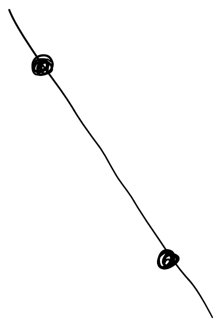
Unifying absolute and relative insertions

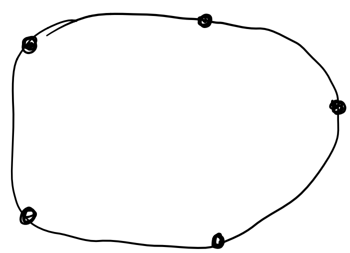
An important topic in enumerative geometry is to count the number of genus g algebraic curve of given homology class in a given space with given conditions.

Two classical ways to put conditions:

- Incidence conditions: the curve has to pass through given subvarieties
- Tangency conditions: the curve has to pass through given subvarieties in a divisor D, and each contact point is of given contact order along D.

e.g. In $\mathbb{P}^2$, through 2 points there is 1 line, through 5 general points there is 1 conic.



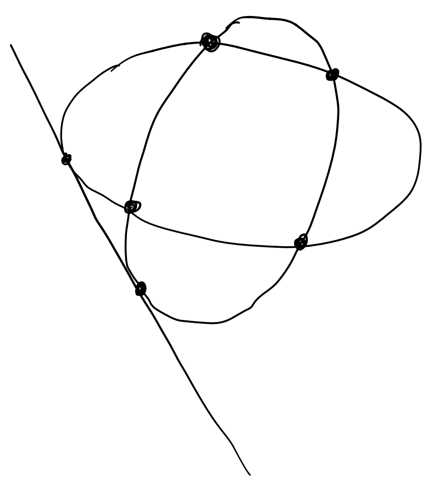


If we ask how many conics there are passing through

Incidence condition

4 general pts and tangent to a given line, there are 2.

Tangency condition



Depending on the context,
sometimes these two types of conditions have similar properties,
↑(GW-H and further, etc.)
Sometimes they are very different.

I want to discuss two things:

1. Our recent attempts to unify these two conditions in GW theory (relative quatum ring, etc.).

2. Some ongoing thoughts/works about how different they are in degeneration formula (attempting to turn deg. formula into sum over stable graphs).

Formally,

X sm. proj.

Given $g \in \mathbb{Z}_{\geq 0}$, $\beta \in H^2(X, \mathbb{Z})$

$\boxed{\text{Absolute Gromov-Witten invariants:}}$

$\mathbb{C}$-multilinear function

$\langle -, -, \dots, - \rangle^X_{g,\beta} : H^*(X)^{\otimes n} \longrightarrow \mathbb{C}$

Count the number of "curves" (stable maps) of genus g class β, passing through given cycles (up to Poincaré dual) specified by $H^*(X)$.

(Absolute) insertion (incidence condition)

e.g. $\langle [pt], [pt] \rangle^{\mathbb{P}^2}_{0, [line]} = 1$

(Absolute) insertions

Generalize to include tangency conditions

X sm. proj. , $D \subset X$ sm. divisor

degree 0   graded by $\mathbb{Z}_{>0}$

Let $\overline{\mathcal{H}} = H^*(X) \oplus \bigoplus_{d>0} H^*(D)$ be a graded vector space

Denote $[\alpha]_d \in \overline{\mathcal{H}}$ the degree d element with cohomology class α ($\alpha \in H^*(X)$, if $d=0$, $\alpha \in H^*(D)$, if $d>0$)

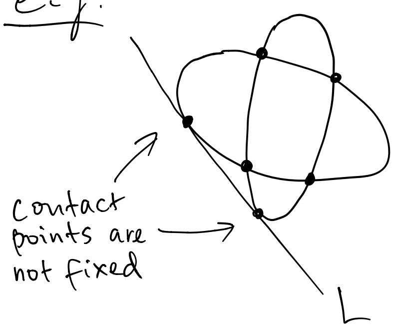
$\boxed{\text{Relative GW}}$: $\mathbb{C}$-multilinear function

$$\langle \underbrace{-, \cdots, -}_{n} \rangle_{g,\beta}^{(X,D)} : \overline{\mathcal{H}} \otimes \cdots \otimes \overline{\mathcal{H}} \longrightarrow \mathbb{C}$$

$\langle [\alpha_1]_{d_1}, \cdots, [\alpha_n]_{d_n} \rangle_{g,\beta}^{(X,D)}$ counts the number of "curves" (relative stable maps) of genus g, class β, passing through cycles $\alpha_1, \cdots, \alpha_n$ of contact order $d_1, \cdots, d_n$ along D, respectively.

$[\alpha]_d$ called $\begin{cases} \text{absolute insertion (if } d=0) \\ \text{(incidence condition)} \\ \text{relative insertion (if } d>0) \\ \text{(tangency condition)} \end{cases}$

e.g.



$$\langle [pt]_0, [pt]_0, [pt]_0, [pt]_0, [1]_2 \rangle^{(\mathbb{P}^2, L)}_{0,\, 2\cdot \text{line}}$$

$$= 2$$

(note: this notation is different than the classical notation

$$\langle \alpha_1, \cdots, \alpha_n \mid \varepsilon_1, \cdots, \varepsilon_n \rangle^{(X,D)}_{g,\beta}$$
)

To unify absolute insertions (incidence conditions) and relative insertions (tangency conditions):

View absolute insertions and relative insertions as simply the degree zero and degree nonzero parts of $\overline{\mathfrak{H}}$, the vector space of all possible insertions.

But this viewpoint has to be compatible with important properties generalizing those in abs. GW thy!

In GW theory of X, there is an important property called **WDVV equation** when genus is 0.

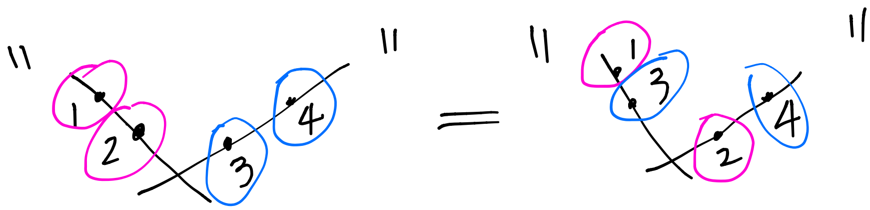
Fix a class $\beta \in H_2(X, \mathbb{Z})$

Let $\{T_i\}$ be a basis of $\boxed{H^*(X)}$ and

$\{T^i\}$ be its dual under the pairing

$$(-,-) = \int_X - \cup -$$

Fix coh classes $\alpha_1, \cdots \alpha_n \in H^*(X)$



$$\sum_{\substack{i,\, \beta_1+\beta_2=\beta, \\ I \sqcup J = \{5,\cdots,n\}}} \langle T_i, \alpha_1, \alpha_2, \alpha_{i_1}, \cdots, \alpha_{i_{|I|}} \rangle^X_{0,\beta_1} \langle T^i, \alpha_3, \alpha_4, \alpha_{j_1}, \cdots, \alpha_{j_{|J|}} \rangle^X_{0,\beta_2}$$

$$= \sum_{\substack{i,\, \beta_1+\beta_2=\beta \\ I \sqcup J = \{5,\cdots,n\}}} \langle T_i, \alpha_1, \alpha_3, \alpha_{i_1}, \cdots, \alpha_{i_{|I|}} \rangle^X_{0,\beta_1} \langle T^i, \alpha_2, \alpha_4, \alpha_{j_1}, \cdots, \alpha_{j_{|J|}} \rangle^X_{0,\beta_2}$$

$\Rightarrow$ quantum cohomology (a deformation of $\boxed{H^*(X)}$)

Quantum cohomology

vector space : $\boxed{H^*(X)}$

pairing : $(\alpha, \beta) = \int_X \alpha \cup \beta$

potential function : $\Phi(t) = \sum_{\substack{n \geq 3 \\ \beta}} \langle \underbrace{t, t, \cdots, t}_{n} \rangle_{0,n,\beta} \frac{q^\beta}{n!}$, $t = t_i T_i$

formal parameter (q^β)

formal parameters (t_i)

product structure : $(T_i * T_j , T_k) := \frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_k}$

unit : Fundamental class $1 \in H^*(X)$

$q = t = 0 \Rightarrow$ recover the cohomology ring

Ex : WDVV $\Rightarrow$ associativity of $*$

Rmk : In this construction, $\boxed{}$ can be replaced by any graded vector sp. with a nondeg. Poincaré pairing and GW invariant

$\langle - - - \rangle$ can be replaced by any multilinear function.

WDVV equation can be stated similarly,

and WDVV $\Leftrightarrow$ Associativity of "$*$"

In absolute GW theory, $\boxed{H^*(X)}$ is the ring of all possible insertions (without descendants).

How do we generalize to relative GW?

<u>An answer (Wu-You-F.)</u>:

We extend the vector space $\overline{\mathfrak{H}}$,

and extend the definition of

relative GWI.

degree 0 $\quad$ graded by $\mathbb{Z}\setminus\{0\}$

$$\mathfrak{H} := H^*(X) \oplus \bigoplus_{d\in\mathbb{Z}\setminus\{0\}} H^*(D)$$

A natural pairing:

$$([\alpha]_i, [\beta]_j) := \begin{cases} \int_X \alpha\wedge\beta & i=j=0 \\ \int_D \alpha\wedge\beta & i+j=0,\ i,j\neq 0 \\ 0 & \text{otherwise} \end{cases}$$

(Wu-You-F.) Generalize relative GW to negative contact order conditions.

We extend $\langle - - - \rangle^{(X,D)}_{g,\beta}$ into:

$\mathbb{C}$-multilinear function

$$\langle \underbrace{-,\dots,-}_{n} \rangle^{(X,D)}_{g,\beta} : \mathfrak{H}\otimes\cdots\otimes\mathfrak{H} \longrightarrow \mathbb{C}$$

without bar
fully graded by $\mathbb{Z}$

(Wu-You-F.) The WDVV equation for $\mathfrak{H}$ holds.

Allow us to construct <u>relative quantum rings</u>:

- vector sp. : $\mathcal{H}$

- invariants : $\langle [\alpha_1]_{i_1}, \cdots, [\alpha_n]_{i_n} \rangle^{(X,D)}_{g,n,\beta}$:

  rel inv. of contact order $(i_1, \cdots, i_n)$, $i_n \in \mathbb{Z}$

- basis of $\mathcal{H}$ $(\{[T_{k,i}]_i\})$ Pick a basis for each grading and put them together

  $t = \sum t_{k,i} [T_{k,i}]_i$

  $\Phi(t) = \sum_{\substack{n \geq 3 \\ \beta}} \langle \underbrace{t, \cdots, t}_{n} \rangle^{(X,D)}_{0,n,\beta} \frac{q^\beta}{n!}$

- $([T_{k,i}]_i * [T_{k',i'}]_{i'}, [T_{k'',i''}]_{i''})$

  $$= \frac{\partial^3 \Phi}{\partial t_{k,i} \partial t_{k',i'} \partial t_{k'',i''}}$$

One can similarly define small rel. quantum ring by restricting parameters:

$$t_{k,i} = 0 \text{ if } i \neq 0 \text{ or } \deg T_{k,i} > 2$$

Rmk: ① rel. quantum ring has two gradings:

contact order and cohomology degree

② The classical ring structure $(t=0, q=0)$:

$$[\alpha_1]_{\geq 0} \cdot [\alpha_2]_{\geq 0} = [\alpha_1 \cup \alpha_2]_{\geq 0}$$

(additive on the contact order!)

$$[\alpha_1]_{<0} \cdot [\alpha_2]_{\geq 0} = [D \cup \alpha_1 \cup \alpha_2]_{\geq 0}$$

$$[\alpha_1]_{<0} \cdot [\alpha_2]_{\geq 0} = [\alpha_1 \cup \alpha_2]_{<0}$$

$$[\alpha_1]_{<0} \cdot [\alpha_2]_{<0} = [D \cup \alpha_1 \cup \alpha_2]_{<0}$$

Example $\mathbb{P}^n$ relative to $\mathbb{P}^{n-1}$

small rel. quantum ring

Let q be the Novikov variable for line class in $\mathbb{P}^n$

hyperplane class

Set $x = [1]_1$, $y = [H]_0$

there is a relation $xy^n = q$ $\left([1]_1 * [H^n] = q\right)$

$$[H^a]_i = [H]_0 * [1]_1^{*a} = y^a x^i$$

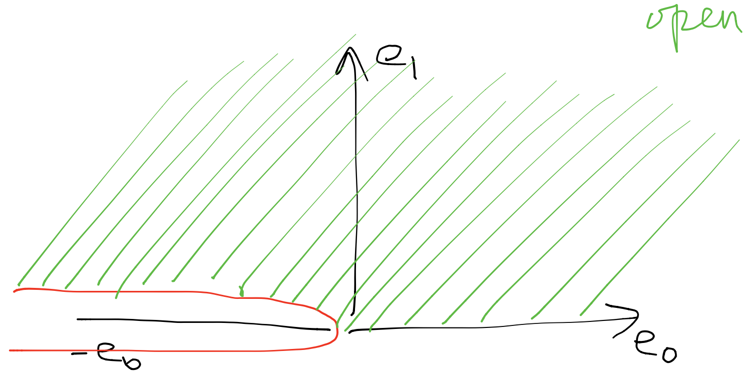
$[H^a]_{-i}$ can be formally written as y^{a+1}/x^i

because $[H^a]_{-i}\, x^i = [H^{a+1}]_0$

$\Rightarrow$ The small rel. quantum ring of $(\mathbb{P}^n, \mathbb{P}^{n-1})$ is the subalgebra of $\mathbb{C}[x, x^{-1}, y]$ generated by

$1, x, y/x, y/x^2, \ldots\ldots$

open cone σ°



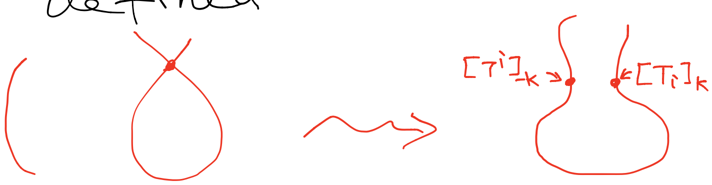
The group ring of σ° ?

Many things do generalize to this framework by replacing $H^*(X)$ with $\mathfrak{H}$

- Givental's Lagrangian cone
- Partial CohFT (only the chain axiom)

But many questions remain:

① Loop axiom in CohFT is not even defined



(all values of k are possible $\Rightarrow$ infinite sum)

② Givental's quantization?

③ Degeneration formula?

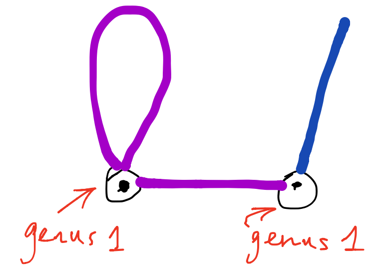
Although we unified abs insertions and relative insertions in a neat way, my thoughts on 2, 3:
Maybe we need to treat abs/rel insertions differently.

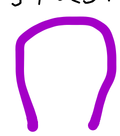
Roughly, Givental's technique on localization:

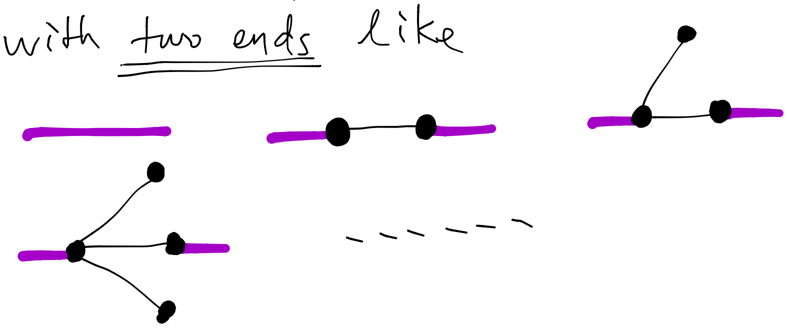
1. An infinite graph sum over all decorated graphs (coming from localization) can be organized into a sum over stable graphs (the dual graph of a stable curve)
2. The stable graph sum is packaged as the action of a quantized operator.

$\boxed{\text{We focus on 1}}$

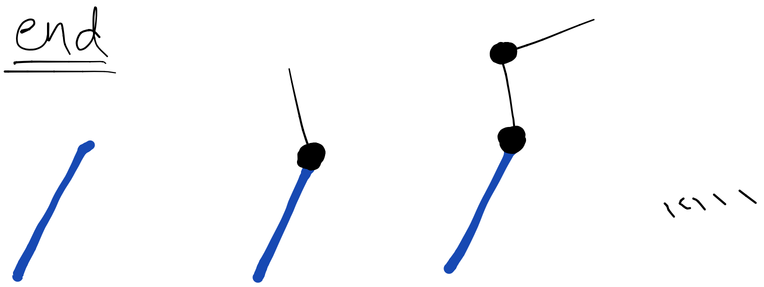
e.g. Fix a stable graph



We sum over all contributions of graphs that stabilize to this, i.e. the edges  represent all possible rational bridges with two ends like



The leg  represent all rational trees with one end

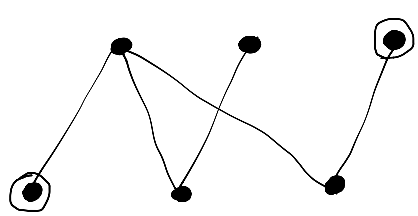


The contribution of all rational bridges $\Rightarrow$ **residues of S-matrix**

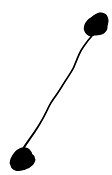
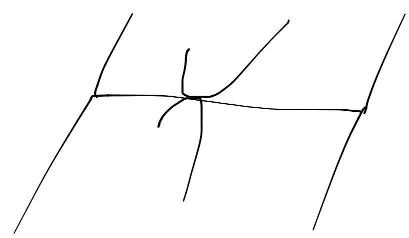
The contribution of all rational trees $\Rightarrow$ **residues of I-function**

<u>Note</u>: Collecting rational components together is a general technique for graph sums.

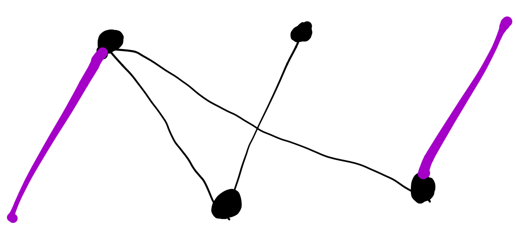
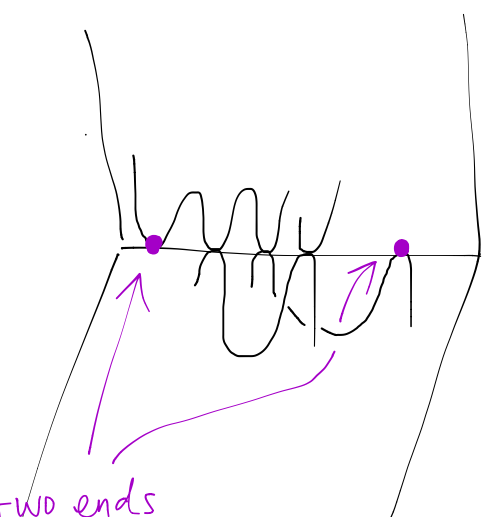
In degeneration formula, we sum over bipartite graphs



where edges correspond to nodes where both components are <u>tangent</u> to a locus

What if we collect all rational bridges/trees ?

the two ends can only be <u>relative markings</u>

In certain degenerations, those partial sum are computable.
(ongoing discussion with Bae, Wu)

Certain "punctured invariant"

$\Rightarrow$ "S-matrix" & "I-function"?