Proof: Approximating $(t_0 + t)^{n-1}$ by an Exponential Function

In Hong Qin's network model of aging, the mortality rate $\mu_{net}(t)$ is given by:

$$\mu_{net}(t) \approx cmn(p\lambda)^n \left(\frac{1-p}{p\lambda} + t\right)^{n-1}$$

where $t_0 = \frac{1-p}{p\lambda}$.

Simplifying this, we have:

$$\mu_{net}(t) \approx cmn(p\lambda)^n (t_0 + t)^{n-1}$$

To understand how this term can be approximated by an exponential function, leading to $\exp(Gt)$ in the $\mu_{net}(t)$ formula, let's follow these steps:

Step-by-Step Approximation

1. **Express t_0 and G in Terms of Network Parameters:**
The virtual age t_0 and rate of increase in mortality G are given by:

$$t_0 = \frac{1-p}{p\lambda}$$

$$G = \frac{(n-1)p\lambda}{1-p}$$

2. **Simplified Form of $\mu_{net}(t)$:**
Using the definition of t_0, we rewrite $\mu_{net}(t)$ as:

$$\mu_{net}(t) \approx cmn(p\lambda)^n \left(\frac{1-p}{p\lambda} + t\right)^{n-1}$$

For small t (i.e., $t \ll t_0$), we can approximate this as:

$$\mu_{net}(t) \approx cmn(p\lambda)^n \left(\frac{1-p}{p\lambda}\right)^{n-1}$$

3. **Approximation by Exponential Function:**
The term $(t_0 + t)^{n-1}$ can be approximated by an exponential function for large t using the following steps:

- Consider the approximation $(t_0 + t) \approx t_0 \cdot \left(1 + \frac{t}{t_0}\right)$. - Using the binomial expansion for large t:

$$(t_0 + t)^{n-1} \approx t_0^{n-1} \left(1 + \frac{t}{t_0}\right)^{n-1}$$

- For large t, $\left(1+\frac{t}{t_0}\right)^{n-1}$ can be approximated using the exponential function:

$$\left(1+\frac{t}{t_0}\right)^{n-1} \approx \exp\left((n-1)\ln\left(1+\frac{t}{t_0}\right)\right)$$

For large t: - $\ln\left(1+\frac{t}{t_0}\right) \approx \frac{t}{t_0}$ because $\ln(1+x) \approx x$ for small x. - Thus:

$$\exp\left((n-1)\ln\left(1+\frac{t}{t_0}\right)\right) \approx \exp\left((n-1)\frac{t}{t_0}\right)$$

So we have:

$$(t_0+t)^{n-1} \approx t_0^{n-1}\exp\left((n-1)\frac{t}{t_0}\right)$$

4. **Incorporate into $\mu_{net}(t)$:**
Substitute this back into the mortality rate expression:

$$\mu_{net}(t) \approx cmn(p\lambda)^n t_0^{n-1}\exp\left((n-1)\frac{t}{t_0}\right)$$

Recall $t_0 = \frac{1-p}{p\lambda}$:

$$\mu_{net}(t) \approx cmn(p\lambda)^n \left(\frac{1-p}{p\lambda}\right)^{n-1}\exp\left((n-1)\frac{tp\lambda}{1-p}\right)$$

Simplify the constants:

$$\mu_{net}(t) \approx R\exp(Gt)$$

where:

$$R = cmn(p\lambda)^n\left(\frac{1-p}{p\lambda}\right)^{n-1}$$

and:

$$G = \frac{(n-1)p\lambda}{1-p}$$

This shows how the original term $(t_0+t)^{n-1}$ can be approximated by an exponential function, leading to the $\exp(Gt)$ term in the $\mu_{net}(t)$ formula.