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Chap 8 of Tennent's Book. An Introduction to Category Theory

1. Motivation.

① The category theory is a branch of mathematics that studies essentially same or common notions and principles in different areas of mathematics. It is very abstract, but provides good guidelines about finding right definitions and right or ~~meaningful~~ meaningful questions to ask in a new mathematical theory.

② The category theory had huge influence on the research on programming languages and the development of practical programming languages, such as Scala, Haskell, and Rust.

③ In this course, we will focus on the influence of the category theory on the semantics research. We will study abstract categorical concepts that ~~have~~ give rise to popular notions appearing in the semantics of programming languages. Another big objective is to understand a result in the category theory (or a categorical formulation of domain theory) that ensures the existence of certain recursively defined domains, such as the following Ω that you saw in the chapter 5 of Reynolds's book:

$$\Omega \simeq (\hat{\Sigma} + \mathbb{Z} \times \Omega + [\mathbb{Z} \to \Omega])_{\perp}.$$

$$\hat{\Sigma} \stackrel{\text{def}}{=} \Sigma + \Sigma.$$

④ We will study only a tiny part of the category theory. If you are excited about it and are willing to read a math book, I recommend Mac Lane's "Categories for the Working Mathematician"

2. Definition of Category.

<u>Definition.</u> A <u>category</u> $\mathcal{C}$ is a tuple $(\mathrm{Obj}, \mathrm{Hom}, \circ, \mathrm{id})$ where

(1) Obj is a collection of ~~objects~~ elements called <u>objects</u>;

(2) for all objects $x, y \in \mathrm{Obj}$, $\mathrm{Hom}[x,y]$ is a collection of elements called <u>morphisms</u> from x to y;

(3) for all objects $x,y,z \in \mathrm{Obj}$, $\circ_{x,y,z}$ (or simply $\circ$) is a map from $\mathrm{Hom}[y,z] \times \mathrm{Hom}[x,y]$ to $\mathrm{Hom}[x,z]$, and is called <u>composition</u>;

(4) for every object $x \in \mathrm{Obj}$, id_x is an element in $\mathrm{Hom}[x,x]$ and is called <u>identity morphism</u>;

(5) these data should satisfy associativity and identity axioms:

[associativity] $\forall w,x,y,z \in \mathrm{Obj}$.
$\forall f \in \mathrm{Hom}[w,x]$.
$\forall g \in \mathrm{Hom}[x,y]$.
$\forall h \in \mathrm{Hom}[y,z]$.

$$h \circ (g \circ f) = (h \circ g) \circ f$$

[identity] $\forall x,y \in \mathrm{Obj}$
$\forall f \in \mathrm{Hom}[x,y]$.

$$f \circ \mathrm{id}_x = \mathrm{id}_y \circ f = f.$$

□

① Although not perfect, a reasonably good intuition is that a category $\mathcal{C}$ is a collection of spaces. The Obj part of $\mathcal{C}$ consists of spaces, (that you encounter in mathematics, such as metric spaces, vector spaces, topological spaces, etc), and the $\mathrm{Hom}[x,y]$ part of $\mathcal{C}$ consists of maps between spaces that preserve the structures of the spaces.