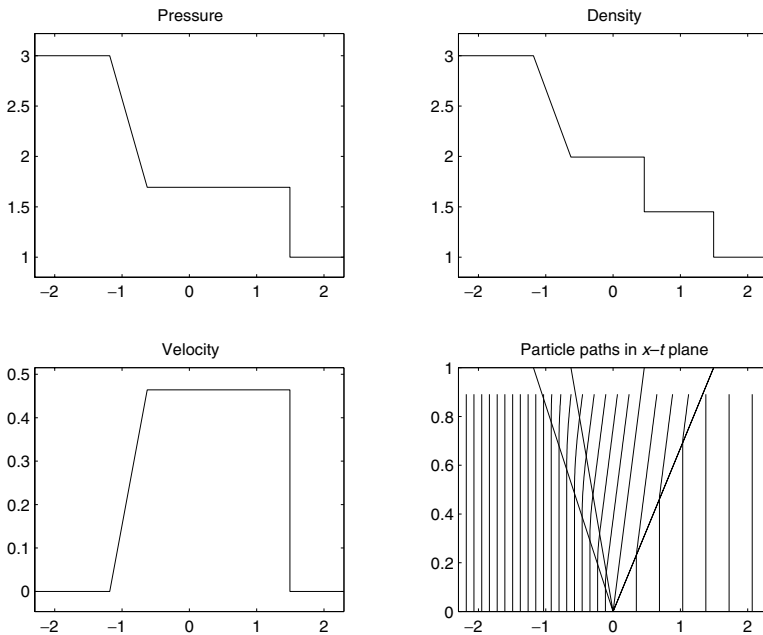
Fig. 14.3. Solution to the Sod shock-tube problem for the Euler equations.

as a test in an influential early comparison of different numerical methods. At any fixed t the points are spaced proportional to $1/\rho$ (the specific volume), so wider spacing indicates lower density. Note that there is a jump in density across the contact discontinuity while the velocity is the same on both sides, and equal to u^*, so particles never cross this contact surface. Note also the decrease in density as particles go through the rarefaction wave and the compression across the shock.

By (14.9), the temperature is proportional to p/ρ. In the Sod problem the temperature is initially uniform, but in the solution there must be a jump in temperature across the contact discontinuity where ρ jumps but p is constant. Although the two gases begin at the same temperature, the gas to the left is cooled by expansion through the rarefaction wave while the gas to the right heats up as it crosses the shock.

When the initial velocity is zero everywhere, the Riemann solution typically consists of one shock and one rarefaction wave, along with a contact discontinuity. This is clear from the structure in the p–u plane shown in Figure 14.2. If the initial velocities are nonzero, then it is possible to obtain solutions that consist of two shocks or two rarefaction waves rather than one of each, depending on the data. This is analogous to what was seen in Chapter 13 for the shallow water equations.

Figure 14.4 shows one other example of a Riemann problem for adiabatic gas dynamics, in which the solution contains two shock waves. In this case the Riemann data is

$$\rho_l = 1, \quad u_l = 3, \quad p_l = 1$$

and

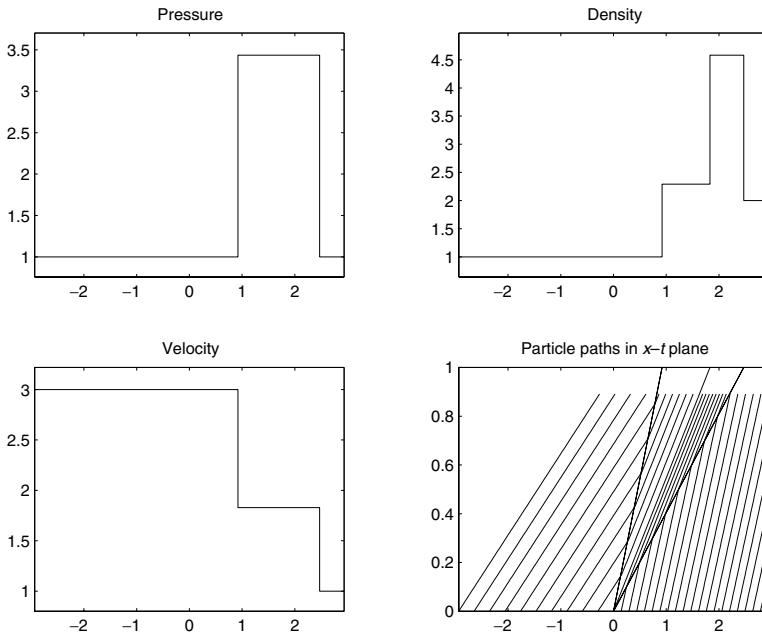$$\rho_r = 2, \quad u_r = 1, \quad p_r = 1.$$

Fig. 14.4. Solution to a Riemann problem for the Euler equations where the gas to the left has a higher velocity and the Riemann solution consists of two shock waves.

In this case the colliding gases lead to an increase in the density and pressure in the intermediate states relative to the initial states. Note that again the contact discontinuity separates particle paths arising from the two initial states.

14.14 Multifluid Problems

In our discussion above, we have assumed that the Riemann problem consists of the same gas on either side of the initial discontinuity, with only a jump in the state of the gas. Many practical problems involve the interaction of more than one gas. As a starting point for developing algorithms for this multifluid or multicomponent situation, we might study a Riemann problem in which two different gases are in contact at $x = 0$ initially. In the simplest case these might be two different ideal gamma-law gases with different values of γ, e.g., a shock-tube problem with the membrane initially separating air ($\gamma_l \approx 1.4$) from a monatomic gas such as argon ($\gamma_r = 5/3$).

In this case the Riemann solution still has the same structure discussed in this chapter and seen in Figure 14.1, with the two gases always in contact at the contact discontinuity. But now the shock or rarefaction wave that is moving into each gas must satisfy the appropriate jump conditions or integral relations for that particular gas. The Riemann problem can still be solved by the procedure outlined in Section 14.11, but the function $\phi_l(p)$ must be defined using γ_l while $\phi_r(p)$ is defined using γ_r. Setting $\phi_l(p) = \phi_r(p)$ and solving for p gives the intermediate pressure p^*. Even in this more general case, the pressure and velocity must be continuous across the contact discontinuity, and only the density has a jump.

To apply a numerical method in the multifluid case we must keep track of the constituent gases so that we know what value of γ to use in each grid cell. This is further complicated by the fact that numerically the sharp interface between the two gases will be smeared due to numerical diffusion, and some cells will contain a mixture of gases. Approaches to handling this with shock-capturing finite volume methods are discussed in Section 16.6. An alternative is to use a front-tracking or moving-mesh method to insure that the fluid interface is also a grid interface, e.g., [131].

14.15 Other Equations of State and Incompressible Flow

We have considered the Riemann problem only for the simplest ideal-gas equation of state (14.23). This is valid for many problems, including aerodynamic problems at subsonic or modest supersonic speeds. In some situations it may be necessary to consider *real-gas* effects, and a more complicated equation of state must be used, either a more complicated analytic expression or perhaps an "equation of state" that is only specified by tabulated values obtained from experiments with the gas or material of interest. The use of more general equations of state can complicate the Riemann solver. See [81], [91], [144], [149], [150], [208], [326], [328], [330], [396] for some discussions of more general Riemann problems.

As an example, consider what happens to a gas if we compress it to the point where the average intermolecular distance begins to approach the size of the molecules. Then the ideal-gas equation of state will no longer be valid, since it is based on a model in which each gas molecule is viewed as a single point and the gas can be compressed indefinitely. Instead we must take into account the fact that the molecules themselves take up some space. This leads to the *covolume equation of state*

$$p = \frac{RT}{v - b}. \tag{14.65}$$

This is written in terms of the specific volume $v = 1/\rho$. For $b = 0$ this agrees with the ideal-gas equation of state (14.9), and $b > 0$ now represents the volume taken up by the molecules themselves.

Additional corrections must be made to this equation of state as v approaches b. When the molecules are very close together the intermolecular attractive forces (van der Waals forces) must be considered, which tend to reduce the pressure. This leads to the *van der Waals equation of state* for dense polytropic gases,

$$p = \frac{RT}{v - b} - \frac{a}{v^2}, \tag{14.66}$$

along with the energy equation

$$e = e(T, v) = c_v T - a/v. \tag{14.67}$$

See, for example, [191], [192], [413]. In extreme cases, the gas undergoes a phase change and becomes a liquid. Phase-change problems can lead to loss of hyperbolicity as described in Section 16.3.2.

The Euler equations of compressible flow can also be used to simulate fluid dynamics in liquids, although in most applications liquids are essentially incompressible and acoustic waves have little effect on the fluid motion. An exception is in the study of violent phenomena such as underwater explosions or cavitation, or certain other problems involving liquid–gas interfaces. For such problems the compressible equations can be used with a suitable equation of state for the liquid. See [65], [88], [138], [212], [370], [449] for some examples.

In some applications acoustic waves in liquids are of interest in their own right, e.g., underwater acoustics, or ultrasound transmission in biological tissue. In these cases the linear acoustic equations can often be used by assuming the liquid is stationary on the time scale of interest in the acoustics problem. Explicit hyperbolic solvers are appropriate in this case.

For most problems involving the dynamics of liquids, the *incompressible Navier–Stokes equations* are used instead. Viscous terms are included, since viscosity generally cannot be ignored in liquids. In these equations the pressure is typically determined by the constraint that the divergence of the velocity field must vanish everywhere ($\nabla \cdot \vec{u} = 0$), since the fluid is incompressible. This is a global constraint that must typically be imposed numerically by solving an elliptic boundary-value problem over the spatial domain each time step. Note that this couples all points in the domain and allows information to travel at infinite speed, as must occur in an incompressible fluid. Applying a force to the fluid at one point will generally cause motion everywhere instantaneously. In reality, information cannot travel at infinite speed, and this motion is in fact accomplished by acoustic waves rapidly bouncing back and forth through the domain at a much higher velocity than the observed fluid motion. See Exercise 3.7 for a similar effect in low Mach number compressible flow, and Example 22.3 for a related example in solid mechanics.

In a liquid, these acoustic waves could be explicitly modeled using the compressible Navier–Stokes equations with appropriate equations of state. However, this is generally very inefficient, since we would need to take extremely small time steps in order to properly resolve these waves. The incompressible equations correctly model the fluid dynamics of interest by capturing the effect of the acoustic waves without explicitly modeling their propagation. High-resolution hyperbolic solvers are often used as a component in these methods for the convective terms, e.g., [9], [23], [53], [56], [266], [329]. These algorithms must then be coupled with implicit solvers for the viscous terms and elliptic solvers in lieu of an equation of state for the pressure. The study of incompressible Navier–Stokes and numerical methods for these equations is a very extensive topic that will not be pursued further here.

The incompressible equations are also useful for gas dynamics at very low Mach numbers, where again the acoustic waves have little effect on the fluid dynamics and the gas density is often nearly constant. Problems in atmospheric flow and low-speed aerodynamics often have this character, for example. Challenging numerical problems arise in simulating low Mach number flow in situations where compressible effects are present but very weak. Ideally one would like to use robust methods that are efficient in the zero Mach number limit (incompressible flow) but behave like high-resolution compressible-flow algorithms as the Mach number increases. See [10], [173], [399], [404], [242], [386], [119] for some discussions of this problem and possible approaches.