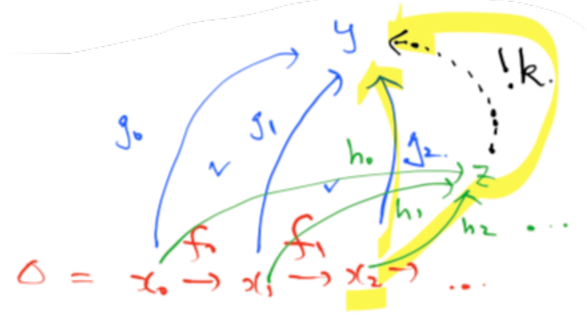
CS 520
Theory of Programming Language

05/05 – 05/12, 2021

Categorical Fixed-Point Thm and Instantiation with $\mathbf{Dom}^{EP}$

1. Side remark.

$$D \cong [D \to D] + \dots \qquad \dots \qquad F(D) \cong D$$



Categories. initial objects. ω-chains.

co-cone.
co-limit.

2. Reminder.

① General strategy — Generalise what we did for fixed-point thm in domain theory.

(1) Domains (PO sets with $\bot$, chain complete).

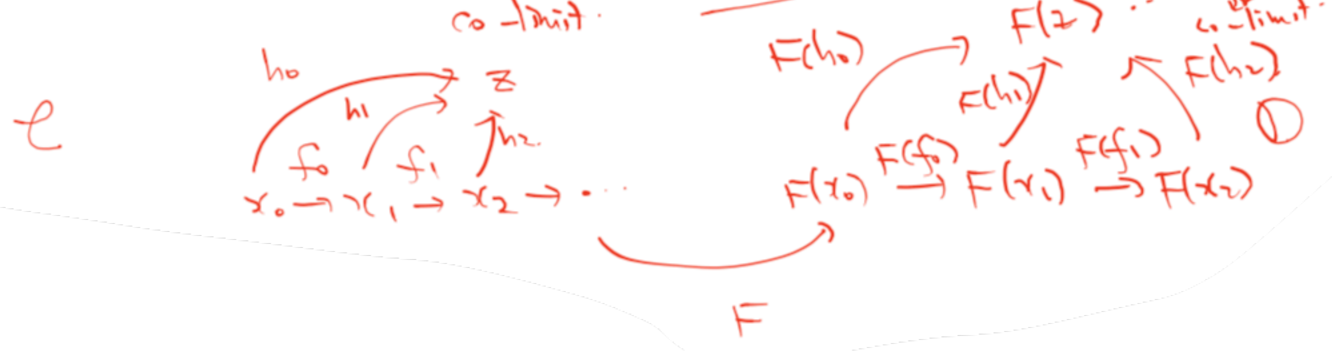
(2) Cont. fns. (mono. fns that preserve lubs of chains).

(3) $\exists$ least pre fixed point.

functors.

preservation of co-limits of ω-chains.

ω-cont functors.



3. Categorical Least Fixed-Point Thm.

[Thm]. $\mathcal{C}$ ..... category with initial object and chain-complete.

$F : \mathcal{C} \to \mathcal{C}$ .... ω-conti. functor.

$\Rightarrow$ $\exists$ an object x_{fix} in $\mathcal{C}$ and a morphism $\eta : F(x_{fix}) \to x_{fix}$ in $\mathcal{C}$ s.t. ... $F(0)$

(1) $\exists\, p : x_{fix} \to F(x_{fix})$ morphism in $\mathcal{C}$ s.t. ... 0

$p \circ \eta = id_{F(x_{fix})}$ and $\eta \circ p = id_{x_{fix}}$

$\begin{bmatrix} 0 \\ F(0) \end{bmatrix}$

exercise. (2) $\forall\, \eta' : F(y) \to y$ morphism in $\mathcal{C}$

$\exists !\, h : x_{fix} \to y$ morphism in $\mathcal{C}$ s.t.

$$\begin{array}{ccc} F(x_{fix}) & \xrightarrow{\eta} & x_{fix} \\ {\scriptstyle F(h)}\downarrow & & \downarrow {\scriptstyle h} \\ F(y) & \xrightarrow{\eta'} & y \end{array}$$

commutes.

$\begin{bmatrix} \text{initiality} \\ (-)_{\perp},\ (-)_{+} \\ \Omega \end{bmatrix}$

(1) Reminder of the proof of LFT in domain theory.

D ... domain, $f : D \to D$ ... cont. fn.

(1) How to construct the least fixed point of f?

$\bot \sqsubseteq f(\bot) \sqsubseteq f^2(\bot) \sqsubseteq f^3(\bot) \sqsubseteq \cdots \cdots \to$ lub ... x_{fix}

(2) Given y s.t. $f(y) \sqsubseteq y$. ($x_{fix} \sqsubseteq y$ ... we should show it).

$\bot \sqsubseteq y$

$f(\bot) \sqsubseteq f(y) \sqsubseteq y$

$f^2(\bot) \sqsubseteq f(y) \sqsubseteq y$ $\cdots \to$ y is upper bound of the chain.

$x_{fix} \sqsubseteq y$. $\square$

② Proof of categorical fixed-point thm.

(1) How to build $x_{fix} \in \mathrm{Obj}(\mathcal{C})$?

ω-chain $= \triangle$ in $\mathcal{C}$.

$$
\begin{tikzcd}
x_0 \arrow[r, "g_0"] \arrow[drr, bend right, "h_0"'] & F(x_0) \arrow[r, "F(g_0)"] \arrow[dr, "h_1"'] & F^2(x_0) \arrow[r, "F^2(g_0)"] \arrow[d, "h_2"] & F^3(x_0) \arrow[r] \arrow[dl, "h_3"] & \cdots \\
 & & x_{fix} & \cdots &
\end{tikzcd}
$$

$\uparrow$ initial object in $\mathcal{C}$.

x_{fix} ... colimit.

(2) Given. $\eta' : F(y) \to y \in \mathrm{Mor}(\mathcal{C})$. To do: Construct $k : x_{fix} \to y$.

$$
\begin{tikzcd}
x_0 \arrow[r, "g_0"] \arrow[d, "a_0"'] & F(x_0) \arrow[d, "F(a_0)"] \\
y & F(y) \arrow[l, "\eta'"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F(x_0) \arrow[r, "F(g_0)"] \arrow[d, "F(a_0)"'] & F^2(x_0) \arrow[d, "F^2(a_0)"] \\
F(y) & F^2(y) \arrow[l, "F(\eta')"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
x_0 \arrow[r, "g_0"] \arrow[d, "a_0"'] & F(x_0) \arrow[r, "F(g_0)"] \arrow[d, "F(a_0)"] & F^2(x_0) \arrow[r, "F^2(g_0)"] \arrow[d, "F^2(a_0)"] & F^3(x_0) \arrow[r] \arrow[d, "F^3(a_0)"] & {} \arrow[d] & \cdots \\
y & F(y) \arrow[l, "\eta'"] & F^2(y) \arrow[l, "F(\eta')"] & F^3(y) \arrow[l, "F^2(\eta')"] & {} \arrow[l] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
x_0 \arrow[r, "g_0"] \arrow[dr] & F(x_0) \arrow[r, "F(g_0)"] \arrow[d, "\eta' \circ F(a_0)"] & F^2(x_0) \arrow[r, "F^2(g_0)"] \arrow[dl, bend left, "\eta' \circ F(\eta') \circ F^2(a_0)"] & F^3(x_0) \arrow[dll, bend left, "\eta' \circ F(\eta') \circ F^2(\eta') \circ F^3(a_0)"] \\
& y & &
\end{tikzcd}
$$

$(y, \ldots)$ co-cone.

(3) How to build $\eta : F(x_{fix}) \to x_{fix}$ and $\rho : x_{fix} \to F(x_{fix})$?

$$\Delta = \begin{tikzcd} x_0 \arrow[r, "g_0"] \arrow[drr, bend right, "h_0"'] & F(x_0) \arrow[r, "F(g_0)"] \arrow[dr, "h_1"] & F^2(x_0) \arrow[r] \arrow[d, "h_2"] & \cdots \\ & & x_{fix} & \cdots \end{tikzcd}$$

$$F(\Delta) = \begin{tikzcd} F(x_0) \arrow[r, "F(g_0)"] \arrow[drr, bend right, "F(h_0)"'] \arrow[ddrr, bend right, "h_1"'] & F^2(x_0) \arrow[r, "F^2(g_0)"] \arrow[dr, "F(h_1)"] \arrow[ddr, bend right, "h_2"'] & F^3(x_0) \arrow[r] \arrow[d, "F(h_2)"] \arrow[dd, bend left, "h_3"] & \cdots \\ & & F(x_{fix}) \arrow[d, dashed, "\eta"] & \\ & & x_{fix} & \cdots \end{tikzcd}$$

$\cdots$ co-limit ($\because$ ω-cont. of F)

co-cone

$$\begin{tikzcd} x_0 \arrow[r, "g_0"] \arrow[drrr, bend right, "k"'] & F(x_0) \arrow[r, "F(g_0)"] \arrow[drr, bend right, "F(h_0)"'] & F^2(x_0) \arrow[r, "F^2(g_0)"] \arrow[dr, "F(h_1)"] & F^3(x_0) \arrow[r] \arrow[d, "F(h_2)"] & {} \\ & & & F(x_{fix}) & \\ & & & x_{fix} \arrow[u, dashed, "\rho"'] & \end{tikzcd}$$

co-cone of Δ.

initiality of x_0 $\cdots$ k.

4. Instantiation with $\underset{\sim}{\mathrm{Dom}}^{EP}$.   (Set, Predom, Dom).

$\vdots$

Ω.

① $[D \to D]$ ... doesn't lead to a functor on Dom.

$f : D \to D'$ ...   we have a pb. for defining morphism action.

$F(f) : [D \to D] \to [D' \to D']$ ... not easy to define.

② Key idea of $\mathbf{Dom}^{EP}$ ... think about special kind of morphisms.

(1) objects ... domains.

(2) morphisms $f : D \to D'$ are strict (⊥-preserving) contin fns from D to D' embeddings.

embedding. s.t. $\exists\ g : D' \to D$ with: proj.

(1)
$$\begin{tikzcd} D \arrow[r, "f"] \arrow[dr, "id_D"'] & D' \arrow[d, "g"] \\ & D \end{tikzcd}$$
commutes.

and (2)
$$\begin{tikzcd} D' \arrow[r, "g"] \arrow[dr, "id_{D'}"'] & D \arrow[d, "f"] \arrow[dl, phantom, "\sqsupseteq"] \\ & D' \end{tikzcd}$$
$g \circ f \sqsubseteq id_{D'}$

g is the proj. for f.

③ $F : \mathbf{Dom}^{EP} \to \mathbf{Dom}^{EP}$

$F(D) = [D \to_c D]$

$F(f) : [D \to_c D] \to [D' \to_c D']$

$(f : D \to D')$ $\quad F(f)(k) = D' \xrightarrow{g} D \xrightarrow{k} D \xrightarrow{f} D' = f \circ k \circ g$.