4. Functor and Natural Transformation

① Intuitively, a functor is a structure-preserving map from one category to another. A natural transformation is a uniform map from one functor to another. In PL terms, a functor is a type constructor. (Recall that in this analogy, an object in a category is a type). And a natural transformation is a polymorphic function.

② Let $\mathcal{C}$ and $\mathbb{D}$ be categories.

Definition. A functor F from $\mathcal{C}$ to $\mathbb{D}$ is a pair of two maps F_{obj} and F_{mor} s.t.

(i) F_{obj} maps objects in $\mathcal{C}$ to objects in $\mathbb{D}$.

(ii) for objects $\underbrace{x, y \in Obj(\mathcal{C})}$ ← objects of $\mathcal{C}$,

$$(F_{mor})_{x,y} \in \Big[\underbrace{Hom_{\mathcal{C}}[x,y]}_{\text{morphisms in } \mathcal{C}} \to \underbrace{Hom_{\mathbb{D}}[F_{obj}(x), F_{obj}(y)]}_{\text{morphisms in } \mathbb{D}}\Big]$$

(iii) F_{mor} preserves $\circ$ and id:

ⓐ for all objects x of $\mathcal{C}$

$$(F_{mor})_{x,x}(id_x) = id_{F_{obj}(x)}$$

ⓑ for all morphisms $f \in Hom_{\mathcal{C}}[x,y]$ and $g \in Hom_{\mathcal{C}}[y,z]$,

$$(F_{mor})_{x,z}(g \circ f) = (F_{mor})_{y,z}(g) \circ (F_{mor})_{x,y}(f). \quad \square$$

we ~~write~~ use F to mean F_{obj} and $(F_{mor})_{x,y}$.

③ To see common examples, we need to understand the product of two categories. When $\mathcal{C}$ and $\mathbb{D}$ are categories, the product category $\mathcal{C} \times \mathbb{D}$ is defined as follows:

- $obj_{(\mathcal{C}\times\mathbb{D})} = \{(x,y) \mid x \in obj(\mathcal{C}) \wedge y \in obj(\mathbb{D})\}$.
- $Hom_{\mathcal{C}\times\mathbb{D}}[(u,v),(x,y)] = \{(f,g) \mid f \in Hom_{\mathcal{C}}[u,x] \wedge g \in Hom_{\mathbb{D}}[v,y]\}$.
- $(f,g) \circ (f',g') = (f \circ f', g \circ g')$
- $Id_{(x,y)} = (id_x, id_y)$.

④ The $\times$, $+$, $\perp$ operators on predomains are in fact functors.

$(-)_\perp$: Predom $\to$ Predom.

$(-)_\perp(P) = P_\perp$

$$(-)_\perp(f) = \lambda x. \begin{cases} \perp & \text{if } x = \perp \\ f(x) & \text{if } x \neq \perp \end{cases}$$

product functor ↘   product of categories ↘

$\times$: Predom $\times$ Predom $\to$ Predom.

$\times(P_0, P_1) = P_0 \times P_1$ ← product of predomains.

$\times(f,g) = \lambda(x,y).(f(x), g(y)) = f \times g$

Reynolds's notation (and standard notation) that we covered in Chap5.

sum functor ↘

$+$: Predom $\times$ Predom $\to$ Predom.

$+(P_0, P_1) = P_0 + P_1$ ← sum of predomains.

$$+(f,g) = \lambda x. \begin{cases} inj_0(f(u)) & \text{if } x = inj_0(u) \\ inj_1(g(v)) & \text{if } x = inj_1(v) \end{cases} = f + g$$

notation that we used in Chap5.

⑤ One other important functor is the identity functor:

Id : Predom $\to$ Predom.

$Id(x) = X$

$Id(f) = f$

⑥ Let F, G be functors from $\mathcal{C}$ to $\mathbb{D}$.

Definition. A natural transformation η from F to G, denoted $\eta : F \dot{\to} G : \mathcal{C} \to \mathbb{D}$ or
$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathbb{D} \end{tikzcd},$$