CPSC 5210 HOMEWORK 3: ELIMINATION METHOD AND MASTER THEOREM

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1 Sum of Series and Logarithmic Properties used in this Homework are as follows:

SUM SERIES:

1. Sum of n numbers: $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$
2. Sum of Squares of n numbers: $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$
3. Geometric Series Sum: $1 + ar + ar^2 + ar^3 + \cdots + ar^n = a[\frac{(1-r^{n+1})}{1-r}]$
4. Sum Series $\sum_{i=1}^{n} i * a^i = \frac{a(na^{n+1}-(n+1)a^n+1)}{(a-1)^2}$
5. Harmonic Series $1 + \frac{1}{2} + \frac{1}{3} \cdots + \frac{1}{n} = ln\ k + \gamma + \epsilon_k \leq (ln\ k) + 1$

LOGARITHMIC PROPERTIES:

1. When $x = a^y; y = \log_a x$
2. $\log_a 1 = 0$
3. $\log_a a = 1$
4. $a^{\log_a x} = x$
5. $\log_a a^x = x$
6. $\log_a(xy) = \log_a x + \log_a y$
7. $\log_a(\frac{x}{y}) = \log_a x - \log_a y$
8. $\log_a x^y = y \log_a x$
9. $\log_a x = (\log_a b) * (\log_b x)$
10. $\log_a x = \frac{\log_b x}{\log_b a}$
11. $\log_a b = \frac{1}{\log_b a}$
12. $\log_a(\frac{1}{b}) = -\log_a b$
13. $\log_{\frac{1}{a}}(b) = -\log_a b$
14. $x^{\log_a y} = y^{\log_a x}$

2 Solutions to HW3 - Elimination Method and Master Theorem

2.1 Compute the Asymptotic Time Complexity of:

$T(n) = T(n-1) + 3 * n^2$;
$T(1) = 0$;
using "Elimination Method".

Solution:

$$T(n) = T(n-1) + 3n^2 \quad (0)$$
$$T(n-1) = T(n-2) + 3(n-1)^2 \quad (1)$$
$$T(n-2) = T(n-3) + 3(n-2)^2 \quad (2)$$
$$\vdots$$
$$T(1) = 0 \quad (k)$$

To compute Analytical Time Complexity, use Elimination Method to get rid of the T-terms on the Right-hand side, so lets add the equations: $\Rightarrow 0 + 1 + 2 + \cdots + k$

$$\begin{aligned}
T(n) &= 3n^2 + 3(n-1)^2 + 3(n-2)^2 + \cdots + 0 \\
&= 3[n^2 + (n-1)^2 + (n-2)^2 + \cdots + 0] \\
&= 3 * \sum_{i=1}^{n} i^2 \\
&= 3 * \frac{n(n+1)(2n+1)}{6} (Refer \rightarrow Page\ 1\ Sum\ Series) \\
&= \frac{1}{2}[n(n+1)(2n+1] \\
&= \frac{1}{2}(n^2+n)(2n+1) \\
&= \frac{1}{2}(2n^3 + 3n^2 + n) \\
\Rightarrow T(n) &= \frac{1}{2}(2n^3 + 3n^2 + n)
\end{aligned}$$

Asymptotic Time Complexity:
Let $f(n) = n^3 + \frac{3}{2}n^2 + \frac{n}{2}$
$\bigcirc : f(n) = n^3 + \frac{3}{2}n^2 + \frac{n}{2}$
$f(n) \leq n^3 + \frac{3}{2}n^2$ (Ignoring the n/2)
$f(n) \leq n^3 + n.n^2$ (when n $\geq$ 3/2)
$f(n) \leq n^3 + n^3$
$f(n) \leq 2n^3$
$\exists\, c_0 = 2; n_0 = 3/2$
$g(n) = n^3 such\ that\ f(n) \leq c_0.g(n) always\ holds\ TRUE$
$\Rightarrow \bigcirc(f(n)) = n^3$

$\Omega : f(n) = n^3 + \frac{3}{2}n^2 + \frac{n}{2}$
$f(n) = n^3 + \frac{n}{2}[3n + 1]$
$f(n) \geq n^3$ (when 3n+1 > 0 ; n > -$\frac{1}{3}$)
$f(n) \geq n^3$
$\exists\, c_1 = 1; n_1 = -1/3$
$g(n) = n^3 such\ that\ f(n) \geq c_1.g(n) always\ holds\ TRUE$
$\Rightarrow \Omega(f(n)) = n^3$
Since, $\bigcirc(f(n)) = \Omega(f(n)) = n^3; \Theta(f(n)) = n^3$

2.2 Application of Master Theorem and Prove using Elimination Method:

$T(n) = 2T(n/4) + \sqrt{n}$

Solution:

Step 1: Master Theorem -

From the given equation, a = 2 ; b = 4 ; $f(n) = n^{1/2}$
$\Rightarrow n^{\log_b a} = n^{\log_4 2} = n^{1/2}$
$\Rightarrow n^{\log_b a} = f(n) = n^{1/2}$
Case 2: Equivalent Complexity Order
$\Rightarrow T(n)) = \Theta(n^{\log_b a} \log_4 n) = \Theta(n^{\log_4 2} \log_4 n) = \Theta(n^{1/2} \log_4 n)$
Asymptotic Time Complexity of $\Rightarrow T(n)) = \Theta(n^{1/2} \log_4 n)$

Step 2: Proof by Elimination Method -

Proof.

$$T(n) = 2T(n/4) + n^{1/2} \tag{0}$$
$$T(n/4) = 2T(n/4^2) + (n/4)^{1/2} \tag{1}$$
$$T(n/4^2) = 2T(n/4^3) + (n/4^2)^{1/2} \tag{2}$$
$$\vdots$$
$$T(n/4^{k-1}) = 2T(n/4^k) + (n/4^{k-1})^{1/2} \tag{k-1}$$
$$T(1) = 1 \tag{k}$$

Get rid of T-terms on Right-hand side:

Add equations $(0) + 2 * (1) + 2^2 * (2) + \cdots + 2^{k-1} * (k-1) + 2^k * k$

Without loss of generality , let us assume $n = 4^k \Rightarrow n^{1/2} = 2^k and \Rightarrow k = \log_4 n$

$$\begin{aligned}
T(n) &= n^{1/2} + 2 * (\frac{n}{4})^{1/2} + 2^2 * (\frac{n}{4^2})^{1/2} + \cdots + 2^{k-1} * (\frac{n}{4^{k-1}})^{1/2} + 2^k \\
&= n^{1/2}[1 + 2.(\frac{1}{4})^{1/2} + 2^2.(\frac{1}{4^2})^{1/2}] + \cdots + 2^{k-1}.(\frac{1}{4^{k-1}})^{1/2}] + 2^k \\
&= n^{1/2}[1 + 2.(\frac{1}{2}) + 4.(\frac{1}{4}) + \cdots + 2^{k-1}.(\frac{1}{2^{k-1}})] + 2^k \\
&= n^{1/2}[1 + 1 + 1 + \cdots + (k-1)] + 2^k \\
&= n^{1/2}(k-1) + 2^k; Because, \sum_{i=1}^{k-1} 1 = (k-1) \\
&= n^{1/2}(k-1) + n^{1/2}; As\ n = 4^k \\
&= n^{1/2}.k - n^{1/2} + n^{1/2} \\
&= n^{1/2}.k \\
&= n^{1/2}\log_4 n
\end{aligned}$$

$\Rightarrow Hence\ Proved\ T(n) = \Theta(n^{1/2}\log_4 n)$

□

2.3 Application of Master Theorem and Prove using Elimination Method:

$T(n) = 2T(n/2) + n^4$

Solution:

Step 1: Master Theorem -

From the given equation, a = 2 ; b = 2 ; $f(n) = n^4$
$\Rightarrow n^{\log_b a} = n^{\log_2 2} = n^1 = n$
$\Rightarrow f(n) > n^{\log_b a}$
Case 3: f(n) Dominates.
Asymptotic Time Complexity $\Rightarrow T(n)) = \Theta(f(n)) = \Theta(n^4)$

Step 2: Proof by Elimination Method -

Proof.

$$
\begin{aligned}
T(n) &= 2T(n/2) + n^4 && (0)\\
T(n/2) &= 2T(n/2^2) + (n/2)^4 && (1)\\
T(n/2^2) &= 2T(n/2^3) + (n/2^2)^4 && (2)\\
&\vdots \\
T(n/2^{k-1}) &= 2T(n/2^k) + (n/2^{k-1})^4 && \text{(k-1)}\\
T(1) &= 1 && \text{(k)}
\end{aligned}
$$

Get rid of T-terms on Right-hand side:
Add equations $(0) + 2 * (1) + 2^2 * (2) + \cdots + 2^{k-1} * (k-1) + 2^k * k$
Without loss of generality , let us assume $n = 2^k \Rightarrow k = \log_2 n$

$$
\begin{aligned}
T(n) &= n^4 + 2*(\frac{n}{2})^4 + 2^2*(\frac{n}{2^2})^4 + \cdots + 2^{k-1}*(\frac{n}{2^{k-1}})^4 + 2^k \\
&= n^4[1 + 2.(\frac{1}{2})^4 + 2^2.(\frac{1}{2^2})^4] + \cdots + 2^{k-1}.(\frac{1}{2^{k-1}})^4] + 2^k \\
&= n^4[1 + \frac{1}{2^3} + \frac{1}{(2^3)^2}) + \cdots + \frac{1}{(2^3)^{k-1}}] + 2^k \\
&= n^4[1 + \frac{1}{8} + \frac{1}{8^2}) + \cdots + \frac{1}{8^{k-1}}] + 2^k \\
&= n^4 * \frac{1(1-(\frac{1}{8})^k}{1-(\frac{1}{8})} + 2^k; From\ Geometric\ Series\ ,\ Refer\ Page\ 1\ for\ the\ formul \\
&= n^4 * \frac{8}{7}(1 - \frac{1}{8^k}) + 2^k \\
&= n^4 * \frac{8}{7}(1 - \frac{1}{(2^k)^3}) + 2^k \\
&= n^4 * \frac{8}{7}(1 - \frac{1}{n^3}) + n \\
&= \frac{8}{7}n^4 - \frac{8}{7}n + n \\
&= \frac{8}{7}n^4 - \frac{1}{7}n
\end{aligned}
$$

$Let f(n) = \frac{8}{7}n^4 - \frac{1}{7}n$

$\bigcirc : f(n) = \frac{8}{7}n^4 - \frac{1}{7}n$

$f(n) \leq \frac{8}{7}n^4 - n^3.n$ (when n $\geq 1/7$)

$f(n) \leq \frac{8}{7}n^4 - n^4$

$f(n) \leq \frac{1}{7}n^4$

$\exists\ c_0 = 1/7; n_0 = 1/7$

$g(n) = n^4 such\ that\ f(n) \leq c_0.g(n) always\ holds\ TRUE$

$\Rightarrow \bigcirc(f(n)) = n^4$

$\Omega : f(n) = \frac{8}{7}n^4 - \frac{1}{7}n$

$f(n) = n^4 + \frac{1}{7}n^4 - \frac{1}{7}n$

$f(n) = n^4 + \frac{1}{7}.n(n^3 - 1)$

$f(n) \geq n^4$ (when $n^3 - 1 > 0; n > 1$)

$\exists\ c_1 = 1; n_1 = 1$

$g(n) = n^4 such\ that\ f(n) \geq c_1.g(n) always\ holds\ TRUE$

$\Rightarrow \Omega(f(n)) = n^4$

$Rightarrow\Theta(f(n)) = n^4$
$\Rightarrow Hence\ Proved\ T(n) = \Theta(f(n)) = \Theta(n^4)$

□

2.4 Application of Master Theorem and Prove using Elimination Method:

$T(n) = T(7n/10) + n$

Solution:

Step 1: Master Theorem -

Given equation can written as $T(n) = T(\frac{n}{10/7}) + n$
From the given equation, a = 1 ; b = 10/7 ; $f(n) = n$
$\Rightarrow n^{\log_b a} = n^{\log_{\frac{10}{7}} 1} = n^0 = 1$
(log of 1 to any base is 0 - Refer to Page 1 for Logarithmic Properties)
$\Rightarrow f(n) > n^{\log_b a}$
Case 3: f(n) Dominates.
Asymptotic Time Complexity $\Rightarrow T(n)) = \Theta(f(n)) = \Theta(n)$

Step 2: Proof by Elimination Method -

Proof.

$$T(n) = T(\frac{n}{10/7}) + n \tag{0}$$

$$T(\frac{n}{10/7}) = T(\frac{n}{(10/7)^2}) + \frac{n}{10/7} \tag{1}$$

$$T(\frac{n}{(10/7)^2}) = T(\frac{n}{(10/7)^3}) + \frac{n}{(10/7)^2} \tag{2}$$

$$\vdots$$

$$T(\frac{n}{(10/7)^{k-1}}) = T(\frac{n}{(10/7)^k}) + \frac{n}{(10/7)^{k-1}} \tag{k-1}$$

$$T(1) = 1 \tag{k}$$

Get rid of T-terms on Right-hand side:
Add equations $(0) + (1) + (2) + \cdots + (k-1) + k$
Without loss of generality , let us assume $n = (\frac{10}{7})^k \Rightarrow k = \log_{\frac{10}{7}}(n)$

$$\begin{aligned}
T(n) &= n + \frac{n}{10/7} + \frac{n}{(10/7)^2} + \cdots + \frac{n}{(10/7)^{k-1}} + 1 \\
&= n[1 + \frac{1}{10/7} + \frac{1}{(10/7)^2} + \cdots + \frac{1}{(10/7)^{k-1}}] + 1 \\
&= n[1 + \frac{7}{10} + (\frac{7}{10})^2 + \cdots + (\frac{7}{10})^{k-1}] + 1 \\
&= n[1.\frac{(1-(\frac{7}{10})^k)}{1-\frac{7}{10}}] + 1(From\ Geometric\ Series\ Refer\ to\ Page\ 1\ for\ the\ formu \\
&= n(\frac{10}{3}(1 - \frac{1}{(10/7)^k})) + 1 \\
&= \frac{10}{3}n(1 - \frac{1}{n}) + 1 \\
&= \frac{10}{3}n - \frac{10}{3} + 1 \\
&= \frac{10}{3}n - \frac{7}{3} \\
T(n) &= \frac{10}{3}n - \frac{7}{3}
\end{aligned}$$

$Let f(n) = \frac{10}{3}n - \frac{7}{3}$
$\bigcirc : f(n) = \frac{10}{3}n - \frac{7}{3}$
$f(n) \leq \frac{10}{3}n - n$ (when n $\geq 7/3$)
$f(n) \leq \frac{7}{3}n$
$\exists\ c_0 = 7/3; n_0 = 7/3$
$g(n) = n such\ that\ f(n) \leq c_0.g(n) always\ holds\ TRUE$
$\Rightarrow \bigcirc(f(n)) = n$

$\Omega : f(n) = \frac{10}{3}n - \frac{7}{3}$
$f(n) = n + \frac{7}{3}n - \frac{7}{3}$
$f(n) = n + \frac{7}{3}(n-1)$
$f(n) \geq n$ (when $n - 1 > 0; n > 1$)
$\exists\ c_1 = 1; n_1 = 1$
$g(n) = n such\ that\ f(n) \geq c_1.g(n) always\ holds\ TRUE$
$\Rightarrow \Omega(f(n)) = n$

$\Rightarrow \Theta(f(n)) = n$
$\Rightarrow Hence\ Proved\ T(n) = \Theta(f(n)) = \Theta(n)$

$\square$

2.5 Application of Master Theorem and Prove using Elimination Method:

$T(n) = 7T(n/2) + n^2$

Solution:

Step 1: Master Theorem -

From the given equation, a = 7 ; b = 2 ; $f(n) = n^2$
$\Rightarrow n^{\log_b a} = n^{\log_2 7}$
$\Rightarrow n^{\log_b a} > f(n)$
Case 1: Recursive Sub problem dominates.
Asymptotic Time Complexity $\Rightarrow T(n)) = \Theta(n^{\log_b a}) = \Theta(n^{\log_2 7})$

Step 2: Proof by Elimination Method -

Proof.

$$T(n) = 7T(\frac{n}{2}) + n^2 \tag{0}$$
$$T(\frac{n}{2}) = 7T(\frac{n}{2^2}) + (\frac{n}{2})^2 \tag{1}$$
$$T(\frac{n}{2^2}) = 7T(\frac{n}{2^3}) + (\frac{n}{2^2})^2 \tag{2}$$
$$\vdots$$
$$T(\frac{n}{2^{k-1}}) = 7T(\frac{n}{2^k}) + (\frac{n}{2^{k-1}})^2 \tag{k-1}$$
$$T(1) = 1 \tag{k}$$

Get rid of T-terms on Right-hand side:
Add equations $(0) + 7 * (1) + 7^2 * (2) + \cdots + 7^{k-1} * (k-1) + 7^k * k$
Without loss of generality , let us assume $n = 2^k \Rightarrow k = \log_2(n)$

$$
\begin{aligned}
T(n) &= n^2 + 7(\frac{n}{2})^2 + 7^2(\frac{n}{2^2})^2 + \cdots + 7^{k-1}(\frac{n}{2^{k-1}})^2 + 7^k \\
&= n^2[1 + 7\frac{1}{2^2} + 7^2\frac{1}{(2^2)^2} + \cdots + 7^{k-1}\frac{1}{(2^2)^{k-1}}] + 7^k \\
&= n^2[1 + 7\frac{1}{4} + 7^2\frac{1}{4^2} + \cdots + 7^{k-1}\frac{1}{4^{k-1}}] + 7^k \\
&= n^2[1 + \frac{7}{4} + (\frac{7}{4})^2 + \cdots + (\frac{7}{4})^{k-1}] + 7^k \\
&= n^2[1(\frac{1-(\frac{7}{4})^k}{1-\frac{7}{4}})] + 7^k (From\ Geometric\ Series\ Refer\ to\ Page\ 1\ for\ the\ for \\
&= n^2[\frac{1-(\frac{7}{4})^k}{-\frac{3}{4}}] + 7^k \\
&= \frac{4}{3}n^2[(\frac{7}{4})^k - 1] + 7^k \\
&= \frac{4}{3}n^2[\frac{7^k}{4^k} - 1] + 7^k \\
&= \frac{4}{3}n^2[\frac{7^k}{n^2} - 1] + 7^k; (Since\ n = 2^k; So => n^2 = 4^k) \\
&= \frac{4}{3}7^k - \frac{4}{3}n^2 + 7^k \\
&= \frac{7}{3}7^k - \frac{4}{3}n^2 \\
&= \frac{7}{3}7^{\log_2 n} - \frac{4}{3}n^2 \\
&= \frac{7}{3}7^{\frac{\log_7 n}{\log_7 2}} - \frac{4}{3}n^2 (From\ Logarithmic\ Properties\ Refer\ to\ Page\ 1\ \#10) \\
&= \frac{7}{3}7^{(\log_7 n)(\frac{1}{\log_7 2})} - \frac{4}{3}n^2 \\
&= \frac{7}{3}n^{\frac{1}{\log_7 2}} - \frac{4}{3}n^2 (From\ Logarithmic\ Properties\ Refer\ to\ Page\ 1\ \#4) \\
&= \frac{7}{3}n^{\log_2 7} - \frac{4}{3}n^2 (From\ Logarithmic\ Properties\ Refer\ to\ Page\ 1\ \#11) \\
\Rightarrow T(n) &= \frac{7}{3}n^{\log_2 7} - \frac{4}{3}n^2
\end{aligned}
$$

$Let f(n) = \frac{7}{3}n^{\log_2 7} - \frac{4}{3}n^2$

$\bigcirc : f(n) = \frac{7}{3}n^{\log_2 7} - \frac{4}{3}n^2$

$f(n) \leq \frac{7}{3}n^{\log_2 7}$ (when n ≥ 0)

$\exists\ c_0 = 7/3; n_0 = 0$
$g(n) = n^{\log_2 7}\ such\ that\ f(n) \leq c_0.g(n) always\ holds\ TRUE$

There is no Lower Bound of $n^{\log_2 7}$

$\Rightarrow Hence\ Proved\ T(n) = \Theta(n^{\log_b a}) = \Theta(n^{\log_2 7})$

□

2.6 Application of Master Theorem and Prove using Elimination Method: $T(n) = 4T(n/3) + n\log_3 n$

Solution:

Step 1: Master Theorem -

From the given equation, a = 4 ; b = 3 ; $f(n) = n\log_3 n$
$\Rightarrow n^{\log_b a} = n^{\log_3 4}$
$\Rightarrow n^{\log_b a} > f(n)$
Case 1: Recursive Sub problem dominates.
Asymptotic Time Complexity $\Rightarrow T(n)) = \Theta(n^{\log_b a}) = \Theta(n^{\log_3 4})$

Step 2: Proof by Elimination Method -

Proof.

$$T(n) = 4T(n/3) + n\log_3 n \quad (0)$$

$$T(n/3) = 4T(n/3^2) + \frac{n}{3}\log_3(n/3) \quad (1)$$

$$T(n/3^2) = 4T(n/3^3) + \frac{n}{3^2}\log_3(n/3^2) \quad (2)$$

$$\vdots$$

$$T(n/3^{k-1}) = 4T(n/3^k) + \frac{n}{3^{k-1}}\log_3(n/3^{k-1}) \quad \text{(k-1)}$$

$$T(1) = 1 \quad \text{(k)}$$

Get rid of T-terms on Right-hand side:
Add equations $(0) + 4 * (1) + 4^2 * (2) + \cdots + 4^{k-1} * (k-1) + 4^k * k$
Without loss of generality , let us assume $n = 3^k \Rightarrow k = \log_3(n)$

$$
\begin{aligned}
T(n) &= n\log_3 n + 4(\frac{n}{3})\log_3(\frac{n}{3}) + 4^2(\frac{n}{3^2})\log_3(\frac{n}{3^2}) + \cdots + 4^{k-1}(\frac{n}{3^{k-1}})\log_3(\frac{n}{3^{k-1}}) + 4^k \\
&= n\log_3 n + 4(\frac{n}{3})[\log_3 n - \log_3 3] + 4^2(\frac{n}{3^2})[\log_3 n - \log_3 3^2] + \cdots \\
&+ 4^{k-1}(\frac{n}{3^{k-1}})[\log_3 n - \log_3 3^{k-1}] + 4^k \\
&= n\log_3 n + 4(\frac{n}{3})\log_3 n - 4(\frac{n}{3})\log_3 3 + 4^2(\frac{n}{3^2})\log_3 n - 4^2(\frac{n}{3^2})\log_3 3^2 + \cdots \\
&+ 4^{k-1}(\frac{n}{3^{k-1}})\log_3 n - 4^{k-1}(\frac{n}{3^{k-1}})\log_3 3^{k-1} + 4^k \\
&= n\log_3 n[1 + \frac{4}{3} + (\frac{4}{3})^2 + \cdots + (\frac{4}{3})^{k-1}] - \\
& n[\frac{4}{3}\log_3 3 + (\frac{4}{3})^2\log_3 3^2 + \cdots + (\frac{4}{3})^{k-1}\log_3 3^{k-1}] + 4^k \\
&= n\log_3 n[1 + \frac{4}{3} + (\frac{4}{3})^2 + \cdots + (\frac{4}{3})^{k-1}] - \\
& n[\frac{4}{3}\log_3 3 + (\frac{4}{3})^2 2\log_3 3 + \cdots + (\frac{4}{3})^{k-1}(k-1)\log_3 3] + 4^k \\
&= n\log_3 n[1(\frac{1-(\frac{4}{3})^k}{1-\frac{4}{3}})] - n[\frac{4}{3}\log_3 3 + (\frac{4}{3})^2 2\log_3 3 + \cdots + (\frac{4}{3})^{k-1}(k-1)\log_3 3] \\
&+ 4^k(From\ Geometric\ Series\ Refer\ to\ Page\ 1\ SumSeries.) \\
&= 3n\log_3 n[(\frac{4}{3})^k - 1] - n[\frac{4}{3}.1 + (\frac{4}{3})^2.2 + \cdots + (\frac{4}{3})^{k-1}.(k-1) + 4^k \\
&(Since \log_3 3 = 1) \\
&= 3n\log_3 n[(\frac{4}{3})^k - 1] - n[\frac{4}{3}.1 + (\frac{4}{3})^2.2 + \cdots + (\frac{4}{3})^{k-1}.(k-1) + 4^k \\
&= 3n\log_3 n[(\frac{4}{3})^k - 1] - n[\frac{4}{3}[\frac{(k-1)(\frac{4}{3})^k - (k)(\frac{4}{3})^{k-1} + 1}{(\frac{4}{3}-1)^2}]] + 4^k \\
&(Refer\ to\ Page\ 1\ SumSeries\#4.) \\
&= 3n\log_3 n(\frac{4}{3})^k - 3n\log_3 n - 9n[\frac{4}{3}[(k-1)(\frac{4}{3})^k - (k)(\frac{4}{3})^{k-1} + 1]] + 4^k \\
&= 3n\log_3 n(\frac{4}{3})^k - 3n\log_3 n - 12n[(k)(\frac{4}{3})^k - (\frac{4}{3})^k - k(\frac{4}{3})^{k-1} + 1] + 4^k \\
&= 3n\log_3 n(\frac{4}{3})^k - 3n\log_3 n - 12n[(k)(\frac{4}{3})^k - (\frac{4}{3})^k - k(\frac{4}{3})^k(\frac{4}{3})^{-1} + 1] + 4^k \\
&= 3n\log_3 n(\frac{4}{3})^k - 3n\log_3 n - 12n[(k)(\frac{4}{3})^k - (\frac{4}{3})^k - k(\frac{4}{3})^k(\frac{3}{4}) + 1] + 4^k \\
&= 3n\log_3 n(\frac{4}{3})^k - 3n\log_3 n - 12n[(k)(\frac{1}{4})(\frac{4}{3})^k - (\frac{4}{3})^k + 1] + 4^k \\
&= 3n\log_3 n(\frac{4}{3})^k - 3n\log_3 n - 3n(k)(\frac{4}{3})^k + 12n(\frac{4}{3})^k + 12n + 4^k
\end{aligned}
$$

$$= 3n(k)(\frac{4}{3})^k - 3n(k) - 3n(k)(\frac{4}{3})^k + 12n(\frac{4}{3})^k + 12n + 4^k$$

$(Substituting\ value\ of\ k = \log_3 n)$

$$= -3n(k) + 12n(\frac{4}{3})^k + 12n + 4^k$$

$$= -3n(k) + 12n(\frac{4^k}{3^k}) + 12n + 4^k$$

$$= -3n(k) + 12n(\frac{4^k}{n}) + 12n + 4^k (Substituting\ value\ of\ 3^k = n)$$

$$= -3n(k) + 12(4^k) + 12n + 4^k$$

$$= 13(4^k) + 12n - 3n(k)$$

$$= 13(4^{\log_3 n}) + 12n - 3n \log_3 n$$

$$= 13(4^{\frac{\log_4 n}{\log_4 3}}) + 12n - 3n \log_3 n (From\ Logarithmic\ Properties - Refer\ page\ 1$$

$$= 13(4)^{\log_4 n (\frac{1}{\log_4 3})} + 12n - 3n \log_3 n$$

$$= 13n^{(\frac{1}{\log_4 3})} + 12n - 3n \log_3 n (From\ Logarithmic\ Properties - Refer\ page\ 1$$

$$= 13n^{\log_3 4} + 12n - 3n \log_3 n (From\ Logarithmic\ Properties - Refer\ page\ 1\#$$

(or)

$$= 13n^{(\log_3 4)} + 12n - 3n \log_3 n (From\ Logarithmic\ Properties - Refer\ page\ 1$$

$$\Rightarrow T(n) = 13n^{(\log_3 4)} + 12n - 3n \log_3 n$$

$Let f(n) = 13n^{(\log_3 4)} + 12n - 3n \log_3 n$

$\bigcirc : f(n) = 13n^{(\log_3 4)} + 12n - 3n \log_3 n$

$f(n) \leq 13n^{(\log_3 4)}$

$g(n) = n^{(\log_3 4)}\ such\ that\ f(n) \leq c_0.g(n) always\ holds\ TRUE$

$\Rightarrow Hence\ Proved\ T(n) = \Theta(n^{\log_b a}) = \Theta(n^{(\log_3 4)})$

$\square$

2.6.1 Application of Master Theorem and Prove using Elimination Method: In this case log is to the base 2: $T(n) = 4T(n/3) + n \log_2 n$

Solution:

Step 1: Master Theorem -

From the given equation, a = 4 ; b = 3 ; $f(n) = n\log_2 n$
$\Rightarrow n^{\log_b a} = n^{\log_3 4}$
$\Rightarrow n^{\log_b a} > f(n)$
Case 1: Recursive Sub problem dominates.
Asymptotic Time Complexity $\Rightarrow T(n)) = \Theta(n^{\log_b a}) = \Theta(n^{\log_3 4})$

Step 2: Proof by Elimination Method -

Proof.

$$T(n) = 4T(n/3) + n\log_2 n \tag{0}$$

$$T(n/3) = 4T(n/3^2) + \frac{n}{3}\log_2(n/3) \tag{1}$$

$$T(n/3^2) = 4T(n/3^3) + \frac{n}{3^2}\log_2(n/3^2) \tag{2}$$

$$\vdots$$

$$T(n/3^{k-1}) = 4T(n/3^k) + \frac{n}{3^{k-1}}\log_2(n/3^{k-1}) \tag{k-1}$$

$$T(1) = 1 \tag{k}$$

Get rid of T-terms on Right-hand side:
Add equations $(0) + 4*(1) + 4^2*(2) + \cdots + 4^{k-1}*(k-1) + 4^k*k$
Without loss of generality , let us assume $n = 3^k \Rightarrow k = \log_3(n)$

$$
\begin{aligned}
T(n) &= n\log_2 n + 4(\frac{n}{3})\log_2(\frac{n}{3}) + 4^2(\frac{n}{3^2})\log_2(\frac{n}{3^2}) + \cdots + 4^{k-1}(\frac{n}{3^{k-1}})\log_2(\frac{n}{3^{k-1}}) + 4^k \\
&= n\log_2 n + 4(\frac{n}{3})[\log_2 n - \log_2 3] + 4^2(\frac{n}{3^2})[\log_2 n - \log_2 3^2] + \cdots \\
&+ 4^{k-1}(\frac{n}{3^{k-1}})[\log_2 n - \log_2 3^{k-1}] + 4^k \\
&= n\log_2 n + 4(\frac{n}{3})\log_2 n - 4(\frac{n}{3})\log_2 3 + 4^2(\frac{n}{3^2})\log_2 n - 4^2(\frac{n}{3^2})\log_2 3^2 + \cdots \\
&+ 4^{k-1}(\frac{n}{3^{k-1}})\log_2 n - 4^{k-1}(\frac{n}{3^{k-1}})\log_2 3^{k-1} + 4^k \\
&= n\log_2 n[1 + \frac{4}{3} + (\frac{4}{3})^2 + \cdots + (\frac{4}{3})^{k-1}] - \\
&n[\frac{4}{3}\log_2 3 + (\frac{4}{3})^2\log_2 3^2 + \cdots + (\frac{4}{3})^{k-1}\log_2 3^{k-1}] + 4^k \\
&= n\log_2 n[1 + \frac{4}{3} + (\frac{4}{3})^2 + \cdots + (\frac{4}{3})^{k-1}] - \\
&n[\frac{4}{3}\log_2 3 + (\frac{4}{3})^2 2\log_2 3 + \cdots + (\frac{4}{3})^{k-1}(k-1)\log_2 3] + 4^k \\
&= n\log_2 n[1(\frac{1-(\frac{4}{3})^k}{1-\frac{4}{3}})] - n[\frac{4}{3}\log_2 3 + (\frac{4}{3})^2 2\log_2 3 + \cdots + (\frac{4}{3})^{k-1}(k-1)\log_2 3] \\
&+ 4^k (From\ Geometric\ Series\ Refer\ to\ Page\ 1\ SumSeries.) \\
&= 3n\log_2 n[(\frac{4}{3})^k - 1] - n\log_2 3[\frac{4}{3}.1 + (\frac{4}{3})^2.2 + \cdots + (\frac{4}{3})^{k-1}.(k-1)] + 4^k \\
&= 3n\log_2 n[(\frac{4}{3})^k - 1] - n\log_2 3[\frac{4}{3}.1 + (\frac{4}{3})^2.2 + \cdots + (\frac{4}{3})^{k-1}.(k-1)] + 4^k \\
&= 3n\log_2 n[(\frac{4}{3})^k - 1] - n\log_2 3[\frac{4}{3}[\frac{(k-1)(\frac{4}{3})^k - (k)(\frac{4}{3})^{k-1} + 1}{(\frac{4}{3}-1)^2}]] + 4^k \\
&(Refer\ to\ Page\ 1\ SumSeries\#4.) \\
&= 3n\log_2 n(\frac{4}{3})^k - 3n\log_2 n - 9n\log_2 3[\frac{4}{3}[(k-1)(\frac{4}{3})^k - (k)(\frac{4}{3})^{k-1} + 1]] + 4^k \\
&= 3n\log_2 n(\frac{4}{3})^k - 3n\log_2 n - 12n\log_2 3[(k)(\frac{4}{3})^k - (\frac{4}{3})^k - k(\frac{4}{3})^{k-1} + 1] + 4^k \\
&= 3n\log_2 n(\frac{4}{3})^k - 3n\log_2 n - 12n\log_2 3[(k)(\frac{4}{3})^k - (\frac{4}{3})^k - k(\frac{4}{3})^k(\frac{4}{3})^{-1} + 1] + 4^k \\
&= 3n\log_2 n(\frac{4}{3})^k - 3n\log_2 n - 12n\log_2 3[(k)(\frac{4}{3})^k - (\frac{4}{3})^k - k(\frac{4}{3})^k(\frac{3}{4}) + 1] + 4^k \\
&= 3n\log_2 n(\frac{4}{3})^k - 3n\log_2 n - 12n\log_2 3[(k)(\frac{1}{4})(\frac{4}{3})^k - (\frac{4}{3})^k + 1] + 4^k \\
&= 3n\log_2 n(\frac{4}{3})^k - 3n\log_2 n - 3n\log_2 3(k)(\frac{4}{3})^k + 12n\log_2 3(\frac{4}{3})^k + 12n\log_2 3 + 4^k
\end{aligned}
$$

$$= 3n(\log_2 3^k)(\frac{4}{3})^k - 3n(\log_2 3^k) - 3n\log_2 3(k)(\frac{4}{3})^k + 12n\log_2 3(\frac{4}{3})^k + 12n\log_2 3$$

$(Substituting\ value\ of\ n = 3^k)$

$$= 3n(k)\log_2 3(\frac{4}{3})^k - 3n(k)\log_2 3 - 3n(k)\log_2 3(\frac{4}{3})^k$$

$$+ 12n\log_2 3(\frac{4}{3})^k + 12n\log_2 3 + 4^k$$

$$= -3n(k)\log_2 3 + 12n\log_2 3(\frac{4}{3})^k + 12n\log_2 3 + 4^k$$

$$= -3n(k)\log_2 3 + 12n\frac{4^k}{3^k}\log_2 3 + 12n\log_2 3 + 4^k$$

$$= -3n(k)\log_2 3 + 12n\frac{4^k}{n}\log_2 3 + 12n\log_2 3 + 4^k$$

$(Substituting\ value\ of\ 3^k = n)$

$$= -3n(k)\log_2 3 + 12(4^k)\log_2 3 + 12n\log_2 3 + 4^k$$

$$= -3n(\log_3 n)(\log_2 3) + 12(4^{\log_3 n})(\log_2 3) + 12n\log_2 3 + 4^{\log_3 n}$$

$$= -3n(\log_2 n) + 12(4^{\log_3 n})(\log_2 3) + 12n\log_2 3 + 4^{\log_3 n}$$

$(From\ Logarithmic\ Properties - Refer\ page\ 1\#9)$

$$= -3n(\log_2 n) + 12(n^{\log_3 4})(\log_2 3) + 12n\log_2 3 + n^{\log_3 4}$$

$(From\ Logarithmic\ Properties - Refer\ page\ 1\#14)$

$$= -3n(\log_2 n) + 12(n^{\log_3 4})(\log_2 3) + 12n\log_2 3 + n^{\log_3 4}$$

$$= n^{\log_3 4} + 12(n^{\log_3 4})(\log_2 3) + 12n\log_2 3 - 3n(\log_2 n)$$

$$= n^{\log_3 4}[1 + 12(\log_2 3)] + 3n[4.\log_2 3 - (\log_2 n)]$$

$$\Rightarrow T(n) = n^{\log_3 4}[1 + 12(\log_2 3)] + 3n[4.\log_2 3 - (\log_2 n)]$$

$Let f(n) = n^{\log_3 4}[1 + 12(\log_2 3)] + 3n[4.\log_2 3 - (\log_2 n)]$

$\bigcirc : f(n) = n^{\log_3 4}[1 + 12(\log_2 3)] + 3n[4.\log_2 3 - (\log_2 n)]$

$f(n) \leq n^{\log_3 4}$

$g(n) = n^{\log_3 4}\ such\ that\ f(n) \leq c_0.g(n) always\ holds\ TRUE$

$\Rightarrow Hence\ Proved\ T(n) = \Theta(n^{\log_b a}) = \Theta(n^{\log_3 4})$

$\square$

2.7 Application of Master Theorem and Prove using Elimination Method: $T(n) = 3T(n/3) + n/\log_3 n$

Solution:

Step 1: Master Theorem -

From the given equation, a = 3 ; b = 3 ; $f(n) = n/\log_3 n$
$\Rightarrow n^{\log_b a} = n^{\log_3 3} = n^1 = n$
$f(n) = n/\log_3 n$ is neither Polynomially bigger or smaller than n and is not equal to $\Theta(n^{\log_b a})$
Use Extended Master Theorem.
Asymptotic Time Complexity $\Rightarrow T(n)) = \Theta(n \log\log n))$

Step 2: Proof by Elimination Method -

Proof.

$$T(n) = 3T(n/3) + n/\log_3 n \tag{0}$$

$$T(n/3) = 3T(n/3^2) + \frac{n/3}{\log_3(n/3} \tag{1}$$

$$T(n/3^2) = 3T(n/3^3) + \frac{n/3^2}{\log_3(n/3^2)} \tag{2}$$

$$\vdots$$

$$T(n/3^{k-1}) = 3T(n/3^k) + \frac{n/3^{k-1}}{\log_3(n/3^{k-1})} \tag{k-1}$$

$$T(1) = 1 \tag{k}$$

Get rid of T-terms on Right-hand side:
Add equations $(0) + 3*(1) + 3^2*(2) + \cdots + 3^{k-1}*(k-1) + 3^k*k$
Without loss of generality , let us assume $n = 3^k \Rightarrow k = \log_3(n)$

$$
\begin{aligned}
T(n) &= \frac{n}{\log_3 n} + 3 * \frac{n/3}{\log_3(n/3)} + 3^2 * \frac{n/3^2}{\log_3(n/3^2)} + \cdots + 3^{k-1} * \frac{n/3^{k-1}}{\log_3(n/3^{k-1})} + 3^k \\
&= \frac{n}{\log_3 n} + \frac{n}{\log_3(n/3)} + \frac{n}{\log_3(n/3^2)} + \cdots + \frac{n}{\log_3(n/3^{k-1})} + 3^k \\
&= n[\frac{1}{\log_3 n} + \frac{1}{\log_3(n/3)} + \frac{1}{\log_3(n/3^2)} + \cdots + \frac{1}{\log_3(n/3^{k-1})}] + 3^k \\
&= n[\frac{1}{\log_3 n} + \frac{1}{\log_3 n - \log_3 3)} + \frac{1}{\log_3 n - \log_3 3^2)} + \cdots + \\
&\frac{1}{\log_3 n - \log_3 3^{(k-1)}}] + 3^k (From\ Logarithmic\ Properties - Refer\ page\ 1\#7) \\
&= n[\frac{1}{\log_3 n} + \frac{1}{\log_3 n - \log_3 3)} + \frac{1}{\log_3 n - 2\log_3 3)} + \cdots + \\
&\frac{1}{\log_3 n - (k-1)\log_3 3}] + 3^k \\
&= n[\frac{1}{k} + \frac{1}{k-1} + \frac{1}{k-2} + \cdots + \frac{1}{k-(k-1)}] + 3^k \\
&(Substituting\ value\ of\ \ \log_3 n = k) \\
&= n[\frac{1}{k} + \frac{1}{k-1} + \frac{1}{k-2} + \cdots + 1] + 3^k \\
&= n[\log k + 1] + 3^k \\
&(From\ Harmonic\ Series\ sum - Refer\ page\ 1) \\
&= n \log k + n + n \\
&= n \log k + 2n \\
&= n \log \log_3 n + 2n \\
\Rightarrow T(n) &= n \log \log_3 n + 2n
\end{aligned}
$$

$$
\begin{aligned}
&Let f(n) = n \log \log_3 n + 2n \\
&\bigcirc : f(n) = n \log \log_3 n + 2n \\
&\quad f(n) \leq n \log \log_3 n \\
&\quad g(n) = \leq n \log \log_3 n\ such\ that\ f(n) \leq c_0.g(n) always\ holds\ TRUE \\
&\Rightarrow Hence\ Proved\ T(n) = \Theta(n \log \log n)
\end{aligned}
$$

$\square$

2.7.1 Application of Master Theorem and Prove using Elimination Method:

When log to the base is 2: $T(n) = 3T(n/3) + n/\log_2 n$

Solution:

Step 1: Master Theorem -

From the given equation, a = 3 ; b = 3 ; $f(n) = n/\log_2 n$
$\Rightarrow n^{\log_b a} = n^{\log_3 3} = n^1 = n$
$f(n) = n/\log_2 n$ is neither Polynomially bigger or smaller than n and is not equal to $\Theta(n^{\log_b a})$
Use Extended Master Theorem.
Asymptotic Time Complexity $\Rightarrow T(n)) = \Theta(n \log_3 \log_3 n))\ (or)\ = \Theta(n \log \log n))$

Step 2: Proof by Elimination Method -

Proof.

$$T(n) = 3T(n/3) + n/\log_2 n \tag{0}$$

$$T(n/3) = 3T(n/3^2) + \frac{n/3}{\log_2(n/3} \tag{1}$$

$$T(n/3^2) = 3T(n/3^3) + \frac{n/3^2}{\log_2(n/3^2)} \tag{2}$$

$$\vdots$$

$$T(n/3^{k-1}) = 3T(n/3^k) + \frac{n/3^{k-1}}{\log_2(n/3^{k-1})} \tag{k-1}$$

$$T(1) = 1 \tag{k}$$

Get rid of T-terms on Right-hand side:
Add equations $(0) + 3 * (1) + 3^2 * (2) + \cdots + 3^{k-1} * (k-1) + 3^k * k$
Without loss of generality , let us assume $n = 3^k \Rightarrow k = \log_3(n)$

$$T(n) = \frac{n}{\log_2 n} + 3 * \frac{n/3}{\log_2(n/3)} + 3^2 * \frac{n/3^2}{\log_2(n/3^2)} + \cdots + 3^{k-1} * \frac{n/3^{k-1}}{\log_2(n/3^{k-1})} + 3^k$$
$$= \frac{n}{\log_2 n} + \frac{n}{\log_2(n/3)} + \frac{n}{\log_2(n/3^2)} + \cdots + \frac{n}{\log_2(n/3^{k-1})} + 3^k$$
$$= n[\frac{1}{\log_2 n} + \frac{1}{\log_2(n/3)} + \frac{1}{\log_2(n/3^2)} + \cdots + \frac{1}{\log_2(n/3^{k-1})}] + 3^k$$
$$= n[\frac{1}{\log_2 n} + \frac{1}{\log_2 n - \log_2 3)} + \frac{1}{\log_2 n - \log_2 3^2)} + \cdots +$$
$$\frac{1}{\log_2 n - \log_2 3^{(k-1)}}] + 3^k (From\ Logarithmic\ Properties - Refer\ page\ 1\#7)$$
$$= n[\frac{1}{\log_2 n} + \frac{1}{\log_2 n - \log_2 3)} + \frac{1}{\log_2 n - 2\log_2 3)} + \cdots +$$
$$\frac{1}{\log_2 n - (k-1)\log_2 3}] + 3^k$$
$$= n[\frac{1}{\log_2 3^k} + \frac{1}{\log_2 3^k - \log_2 3)} + \frac{1}{\log_2 3^k - 2\log_2 3)} + \cdots +$$
$$\frac{1}{\log_2 3^k - (k-1)\log_2 3}] + 3^k$$
$$(Substituting\ value\ of\ \log_3 n = k)$$
$$= n[\frac{1}{k\log_2 3} + \frac{1}{k\log_2 3 - \log_2 3)} + \frac{1}{k\log_2 3 - 2\log_2 3)} + \cdots +$$
$$\frac{1}{k\log_2 3 - (k-1)\log_2 3}] + 3^k$$
$$= \frac{n}{\log_2 3}[\frac{1}{k} + \frac{1}{k-1)} + \frac{1}{k-2} + \cdots + \frac{1}{k-(k-1)}] + 3^k$$
$$= \frac{n}{\log_2 3}[\frac{1}{k} + \frac{1}{k-1} + \frac{1}{k-2} + \cdots + 1] + 3^k$$
$$= \frac{n}{\log_2 3}[\log_2 k + 1] + 3^k$$
$$(From\ Harmonic\ Series\ sum - Refer\ page\ 1)$$
$$= \frac{n}{\log_2 3}\log_2 k + \frac{n}{\log_2 3} + n$$
$$= n\log_3 k + \frac{n}{\log_2 3} + n (From\ Logarithmic\ Properties - Refer\ page\ 1\#10)$$
$$= n\log_3 k + n(\log_3 2) + n (From\ Logarithmic\ Properties - Refer\ page\ 1\#11$$
$$= n\log_3 \log_3 n + n(\log_3 2) + n$$
$$\Rightarrow T(n) = n\log_3 \log_3 n + n(\log_3 2) + n$$

$Let f(n) == n \log_3 \log_3 n + n(\log_3 2) + n$

$\bigcirc : f(n) = n \log_3 \log_3 n + n(\log_3 2) + n$

$\quad f(n) \leq n \log_3 \log_3 n$

$\quad g(n) =\leq n \log_3 \log_3 n \ such \ that \ f(n) \leq c_0.g(n) always \ holds \ TRUE$

$\Rightarrow Hence \ Proved \ T(n) = \Theta(n \log_3 \log_3 n) \ (or) \ \Theta(n \log \log n)$

$\square$