Exercise. Construct co-limiting co-cones of ω-chains in the category of Set. Do the same thing in the poset category $(2^{\mathbb{N}}, \subseteq)$ and in the category of predomains.
(partially ordered set)

3. ω-Continuous Functor.

① Let $\mathcal{C}$ and $\mathbb{D}$ be categories that have co-limiting co-cones for all ω-chains. We will call such categories as ~~ω-complete~~ chain-~~ω~~-complete categories.
(or in other words, co-limits)

② A functor $F : \mathcal{C} \to \mathbb{D}$ is ω-continuous if it maps a co-limiting co-cone (co-limit) of an ω-chain to a co-limiting co-cone (co-limit) of an ω-chain, that is,

for every ω-chain $\Delta = (\{x_i\}_i, \{f_i\}_i)$ in $\mathcal{C}$,
for every co-limiting co-cone $(x, \{g_i\}_i)$ (co-limit) of Δ,

$(F(x), \{F(g_i)\}_i)$ is a ~~co-limiting~~ co-limit of $F(\Delta) = (\{F(x_i)\}_i, \{F(f_i)\}_i)$ in $\mathbb{D}$.

③ Intuitively, this ω-continuity of F means that F preserves the least upper bound of an increasing chain.

④ Note that a functor always maps a co-cone of an ω-chain to a co-cone of an ω-chain. Visually,

$$\begin{tikzcd} x_0 \arrow[r, "f_0"] \arrow[drr, "g_0"'] & x_1 \arrow[r, "f_1"] \arrow[dr, "g_1"] & x_2 \arrow[r, "f_2"] \arrow[d, "g_2"] & \cdots \\ & & x & \end{tikzcd} \qquad \begin{tikzcd} F(x_0) \arrow[r, "F(f_0)"] \arrow[drr, "F(g_0)"'] & F(x_1) \arrow[r, "F(f_1)"] \arrow[dr, "F(g_1)"] & F(x_2) \arrow[r, "F(f_2)"] \arrow[d, "F(g_2)"] & \cdots \\ & & F(x) & \end{tikzcd}$$

Category $\mathcal{C}$ Category $\mathbb{D}$.

If the diagram in $\mathcal{C}$ commutes, the diagram in $\mathbb{D}$ also commutes. This is because the preservation of $\circ$ and id by a functor implies that the functor maps every commuting diagram to a commuting diagram.

The situation is similar to the fact that a monotone function f from a predomain to a predomain maps an upper bound of a chain to an upper bound of a chain.

⑤ However, the functoriality of F doesn't ensure that if $(x, \{g_i\}_i)$ is co-limiting, so is $(F(x), \{F(g_i)\}_i)$. When F satisfies this addition property, we say that F is ω-continuous.

Exercise.
Show that the functor from Set to Set

$$F(S) = \mathbb{Z} \times S$$
$$F(f) = id_{\mathbb{Z}} \times f = \lambda(n,s).\,(n, f(s))$$

is ω-continuous.

4. Fixed Point Theorem.

[Thm] Let $\mathcal{C}$ be a category with an initial object x_0. Assume that $\mathcal{C}$ is chain-~~ω~~-complete. (i.e., every ω-chain Δ in $\mathcal{C}$ has a co-limit). Then, for every ω-continuous functor $F : \mathcal{C} \to \mathcal{C}$, there exists an object x_{fix} in $\mathcal{C}$ and an ~~iso~~morphism $\eta : F(x_{fix}) \to x_{fix}$ s.t.

i) η is an isomorphism, i.e., $\exists\, \phi : x_{fix} \to F(x_{fix})$ s.t.
(a morphism)
$\eta \circ \phi = id_{x_{fix}}$ and $\phi \circ \eta = id_{F(x_{fix})}$

ii) for every morphism $\eta' : F(y) \to y$, there exists a unique morphism $\rho : x_{fix} \to y$ s.t.
$$\begin{tikzcd} F(x_{fix}) \arrow[r, "\eta"] \arrow[d, "F(\rho)"'] & x_{fix} \arrow[d, "\rho"] \\ F(y) \arrow[r, "\eta'"] & y \end{tikzcd}$$ □