Lesson 9C: Review of HW3

4.1(a) $\begin{cases} T(n) = 2T\left(\frac{n}{2}\right) + n^4 \\ T(1) = 1 \end{cases}$

Proof:

multiply LHS to cancel out

put in ( )

$$⓪\ T(n) = 2T\left(\frac{n}{2}\right) + n^4$$

$$①\ 2T\left(\frac{n}{2}\right) = 2T\left(\frac{n}{4}\right) + \left(\frac{n}{2}\right)^4$$

$$②\ 2T\left(\frac{n}{4}\right) = 2T\left(\frac{n}{8}\right) + \left(\frac{n}{4}\right)^4$$

$$③\ 2T\left(\frac{n}{8}\right) = 2T\left(\frac{n}{16}\right) + \left(\frac{n}{8}\right)^4$$

$$\vdots$$

$$(k-1)\ 2T\left(\frac{n}{2^{k-1}}\right) = 2T\left(\frac{n}{2^k}\right) + \left(\frac{n}{2^{k-1}}\right)^4$$

$$(k)\ T(1) = 1$$

without loss of generality,

$$2^k = n \quad \text{or} \quad k = \log_2 n$$

$$0 \cdot 2^0 + 1 \cdot 2^1 + \cdots + k-1 \cdot 2^{k-1} + k \cdot 2^k$$

$$\Rightarrow n^4\left(1 + \frac{1}{8} + \frac{1}{8^2} + \cdots + \frac{1}{8^{k-1}}\right) + 2^k$$

$$\Rightarrow n^4 \cdot \left(\frac{1-\left(\frac{1}{8}\right)^k}{1-\frac{1}{8}}\right) + n$$

$$= \frac{8n^4}{7}\left(1 - \frac{1}{8^k}\right) + n$$

$$= \frac{8n^4}{7}\left(1 - \frac{1}{n^3}\right) + n$$

$$\frac{1}{8^k} = \frac{1}{(2^k)^3}$$

$$= \frac{1}{(n)^3}$$

$$= \frac{8n^4}{7} - \frac{8n}{7} + n$$

next question

don't have to multiply to cancel out

(0) $T(n) = T\left(\frac{7}{10}n\right) + n$

(1) $T\left(\frac{7n}{10}\right) = T\left(\frac{49}{100}n\right) + \left(\frac{7}{10}\right)n$

(2) $T\left(\frac{7n^2}{10^2}\right) = T\left(\frac{7^3}{10^3}n\right) + \left(\frac{7^2}{10^2}\right)n$

$\vdots$

(k-1) $T\left(\frac{7n^{k-1}}{10^{k-1}}\right) = T\left(\frac{7^k}{10^k}n\right) + \left(\frac{7^{k-1}}{10^{k-1}}\right)n$

(k) $T(1) = 1$

$\left(\frac{10}{7}\right)^k = n$ or $k = \log_{\frac{10}{7}} n$

k-1 included but not written

(0) + (1) + (2) + ... + (k)

$T(n) = n\left[1 + \frac{7}{10} + \left(\frac{7}{10}\right)^2 + \cdots + \left(\frac{7}{10}\right)^{k-1}\right]$

$+1$

$$= n\left[\frac{1-\left(\frac{7}{10}\right)^k}{1-\frac{7}{10}}\right]+1$$

$$= \frac{10}{3}n\left[1-\frac{1}{n}\right]+1$$

$$= \frac{10}{3}n - \frac{10}{3}+1 = \frac{10}{3}n - \frac{7}{3}$$

<u>next</u>

<u>Proof:</u>

$$⓪\ T(n) = 7T\left(\frac{n}{2}\right)+n^2$$

$$7\ ①\ T\left(\frac{n}{2}\right) = 7\left(\frac{n}{4}\right)+\left(\frac{n}{2}\right)^2$$

$$7\ ②\ T\left(\frac{n}{4}\right) = 7\left(\frac{n}{8}\right)+\left(\frac{n}{2}\right)^3$$

$$7\ (k-1)\ T\left(\frac{n}{2^{k-1}}\right) = 7T\left(\frac{n}{2^k}\right)+\left(\frac{n}{2^{k-1}}\right)^2$$

$$7\ (k)\ T(1)=1$$

Assume $2^k = n$ or $k=\log_2 n$

$$= n^2\left(1+\frac{7}{4}+\left(\frac{7}{4}\right)^2+\cdots+\left(\frac{7}{4}\right)^k\right)+7^k$$

$$= n^2\left(\frac{1-\left(\frac{7}{4}\right)^{k+1}}{1-\frac{7}{4}}\right)+7^k$$

$$= n^2\left(\frac{1-\left(\frac{7}{4}\right)^{k+1}}{1-7/4}\right)+7^k$$

$$= \frac{4}{3}n^2\left(\frac{7^{k+1}}{4^{k+1}}\right)+n^{\log_2 7}$$

$$\left(2^{\log_2 7}\right)^k$$

$$=\left(2^k\right)^{\log_2 7}$$

$$=(n)^{\log_2 7}$$

$$a = b^{\log_b a}$$

$$= \frac{\cancel{4}}{3}n\cdot\frac{7}{\cancel{4}}\cdot\frac{7^k}{4^k}+n^{\log_2 7}$$

$$= \frac{7}{3}\cdot\frac{7^k}{4^k}+n^{\log_2 7}$$

$$= \frac{7}{3}\cancel{n}\left(\frac{\cancel{n}\cdot\log_2 n}{\cancel{n^2}}\right)+n^{\log_2 7}$$

$$= \frac{7}{3}\left(\log_2 n\right)+n^{\log_2 7} \quad O(n^2)$$

next 4,3 a)

$$\begin{cases} T(n) = 4\cdot T(\frac{n}{3}) + n\log n \\ T(1) = 1. \end{cases}$$

<u>proof</u>:

⓪ $T(n) = 4T(\frac{n}{3}) + n\log n$

① $4T(\frac{n}{3}) = 4T(\frac{n}{3^2}) + (\frac{n}{3})\log(\frac{n}{3})$

$= 4T(\frac{n}{3^2}) + (\frac{n}{3^2})\log_3(n/3)$

$= 4T(\frac{n}{3^2}) + (\frac{n}{3^2})\log(n-1)$

② $4T(\frac{n}{3^2}) = 4T(\frac{n}{3^3}) + (\frac{n}{3^2})\log(n-2)$

③ $4T(\frac{n}{3^3}) = 4T(\frac{n}{3^4}) + (\frac{n}{3^3})\log(n-3)$

$\vdots$

$k-1$ $4T(\frac{n}{3^{k-1}}) = 4T(\frac{n}{3^k}) + \frac{n}{3^{k-1}}(\log n \cdot k - 1)$

k $T(\frac{n}{3^k}) = T(1) = 1,$

$k = \log_3 n \quad , \quad n = 3^k$

$$0 + 1 \cdot 4 + 2 \cdot 4^2 + \cdots + k \cdot 4^k$$

$$T(n) = n \cdot \log_3 n + \frac{4}{3} n (\log_3 n - 1)$$
$$+ \left(\frac{4}{3}\right)^2 n \cdot (\log_3 n - 2)$$
$$+ \left(\frac{4}{3}\right)^3 n \cdot (\log_3 n - 3)$$
$$\vdots$$
$$+ \left(\frac{4}{3}\right)^{k-1} n \cdot (\log_3 n - k + 1)$$
$$+ 4^k$$

$$= n \cdot \log_3 n \left[\left(\frac{4}{3}\right)^0 + \left(\frac{4}{3}\right)^1 + \cdots + \left(\frac{4}{3}\right)^{k-1} \right]$$
$$- \left[\left(\frac{4}{3}\right) \cdot 1 + \left(\frac{4}{3}\right)^2 \cdot 2 + \left(\frac{4}{3}\right)^3 \cdot 3 + \cdots + \left(\frac{4}{3}\right)^{k-1} \cdot (k-1) \right]$$

power sequence and geometric sequence

4.3b

from last problem

(0) $T(n) = 3T\left(\frac{n}{3}\right) + \frac{n}{\log_3 n}$

(1) $3T\left(\frac{n}{3}\right) = 3T\left(\frac{n}{3^2}\right) + \left(n/3\right)/\left(\log_3 n - 1\right)$

(2) $3T\left(\frac{n}{3^2}\right) = 3T\left(\frac{n}{3^3}\right) + \left(\frac{n}{3^2}\right)/\left(\log_3 n - 2\right)$

⋮

(k-1) $3T\left(\frac{n}{3^{k-1}}\right) = 3T\left(\frac{n}{3^k}\right) + \left(\frac{n}{3^{k-1}}\right)/\left(\log_3 n - k + 1\right)$

(k) $T(1) = 1$