Answer.

letrec add $\equiv \lambda t.$ sumcase t
of $(\lambda n.n$,
$\lambda t'.\ (\mathrm{add}\ (t'.0)) + (\mathrm{add}\ (t'.1)))$

in add.

4. Denotational Semantics.

① Recall the denotational semantics for the lambda calculus and the eager evaluation, in particular domains used and the form of the semantics function $[\![-]\!]$.

Isomorphisms.

$$D \overset{\text{def}}{=} V_\perp$$

$$V \underset{c}{\simeq} V \to D, \text{ more precisely, } V \underset{\psi}{\overset{\phi}{\rightleftarrows}} V \to D$$

$$[\![-]\!] \in [\langle exp\rangle \to V^{\langle var\rangle} \to D].$$

Since we added four new kinds of values, we should change $\perp$ so that the isomorphism says V consists of not just continuous functions but also those new values. Also, we have to change to account for errors and failures of the runtime typechecker.

② We use the following V and V_* (which corresponds to D above) and change the form of $[\![-]\!]$ accordingly.

$$V_* \overset{\text{def}}{=} (V + \{\mathrm{error}, \mathrm{typeerror}\})_\perp$$

$$V \underset{\psi}{\overset{\phi}{\rightleftarrows}} V_{int} + V_{bool} + V_{fun} + V_{tuple} + V_{alt}.$$

$$V_{int} \overset{\text{def}}{=} \mathbb{Z} \qquad V_{bool} \overset{\text{def}}{=} \mathbb{B} = \{tt, ff\}. \qquad V_{fun} \overset{\text{def}}{=} V \underset{c}{\to} V_*.$$

$$V_{tuple} \overset{\text{def}}{=} \bigcup_{n=0}^{\infty} \underset{\sim}{V^n}. \qquad V_{alt} \overset{\text{def}}{=} \mathbb{N} \times V$$

$\underbrace{V^n = V \times \cdots \times V}$ n-ary product of V.

i) Don't be confused between V here and V in ①. They are different. Also, note the parallel between the definition of $\langle cfm\rangle$ and the isomorphism for V here.

This isomorphism is in a sense a denotational way of saying there are five kinds of canonical forms. (or values).

ii) V_* extends V not just with $\perp$ but also with error and typeerror, so that the semantics can express such errors.

iii) In general, the (denotational) semantics of an eager functional language has the form:

domain constructor like $(-)_\perp$ and $(-)_*$.

$$[\![-]\!] \in [\langle exp\rangle \to V^{\langle var\rangle} \to T(V)]$$

and interprets functions using

$$V \underset{c}{\to} T(V).$$

This indicates that variables always get bound to values / canonical forms, not to arbitrary computations.

③ The actual definition of $[\![-]\!]$ is involved, but in a sense straightforward. The only things to be noteworthy are the uses of f_* and $g_{\theta *}$ for $\theta \in \{int, bool, fun, tuple, alt\}$.

$$f \in [V \underset{c}{\to} V_*]$$

$$f_* \in [V_* \underset{c}{\to} V_*] \qquad f_*(a) = \begin{cases} f(b) & \text{if } a = \langle 0, b\rangle \\ a & \text{otherwise.} \end{cases}$$

1st component of $V + \{\mathrm{error}, \mathrm{typeerror}\}$