Homework 5 of CS520 Theory of Programming Languages

You do not need to submit this homework. But I strongly encourage you to study the questions before the exam. The numbers in the questions refer to exercise questions in the textbook of the course, i.e. "Theories of Programming Languages" by John C. Reynolds.

Question 1

Solve 11.11. While solving this question, you might find it useful to use lists. Lists are explained in Section 11.4 of the textbook. If you do not want to read the textbook but still are interested in using lists, a quick solution is to view constructs for lists as syntactic sugars, as shown below:

$$\begin{aligned}
\mathbf{nil} &\overset{\text{def}}{=} @\ 0\ 0 \\
e_0 :: e_1 &\overset{\text{def}}{=} @\ 1\ \langle e_0,\, e_1\rangle \\
\mathbf{listcase}\ e\ \mathbf{of}\ (e_0,\, e_1) &\overset{\text{def}}{=} \mathbf{sumcase}\ e\ \mathbf{of}\ \Big((\lambda v.\, e_0),\ (\lambda v.\, (e_1\ v.0)\ v.1)\Big) \quad \text{for a fresh variable } v
\end{aligned}$$

Intuitively, **nil** denotes the empty list, and $e_0 :: e_1$ denotes the list with e_0 as its head and e_1 as its tail. The last performs the pattern match. If e is the empty list, we evaluate e_0. Otherwise, we apply e_1 to two parameters, one for the head of e and the other for the tail of e.

Question 2

Solve 11.12.

Question 3

Solve 12.1. When solving Part (a), assume that the direct semantics uses the following domains and predomains:

$$V_* = (V + \{\mathit{err}, \mathit{typerr}\})_\perp, \qquad V \simeq V_{\mathit{int}} + V_{\mathit{bool}} + V_{\mathit{fun}} + V_{\mathit{tuple}} + V_{\mathit{alt}},$$

$$V_{\mathit{int}} = \mathbb{Z}, \qquad V_{\mathit{bool}} = \mathbb{B},$$

$$V_{\mathit{fun}} = [V \to_c V_*], \qquad V_{\mathit{tuple}} = \bigcup_{n=0}^{\infty} V^n, \qquad V_{\mathit{alt}} = \mathbb{N} \times V.$$

Also, when solving Part (b), assume that the continuation semantics uses the following domains and predomains:

$$V_* = (V + \{\mathit{err}, \mathit{typerr}\})_\perp, \qquad V \simeq V_{\mathit{int}} + V_{\mathit{bool}} + V_{\mathit{fun}} + V_{\mathit{tuple}} + V_{\mathit{alt}} + V_{\mathit{cont}}$$

$$V_{\mathit{int}} = \mathbb{Z}, \qquad V_{\mathit{bool}} = \mathbb{B}, \qquad V_{\mathit{fun}} = [V \to_c V_{\mathit{cont}} \to_c V_*],$$

$$V_{\mathit{tuple}} = \bigcup_{n=0}^{\infty} V^n, \qquad V_{\mathit{alt}} = \mathbb{N} \times V, \qquad V_{\mathit{cont}} = V \to_c V_*.$$

Model answers

Answer to question 1

Here is my implementation of the required function *codecheck* in the question.

$$\textbf{letrec}\ \mathit{in\_list} \equiv \lambda n.\,\lambda l.\,\textbf{listcase}\ l\ \textbf{of}\ \Big(\textbf{false},\ \lambda n'.\,\lambda l'.\,\textbf{if}\ (n = n')\ \textbf{then}\ \textbf{true}\ \textbf{else}\ (\mathit{in\_list}\ n\ l')\Big)\ \textbf{in}$$
$$\Big(\lambda F.\ \textbf{letrec}\ g \equiv \Big(\lambda l.\,\lambda n.\,\textbf{if}\ (\mathit{in\_list}\ n\ l)\ \textbf{then}\ \textbf{error}\ \textbf{else}\ (F\ (g\ (n :: l))\ n)\Big)\ \textbf{in}\ (g\ \textbf{nil})\Big)$$

Answer to question 2

[Part (a)] To show the claimed isomorphism between some domain D and another D' in Part (a), we need to define two continuous functions $\phi : D \to D'$ and $\psi : D' \to D$, and to show that $\phi \circ \psi$ is the identify function on D' and $\psi \circ \phi$ is the identify function on D. For each of the three isomorphisms in the question, we just describe what these continuous functions are. If you have time, try to check whether these functions are indeed continuous and also they are mutually inverses. In the first two cases, both of the continuous functions ϕ and ψ are identify functions. For the third case, we use the following continuous functions:

$$\phi : [P \to_c D] \to [P_\perp \to_c^s D]$$
$$\phi(f)(a) \stackrel{\text{def}}{=} \begin{cases} \perp & \text{if } a = \perp \\ f(a) & \text{otherwise} \end{cases}$$

$$\psi : [P_\perp \to_c^s D] \to [P \to_C D]$$
$$\psi(f)(a) \stackrel{\text{def}}{=} f(a)$$

Here we write $D \to_c^s D'$ to denote the space of strict continuous functions from a domain D to another domain D'.

[Part (b)] Let us write $V^{s,n}$ and $\bigoplus$ for the smash product versions of V^n and $\sum$. Also, instead of $+_{smash}$ and $\times_{smash}$, we write $+_s$ and $\times_s$. The the required isomorphism is:

$$V_* \simeq \Big(\mathbb{Z}_\perp +_s \mathbb{B}_\perp +_s (V_* \to_c^s V_*) +_s \bigoplus_{n=0}^{\infty} V_*^{s,n} +_s (\mathbb{N}_\perp \times_s V_*)\Big)$$

Answer to question 3

[Part (a)] Here is the derivation of the semantics of the let expression in the direct semantics.

$$\begin{aligned}
[\![\textbf{let}\ v \equiv e\ \textbf{in}\ e']\!]_d\,\eta &= [\![((\lambda v.\, e')\ e)]\!]_d\,\eta \\
&= \big(\lambda f.\,(\lambda a.\,f\ a)_*\ ([\![e]\!]_d\,\eta)\big)_{\mathit{fun}*}\big(\,[\![(\lambda v.\,e')]\!]_d\,\eta\big) \\
&= \big(\lambda f.\,(\lambda a.\,f\ a)_*\ ([\![e]\!]_d\,\eta)\big)_{\mathit{fun}*}\big(\langle 0,\ \psi\langle 2,\ \lambda b.\ [\![e']\!]_d\,[\eta \mid v : b]\rangle\rangle\big) \\
&= \Big(\lambda a.\,\big(\lambda b.\ [\![e']\!]_d\,[\eta \mid v : b]\big)\ a\Big)_*\ ([\![e]\!]_d\,\eta) \\
&= \Big(\lambda a.\ [\![e']\!]_d\,[\eta \mid v : a]\Big)_*\ ([\![e]\!]_d\,\eta).
\end{aligned}$$

[Part (b)] Here is our derivation in the continuation semantics.

$$\begin{aligned}
[\![\mathbf{let}\ v \equiv e\ \mathbf{in}\ e']\!]\,\eta\,\kappa &= [\![((\lambda v.\, e')\ e)]\!]\,\eta\,\kappa \\
&= [\![\lambda v.\, e']\!]\,\eta\,\Big(\lambda f.\ [\![e]\!]\,\eta\,(\lambda a.\,(f\ a)\ \kappa)\Big)_{\mathit{fun}} \\
&= \Big(\lambda f.\ [\![e]\!]\,\eta\,(\lambda a.\,(f\ a)\ \kappa)\Big)_{\mathit{fun}}\ \Big(\psi\langle 2,\ \lambda b.\,\lambda\kappa'.\ [\![e']\!]\,[\eta|v:b]\,\kappa'\rangle\Big) \\
&= [\![e]\!]\,\eta\,\Big(\lambda a.\,((\lambda b.\,\lambda\kappa'.\ [\![e']\!]\,[\eta|v:b]\,\kappa')\ a)\ \kappa\Big) \\
&= [\![e]\!]\,\eta\,\Big(\lambda a.\ [\![e']\!]\,[\eta|v:a]\,\kappa\Big).
\end{aligned}$$

[Part (c)] Consider an environment η and a continuation κ. Let z be the value that satisfies

$$[\![e]\!]\,\eta\kappa' = \kappa'\,z \text{for all continuations } \kappa'.$$

Note that the choice of z depends only on η, not κ nor κ'. We have to show that

$$[\![\mathbf{let}\ v \equiv e\ \mathbf{in}\ \lambda v'.\, e']\!]\,\eta\,\kappa = [\![\lambda v'.\,\mathbf{let}\ v \equiv e\ \mathbf{in}\ e']\!]\,\eta\,\kappa.$$

We simplify the left-hand side of this equation by unpacking the semantic definitions, applying the (mathematical) functions and using z.

$$\begin{aligned}
[\![\mathbf{let}\ v \equiv e\ \mathbf{in}\ \lambda v'.\, e']\!]\,\eta\,\kappa &= [\![e]\!]\,\eta\,\Big(\lambda a.\ [\![\lambda v'.\, e']\!]\,[\eta|v:a]\,\kappa\Big) \\
&= [\![e]\!]\,\eta\,\Big(\lambda a.\,\kappa\,(\psi\langle 2,\ (\lambda b.\,\lambda\kappa'.\ [\![e']\!]\,[\eta|v:a|v':b]\,\kappa')\rangle)\Big) \\
&= \Big(\lambda a.\,\kappa\,(\psi\langle 2,\ (\lambda b.\,\lambda\kappa'.\ [\![e']\!]\,[\eta|v:a|v':b]\,\kappa')\rangle)\Big)\ z \\
&= \kappa\,\Big(\psi\Big\langle 2,\ (\lambda b.\,\lambda\kappa'.\ [\![e']\!]\,[\eta|v:z|v':b]\,\kappa')\Big\rangle\Big).
\end{aligned}$$

Similarly, we simplify the right-hand side of the equation that we want to prove.

$$\begin{aligned}
[\![\lambda v'.\,\mathbf{let}\ v \equiv e\ \mathbf{in}\ e']\!]\,\eta\,\kappa &= \kappa\,\Big(\psi\Big\langle 2,\ (\lambda b.\,\lambda\kappa'.\ [\![\mathbf{let}\ v \equiv e\ \mathbf{in}\ e']\!]\,[\eta|v':b]\,\kappa')\Big\rangle\Big) \\
&= \kappa\,\Big(\psi\Big\langle 2,\ (\lambda b.\,\lambda\kappa'.\ [\![e]\!]\,[\eta|v':b]\,(\lambda a.\ [\![e']\!]\,[\eta|v':b|v:a]\,\kappa'))\Big\rangle\Big) \\
&= \kappa\,\Big(\psi\Big\langle 2,\ (\lambda b.\,\lambda\kappa'.\ [\![e]\!]\,\eta\,(\lambda a.\ [\![e']\!]\,[\eta|v:a|v':b]\,\kappa'))\Big\rangle\Big) \\
&= \kappa\,\Big(\psi\Big\langle 2,\ (\lambda b.\,\lambda\kappa'.\,(\lambda a.\ [\![e']\!]\,[\eta|v:a|v':b]\,\kappa')\ z)\Big\rangle\Big) \\
&= \kappa\,\Big(\psi\Big\langle 2,\ (\lambda b.\,\lambda\kappa'.\ [\![e']\!]\,[\eta|v:z|v':b]\,\kappa')\Big\rangle\Big).
\end{aligned}$$

In the third equation, we use two facts. First, v' does not occur free in e, so that $[\![e]\!]\,\eta = [\![e]\!]\,[\eta|v':b]$. Second, v and v' are different, so that $[\eta|v:a|v':b] = [\eta|v':b|v:a]$. Note that both simplifications lead to the same mathematical formula, which implies the desired equality.

[Part (d)] Here is a simple example:

$$e \equiv v', \qquad e' \equiv v.$$

Then,

$$\Big(\mathbf{let}\ v \equiv e\ \mathbf{in}\ \lambda v'.\, e'\Big) \equiv \Big(\mathbf{let}\ v \equiv v'\ \mathbf{in}\ \lambda v'.\, v\Big)$$

which means the same thing as $\lambda v''.\, v'$. But

$$\Big(\lambda v'.\,\mathbf{let}\ v \equiv e\ \mathbf{in}\ e'\Big) \equiv \Big(\lambda v'.\,\mathbf{let}\ v \equiv v'\ \mathbf{in}\ v\Big)$$

which means the same thing as $\lambda v''.\, v''$. Thus, they are different.