

Exercise. Construct ~~the~~ co-limiting co-cones of  $\omega$ -chains in the category of Set. Do the same thing in the poset category  $(\mathbb{Z}^N, \leq)$ , and in the category of predomains (partially ordered set).

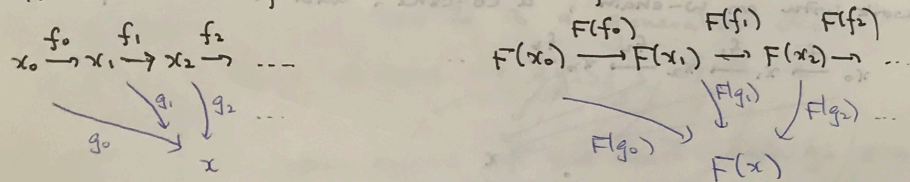
### 3. $\omega$ -Continuous Functor.

① Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories that have co-limiting co-cones for all  $\omega$ -chains. We will call such categories as ~~chain~~ <sup>(in other words, co-limits)</sup> ~~co-complete~~ <sup>chain</sup> ~~co-complete~~ categories.

② A functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is  $\omega$ -continuous if it maps a co-limiting co-cone of an  $\omega$ -chain to a co-limiting co-cone of an  $\omega$ -chain, that is, for every  $\omega$ -chain  $\Delta = (\{x_i\}, \{f_i\})$  in  $\mathcal{C}$ , for every co-limiting co-cone  $(x, \{g_i\})$  of  $\Delta$ ,  $(F(x), \{F(g_i)\})$  is a ~~co-limiting~~ co-limit of  $F(\Delta) = (\{F(x_i)\}, \{F(f_i)\})$  in  $\mathcal{D}$ .

③ Intuitively, this  $\omega$ -continuity of  $F$  means that  $F$  preserves the least upper bound of an increasing chain.

④ Note that a functor  $F$  always maps a co-cone of an  $\omega$ -chain to a co-cone of an  $\omega$ -chain. Visually,



Category  $\mathcal{C}$

Category  $\mathcal{D}$ .

If the diagram in  $\mathcal{C}$  commutes, the diagram in  $\mathcal{D}$  also commutes. This is because the preservation of  $\circ$  and  $\text{id}$  by a functor implies that the functor maps every commuting diagram to a commuting diagram.

The situation is similar to the fact that a monotone function  $f$  from a predomain to a predomain maps an upper bound of a chain to an upper bound of a chain.

⑤ However, the functoriality of  $F$  doesn't ensure that if  $(x, \{g_i\})$  is co-limiting, so is  $(F(x), \{F(g_i)\})$ . When  $F$  satisfies this addition property, we say that  $F$  is  $\omega$ -continuous.

### Exercise.

Show that the functor from Set to Set

$$F(S) = \mathbb{Z} \times S$$

$$F(f) = \text{id}_{\mathbb{Z}} \times f = \lambda(n, s). (n, f(s))$$

is  $\omega$ -continuous.

### 4. Fixed Point Theorem.

[Thm] Let  $\mathcal{C}$  be a category with an initial object  $x_0$ . Assume that  $\mathcal{C}$  is ~~chain~~ <sup>chain</sup> ~~co-complete~~ <sup>co-complete</sup>. (i.e., every  $\omega$ -chain  $\Delta$  in  $\mathcal{C}$  has a co-limit).

Then, for every  $\omega$ -continuous functor  $F: \mathcal{C} \rightarrow \mathcal{C}$ , there exists an object  $x_{\text{fix}}$  in  $\mathcal{C}$  and an ~~isomorphism~~ morphism  $\eta: F(x_{\text{fix}}) \rightarrow x_{\text{fix}}$  s.t.

i)  $\eta$  is an isomorphism, i.e.,  $\exists \phi: x_{\text{fix}} \rightarrow F(x_{\text{fix}})$  s.t. a morphism

$$\eta \circ \phi = \text{id}_{x_{\text{fix}}} \text{ and } \phi \circ \eta = \text{id}_{F(x_{\text{fix}})}$$

ii) for every morphism  $\eta': F(y) \rightarrow y$ , there exists a unique morphism  $\rho: x_{\text{fix}} \rightarrow y$  s.t.

$$\begin{array}{ccc} F(x_{\text{fix}}) & \xrightarrow{\eta} & x_{\text{fix}} \\ F(\rho) \downarrow & & \downarrow \rho \\ F(y) & \xrightarrow{\eta'} & y \end{array} \quad \square$$