is a collection of morphisms in $\mathcal{D}$ indexed by objects in $\mathcal{C}$

$$\left(\eta_x : F(x) \to G(x)\right)_{x \in Obj(\mathcal{C})}.$$

such that

(i) for each object x in $\mathcal{C}$,

η_x is a $\mathcal{D}$-morphism from $F(x)$ to $G(x)$;

(ii) for every morphism $f : x \to y$ in $\mathcal{C}$ (i.e. $f \in Hom_{\mathcal{C}}[x, y]$),

$$\begin{tikzcd} F(x) \arrow[r, "\eta_x"] \arrow[d, "F(f)"'] & G(x) \arrow[d, "G(f)"] \\ F(y) \arrow[r, "\eta_y"] & G(y) \end{tikzcd}$$

→ This is called naturality condition

⑦ One intuition is that F and G are type constructors and η is a polymorphic function from F to G. Whenever we are given a type x, we have an instantiation of η at x that is a function from the type $F(x)$ to the type $G(x)$.

The condition (i) says that η should typecheck. The condition (ii) says that ~~what~~ what η does at x should be identical in a sense to what it does at y.

In other words, η should not depend much on its type parameter x. This is called uniformity.

⑧ Here are a few examples:

unit : $Id \stackrel{\cdot}{\to} (-)_\perp$: Predom → Predom.

$unit_P \in [P \to P_\perp]$

$unit_P(x) = x$.

Let fst be a functor from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{C}$ defined by

objects of categories

morphisms of categories

$Fst(x, y) = x$

$Fst(f, g) = f$.

Then,

$\pi_0 : (-)\times(-) \stackrel{\cdot}{\to} Fst$: Predom × Predom → Predom.

$(\pi_0)_{P,P'} \in [P \times P' \to_c P]$

$(\pi_0)_{P,P'}(a, b) = a$.

exercise. Show that unit and π_0 are indeed natural transformations.

exercise. π_1, inj_0, inj_1 are also natural transformations if we pick appropriate functors. Find such functors.

⑨ Notice that all of unit, π_0, π_1, inj_0, inj_1 do something very straightforward intuitively. They don't do any clever tricks. Naturality condition says in a sense that a natural transformation doesn't do anything clever.

In more positive terms, it says that a natural transformation only performs canonical operations.