CS 520

Theory of Programming Language

04/28 – 05/05, 2021

Categorical Fixed-Point Theorem and Recursively Defined Domains.

1. Reminder / Overview.

① 3-views on Categories, functors.

(1) (spaces, structure-pres. fns.)

(2) generalised Partial order. (elements, $\sqsubseteq$). , generalised mono. fns.

(3). (types, well-typed fns), type constructors, polymorphic fns.

parametrically.

(2) Recursively defined domains. Why exist? How to build them?

$$\Omega \cong \underbrace{(\hat{\Sigma} + \mathbb{Z} \times \Omega + (\mathbb{Z} \to \Omega))_\perp}_{F(\Omega)} \quad \cdots\cdots \quad \Omega \cong F(\Omega)$$

$$\hat{\Sigma} = \Sigma + \Sigma.$$

special property of Ω.

[Ω ... cont. initial alg. $(-)_\amalg$, $(-)_+$]

(3) Repeat the setup and proof/const. of the least fixed point thm in domain theory, but in the setting of category theory.

2. Key actors in our result (cat. least fix. thm).

① Recall key actors in the LFT in domain theory. / ② Key actors in the Cat. LFT.

(1) D .... domain (D partially-ord. set, $\bot$, chain-complete).

$\mathcal{C}$ .... category (✓ , initial object, chain-completeness) ω-chain, co-cone, colimiting co-cone, colimit

(2) $f: D \to D$ ... conti. (f monotone, preserves the lub of every chain).

F .... functor ((pre). ✓ , ω-continuous)

(3) (1) $\wedge$ (2) $\Rightarrow$ $\exists$ the least fixed point $x_{fix} \in D$ of f.

ⓐ $f(x_{fix}) = x_{fix}$

ⓑ $\forall y \in D.\ f(y) \sqsubseteq y \Rightarrow x_{fix} \sqsubseteq y$ ✓

$\exists$ an obj. $x_{fix} \in \mathcal{C}$ s.t.

ⓐ $\eta : F(x_{fix}) \longrightarrow x_{fix}$ in $\mathcal{C}$ s.t.

η is an isomorphism. ($\eta' : x_{fix} \to F(x_{fix})$ s.t. $\eta' \circ \eta = id$, $\eta \circ \eta' = id$)

ⓑ $\forall y \in Obj(\mathcal{C})\ \forall \rho : F(y) \to y$ ...morph. in $\mathcal{C}$ ✓

$\exists$ a uniq. morphism $h : x_{fix} \to y$ s.t.

*
$$\begin{tikzcd} F(x_{fix}) \arrow[r, "\eta"] \arrow[d, "F(h)"'] & x_{fix} \arrow[d, "h"] \\ F(y) \arrow[r, "\rho"] & y \end{tikzcd}$$
commutes.

③ Instantiation for domain theory ... embedding/proj. pair.

3. ω-chain, co-cone, co-limiting co-cone (colimit).

$\mathcal{C}$ … category. in $\mathcal{C}$

① An ω-chain Δ is $\{ f_i : x_i \to x_{i+1} \}_{i=0,1,2,\cdots}$ (… a countable family of morphisms)

$$\Delta = \quad x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} x_2 \xrightarrow{f_2} x_3 \to \quad \cdots\cdots$$

e.g. Cat.

$$\{0\} \xrightarrow{f_0} \{0,1\} \xrightarrow{f_1} \{0,1,2\} \xrightarrow{f_2} \{0,1,2,3\} \quad \cdots \; \omega\text{-chain.}$$

② A <u>co-cone</u> of an ω-chain $\Delta = \{ f_i : x_i \to x_{i+1} \}_{i \geq 0}$ is a pair of an object y and a family of morphisms $\{ g_i : x_i \to y \}_{i \geq 0}$ s.t.

$\in \mathcal{C}$.

$$\begin{tikzcd} x_i \arrow[r, "f_i"] \arrow[dr, bend right, "g_i"'] & x_{i+1} \arrow[d, "g_{i+1}"] \\ & y \end{tikzcd}$$

commutes for every i. ☆

e.x: prove the below diagram commutes using the above diagrams.

$$\begin{tikzcd} x_i \arrow[r, "f_i"] \arrow[drr, bend right, "g_i"'] & x_{i+1} \arrow[r, "f_{i+1}"] \arrow[dr, "g_{i+1}"] & x_{i+2} \arrow[r, "f_{i+2}"] \arrow[d, "g_{i+2}"] & x_{i+3} \arrow[dl, bend left, "g_{i+3}"] \\ & & y & \end{tikzcd}$$

[co-limit]

③ A co-cone $(y, \{g_i : x_i \to y\}_i)$ of Δ is <u>co-limiting</u> if

for all co-cones $(z, \{h_i : x_i \to z\}_i)$ of Δ,

there exists a unique morphism $k : y \to z$ s.t

$$\begin{tikzcd} x_i \arrow[r, "g_i"] \arrow[dr, bend right, "h_i"'] & y \arrow[d, "k"] \\ & z \end{tikzcd}$$

commutes for every i.

$\forall i \, \forall a \in x_i \, \forall b \in x_{i+1}$

$f_i(a) = b \Rightarrow (i, a) \sim (i+1, b)$

$\left(\bigcup_{i=0}^{\infty} \{i\} \times x_i \right) / \sim$

$f_0, f_1, \dots$

e.x (1). Inj … sets, injections. $x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} x_2 \xrightarrow{f_2} \cdots$

(2) Set . . sets, fns.

What are co-limits of those chains?

4. ω-cont. functors.

$\mathcal{C}, \mathcal{D}$ ... categories. $F : \mathcal{C} \to \mathcal{D}$ is a functor.

① F is ω-continuous if for every ω-chain $\Delta = \{f_i : x_i \to x_{i+1}\}_{i \geq 0}$ in $\mathcal{C}$, for every co-limit $(y, \{g_i : x_i \to y\}_{i \geq 0})$ of Δ in $\mathcal{C}$,

$(F(y), \{F(g_i) : F(x_i) \to F(y)\}_{i \geq 0})$ is

a co-limit of $\underbrace{F(\Delta)}$ in $\mathcal{D}$.

$\{F(f_i) : F(x_i) \to F(x_{i+1})\}_{i \geq 0}$

[co-limits of ω-chains in $\mathcal{C}$ get mapped to co-limits of ω-chains in $\mathcal{D}$ by F].

② F ... functor.

$$\Delta = \begin{tikzcd} x_0 \arrow[r, "f_0"] \arrow[drr, bend right, "g_0"'] & x_1 \arrow[r, "f_1"] \arrow[dr, "g_1"] & x_2 \arrow[r, "f_2"] \arrow[d, "g_2"] & \cdots \\ & & y & \end{tikzcd}$$

.... co-cone in $\mathcal{C}$.

$$F(\Delta) = \begin{tikzcd} F(x_0) \arrow[r, "F(f_0)"] \arrow[drr, bend right, "F(g_0)"'] & F(x_1) \arrow[r, "F(f_1)"] \arrow[dr, "F(g_1)"] & F(x_2) \arrow[r, "F(f_2)"] \arrow[d, "F(g_2)"] & \cdots \\ & & F(y) & \end{tikzcd}$$

.... co-cone in $\mathcal{D}$.

⌜Domain theory: $a_0 \sqsubseteq a_1 \sqsubseteq a_2 \sqsubseteq \cdots$ $\qquad f(a_0) \sqsubseteq f(a_1) \sqsubseteq \cdots$

$\sqsubseteq \quad \sqcap\!\!\mid \quad \cdots \qquad\qquad \sqsubseteq \quad \sqcap\!\!\mid$

$b \qquad\qquad\qquad f(b)$ ⌟