

① Intuitively, the theorem says that x_{fix} is ~~the~~ a least fixed point of F . The condition i) says that x_{fix} is a fixed point. The condition ii) says that it is the least fixed point.

② The proof is complex, but not that difficult. Very similar to the proof of the ~~usual~~ ^{standard} fixed point theorem of the domain theory. We will study only some parts of the proof.

③ The key part of the proof is to construct x_{fix} . Here we use the initial object x_0 of \mathcal{C} , the functoriality of F , and the chain completeness of \mathcal{C} . (They correspond to 1, the monotonicity of f and the chain completeness of a pre-domain D in the proof of the fixed-point theorem in domain theory).

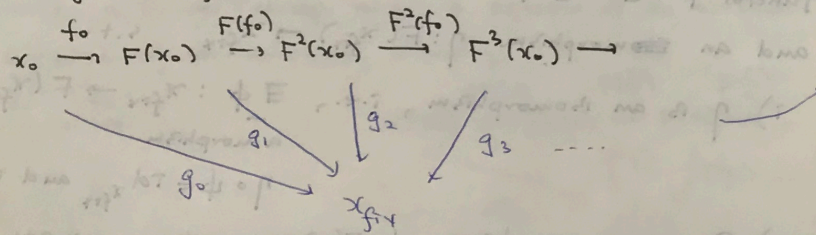
~~def.~~ We construct a chain: $\Delta = x_0 \xrightarrow{f_0} F(x_0) \xrightarrow{F(f_0)} F^2(x_0) \xrightarrow{F^2(f_0)} F^3(x_0) \xrightarrow{F^3(f_0)} \dots$

unique morphism from the initial object x_0 to $F(x_0)$

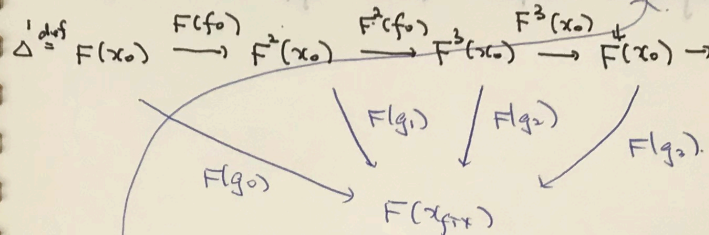
$F(f_0): F(x_0) \rightarrow F^2(x_0)$

$F(F(f_0)): F^2(x_0) \rightarrow F^3(x_0)$

Since \mathcal{C} is chain-complete, there exists a co-limit $(x_{fix}, \{g_i\})$ of the chain Δ that we just built.

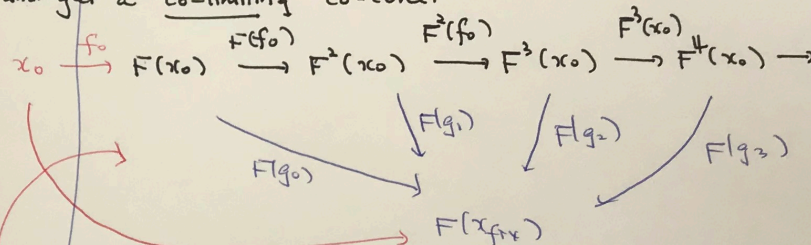


④ Next we build $\eta: F(x_{fix}) \rightarrow x_{fix}$. Here we use the ω -continuity of F . Apply F to the diagram. This gives us



Since F is ω -continuous, this is a co-limiting co-cone of the chain Δ' , which is the x_0 -truncated version of Δ .

⑤ Because x_0 is the initial object, we can add x_0 to the diagram and get a co-limiting co-cone.



g_0'
↑
unique morphism from x_0 to $F(x_{fix})$.

this triangle commutes because of the initiality of x_0 .

Now we have two co-limits, x_{fix} and $F(x_{fix})$, of the same chain Δ . One general result (which is easy to show) is that two co-limits of a chain are isomorphic, which means, in our case that there exists morphisms $\eta: F(x_{fix}) \rightarrow x_{fix}$ and $\psi: x_{fix} \rightarrow F(x_{fix})$ s.t.

$$\psi \circ \eta = id_{F(x_{fix})} \text{ and } \eta \circ \psi = id_{x_{fix}}$$

We just proved i) of the theorem. We leave the proof of ii) as an exercise.