Thus, the monotonicity already holds for F in a sense.

③ ~~S~~ OK, what remain? We still need to generalize

i) chains

ii) least upper bounds (or limits) of chains.

iii) limit-preservation.

④ An ω-chain in a category $\mathcal{C}$ is a countably infinite sequence of objects $(x_0, x_1, \cdots)$ of $\mathcal{C}$ and a collection of morphisms $\{ f_i : x_i \to x_{i+1} \}_{i \geq 0}$ in $\mathcal{C}$. The best way to understand this is to imagine the following figure:

$$x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} x_2 \xrightarrow{f_2} x_3 \to \cdots$$

We use Δ to denote an ω-chain

⑤ A co-cone of an ω-chain $\Delta = \{(x_i, f_i)\}_{i \geq 0}$ is a pair of object x and a collection of morphisms $\{g_i : x_i \to x\}_i$ such that for all $i \geq 0$, $g_{i+1} \circ f_i = g_i$, i.e., in picture,

$$\begin{tikzcd} x_i \arrow[r, "f_i"] \arrow[dr, "g_i"'] & x_{i+1} \arrow[d, "g_{i+1}"] \\ & x \end{tikzcd}$$

commutes.

⑥ Intuitively, an ω-chain Δ is a generalized chain, and the object x of a co-cone $(x, \{g_i\}_i)$ of Δ is a generalized upper bound of the chain. Each $g_i : x_i \to x$ ~~says why x is~~ provides a way to view that x ~~provides a reason that~~ is ~~small~~ ~~at least~~ larger than or equal to x_i. Meanwhile, each $f_i : x_i \to x_{i+1}$ provides a way to view that x_{i+1} is larger than or equal to x_i. The commutativity requirement says that these two views should be compatible.

⑥ A co-cone $(x, \{g_i\}_i)$ of an ω-chain $\Delta = (\{x_i\}, \{f_i\})$ is co-limiting if for every co-cone $(x', \{g_i'\})$ of Δ, there exists a unique morphism $h : x \to x'$ s.t. for all i, $g_i' = h \circ g_i$, in a diagram,

$$\begin{tikzcd} x_i \arrow[dr, "g_i"] \arrow[ddr, bend right, "g_i'"'] & \\ & x \arrow[d, dashed, "!h"] \\ & x' \end{tikzcd}$$

Intuitively, the very existence of h says that x is smaller than or equal to x'. The commutativity (and the uniqueness) says that h's explanation about why x is smaller than or equal to x' ~~comes~~ follows automatically and canonically from the g_i and the g_i'.

⑦ A co-limiting co-cone (often simply called co-limit) $(x, \{g_i\})$ of Δ is a generalisation of the least upper bound of a chain. I usually imagine the following visual image whenever I work with an ω-chain, a co-cone, and a co-limiting co-cone.