CS 520

Theory of Programming Language

04/19 – 04/28, 2021

An Introduction to Category Theory (Chap 8 of Tennent's book)

1. Motivation.

① Impact of CT on PL ..... (a) Lang. design. Scala, Rust, Haskell, Java. (Monad, Functor, Generics ...)

CT

(b). Research on Semantics.

- concepts derived from CT.
- Allows us to construct complex domains.

output ... nontermination.

$$\Omega \simeq (\hat{\Sigma} + \mathbb{Z} \times \Omega + [\mathbb{Z} \to \Omega])_{\perp}$$

input.

$$\Sigma = [\langle var \rangle \to \mathbb{Z}]$$

$$\hat{\Sigma} = \Sigma + \Sigma$$

normal term. abnormal term.

$$D \simeq [[D \to_c D] + \mathbb{Z} \; \ldots]$$

..... one of big achievements in semantics research.

(Any set X. $\not\simeq (X \to X) \cup \ldots$)

2. Definition of Category.

- objects , morphisms.

intuition ① spaces. — structure-preserving fns between spaces.

② types. — functions in a PL.

example.

sets. $X, Y, \cdots$ — functions. $f: X \to Y$, $g: Y \to Z$.

vector spaces $\mathbb{R}^n$, X, Y — linear maps.

Def. A category is a tuple $(\underset{\text{capital}}{\overset{\mathrm{Obj}}{\mathrm{obj}}}, \mathrm{Hom}, \circ, \mathrm{id})$ s.t.

① obj is a collection (whose elements are called <u>objects</u>),

② $\forall x, y \in \mathrm{obj}$, $\mathrm{Hom}[x, y]$ is a collection (whose elements are called <u>morphisms</u> <u>from x to y</u>),

③ $\forall x, y, z \in \mathrm{obj}$,

$\circ$ ..... $\circ_{x,y,z} : \mathrm{Hom}[y, z] \times \mathrm{Hom}[x, y] \longrightarrow \mathrm{Hom}[x, z]$ is a fn. on morphisms,

④ $\forall x \in \mathrm{obj}$,

id ..... $\exists\ \mathrm{id}_x \in \mathrm{Hom}[x, x]$, and

⑤ the above data satisfy associativity and identity conditions:

$\forall x, y \in \mathrm{obj}$ $\forall f \in \mathrm{Hom}[x, y]$ ($f : x \to y$) $\quad f \circ \mathrm{id}_x = f = \mathrm{id}_y \circ f$. [identity].

$\forall x, y, u, v \in \mathrm{obj}$ $\forall f : x \to y,\ g : y \to u,\ h : u \to v$. [associativity].

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

E.g. ① **Set** ... category of sets. Obj = a collection of all sets. (small sets)

Hom[x, y] = a collection of all fns from x to y

$\circ$ ... function composition , id_x ... identity fn on x

② **Predom** ... category of predomains.

- objects ... predomains.
- morphisms ... continuous fns.
- $\circ$ ... fn comp. id ... identity fn.

Dom ... category of domains.

... domains.

" ... } all same.

...

Dom. ... category of domains with strict conti. fns.

Obj ... domains.

morphisms ... strict conti. fns.

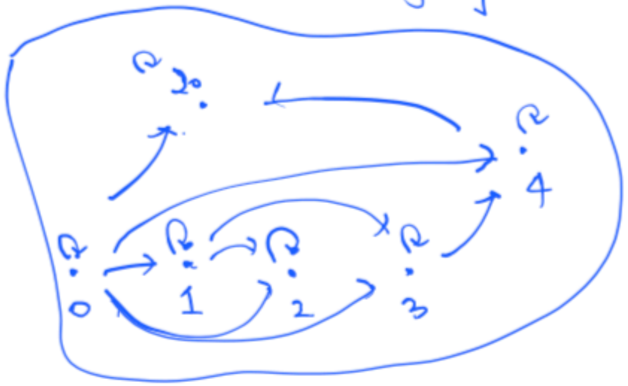
... same.

③ Given a partially ordered set $(P, \sqsubseteq)$.

$(P, \sqsubseteq)$ can be viewed as a category : objects ... elements of P.

$$\mathrm{Hom}[x,y] = \begin{cases} \{*\}. & \text{if } x \sqsubseteq y \\ \phi. & \text{otherwise.} \end{cases}$$

$(\mathbb{N}, \leq) \to$ category



$id_x \in \mathrm{Hom}[x,x]$. ... $Id_x = *$ (why ok? $\sqsubseteq$ reflexive. $x \sqsubseteq x$. $\therefore \mathrm{Hom}[x,x] = \{*\}$)

$$\overset{\phi\ =\ \{*\}\ =\ \text{or}\ \{*\}\ =\ \phi}{\circ : \mathrm{Hom}[y,z] \times \mathrm{Hom}[x,y]} \longrightarrow \underset{\sim}{\mathrm{Hom}[x,z]}.$$

if $y \sqsubseteq z$ and $x \sqsubseteq y$, $(x \sqsubseteq z)$ $\{*\}$. $\circ$ .... constant fn. $\circ(*,*) = *$.

$k : X \to Y$ but $X = \phi$. k is the empty fn. $C_{\{\}}$.

[if $\neg(y \sqsubseteq z$ and $x \sqsubseteq y)$, $\mathrm{Hom}[y,z] \times \mathrm{Hom}[x,y] = \phi$.
$\circ$ .... empty fn.

3. Notation: ① $f : x \to y$ or $x \xrightarrow{f} y$ to mean $f \in \mathrm{Hom}[x, y]$.

② commutative diagram

$$
\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, "h"'] & y \arrow[d, "g"] \\
u \arrow[r, "k"'] & v
\end{tikzcd}
$$

..... * $g \circ f = k \circ h$

$$
\begin{tikzcd}
x \arrow[r, "f"] \arrow[dr, "k"'] \arrow[dr, bend right=60] & y \arrow[d, "g"] \arrow[d, bend left=60, <-, phantom] \\
 & v
\end{tikzcd}
$$

.... $g \circ f = k$

4. Terminal objects, initial objects, products of two objects, Co-products of two objects.

① names given to objects in a cat. that satisfy some nice properties.
wellknown

related to constructions that can be carried out in a category.

② $\mathcal{C}$ … category. $x \in \mathrm{Obj}(\mathcal{C})$

[Def] x is an initial object if $\forall y \in \mathrm{Obj}(\mathcal{C})$, there exists a unique morphism from x to y (i.e., $|\mathrm{Hom}[x,y]| = 1$).

[Def] x is a final object if $\forall y \in \mathrm{Obj}(\mathcal{C})$, there exists a unique morphism from y to x. (i.e. $|\mathrm{Hom}[y,x]| = 1$).

because the empty fn is the only fn. $\hookrightarrow \phi$.

… singleton set $\{*\}$. $x \longrightarrow \{*\}$.

exercise. Set. … find out what initial and final objects are.

$(P, \sqsubseteq)$ … "

initial. final.

$x \sqsubseteq y$. least element. greatest element. $y \sqsubseteq x$. $(y \in P)$

③ $\mathcal{C}$ …… category, $x, y, z \in \mathrm{Obj}(\mathcal{C})$.

[Def] z is a product of x and y if.

$\exists$ morphisms $\pi_0 : z \to x$, $\pi_1 : z \to y$ s.t.

$\Big[$ $\forall\, u \in \mathrm{Obj}(\mathcal{C})$, $^\forall f : u \to x$, $^\forall g : u \to y$

there exists a unique morphism $k : u \to z$ s.t.

$f = \pi_0 \circ k$ and $g = \pi_1 \circ k$.

↳

$$\begin{tikzcd} x & & y \\ & z \arrow[ul, bend right, "\pi_0"'] \arrow[ur, bend left, "\pi_1"] & \\ & u \arrow[uul, bend left, "f"] \arrow[u, dashed, "!k"'] \arrow[uur, bend right, "g"'] & \end{tikzcd}$$

$z = x \times y$

ex. Set ... What is a product of x and y?

$(P, \sqsubseteq)$ ... What is a product of x and y $(x, y \in P)$?

greatest lower bound of x and y.