

### 3. Initial and terminal objects. Product and Co-product. (or Sum)

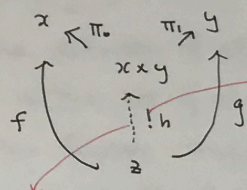
① In a category  $\mathcal{C}$ , we can build a new object out of existing ones. Often this new object satisfies one of the well-known conditions, and has a well-known name. We will study a few such names.

② Consider a category  $\mathcal{C}$  and its objects  $x, y$ .  
 (i) An object  $z$  is a product of  $x$  and  $y$ , often written as  $z = x \times y$ , if there are morphisms  $\pi_0 \in \text{Hom}[z, x]$  and  $\pi_1 \in \text{Hom}[z, y]$  s.t.  
 (often written as  $\pi_i : x \times y \rightarrow x$ )  
 (same as  $f \in \text{Hom}[w, x]$ )

for all objects  $w$  and morphisms  $f: w \rightarrow x$  and  $g: w \rightarrow y$ ,

(there exists a unique morphism  $h: w \rightarrow x \times y$  with  $f = \pi_0 \circ h$  and  $g = \pi_1 \circ h$ .)

in picture.

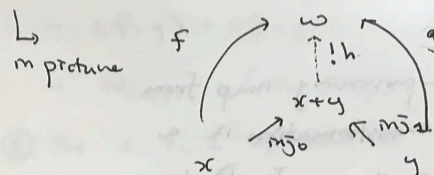


(we often write  $h$  as  $\langle f, g \rangle$ .  
 Reynolds writes it as  $f \otimes g$   
 means uniqueness.

(ii) An object  $z$  is a co-product (or sum) of  $x$  and  $y$ , often written as  $z = x + y$ , if there are morphisms  $m_0: x \rightarrow x + y$  and  $m_1: y \rightarrow x + y$  s.t.

for all objects  $w$  and morphisms  $f: x \rightarrow w$  and  $g: y \rightarrow w$ , there exists a unique morphism  $h: x + y \rightarrow w$

with  $f = h \circ m_0$  and  $g = h \circ m_1$  (often written as  $\langle f, g \rangle$ . Reynolds writes it as  $f \otimes g$ )



(iii) An object  $x$  is initial if for every object  $w$ , there exists the unique morphism  $h: x \rightarrow w$ .

(iv) An object  $x$  is final if for every object  $w$ , there exists the unique morphism  $h: w \rightarrow x$ .

(often written as  $!w:$ )  
 or!

③ Note that all of these ~~notions~~ are defined purely in terms of morphisms, without referring to elements of objects. This is a bit like specifying a property of an abstract data type in terms of ~~properties~~ its operations, not in terms of its implementations.

④ In the category  $\text{Predom}$ , the product of two predomains  $P_0$  and  $P_1$  that we studied in Chap 5 is indeed a categorical product in (i). Also, the sum of  $P_0$  and  $P_1$  in chap 5 is indeed a categorical sum or co-product. The predomain of the singleton set  $(\{*\}, E)$  is a terminal object. The predomain of the empty set  $(\{\}, E)$  is an initial object.

⑤ Consider the category corresponding to a partially ordered set  $(X, E)$ . Then, an object  $x \in X$  is initial if and only if it is the ~~smallest~~ least element in  $X$ . It is final if and only if it is the greatest element. For objects  $x, y$  in  $X$ ,  $x + y$  is the least upper bound of  $x$  and  $y$ , and  $x \times y$  is the greatest lower bound of  $x$  and  $y$ .

Exercise. Prove ④ and ⑤.