⑤ As before, understanding these domains is the key to understand the denotational semantics of the lambda calculus under three different notions of computations,

i) the contraction relation,

ii) the normal-order evaluation

iii) the eager evaluation.

The equations for (D_2, V_2) and (D_3, V_3) indicate that under ii) and iii), we need to differentiate expressions that don't terminate and those that do and become lambda expressions. D_2 and D_3 are domains for all expressions, and V_2 and V_3 are domains for the latter kind of expressions. Sometimes V_2 and V_3 are called <u>domains for values</u>, and D_2 and D_3 <u>domains for computations</u>. Note that we don't make this kind of distinction for D_1. For instance, in D_2 and D_3, $\bot$ is different from the constant function that always returns $\bot$ (i.e. $\lambda a.\bot$). On the other hand, in D_1, they are regarded the same. (In D_1, $\lambda a.\bot$ is the least element up to the isomorphism $D_1 \cong [D_1 \to_c D_1]$.).

Why is D_1's way of defining $\bot$ different from D_2 and D_3's? Because the normal-order evaluation and the eager evaluation do not reduce subexpressions under the lambda abstraction $(\lambda x.e)$ whereas
↑ not reduced.
the contraction relation do reduce such subexpressions.

⑥ Now let's try to understand the difference between (D_2, V_2) and (D_3, V_3). : Rewriting isomorphisms and definitions slightly can help us to see this difference more easily.

$$D_2 \stackrel{\text{def}}{=} (V_2)_\bot \qquad V_2 \cong [\underline{D_2} \to_c D_2]$$

$$D_3 \stackrel{\text{def}}{=} (V_3)_\bot \qquad V_3 \cong [V_3 \to_c D_3]$$

The key difference lives <u>here</u>. The argument domain for the normal-order evaluation is that for computations, while the argument domain for the eager evaluation is the one for values. This difference comes from the fact that in the eager evaluation, we pass only canonical forms (which are <u>lambda expressions</u>) to functions as arguments, while in the normal-order evaluation, we pass any expressions as function arguments. So, when we use the normal-order evaluation, variables may be bound to expressions that may denote any computations. But if we use the eager evaluation, variables should be bound to lambda expressions or more generally canonical forms that denote values.

⑦ Here is the denotational semantics for the contraction relation.

$$D_1 \underset{\psi}{\overset{\phi}{\rightleftarrows}} [D_1 \to_c D_1]$$

$(= [\langle var\rangle \to D_1]$ ordered pointwise$)$

$$[\![-]\!] \in [\langle exp\rangle \to [\underbrace{D_1^{\langle var\rangle}}_{\ni\, \eta} \to_c D_1]]$$

η … called <u>environment</u>

$$[\![v]\!]\eta = \eta(v)$$

$$[\![e_0 e_1]\!]\eta = \phi([\![e_0]\!]\eta)([\![e_1]\!]\eta)$$