# CS 520

## Theory of Programming Language

04/07 – 04/14, 2021

1. Reminder.

① PL with fail, input-output.

② $[\![-]\!] : \langle comm \rangle \to [\Sigma \to \Omega]$.

$$\Omega \simeq (\hat{\Sigma} + \mathbb{Z} \times \Omega + [\mathbb{Z} \to \Omega])_\perp$$

output, input, non-termination.

where $\hat{\Sigma} = \Sigma + \Sigma$.

normal term. abnormal term.

(3) Predomain constructions : $+$, $\times$

$(P_0, \sqsubseteq_0)$, $(P_1, \sqsubseteq_0)$

$$P_0 + P_1 = \{\langle i, x\rangle \mid i \in \{0,1\}, x \in P_i\}.$$

$$P_0 \times P_1 = \{\langle x, y\rangle \mid x \in P_0, y \in P_1\}.$$ componentwise order.

$\iota_0 : P_0 \longrightarrow P_0 + P_1$, $\iota_1 : P_1 \longrightarrow P_0 + P_1$

$\pi_0 : P_0 \times P_1 \longrightarrow P_0$, $\pi_1 : P_0 \times P_1 \longrightarrow P_1$

$(P, \sqsubseteq)$, $(Q, \sqsubseteq)$.

$$\begin{tikzcd} & P_0 \arrow[dr, bend left, "g_0"] & \\ P \arrow[ur, bend left, "f_0"] \arrow[dr, bend right, "f_1"'] & & Q \\ & P_1 \arrow[ur, bend right, "g_1"'] & \end{tikzcd}$$

$\langle f_0, f_1\rangle : P \longrightarrow P_0 \times P_1$

$\langle f_0, f_1\rangle(x) = \langle f_0(x), f_1(x)\rangle$

$[g_0, g_1] : P_0 + P_1 \longrightarrow Q$

case. $\cdots [g_0, g_1]\langle i, x\rangle = g_i(x)$

$f_0' \times f_1'$, $f_0' + f_1'$, $f_0' : P_0 \longrightarrow P_0'$, $f_1 : P_1 \longrightarrow P_1'$

$: P_0 \times P_1 \longrightarrow P_0' \times P_1'$

2. Closer look at the def'n of $\Omega$.

① $(\Omega, \sqsubseteq)$ predomain s.t.

$$\Omega \underset{\boxed{\phi}}{\overset{\psi}{\rightleftarrows}} (\hat{\Sigma} + \mathbb{Z} \times \Omega + [\mathbb{Z} \to \Omega])_\perp$$

conti. fns.

$$\hat{\Sigma} = \Sigma + \Sigma.$$

with $\phi \circ \psi = id_\Omega$ and $\psi \circ \phi = id_{(\hat{\Sigma} + \cdots)_\perp}$

* minimality ..... (continuous initial algebra) ... we can do something similar to syntax-directed definition

meaning?

$\forall P \xleftarrow[\phi']{} (\hat{\Sigma} + \mathbb{Z} \times P + [\mathbb{Z} \to P])_\perp$ cont. fn.

$\exists!$ cont. fn. $f : \Omega \to P$ s.t.

$$\begin{tikzcd}
(\hat{\Sigma} + \mathbb{Z} \times \Omega + [\mathbb{Z} \to \Omega])_\perp \arrow[r, "(id_{\hat{\Sigma}} + id_{\mathbb{Z}} \times f + {[\mathbb{Z} \to f]})_\perp"] \arrow[d, "\phi"] & (\hat{\Sigma} + \mathbb{Z} \times P + [\mathbb{Z} \to P])_\perp \arrow[d, "\phi'"] \\
\Omega \arrow[r, "f"] & P
\end{tikzcd}$$

$$[\mathbb{Z}\to f] : [\mathbb{Z}\to \check{\Omega}] \to [\mathbb{Z} \to \check{P}]$$

$$[\mathbb{Z}\to f](g) = f\circ g$$

$$k_\perp : Q_\perp \to R_\perp \qquad ( k : Q \to R )$$

$$k_\perp(\perp) = \perp$$

$$k_\perp(x) = k(x) \quad \text{when } x \in Q$$

(2)

$$\left[\Omega \underset{\phi}{\overset{\psi}{\rightleftarrows}} \left[\hat{\Sigma} + \mathbb{Z}\times\Omega + [\mathbb{Z}\to\Omega]\right]_\perp ,\quad \hat{\Sigma} = \Sigma + \Sigma\right]$$

$\bar{\iota}_{term} : \Sigma \longrightarrow \Omega$. , $\bar{\iota}_{abort} : \Sigma \to \Omega$

$(P+Q) \neq P \cup Q$.

$\checkmark\ \bar{\iota}_{term}(6) = \phi(\langle 0, \langle 0, 6\rangle\rangle)$. $\checkmark\ \bar{\iota}_{abort}(6) = \phi(\langle 0, \langle 1, 6\rangle\rangle)$

$\| \quad \{0\}\times P \cup \{1\}\times Q$.

$\bar{\iota}_{out} : \mathbb{Z}\times\Omega \longrightarrow \Omega$ , $\bar{\iota}_{in} : [\mathbb{Z}\to\Omega] \to \Omega$

$\checkmark\ \bar{\iota}_{out}(n, w) = \phi\langle 1, \langle n, w\rangle\rangle$ $\quad \bar{\iota}_{in}(g) = \phi(\langle 2, g\rangle)$

$\bar{\iota}_\perp : \{\perp\} \longrightarrow \Omega$.

$\checkmark\ \bar{\iota}_\perp(\perp) = \phi(\perp)$.

(3) "syntax-directed definition"

$f : \Omega \to P$.

- case analysis ... specify $f$ for these five cases.

- recursion/induction.

$[\![ c_1 ; c_2 ]\!]$.

$[\![ c_1 ]\!] \in [\Sigma \to \Omega]$

$[\![ c_2 ]\!] \in [\Sigma \to \Omega]$.

$[\![ c_2 ]\!]$ " $\circ$ " $[\![ c_1 ]\!]$

↳ not good.

$[\![ c_2 ]\!]_* \in [\Omega \to \Omega]$.

??

Assume $g : \underline{\Sigma} \longrightarrow \Omega$

We want to define $g_* : \underline{\Omega} \to \Omega$.

$g_*(\bar{u}_{term}(b)) = g(b)$.

✓ $g_*(\bar{u}_{abort}(b)) = \bar{u}_{abort}(b)$. ✓

✓ $g_*(\bar{u}_{out}(n, w)) = \bar{u}_{out}(n, g_*(w))$.

✓ $g_*(\bar{u}_{in}(k)) = \bar{u}_{in}(\lambda n \in \mathbb{Z}.\ g_*(k(n)))$

$\qquad k : \mathbb{Z} \to \underline{\Omega}. \qquad\qquad k(n) : \underline{\Omega}$

$g_*(\bar{u}_\bot(\bot)) = \bar{u}_\bot(\bot)$.

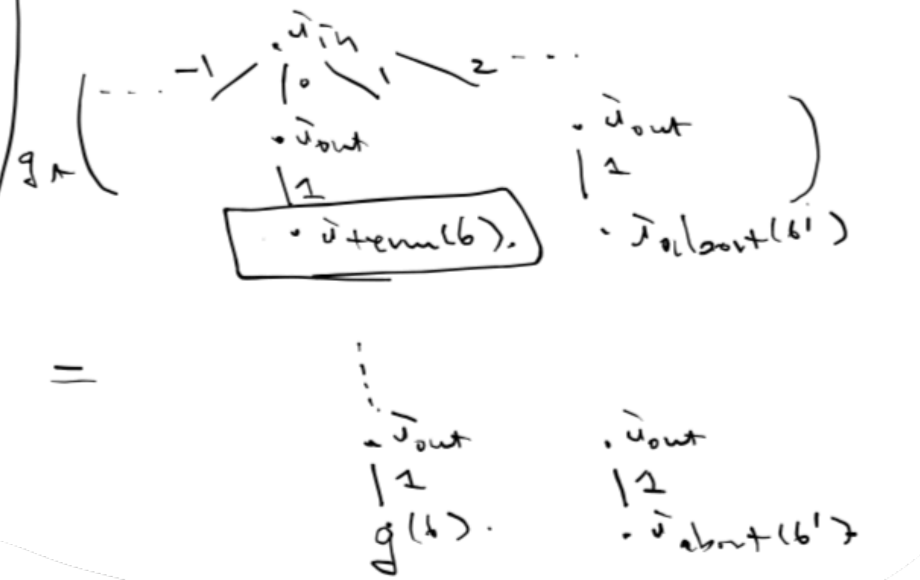
*: $\underline{\Omega}$

$\rotatebox{90}{$\in$}$

$g_*(\bar{u}_{in}(\lambda n.\ \bar{u}_{out}(n+1, \bar{u}_{term}(b))))$ ✓

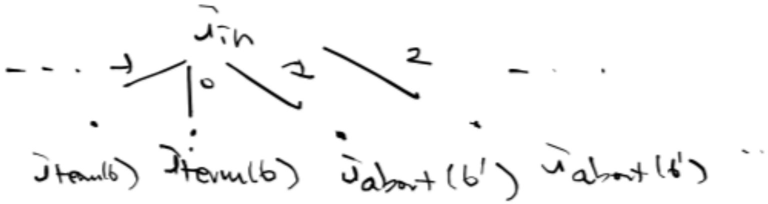
$\rotatebox{90}{$=$}$

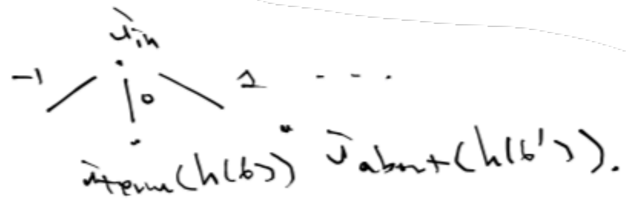
$\bar{u}_{in}(\lambda n.\ \bar{u}_{out}(n+1, g(b)))$



Ex. Given: $h : \Sigma \to \Sigma$.

$h_+ : \Omega \to \Omega$.

$h_+ \Big($ 

$\Big) =$ 


$$h_+(\bar{u}_{abort}(b)) = \bar{u}_{abort}(h(b)) \qquad h_+(\bar{u}_{term}(b)) = \bar{u}_{term}(h(b))$$

$$h_+(\bar{u}_{out}(\langle n, \omega\rangle)) = u_{out}(\langle n, h_+(\omega)\rangle) \qquad h_+(\bar{u}_{in}(k)) = \bar{u}_{in}(\lambda n.\ h_+(k(n)))$$

$$h_+(\bar{u}_\perp(\perp)) = \bar{u}_\perp(\perp)$$

4. Semantics of the lang.

$$[\![-]\!] : \langle \text{comm} \rangle \to [\Sigma \to \Omega]$$

$$[\![x := e]\!]\sigma = \iota_{\text{term}}\left([\sigma \mid x : [\![e]\!]\sigma]\right).$$

$$[\![\text{fail}]\!]\sigma = \iota_{\text{abort}}(\sigma).$$

$$[\![c_1 ; c_2]\!]\sigma = [\![c_2]\!]_* \left(\underbrace{[\![c_1]\!]\sigma}_{\Omega.}\right)$$

$$[\![\text{if } b \text{ then } c_1 \text{ else } c_2]\!]\sigma = \text{if } [\![b]\!]\sigma = tt \text{ then } [\![c_1]\!]\sigma \text{ else } [\![c_2]\!]\sigma$$

$$[\![\text{while } b \text{ do } c]\!]\sigma = Y(F)\,\sigma.$$

$$F : (\Sigma \to \Omega) \to [\Sigma \to \Omega].$$

$$F(f)(\sigma) = \text{if } [\![b]\!]\sigma = tt \text{ then } f_*\left([\![c]\!]\sigma\right) \text{ else } \iota_{\text{term}}(\sigma)$$

$$[\![ ?x ]\!]\sigma = \iota_{in}(\lambda n \in \mathbb{Z}.\ \iota_{term}([\sigma \mid x : n]))$$

$$[\![ !e ]\!]\sigma = \iota_{out}(\langle [\![ e ]\!]\sigma ,\ \iota_{term}(\sigma) \rangle)$$

$$[\![ \mathbf{newvar}\ x := e\ \mathbf{in}\ c ]\!]\sigma = \underbrace{(\lambda \sigma'.\ [\sigma' \mid x : \sigma(x)])_{\dagger}}_{}\left([\![ c ]\!].\ ([\sigma \mid x : [\![ e ]\!]\sigma])\right)$$

original value of the global var $x$.

recommended exercise. — solve the exercises at the end of note 5.

no present $\longrightarrow$ $[\![ \mathbf{newvar}\ x := e\ \mathbf{in}\ c ]\!]$

$\neq [\![ \mathbf{newvar}\ x' := e\ \mathbf{in}\ c/x \to x' ]\!]$

where $x' \notin FV(e, c)$.