X sm. proj.

Given $g \in \mathbb{Z}_{\geq 0}$, $\beta \in H_2(X, \mathbb{Z})$

<u>Gromov-Witten invariants:</u>

$\mathbb{C}$-multilinear function

$$\langle -, -, \dots, - \rangle^X_{g,\beta} : H^*(X)^{\otimes n} \longrightarrow \mathbb{C}$$

Count the number of "curves" (stable maps) of genus g class β, passing through given cycles (up to Poincaré dual) specified by $H^*(X)$.

They are deformation invariant.

e.g. $\langle [pt], [pt] \rangle^{\mathbb{P}^2}_{0, [line]} = 1$

How are Gromov-Witten invariants defined?

<u>Very brief story:</u>

Consider the moduli space of embedded sm. curves of genus g and class β in X with n marked points (not allowed to collide)

$$M_{g,n}(X,\beta)^{\circ}$$

It is $\boxed{\text{not proper}}$ and we cannot have a reasonable intersection theory to get numbers.

We allow more objects than embedded curves (so-called stable maps) to form a larger moduli (stack).

$$M_{g,n}(X,\beta)^{\circ} \subset \underline{\overline{M}_{g,n}(X,\beta)}$$

$\overline{M}_{g,n}(X,\beta)$ is $\boxed{\text{proper}}$.

<u>Roughly</u>, $\overline{M}_{g,n}(X,\beta) = \{ (C, p_1,\cdots,p_n \in C, f: C \to X)$, s.t

C at worst nodal,

$f_*[C] = \beta$, plus some stability cond. $\}$

There is a cycle

$$[\overline{M}_{g,n}(X,\beta)]^{vir} \in A_{r}(\overline{M}_{g,n}(X,\beta)) \quad \text{for certain } r \geq 0$$

virtual/expected dim

coming from the natural deformation-obstruction theory on the moduli.

There are n evaluation maps

$$ev_1, \cdots, ev_n : \overline{M}_{g,n}(X,\beta) \longrightarrow X$$

mapping to the images of the corresponding marked pts.

$$\alpha_1, \cdots, \alpha_n \in H^*(X)$$

$$\langle \alpha_1, \cdots, \alpha_n \rangle^X_{g,\beta} := \text{``} \int_{[\overline{M}_{g,n}(X,\beta)]^{vir}} ev_1^*\alpha_1 \cdot \cdots \cdot ev_n^*\alpha_n \text{''}$$

$$= \underline{\pi_*}\left([\overline{M}_{g,n}(X,\beta)]^{vir} \cap ev_1^*\alpha_1 \cap \cdots \cap ev_n^*\alpha_n \right)$$

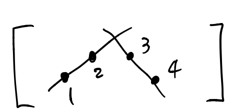
$$\left(\pi : \overline{M}_{g,n}(X,\beta) \longrightarrow \{\bullet\} \right)$$

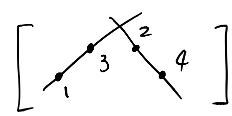
need properness of $\overline{M}_{g,n}(X,\beta)$

WDVV equations, quantum cohomology and Dubrovin connections

When $g=0$, there are the WDVV equations.

$$\overline{M}_{0,n+4}(X,\beta) \xrightarrow{\pi} \overline{M}_{0,4}$$





$$\begin{matrix} \| & & \| \\ D_{12|34} & = & D_{13|24} \end{matrix}$$

Pulling back $D_{12|34} = D_{13|24}$ via π yields a relation in $A_*(\overline{M}_{0,n+4}(X,\beta))$, leading to a relation of Gromov-Witten invariants.

(for details, see for e.g. Fulton-Pandharipande arXiV: 960811)

WDVV equations

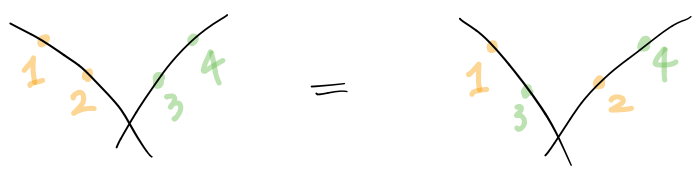
$\{T_i\}$ basis of $H^*(X)$, $\{T^i\}$ dual basis

Fix coh. classes $\alpha_1, \cdots, \alpha_n \in H^*(X)$

$$\sum_{\substack{I \cup J = \{5,6,\cdots,n\} \\ \beta_1+\beta_2=\beta}} \langle \alpha_1, \alpha_2, \alpha_I, T_i \rangle^X_{0,\beta_1} \langle T^i, \alpha_J, \alpha_3, \alpha_4 \rangle^X_{0,\beta_2}$$

$$= \sum_{\substack{I \cup J = \{5,\cdots,n\} \\ \beta_1+\beta_2=\beta}} \langle \alpha_1, \alpha_3, \alpha_I, T_i \rangle^X_{0,\beta_1} \langle T^i, \alpha_J, \alpha_2, \alpha_4 \rangle^X_{0,\beta_2}$$

(α_I denotes $\alpha_{i_1}, \alpha_{i_2}, \cdots, \alpha_{i_k}$ for $I = \{i_1, \cdots, i_k\}$)



WDVV equations give rise to some structures

WDVV $\Rightarrow$ Quantum cohomology

associativity $\Updownarrow$ flatness

Dubrovin connection

(a (family) of Frobenius algebra(-s))

(a local system)

(vector bdl + flat connection)

Frobenius manifold

Frobenius algebra

A (commutative) Frobenius algebra$/k$

- a k-vector space A
  equipped with the structure of
  a unital commutative algebra (multiplitation is commutative and associative)

- non-degenerate symmetric pairing
  $b: A \otimes A \to k$ s.t. $\forall a_1, a_2, a_3 \in A$
  $b(a_1 a_2, a_3) = b(a_1, a_2 a_3)$

e.g. WDVV equations give rise to a Frobenius algebra (quantum cohomology) which is a deformation of the cohomology ring $H^*(X)$

Quantum cohomology

Assume $H^*(X)$ only has even degree classes

- Novikov variables: $\left\{ q^\beta,\ \beta \right.$ positive linear combination of effective curve classes in $H_2(X)$ $\left.\right\}$

Freely generate an algebra and do a completion, denoted by $\mathbb{C}[\![q]\!]$

- $\{T_i\}$ a basis of $H^*(X)$

Introduce formal parameters t_i corresponding to each T_i

$\boxed{\text{vector space}}$: $H^*(X) \otimes_{\mathbb{C}} \mathbb{C}[\![q]\!] \otimes_{\mathbb{C}} \mathbb{C}[\![t_1, \cdots, t_{\dim H^*(X)}]\!]$

$\boxed{\text{pairing}}$: $(\alpha, \beta) = \int_X \alpha \cup \beta$, extending by linearity w.r.t. q, t variables.

potential function: $\Phi(t) = \sum_{\substack{n \geq 3 \\ \beta}} \langle \underbrace{t, t, \cdots, t}_{n} \rangle_{0,\beta}^X \dfrac{q^\beta}{n!}$,

$\boxed{\text{product structure}}$: $(T_i * T_j , T_k) := \dfrac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_k}$

$\boxed{\text{unit}}$: Fundamental class $1 \in H^*(X)$

By definition, $(T_i * T_j, T_k) = (T_i, T_j * T_k)$

$q = t = 0 \Rightarrow$ recover the cohomology ring

Ex: WDVV $\Rightarrow$ associativity of $*$

People usually view quantum cohomology as a formal family of Frobenius algebras over the ring $\mathbb{C}[\![q]\!]$ parametrized by the base $\mathrm{Spf}\, \mathbb{C}[\![t_1, \cdots, t_{\dim H^*(X)}]\!]$ (a formal neighborhood of $\mathbb{C}^{\dim H^*(X)}$ at 0)

because ignoring the convergence issue, each value of t_i gives us a Frobenius algebra over $\mathbb{C}[\![q]\!]$.

(also notice that $\mathbb{C}^{\dim H^*(X)}$ is isomorphic to $H^*(X)$ as $\mathbb{C}$-vector space)

Frobenius manifold

Def: An (even) complex Frobenius manifold is a quadruple $(M, g, A, 1)$ where

- M is a cplx n-dim mfd
- g is a flat holomorphic Riemanian metric
- A is a holomorphic symmetric tensor
  $$A : TM \otimes TM \otimes TM \longrightarrow \mathcal{O}_M$$
- 1 is a holomorphic vector field

such that

– M is covered by open sets U each equipped with a commuting basis of g-flat holomorphic vector fields

$X_1, \cdots, X_m \in \Gamma(U, TM)$,

and a holomorphic potential function

$\Phi \in \Gamma(U, \mathcal{O}_U)$ s.t.

$$A(X_i, X_j, X_k) = X_i X_j X_k (\Phi)$$

- The operator $*$ defined by

  $g(X * Y, Z) = A(X, Y, Z)$

  is associative

- 1 is a g-flat unit vector field

The base M can be a (smooth) formal R-scheme (R is a commutative ring)

$\longrightarrow$ formal Frobenius mfd over R

Quantum cohomology gives rise to

a formal Frobenius mfd over $\mathbb{C}[\![q]\!]$

$$\mathbb{C}[\![q]\!] \otimes \underline{\underline{H^*(X)}} \otimes \mathbb{C}[\![t_1, \dots, t_{\dim H^*(X)}]\!]$$

$\mathbb{C}$- vector space spanned by

$T_1, \dots, T_{\dim H^*(X)}$

$$\frac{\partial}{\partial t_1} \quad \cdots \quad \frac{\partial}{\partial t_{\dim H^*(X)}}$$

Coeff. ring	tangent bundle	base (formal) scheme

Dubrovin connection

Let $X_1, \cdots, X_m$ be the commuting basis of g-flat holomorphic vector field

(think about $\frac{\partial}{\partial t_1}, \cdots, \frac{\partial}{\partial t_{\dim H^*(X)}}$)

Levi-Civita connection ∇:

$$\nabla_Y \left(\sum_{i=1}^{m} a_i X_i \right) = \sum_{i=1}^{m} Y(a_i) \cdot X_i$$

(taking derivatives directly on coefficients)

For $z \in \mathbb{C}^*$, define the Dubrovin connection

$$\nabla_{z,Y}(X) = \nabla_Y X - \frac{1}{z} Y * X$$

Exercise: ∇_z is flat $\Leftrightarrow$ $*$ is associative

Suffice to check on the commuting g-flat basis (because of linearity of torsion tensor)

$$\nabla_{z,X_i}\nabla_{z,X_j}X_k - \nabla_{z,X_j}\nabla_{z,X_i}X_k - \underbrace{\cancel{\nabla_{z,[X_i,X_j]}X_k}}_{=0 \text{ b/c commuting}} = 0$$

$$\cancel{\nabla_{X_i}\nabla_{X_j}X_k} - \cancel{\frac{1}{z}X_i * \underbrace{\nabla_{X_j}X_k}_{=0}} - \cancel{\frac{1}{z}\nabla_{X_i}X_j * X_k}_{=0}$$

$$+ \frac{1}{z^2}X_i * X_j * X_k$$

$$= \cancel{\nabla_{X_j}\nabla_{X_i}X_k} - \cancel{\frac{1}{z}X_j * \underbrace{\nabla_{X_i}X_k}_{=0}} - \cancel{\frac{1}{z}\nabla_{X_j}X_i * X_k}_{=0}$$

$$+ \frac{1}{z^2}X_j * X_i * X_k$$

Sometimes people let z become a variable and define Frobenius structure on the whole $\underbrace{M}_{\text{original Frob. mfd.}} \times \underbrace{\mathbb{C}^*}_{z\text{-parameter}}$.

This requires M to be the so-called conformal Frobenius manifold. But I won't have time to explain. Details see Rahul & YP's unfinished book (can be found on Rahul's homepage).

Kontsevich conjectures that for a blow up $Bl_Y X$ (X sm. proj, $Y \subset X$ sm.), the Frobenius mfd from the Gromov-Witten theory of $Bl_Y X$ has a product decomposition over generic point, and factors of the product relate to the Frobenius mfd of X and Y. This seems to be one of the keys in his approach to the rationality of hypersurfaces.