CS 520

Theory of Programming Language

03/17 – 03/31, 2021

Chap 2 : The simple Imperative Language.

1. Overview / Motivation.

① Many high-level design principles for PL.

... imperative comp. , applicative comp. , OO principle , ...

state ..... variables.

computation proceeds by read / write of variables.

function application as a main mechanism for computation.

② $y := 0$; $x := 0$; $\big(\text{while } (x < n) \text{ do } (y := y + x); (x := x + 1)\big)$ ... example

(while true do skip)

③ Domain theory ... denotational semantics in the presence of while.

Syntactic sugar.

Quality of denotational semantics. (soundness, full abstraction, adequacy)

2. Syntax. ..... abstract syntax, abstract grammar.

$$\langle intexp \rangle ::= \ldots \mid \langle var \rangle \mid \ldots$$

$$\langle boolexp \rangle ::= \ldots \mid \langle intexp \rangle = \langle intexp \rangle \mid \neg \langle boolexp \rangle \mid \ldots$$

$$\langle comm \rangle ::= \textbf{skip} \mid \langle var \rangle := \langle intexp \rangle \mid \langle comm \rangle ; \langle comm \rangle$$

$$\mid \textbf{if}\ \langle boolexp \rangle\ \textbf{then}\ \langle comm \rangle\ \textbf{else}\ \langle comm \rangle$$

$$\mid \textbf{while}\ \langle boolexp \rangle\ \textbf{do}\ \langle comm \rangle.$$

Syntax-directed def. structural induction all available because we are initial algebra / finite derivation trees.

3. Denotational semantics ... What is a potential issue?

$[\![-]\!] : \langle comm \rangle \to S$

$[\![\text{while } b \text{ do } c]\!] = [\![\text{if } b \text{ then } (c; \text{while } b \text{ do } c) \text{ else skip}]\!]$.

$= \underbrace{\ldots\ldots}\ [\![\text{while } b \text{ do } c]\!]\ \underbrace{\ldots\ldots}\ [\![\text{while } b \text{ do } c]\!]\ \underbrace{\ldots}$

$= F([\![\text{while } b \text{ do } c]\!])$.

" is a fixed point of the function F.

can be defined by a program.

domain theory. ... approxi. theory of computability that focuses on the limit of communication. (in order to produce a finite amount of inf. in the output, a program only needs a finite amount of info. in the input.)

$g : \mathbb{N} \to \mathbb{N}$

$g(x) = x+1$. ... no fixed point.

4. Domain theory. .... when does a function F have a fixed point?

① High level heuristic.

- Set with some structure, $\sqsubseteq$ ..... approx. order.
- Functions between such sets that preserve this structure.

② Definitions. .... structure.

<u>Def.</u> A partial order on a set S is a binary relation $\sqsubseteq$ on S s.t. $(\sqsubseteq \subseteq S \times S)$

1) (reflexive) $x \sqsubseteq x$ for all $x \in S$;
   $(x, x) \in \sqsubseteq$.
2) (transitive) $x \sqsubseteq y \wedge y \sqsubseteq z \Rightarrow x \sqsubseteq z$ for all $x, y, z \in S$;
3) (anti-symmetry) $x \sqsubseteq y \wedge y \sqsubseteq x \Rightarrow x = y$ for all $x, y \in S$.

A set S with a partial order $\sqsubseteq$ is called poset or partially ordered set.

examples. (a) $\{\bot, \top\}$, $\bot \sqsubseteq \bot$, $\top \sqsubseteq \top$, $\bot \sqsubseteq \top$ ......

$\top$
$|$
$\bot$

↳ domain

(b) $\mathbb{N}^\top = $ $\top$ ⋮ 1 2 1 1 1 0

$\left(\begin{array}{l} x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \cdots \\ \bot \sqsubseteq \bot \sqsubseteq \bot \sqsubseteq \cdots \end{array} \right)$

$\underbrace{0 \sqsubseteq 1 \sqsubseteq 2 \sqsubseteq \cdots}_{\top}$

$(3 \sqsubseteq 3 \sqsubseteq 3 \sqsubseteq \cdots)$ $\top$ 3

$3 \not\sqsubseteq 4$

predomain

not domain

(c) $\mathbb{Z} = \ldots \; -3 \; -2 \; -1 \; 0 \; 1 \; 2 \; \ldots$

$(x \sqsubseteq y$ iff $x = y)$

$\left[\begin{array}{l} x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \cdots \quad \ldots \; lub = x_0 \\ x_0 = x_1 = x_2 = x_3 \; \cdots \end{array} \right]$

.... discrete partial order.

(d) $\mathbb{Z}_\bot = \ldots \; -3 \; -2 \; -1 \; 0 \; 1 \; 2 \; \ldots$

$\bot$

domain

$\langle\rangle \sqsubseteq \langle 1\rangle \sqsubseteq \langle 1,2\rangle \sqsubseteq \langle 1,2,3\rangle \sqsubseteq \cdots$

$\langle 1,2,3,4,\cdots\rangle$

(e) $\mathbb{Z}^{*\,\bullet\,\omega} = \{ \langle\rangle, \langle 1\rangle, \langle 2,3\rangle, \langle 4,1,2\rangle, \ldots \}$

domain

$\alpha \sqsubseteq \beta$ iff α is a prefix of β.

$\underline{\langle 1,2\rangle} \sqsubseteq \langle \underline{1,2},3,1\rangle$ $\qquad \langle 1,2\rangle \not\sqsubseteq \langle 2,3,1\rangle$

domain

ϕ

(f) $2^{\mathbb{Z}} (= P(\mathbb{Z})) = \{ \{1,2\}, \{-1,-4\}, \mathbb{Z}, \text{odd}, \ldots \}$

$x_0 \sqsubseteq x_1 \sqsubseteq \cdots$

$\bigsqcup_n x_n = \bigcup_n x_n$

$A \sqsubseteq B$ iff $A \subseteq B$.

Ex ① which one is a pre-domain? which one is a domain?

② find a way to convert it to a predomain or domain quickly.

* method for checking whether $(S, \sqsubseteq)$ is a predomain

.... non-existence of interesting chain

$\left[x_1 \sqsubseteq x_2 \sqsubseteq x_3 \sqsubseteq \underbrace{x_4 \sqsubseteq x_4 \sqsubseteq x_4 \cdots}_{x_4} \right]$

Def. A poset $(S, \sqsubseteq)$ is a predomain if every chain in S has $\boxed{\text{the}}$ least upper bound.

$(P, \sqsubseteq)$

$$\left[\begin{array}{l} (x_n)_{n\in\mathbb{N}} = (x_0, x_1, x_2, \cdots) \quad (x_0 \in S) \\ \text{s.t.} \quad x_0 \sqsubseteq x_1 \sqsubseteq x_2 \cdots \end{array}\right]$$

Key players.

$$\bigsqcup_{n\in\mathbb{N}} x_n = \bigsqcup_{n=0}^{\infty} x_n = y \in S \text{ s.t. }$$

upper bound.

1). $x_n \sqsubseteq y$ for all n.

2). $\forall y' \in S$. if $x_n \sqsubseteq y'$ for all n, then $y \sqsubseteq y'$.

least.

A predomain $(S, \sqsubseteq)$ is a domain $(D, \sqsubseteq)$ if it has the least element, which we denote by $\bot$.

$(\bot \sqsubseteq x$ for all $x \in S.)$

<u>Def</u> $(P_1, \sqsubseteq)$, $(P_2, \sqsubseteq)$ ... predomains.

$f : P_1 \longrightarrow P_2$.

① f is monotone if it preserves the $\sqsubseteq$ relation.

$$\left(\forall x_1, x_2 \in P_1.\quad x_1 \sqsubseteq_{P_1} x_2 \Rightarrow f(x_1) \sqsubseteq_{P_2} f(x_2) \right)$$

② When f is monotone, we say that f is continuous if.

it preserves the least upper bound of every chain in P_1

($\forall$ chain $(x_n)_n$ in P_1 ... $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \cdots$

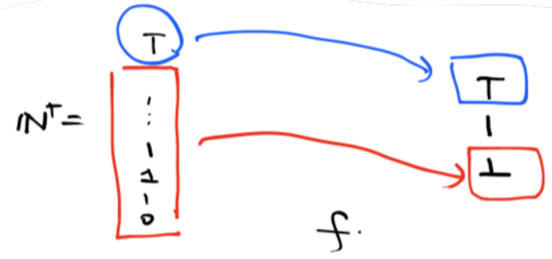
the chain $(f(x_n))_n$ in P_2 has $f(\bigsqcup_n x_n)$ as its least upper bound.

$$f(\bigsqcup_n x_n) = \bigsqcup_n f(x_n)$$)

③ Suppose $(P_1, \sqsubseteq)$, $(P_2, \sqsubseteq)$ are domains.
$\qquad\qquad\quad \bot_1 \qquad\quad \bot_2$.

A function $f : P_1 \to P_2$ is strict if $f(\bot_1) = \bot_2$.

e.g.



f is not continuous.

$x_0 = 0,\ x_1 = 2,\ x_2 = 4,\ x_3 = 6,\ \ldots$

$\bigsqcup_n x_n = \top \qquad f(\bigsqcup_n x_n) = f(\top) = \top$

$f(x_n) = f(2n) = \bot. \qquad \bigsqcup_n f(x_n) = \bigsqcup_n \bot$

$= \underline{\bot}$

$$\mathbb{Z}_\bot = \begin{array}{c} \cdots\ -1\ \ 0\ \ 1\ \cdots \\ \bot \end{array} \xrightarrow{f \ \text{monotone}} P \ \text{predomain}$$

$\rightsquigarrow$ f is automatically cont.

$$f(\underbrace{\textstyle\bigsqcup_n x_n}) \overset{?}{=} \bigsqcup_n f(x_n).$$

absence of interesting chains. $\longrightarrow$ $\parallel$

$$f(x_{m_0}). \ \sqsubseteq \ \bigsqcup_n f(x_n)$$

$\uparrow$ mono.

$$f(\bigsqcup_n x_n) \ \sqsupseteq \ \bigsqcup_n f(x_n)$$

$\uparrow$ true for all mono. fns.

(f.)

$$\left(\begin{array}{l} x_m \sqsubseteq \bigsqcup_n x_n. \\ f(x_m) \sqsubseteq \underbrace{f(\bigsqcup_n x_n)}_{\text{upper bound.}} \quad \text{for all } m \\ \bigsqcup_m f(x_m) \sqsubseteq f(\bigsqcup_n x_n). \end{array} \right)$$

Thm [Least Fixed Point Thm].

$(D, \sqsubseteq, \bot)$ ... domain

$f : D \to D$ ... continuous fn.

$\Rightarrow \exists x \in D$ s.t. 1) $f(x) = x$

2) $\forall y.\ f(y) = y \Rightarrow x \sqsubseteq y.$

x is the least fixed point of f.

Proof. We construct x as follows.

① $\underbrace{\bot \sqsubseteq f(\bot)}_{\text{true}} \sqsubseteq f^2(\bot) \sqsubseteq \cdots \cdots$ … chain. $(f^n(\bot))_n$. … $\bot$.

f is mono. $f(\bot) \sqsubseteq f(f(\bot))$

② least upper bound $x = \bigsqcup_n f^n(\bot)$ lub. $(f^{n+1}(\bot))_n$

Is x a fixed point? Yes.

$$f(x) = f\left(\bigsqcup_{n=0}^{\infty} f^n(\bot)\right) = \bigsqcup_{n=0}^{\infty} f(f^n(\bot)) = \bigsqcup_{n=0}^{\infty} f^{n+1}(\bot) = \bigsqcup_{n=0}^{\infty} f^n(\bot) = x.$$

least? Pick $y \in D$ s.t. $f(y) = y$.

y is an upper bound of $(f^n(\bot))_n$.

$\bot \sqsubseteq y$.

$f(\bot) \sqsubseteq f(y) = y$.

$f^2(\bot) \sqsubseteq f(y) = y$.

Since x is the least upp. bound of the chain:

$x \sqsubseteq y$.

$\square$