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Chap 10 of Tennent's book. Recursively Defined Domains

1. Motivation.

① One reason that we studied category theory is to understand a general principle behind the construction of recursively defined domains, such as the following Ω that you encountered before:

$$\Omega \cong (\hat{\Sigma} + \mathbb{Z}\times\Omega + (\mathbb{Z}\to\Omega))_\perp$$
$$\hat{\Sigma} \stackrel{\text{def}}{=} \Sigma \cup \{\text{abort}\}\times\Sigma \quad (\cong \Sigma+\Sigma)$$

② If we write the RHS (right-hand side) of the above isomorphism as $F(\Omega)$, the formula says:

$$\Omega \cong F(\Omega).$$

that is, Ω is a fixed point of F. In fact, Ω is not just a fixed point, but the best fixed point. where "the best" means something very similar to "the least" in the standard fixed point theorem of the domain theory.

③ We will generalise the standard least fixed point theorem of the domain theory and obtain a general categorical least fixed point theorem. ~~Our~~ This generalization closely follows the intuition that categories are generalized partially ordered sets (and functors are generalized monotone functions). Then, we will instantiate our generalisation with a particular category constructed out of domains and a particular kind of continuous functions called embeddings. (omega, meaning the first countable ordinal).

2. ω-Chain ~~[illegible]~~ and Co-limit of ω-chain.

① Let's start by remembering ingridients that we needed when expressing the standard least fixed point theorem of the domain theory. ~~[illegible]~~

i). A partially ordered set D has the least element.

ii) Every chain in D has the least upper bound.

iii) A function f on D is continuous (i.e., monotone and chain-limit-preserving).

Then, the theorem says that f has the least fixed point x_0. That is, $f(x_0)=x_0$ and for all y s.t. $f(y)\sqsubseteq y$, $x_0 \sqsubseteq y$.

property that is a bit stronger than x_0 being the least fixed point.

② In the categorical generalisation of the theorem,

i) D becomes a category $\mathcal{C}$;

ii) $f \in [D\to D]$ becomes a functor $F:\mathcal{C}\to\mathcal{C}$;

iii) x_0 becomes an object in $\mathcal{C}$;

iv) the least element of D corresponds to the initial object of $\mathcal{C}$.

Note that the monotonicity of f (preservation of the $\sqsubseteq$ relation) translates to F's morphism map being type-checked wrt. its object map. (i.e. for all $g: x\to y$, $F(g): F(x)\to F(y)$). This translated property is a part of the conditions for F being a functor.