strict (i.e. $\bot$-preserving) continuous functions f from D to D' s.t. there exists a continuous function $g : D' \to D$ with

$$g \circ f = Id_D \quad \text{and} \quad f \circ g \sqsubseteq Id_D.$$

(iii) $\circ$ is the usual function composition.

(iv) Id_D is the identity function on D.

⑤

Compared with the category Dom of domains and continuous functions, this category Dom^{EP} has a rather unusual notion of morphisms. In Dom^{EP}, a morphism $f : D \to D'$ should be not just continuous, but also strict. More importantly, it should have $g : D' \to D$ s.t.

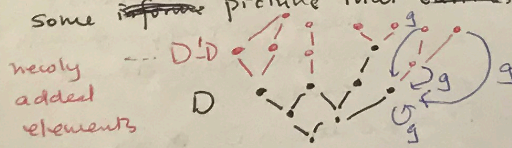
$$g \circ f = Id_D \quad \text{and} \quad f \circ g \sqsubseteq Id_D,$$

in picture.

$$\begin{tikzcd} D \arrow[r, "f"] \arrow[dr, "Id_D"'] & D' \arrow[d, "g"] \\ & D \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} D' \arrow[r, "g"] \arrow[dr, "Id_D"'] & D \arrow[d, "f"] \arrow[dl, phantom, "\sqsupseteq"] \\ & D' \end{tikzcd}$$

this diagram means the above inequality

Intuitively, the existence of such g means that D' is built by putting additional elements above existing elements of D, f is just the embedding of D into this larger D', and g maps all additional elements to their best underapproximations in D. Here is some ~~informal~~ picture that ~~????~~ shows this intuition.



g ---- g maps elements of D' to their best underapproximations in D.

What this means is that a morphism $f : D \to D'$ is really saying that D' is larger than D.

⑥ A morphism $f : D \to D'$ in Dom^{EP} is called <u>embedding</u> and a corresponding $g : D \to D'$ is called <u>projection</u>.

[[If you happen to take a course on program analysis, this pair of embedding f and projection g is closely related to the Galois connection there]]

Note that the definition doesn't say that there exists only one projection g for a given embedding f. That is, there may be multiple projections. But this doesn't happen.

<u>Lemma)</u> For every embedding $f : D \to D'$ and projections $g_1 : D' \to D$ and $g_2 : D' \to D$ for f, we have that

$$g_1 = g_2.$$

<u>Proof.</u> We will show that $g_1 \sqsubseteq g_2$. A similar argument can show the opposite inequality.

$$g_1 = Id_D \circ g_1 = (g_2 \circ f) \circ g_1 = g_2 \circ (f \circ g_1) \sqsubseteq g_2 \circ Id_{D'} = g_2. \quad \square$$

We write f^P to denote the unique projection for an embedding f.

⑦ The category Dom^{EP} has an initial object, which is a singleton domain $(\{\bot\}, \sqsubseteq)$ ($x \sqsubseteq y$ iff $x = y$.) It also has a co-limit for any ω-chain. We will not prove this. But we point out one important property of these co-limits.