

# CS 520

## Theory of Programming Language

04/28 – 05/05, 2021

# Categorical Fixed-Point Theorem and Recursively Defined Domains.

1. Reminder. / Overview.

① 3-views on Categories, functors.

(1) (spaces, structure-pres. fns.)

\* (2) generalised partial order. (elements,  $\leq$ ), generalised mono. fns.

(3). (types, well-typed fns), type constructors, polymorphic fns.  
parametrically.

② Recursively defined domains. Why exist? How to build them?

$$\underline{\Omega} \simeq (\hat{\Sigma} + \mathbb{Z} \times \underline{\Omega} + (\mathbb{Z} \rightarrow \underline{\Omega}))_{\perp}$$

$$\hat{\Sigma} = \Sigma + \Sigma.$$

$F(\Omega)$

.....  $\Omega \simeq F(\Omega)$

↓  
special property of  $\Omega$ .

[  $\Omega \dots$  cont. initial alg. ]  
 $(-)_{\perp}, (-)_{+}$

③ Repeat the setup and proof/const. of the least fixed point thm in domain theory, but in the setting of category theory.

2. Key actors in our result. (at least fix. thm).

① Recall key actors in the LFT in domain theory. ② Key actors in the Cat. LFT.

(1)  $D \dots$  domain ( $D$  partially-ord. set,  $\perp$ , chain-complete).  
 $\mathcal{C} \dots$  category. ( $\mathcal{C}$  ...  $\omega$ -chain, co-cone,  $\omega$ -continuity, initial object, chain-completeness, colimit)

(2)  $f: D \rightarrow D \dots$  conti. ( $f$  monotone, preserves the lub of every chain).  
 $F \dots$  functor. ( $F$  ...  $\omega$ -continuous)

(3) (1)  $\wedge$  (2)  $\Rightarrow \exists$  the least fixed point  $x_{fix} \in D$  of  $f$ .

①  $f(x_{fix}) = x_{fix}$

②  $\forall y \in D. f(y) \sqsubseteq y \Rightarrow x_{fix} \sqsubseteq y$

$\exists$  an obj.  $x_{fix} \in \mathcal{C}$  s.t.

①  $\eta: F(x_{fix}) \rightarrow x_{fix}$  in  $\mathcal{C}$  s.t.

$\eta$  is an isomorphism.  $\rightarrow (\eta': x_{fix} \rightarrow F(x_{fix})$   
s.t.  $\eta' \circ \eta = id$

②  $\forall y \in \text{Obj}(\mathcal{C}) \forall \rho: F(y) \rightarrow y$  morph. in  $\mathcal{C}$   $\eta \circ \eta' = id$

$\exists$  a uniq. morphism  $h: x_{fix} \rightarrow y$  s.t.

$$\begin{array}{ccc} F(x_{fix}) & \xrightarrow{\eta} & x_{fix} \\ F(h) \downarrow & & \downarrow h \\ F(y) & \xrightarrow{\rho} & y \end{array} \quad \text{commutes.}$$

③ Instantiation for domain theory. ... embedding/proj. pair.

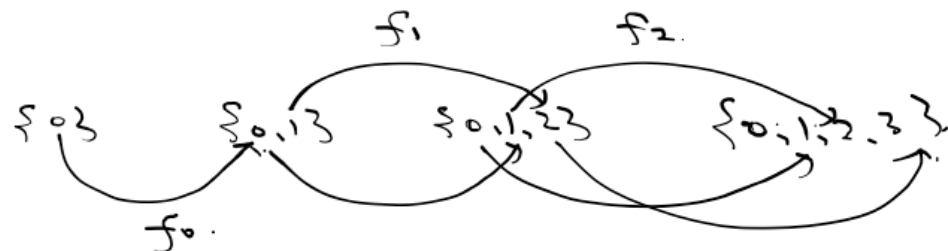
3.  $\omega$ -chain, co-cone, co-limiting co-cone (colimit).

$\mathcal{C}$  ... category.  $\bar{n} \in \mathcal{C}$ .

①  $\underbrace{\omega\text{-chain}}_{An} \triangle \bigvee_{i \in \mathbb{N}} B_i$ .  $\{ f_i: x_i \rightarrow x_{i+1} \}_{i=0,1,2,\dots}$  (--- a family of *countable* morphisms)

$$\Delta = x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} x_2 \xrightarrow{f_2} x_3 \rightarrow \dots$$

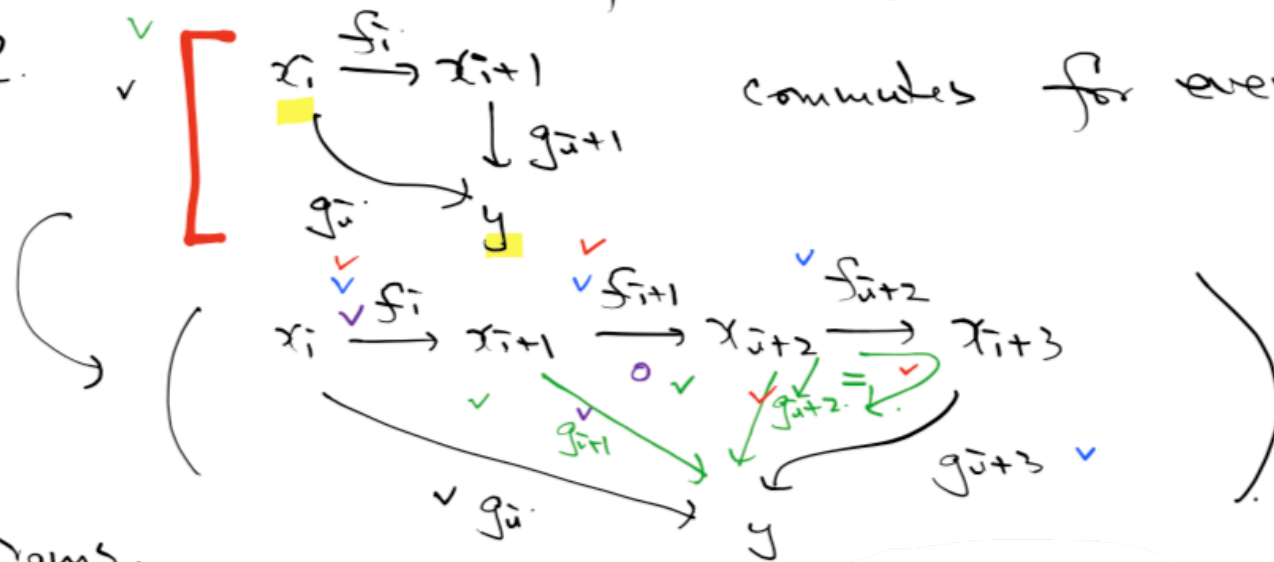
e.g. Cat.



...  $\omega$ -chain.

②. A co-cone of an  $\omega$ -chain  $\Delta = \{ \underline{f_i}: x_i \rightarrow x_{i+1} \}_{i \geq 0}$  is a pair of an object  $y$  and a family of morphisms  $\{ \underline{g_i}: x_i \rightarrow y \}_{i \geq 0}$  s.t.  $\cong \varphi$ .  $\checkmark$  commutes for every  $i$ .  $\star$

ex.: prove the below diagram commutes using the above diagrams.



② A co-cone  $(y, \{g_i: x_i \rightarrow y\}_i)$  of  $\Delta$  is co-limiting if [co-limit]  
 for all co-cones  $(z, \{h_i: x_i \rightarrow z\}_i)$  of  $\Delta$ ,  
 there exists a unique morphism  $k: y \rightarrow z$ . s.t



commutes for every  $i$ .

$$\left( \bigcup_{i=0}^{\infty} \{i\} \times x_i \right) / \sim$$

$$\begin{aligned}
 &\forall i \forall a \in x_i \forall b \in x_{i+1} \\
 &f_i(a) = b \Rightarrow (i, a) \sim (i+1, b) \\
 &f_0, f_1, \dots
 \end{aligned}$$

$$x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} x_2 \xrightarrow{f_2} \dots$$

What are co-limits of those chains?

- e.x (i) Inj. ... sets, injections.  
 (2) Set ... sur, fns.

4.  $\omega$ -cont. functors.

$\mathcal{C}, \mathcal{D}$  ... categories.

$F: \mathcal{C} \rightarrow \mathcal{D}$  is a functor.

①  $F$  is  $\omega$ -continuous if for every  $\omega$ -chain  $\Delta = \{f_i: x_i \rightarrow x_{i+1}\}_{i \geq 0}$  in  $\mathcal{C}$ ,  
for every  $\omega$ -limit  $(y, \{g_i: x_i \rightarrow y\}_{i \geq 0})$  of  $\Delta$  in  $\mathcal{C}$ ,

$(F(y), \{F(g_i): F(x_i) \rightarrow F(y)\}_{i \geq 0})$  is

a  $\omega$ -limit of  $F(\Delta)$  in  $\mathcal{D}$ .

$\{F(f_i): F(x_i) \rightarrow F(x_{i+1})\}_{i \geq 0}$ .

[  $\omega$ -limits of  $\omega$ -chains  $\Delta$  in  $\mathcal{C}$  get mapped to  $\omega$ -limits of  $\omega$ -chains in  $\mathcal{D}$  by  $F$  ].



②  $F \dots$  functor.

$$\Delta = \begin{array}{ccccccc} x_0 & \xrightarrow{f_0} & x_1 & \xrightarrow{f_1} & x_2 & \xrightarrow{f_2} & \dots \\ & & \searrow g_1 & & \searrow g_2 & & \\ & & & & & & y \\ & \searrow g_0 & & & & & \end{array}$$

$\dots$  co-cone in  $\mathcal{C}$ .

$$F(\Delta) = \begin{array}{ccccccc} F(x_0) & \xrightarrow{F(f_0)} & F(x_1) & \xrightarrow{F(f_1)} & F(x_2) & \xrightarrow{F(f_2)} & \dots \\ & & \searrow F(g_1) & & \searrow F(g_2) & & \\ & & & & & & F(y) \\ & \searrow F(g_0) & & & & & \end{array}$$

$\dots$  co-cone in  $\mathcal{D}$ .

Domain Theory:

$$a_0 \sqsubseteq a_1 \sqsubseteq a_2 \sqsubseteq \dots$$

$\sqcup$        $\prod$        $\dots$   
 $b$

$$f(a_0) \sqsubseteq f(a_1) \sqsubseteq \dots$$

$\sqcup$        $\prod$   
 $f(b)$