X sm. proj.

Given $g \in \mathbb{Z}_{\geq 0}$, $\beta \in H_2(X, \mathbb{Z})$

<u>Gromov-Witten invariants:</u>

$\mathbb{C}$-multilinear function

$$\underbrace{\langle -, -, \cdots, - \rangle}_{n}{}^{X}_{g,\beta} : H^*(X)^{\otimes n} \longrightarrow \mathbb{C}$$

"Count" the number of "curves" (stable maps) of genus g class β, passing through given cycles (up to Poincaré dual) specified by $H^*(X)$.

They are deformation invariant.

e.g. $\langle [pt], [pt] \rangle^{\mathbb{P}^2}_{0, [line]} = 1$

$$\langle [pt], [pt], [pt], [pt], [pt] \rangle^{\mathbb{P}^2}_{0,2} = 1$$

How are Gromov-Witten invariants defined?

Very brief story:

Consider the moduli space of embedded sm. curves of genus g and class β in X with n marked points (not allowed to collide)

$$M_{g,n}(X,\beta)^{\circ}$$

It is $\boxed{\text{not proper}}$ and we cannot have a reasonable intersection theory to get numbers.

We allow more objects than embedded curves (so-called stable maps) to form a larger moduli (stack).

$$M_{g,n}(X,\beta)^{\circ} \subset \overline{M}_{g,n}(X,\beta)$$

$\overline{M}_{g,n}(X,\beta)$ is $\boxed{\text{proper}}$.

<u>Roughly</u>, $\overline{M}_{g,n}(X,\beta) = \left\{ (C, p_1, \dots, p_n \in C,\ f: C \to X), \text{ s.t. } \begin{array}{l} C \text{ at worst nodal}, \\ f_*[C] = \beta, \text{ plus some stability cond.} \end{array} \right\}$

There is a cycle

virtual/expected dim

$$[\overline{M}_{g,n}(X,\beta)]^{vir} \in A_r(\overline{M}_{g,n}(X,\beta)) \quad \text{for certain } r \geq 0$$

coming from the natural deformation – obstruction theory on the moduli.

There are n evaluation maps

$$ev_1, \cdots, ev_n : \overline{M}_{g,n}(X,\beta) \longrightarrow X$$

mapping to the images of the corresponding marked pts.

$\alpha_1, \cdots, \alpha_n \in H^*(X)$

$$\langle \alpha_1, \cdots, \alpha_n \rangle^X_{g,\beta} := \text{“} \int_{[\overline{M}_{g,n}(X,\beta)]^{vir}} ev_1^*\alpha_1 \cdot \cdots \cdot ev_n^*\alpha_n \text{”}$$

$$= \pi_*\left([\overline{M}_{g,n}(X,\beta)]^{vir} \cap ev_1^*\alpha_1 \cap \cdots \cap ev_n^*\alpha_n \right)$$

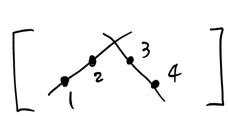
$$\left(\pi : \overline{M}_{g,n}(X,\beta) \longrightarrow \{\bullet\} \right)$$

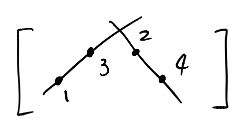
need properness of $\overline{M}_{g,n}(X,\beta)$

WDVV equations, quantum cohomology and Dubrovin connections

When $g=0$, there are the <u>WDVV equations</u>.

$$\overline{M}_{0,n+4}(X,\beta) \xrightarrow{\pi} \overline{M}_{0,4}$$





$$D_{12|34} = D_{13|24}$$

Pulling back $D_{12|34} = D_{13|24}$ via π yields a relation in $A_*(\overline{M}_{0,n+4}(X,\beta))$, leading to a relation of Gromov-Witten invariants.

(for details, see for e.g. Fulton-Pandharipande arXiv: 960811)

WDVV equations

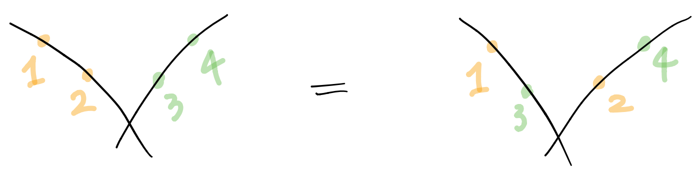
$\{T_i\}$ basis of $H^*(X)$, $\{T^i\}$ dual basis

Fix coh. classes $\boxed{\alpha_1, \cdots, \alpha_n} \in H^*(X)$

$$\sum_{\substack{I \cup J = \{5,6,\cdots,n\} \\ \beta_1+\beta_2=\beta}} \langle \alpha_1, \alpha_2, \alpha_I, T_i \rangle^X_{0,\beta_1} \langle T^i, \alpha_J, \alpha_3, \alpha_4 \rangle^X_{0,\beta_2}$$

$$= \sum_{\substack{I \cup J = \{5,\cdots,n\} \\ \beta_1+\beta_2=\beta}} \langle \alpha_1, \alpha_3, \alpha_I, T_i \rangle^X_{0,\beta_1} \langle T^i, \alpha_J, \alpha_2, \alpha_4 \rangle^X_{0,\beta_2}$$

(α_I denotes $\alpha_{i_1}, \alpha_{i_2}, \cdots, \alpha_{i_k}$ for $I = \{i_1, \cdots, i_k\}$)



WDVV equations give rise to some structures

WDVV $\Longrightarrow$ Quantum cohomology

associativity

$\Updownarrow$

flatness

Dubrovin connection

(a (family) of Frobenius algebra(-s))

$\rightsquigarrow$

(a local system)

(vector bdl + flat connection)

Frobenius manifold

Frobenius algebra

A (commutative) Frobenius algebra$/k$

- a k-vector space A equipped with the structure of a unital commutative algebra

(multiplitation is commutative and associative)

- non-degenerate symmetric pairing

$b : A \otimes A \to k$ s.t. $\forall a_1, a_2, a_3 \in A$

$b(a_1 a_2, a_3) = b(a_1, a_2 a_3)$

e.g. WDVV equations give rise to a Frobenius algebra (quantum cohomology) which is a deformation of the cohomology ring $H^*(X)$

Quantum cohomology

Assume $H^*(X)$ only has even degree classes

- Novikov variables: $\{ q^\beta ,\ \beta$ positive linear combination of effective curve classes in $H_2(X) \}$

Freely generate an algebra and do a completion, denoted by $\mathbb{C}[\![q]\!]$

- $\{T_i\}$ a basis of $H^*(X)$

Introduce formal parameters t_i corresponding to each T_i

$\boxed{\text{vector space}}$: $H^*(X) \otimes_{\mathbb{C}} \mathbb{C}[\![q]\!] \otimes_{\mathbb{C}} \mathbb{C}[\![t_1, \cdots, t_{\dim H^*(X)}]\!]$

$\boxed{\text{pairing}}$: $(\alpha, \beta) = \int_X \alpha \cup \beta$, extending by linearity w.r.t. q, t variables.

potential function : $\Phi(t) = \sum_{\substack{n \geq 3 \\ \beta}} \langle \underbrace{t, t, \cdots, t}_{n} \rangle^X_{0,\beta} \frac{q^\beta}{n!}$, $t = t_1 T_1 + \cdots + t_{\dim H^*(X)} T_{\dim}$

$\boxed{\text{product structure}}^*$: $(T_i * T_j , T_k) := \dfrac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_k}$

$\boxed{\text{unit}}$: Fundamental class $1 \in H^*(X)$

By definition, $(T_i * T_j, T_k) = (T_i, T_j * T_k)$

$q = t = 0 \Rightarrow$ recover the cohomology ring

Ex: WDVV $\Leftrightarrow$ associativity of $*$

People usually view quantum cohomology as a formal family of Frobenius algebras over the ring $\mathbb{C}[\![q]\!]$ parametrized by the base $\mathrm{Spf}\,\mathbb{C}[\![t_1, \cdots, t_{\dim H^*(X)}]\!]$ (a formal neighborhood of $\mathbb{C}^{\dim H^*(X)}$ at 0)

because ignoring the convergence issue, each value of t_i gives us a Frobenius algebra over $\mathbb{C}[\![q]\!]$.

(also notice that $\mathbb{C}^{\dim H^*(X)}$ is isomorphic to $H^*(X)$ as $\mathbb{C}$-vector space)

Frobenius manifold

<u>Def</u>: An (even) complex Frobenius manifold is a quadruple $(M, g, A, 1)$ where

- M is a cplx n-dim mfd
- g is a flat holomorphic non-deg. quadratic form
- A is a holomorphic symmetric tensor
  $A : TM \otimes TM \otimes TM \longrightarrow \mathcal{O}_M$
- $\mathbb{1}$ is a holomorphic vector field

such that

— M is covered by open sets U each equipped with a commuting basis of g-flat holomorphic vector fields

$X_1, \cdots, X_m \in \Gamma(U, TM)$,

and a holomorphic potential function

$\Phi \in \Gamma(U, \mathcal{O}_U)$ s.t.

$$A(X_i, X_j, X_k) = X_i X_j X_k (\Phi)$$

- The operator $*$ defined by

  $g(X * Y, Z) = A(X, Y, Z)$

  is associative

- 1 is a g-flat unit vector field

The base M can be a (smooth)

formal R-scheme (R is a commutative ring)

$\longrightarrow$ formal Frobenius mfd over R

Quantum cohomology gives rise to a formal Frobenius mfd over $\mathbb{C}[[q]]$

$$\mathbb{C}[[q]] \otimes H^*(X) \otimes \mathbb{C}[[t_1, \dots, t_{\dim H^*(X)}]]$$

$H^*(X)$ $\downarrow$ $\mathbb{C}$-vector space spanned by $T_1, \dots, T_{\dim H^*(X)}$

$$T_1 \mapsto \frac{\partial}{\partial t_1}, \quad \dots, \quad T_{\dim H^*(X)} \mapsto \frac{\partial}{\partial t_{\dim H^*(X)}}$$

coeff. ring | tangent bundle | base (formal) scheme

Dubrovin connection

Let $X_1, \cdots, X_m$ be the commuting basis of g-flat holomorphic vector field

(think about $\frac{\partial}{\partial t_1}, \cdots, \frac{\partial}{\partial t_{\dim H^*(X)}}$)

Levi-Civita connection ∇:

$$\nabla_Y\left(\sum_{i=1}^m a_i X_i\right) = \sum_{i=1}^m Y(a_i)\cdot X_i$$

(taking derivatives directly on coefficients)

For $z \in \mathbb{C}^*$, define the Dubrovin connection

$$\nabla_{z,Y}(X) = \nabla_Y X - \frac{1}{z} Y * X$$

Exercise: ∇_z is flat $\Leftrightarrow$ $*$ is associative

Suffice to check on the commuting g-flat basis (because of linearity of torsion tensor)

$$\nabla_{z,X_i}\nabla_{z,X_j}X_k - \nabla_{z,X_j}\nabla_{z,X_i}X_k - \underbrace{\cancel{\nabla_{z,[X_i,X_j]}X_k}}_{=0 \text{ b/c commuting}} = 0$$

$$\cancel{\nabla_{X_i}\nabla_{X_j}X_k} - \underbrace{\cancel{\frac{1}{z}X_i * \nabla_{X_j}X_k}}_{=0} - \underbrace{\cancel{\frac{1}{z}\nabla_{X_i}X_j * X_k}}_{=0}$$

$$+ \frac{1}{z^2}X_i * X_j * X_k$$

$$= \cancel{\nabla_{X_j}\nabla_{X_i}X_k} - \underbrace{\cancel{\frac{1}{z}X_j * \nabla_{X_i}X_k}}_{=0} - \underbrace{\cancel{\frac{1}{z}\nabla_{X_j}X_i * X_k}}_{=0}$$

$$+ \frac{1}{z^2}X_j * X_i * X_k$$

An example (small quantum ring of $\mathbb{P}^n$):

small quantum ring: Setting $t_i = 0$

(note q^β are still not zero)

virtual dimension:

$$[\overline{M}_{g,m}(X,\beta)]^{vir} \in A_r(\overline{M}_{g,m}(X,\beta))$$

$$r = (1-g)(\dim_{\mathbb{C}} X - 2) + m + \int_\beta c_1(TX)$$

When setting $t_i = 0$, the coefficients in the quantum product become

$$\left.\frac{\partial \Phi}{\partial t_i \partial t_j \partial t_k}\right|_{t=0} = \sum_\beta \langle T_i, T_j, T_k \rangle_{0,\beta} \, q^\beta$$

In the case of $\mathbb{P}^n$, let $H \in H^2(\mathbb{P}^n)$ be the hyperplane class.

The fact that $t_i = 0, q = 0$ recover the classical cohomology ring (needs proof. See Fulton-Pandharipande's note) implies that

$$H^i * H^j = \begin{cases} H^{i+j} + O(q) & i+j \leq n \\ O(q) & i+j > n \end{cases}$$

Now consider $\left.\dfrac{\partial \Phi}{\partial t_i \partial t_j \partial t_k}\right|_{t=0} = \sum_d \langle T_i, T_j, T_k \rangle_{0,d}^{\mathbb{P}^n} q^d$

Virtual dimension of $\overline{M}_{0,3}(\mathbb{P}^n, d)$:

$$(n-2) + 3 + d \cdot (n+1) = (d+1)(n+1)$$

But the (complex) degree of a coh. cycle is at most n ! $\Rightarrow$ the curve degree $d = 0$ or 1

$$H * H^n = 1 \cdot q$$

$\longrightarrow \langle H, H^n, H^n \rangle_{0,1}^{\mathbb{P}^n}$

= number of lines passing thru 2 points

$$\boxed{QH^*_{sm}(\mathbb{P}^n) \cong \mathbb{C}[H][\![q]\!] / (H^{n+1} - q)}$$

Sometimes people let z become a variable and define Frobenius structure on the whole $\underbrace{M}_{\text{the original Frobenius mfd}} \times \underbrace{\mathbb{C}^*}_{z\text{-parameter}}$.

This requires M to be the so-called conformal Frobenius manifold. But I won't have time to explain. Details see Rahul & YP's unfinished book (can be found on Rahul's homepage).

Kontsevich conjectures that for a blow up $Bl_Y X$ (X sm. proj, $Y \subset X$ sm.), the Frobenius mfd from the Gromov-Witten theory of $Bl_Y X$ has a product decomposition over generic point, and the factors of the product relate to the Frobenius mfd of X and Y. This seems to be one of the keys in his approach to the rationality of hypersurfaces.