① Intuitively, the theorem says that x_{fix} is a least fixed point of F. The condition i) says that x_{fix} is a fixed point. The condition ii) says that it is the least fixed point.

② The proof is complex, but not that difficult. Very similar to the proof of the ~~usual~~ standard fixed point theorem of the domain theory. We will study only some parts of the proof.

③ The key part of the proof is to construct x_{fix}. Here we use the initial object x_0 of $\mathcal{C}$, the functoriality of F, and the chain completeness of $\mathcal{C}$. (They correspond to $\perp$, the monotonicity of a continuous fn. f and the chain completeness of a predomain D in the proof of the fixed-point theorem in domain theory).

We construct a chain:

$$\Delta \stackrel{def.}{=} x_0 \xrightarrow{f_0} F(x_0) \xrightarrow{f_1} F^2(x_0) \xrightarrow{f_2} F^3(x_0) \xrightarrow{f_3} \cdots$$

unique morphism from the initial object x_0 to $F(x_0)$

$F(f_0) : F(x_0) \to F^2(x_0)$

$F(F(f_0)) : F^2(x_0) \to F^3(x_0)$

$F^3(f_0) : F^3(x_0) \to F^4(x_0)$

Since $\mathcal{C}$ is chain-complete, there exists a co-limit $(x_{fix}, \{g_i\})$ of the chain Δ that we just built.

$$\begin{tikzcd}
x_0 \arrow[r, "f_0"] \arrow[drr, "g_0"'] & F(x_0) \arrow[r, "F(f_0)"] \arrow[dr, "g_1"] & F^2(x_0) \arrow[r, "F^2(f_0)"] \arrow[d, "g_2"] & F^3(x_0) \arrow[r] \arrow[dl, "g_3"] & \cdots \\
& & x_{fix} & &
\end{tikzcd}$$

④ Next we build $\eta : F(x_{fix}) \to x_{fix}$. Here we use the ω-continuity of F. Apply F to the diagram. This gives us

$$\begin{tikzcd}
\Delta' \stackrel{def}{=} F(x_0) \arrow[r, "F(f_0)"] \arrow[drr, "F(g_0)"'] & F^2(x_0) \arrow[r, "F^2(f_0)"] \arrow[dr, "F(g_1)"] & F^3(x_0) \arrow[r, "F^3(x_0)"] \arrow[d, "F(g_2)"] & F^4(x_0) \arrow[r] \arrow[dl, "F(g_3)"] & {} \\
& & F(x_{fix}) & &
\end{tikzcd}$$

Since F is ω-continuous, this is a co-limiting co-cone of the chain Δ', which is the x_0-truncated version of Δ.

⑤ Because x_0 is the initial object, we can add x_0 to the diagram and get a co-limiting co-cone.

$$\begin{tikzcd}
x_0 \arrow[r, "f_0"] \arrow[drrr, bend right, "g_0'"'] & F(x_0) \arrow[r, "F(f_0)"] \arrow[drr, "F(g_0)"] & F^2(x_0) \arrow[r, "F^2(f_0)"] \arrow[dr, "F(g_1)"] & F^3(x_0) \arrow[r, "F^3(x_0)"] \arrow[d, "F(g_2)"] & F^4(x_0) \arrow[r] \arrow[dl, "F(g_3)"] & {} \\
& & & F(x_{fix}) & &
\end{tikzcd}$$

g_0' ↑ unique morphism from x_0 to $F(x_{fix})$.

this triangle commutes because of the initiality of x_0.

Now we have two co-limits, x_{fix} and $F(x_{fix})$, of the same chain Δ. One general result (which is easy to show) is that two co-limits of a chain are isomorphic, which means in our case that there exists morphisms $\eta : F(x_{fix}) \to x_{fix}$ and $\psi : x_{fix} \to F(x_{fix})$ s.t.

$$\psi \circ \eta = id_{F(x_{fix})} \quad \text{and} \quad \eta \circ \psi = id_{x_{fix}}.$$

We just proved i) of the theorem. We leave the proof of ii) as an exercise.