Lemma 2 Consider the following co-cone of an ω-chain Δ:

$$
\Delta = \left(
\begin{tikzcd}
D_0 \arrow[r, "f_0"] \arrow[drr, bend right, "h_0"'] & D_1 \arrow[r, "f_1"] \arrow[dr, "h_1"] & D_2 \arrow[r, "f_2"] \arrow[d, "h_2"] & \cdots \\
 & & D' &
\end{tikzcd}
\right)
$$

Let h_i^P be the projection for the embedding h_i.

Then, D' is co-limiting iff $\bigsqcup_{i=0}^{\infty} (h_i \circ h_i^P) = id_{D'}$.

(intuitively means least upper bound)

(note that $h_i \circ h_i^P \sqsubseteq id_{D'}$. We can easily show that $\{h_i \circ h_i^P\}$ is an increasing chain in $[D' \to_c D']$. The underlined condition says that the least upper bound of the chain is id.)

□

This lemma says that the order on morphisms plays an important role in deciding whether D' is a co-limit or not. One important consequence is the following lemma.

~~Lemma: A functor $F: Dom^{EP} \to Dom^{EP}$ is ω continuous if for all domains D, D', F induces a continuous function from $Hom_{Dom^{EP}}[D, D']$ to $Hom_{Dom^{EP}}[F(D), F(D')]$ where and are ordered by the usual pointwise order of the function space.~~

↑ ignore this

Lemma 3 A functor $F: Dom^{EP} \to Dom^{EP}$ is ω-continuous if for every co-cone of an ω-chain Δ

$$
\begin{tikzcd}
D_0 \arrow[r, "f_0"] \arrow[drr, "h_0"'] & D_1 \arrow[r, "f_1"] \arrow[dr, "h_1"] & D_2 \arrow[r, "f_2"] \arrow[d, "h_2"] & \cdots \\
 & & D' &
\end{tikzcd}
$$

$\bigsqcup_{i=0}^{\infty} h_i \circ h_i^P = id_{D'}$ implies $\bigsqcup_{i=0}^{\infty} F(h_i) \circ F(h_i)^P = id_{F(D')}$.

Proof.

This is a direct consequence of Lemma 2. □

8 Lemma 3 is our tool to check the ω-continuity of a functor F on Dom^{EP}. If this check passes, by the fixed point theorem, we know that there exists a domain D s.t. $F(D) \cong D$.

(i) Our functor $F(\Omega) = (\Sigma + \Sigma + \mathbb{Z} \times \Omega + [\mathbb{Z} \to \Omega])_\perp$ is an example of such a functor.

(ii) Another famous example is the following G that defines the function space.

$G(D) = [D \to_c D]$ ··· the domain of continuous functions in D.

For every $f: D \to D'$,

$$G(f) : \underbrace{G(D)}_{[D \to_c D]} \to \underbrace{G(D')}_{[D' \to_c D']}$$

$$G(f)(h) = f \circ h \circ f^P$$

(we are using the projection of f here. If f were just a continuous fn, we couldn't do it.)