4. Deriving a First-order Semantics

(Semantic version of defunctionalisation)

① The continuation semantics and the direct semantics both use functions so heavily, sometimes even higher-order functions, i.e., functions that take functions as parameters. Can we define a semantics that avoids the use of such functions, or at least minimises the use of higher-order functions?

More concretely, recall the definitions of predomains and domains involved in the continuation semantics.

$$V_* = (V + \{\mathrm{err}, \mathrm{typerr}\})_\perp$$

$$V \underset{\psi}{\overset{\phi}{\rightleftarrows}} (V_{int} + V_{bool} + V_{fun} + V_{tuple} + V_{alt} + V_{cont})$$

$$V_{int} = \mathbb{Z}$$

$$V_{bool} = \mathbb{B}$$

$$V_{fun} = [V \to_c [V_{cont} \to_c V_*]]$$

$$V_{tuple} = \bigcup_{n=0}^{\infty} V^n$$

$$V_{alt} = \mathbb{N} \times V$$

$$V_{cont} = [V \to_c V_*].$$

If we substitute the definition of V_{cont} in the definition of V_{fun}, we get

$$V_{fun} = [V \to_c [V \to_c V_*] \to_c V_*]$$

So, elements in V_{fun} are higher-order functions. We would like to have a semantics that avoids using such higher-order functions. Such a semantics is called first-order.

② Before answering the question raised in ①, let me say a few words about why we are interested in such a first-order semantics. The first reason is a bit theoretical. It is that defining such first-order semantics involves solving much simpler and easier recursive domain equations. In our original continuation semantics, we assumed that V is a solution of the following recursive (pre)domain equation:

$$V \cong (\mathbb{Z} + \mathbb{B} + [V \to_c [V \to_c (V + \{\mathrm{err}, \mathrm{typerr}\})_\perp] \to_c (V + \{\mathrm{err}, \mathrm{typerr}\})_\perp]$$
$$+ \bigcup_{n=0}^{\infty} V^n$$
$$+ \mathbb{N} \times V$$
$$+ [V \to_c (V + \{\mathrm{err}, \mathrm{typerr}\})_\perp])$$

Note that V appears on the both sides of $\to$. The occurrences of V underlined with two blue lines = make this recursive predomain equation very difficult to solve. We should use the categorical fixed point theorem and the category of domains with embeddings (which we covered before) to solve this equation.

On the other hand, in the first-order semantics, we have a recursive predomain equation that is much easier to solve. It doesn't have those tricky recursive occurrences of $\hat{V}$ (a predomain being defined) that appear on the left argument side of $\to$,