3. Initial and terminal objects. Product and Co-product. (or Sum)

① In a category $\mathcal{C}$, we can build a new object out of existing ones. Often this new object satisfies one of the well-known conditions, and has a well-known name.

We will study a few such names.

② Consider a category $\mathcal{C}$ and its objects x, y.

(i) An object z is a <u>product</u> of x and y, often written as $z = x \times y$, if there are morphisms $\pi_0 \in \mathrm{Hom}[z, x]$ (often written as $\pi_0 : x \times y \to x$.) and $\pi_1 \in \mathrm{Hom}[z, y]$ s.t. (often written as $\pi_1 : x \times y \to y$)

for all objects w and morphisms $f : w \to x$ (same as $f \in \mathrm{Hom}[w, x]$) and $g : w \to y$,

(there exists a unique morphism $h : w \to x \times y$ with $f = \pi_0 \circ h$ and $g = \pi_1 \circ h$.

(We often write h as $\langle f, g \rangle$. Reynolds writes it as $f \otimes g$)

in picture.

$$
\begin{tikzcd}
x & x \times y \arrow[l, "\pi_0"'] \arrow[r, "\pi_1"] & y \\
 & z \arrow[ul, bend left, "f"] \arrow[u, dashed, "!h"] \arrow[ur, bend right, "g"'] &
\end{tikzcd}
$$

! means uniqueness.

...... means existence

□

(ii) An object z is a <u>co-product</u> (or <u>sum</u>) of x and y, often written as $z = x + y$, if there are morphisms $\mathrm{inj}_0 : x \to x + y$ and $\mathrm{inj}_1 : y \to x + y$ s.t.

(for all objects w and morphisms $f : x \to w$ and $g : y \to w$, there exists a unique morphism $h : x + y \to w$ with $f = h \circ \mathrm{inj}_0$ and $g = h \circ \mathrm{inj}_1$

(often written as $[f, g]$. Reynolds writes it as $f \oplus g$)

↳ in picture

$$
\begin{tikzcd}
 & w & \\
 & x + y \arrow[u, dashed, "!h"] & \\
x \arrow[ur, "\mathrm{inj}_0"] \arrow[uur, bend left, "f"] & & y \arrow[ul, "\mathrm{inj}_1"'] \arrow[uul, bend right, "g"']
\end{tikzcd}
$$

□

(iii) An object x is <u>initial</u> if for every object w, there exists the unique morphism $h : x \to w$. □

(iv) An object x is <u>final</u> if for every object w, there exists the unique morphism $h : w \to x$.

(often written as $!_w$ or $!$). □

③ Note that all of these ~~objects~~ notions are defined purely in terms of morphisms, without referring to elements of objects. This is a bit like specifying a property of an abstract data type in terms of ~~propert~~ its operations, not in terms of its implementations.

④ In the category Predom, the product of two predomains P_0 and P_1 that we studied in Chap 5 is indeed a categorical product in (i). Also, the sum of P_0 and P_1 in chap 5 is indeed a categorical sum or co-product. The predomain of the singleton set $(\{*\}, \sqsubseteq)$ is a terminal object. The predomain of the empty set $(\{\}, \sqsubseteq)$ is an initial object.

⑤ Consider the category corresponding to a partially ordered set $(X, \sqsubseteq)$. Then, an object $x \in X$ is initial if and only if it is the ~~smallest~~ least element in X. It is final if and only if it is the greatest element. For objects x, y in X, $x + y$ is the least upper bound of x and y, and $x \times y$ is the greatest lower bound of x and y.

<u>Exercise.</u> Prove ④ and ⑤.