# Lecture 13

# Compactness for first-order logic

The compactness theorem, non-standard models of arithmetic

Introduction to Logic for Computer Science

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These slides are minor variants of those made by Prof Worrell and Dr Haase for their logic course at Oxford.

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- (1) all finite subsets of S are satisfiable
- $(2) \qquad \Rightarrow \text{ all finite subsets of } \mathcal{T} \text{ are satisfiable}$
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- $(6) \qquad \Rightarrow \mathcal{S} \text{ is satisfiable.}$

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Ex: Explain why it is ok to rename. You need to prove the following.

- Every finite subset of S before renaming is satisfiable if and only if every finite subset of S after renaming is satisfiable.
- The whole of  $\mathcal S$  before renaming is satisfiable if and only if the whole of  $\mathcal S$  after renaming is satisfiable.

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 $(5) \Rightarrow (6)$ . Ex: Why does this step hold?

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#### Lemma

Let F be a  $\sigma$ -sentence over some signature  $\sigma$  such that F has a model  $\mathcal{A}_n$  with  $|U_{\mathcal{A}_n}| = n$  for every n > 1. Then, F has a model with an infinite universe.

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**Proof:** Introduce a fresh binary predicate R. For n > 1, define

$$G_n = \forall x \neg R(x, x) \land \exists x_1 \ldots \exists x_n \bigwedge_{1 \leq i < j \leq n} R(x_i, x_j).$$

Then,  $B \models G_n$  implies  $|U_{\mathcal{B}}| \geq n$ .

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$$F_n := F \wedge G_n$$
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Every finite subset of S is satisfiable. (Ex2: Why?)

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Every finite subset of S is satisfiable. (Ex2: Why?) By compactness, S has a model B. Then,  $|U_B|$  is infinite, and  $B \models F$ . (Ex3: Why?)

Let  $\sigma=\langle 0,s,+,\cdot,=\rangle$  be the sig. of arithmetic. Can we find a possibly infinite set of  $\sigma$ -formulas whose only model up to isomorphism is the classical arithmetic, i.e. the standard structure on  $\mathbb{N}$ ?

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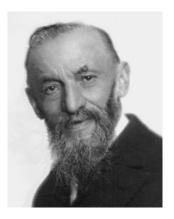


Figure: Giuseppe Peano (1858 - 1932)

Let  $\sigma = \langle 0, s, +, \cdot, = \rangle$ . Peano axioms axiomatise elementary facts about arithmetic in first-order logic with equality:

$$\forall x \neg (s(x) = 0), \qquad \forall x \forall y (s(x) = s(y) \rightarrow x = y),$$
  
 
$$\forall x (x + 0 = x), \qquad \forall x \forall y (x + s(y) = s(x + y)),$$
  
 
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But, cannot quantify over sets in first-order logic. We instead resort to infinitely many axioms, one for each formula  $\phi(x, y_1, \dots, y_K)$ :

$$\forall y_1 \dots y_k \left( \left( \phi(0) \wedge \forall x \Big( \phi(x) \to \phi(s(x)) \Big) \right) \to \forall x \phi(x) \right).$$

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Let  $S_{PA}$  be the union of all formulas above. Then, "classical arithmetic" is a model of  $S_{PA}$ .

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The compactness theorem holds for first-order logic with equality.

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$$\mathcal{C} = \{ \neg (\mathbf{c} = \mathbf{s}^i(0)) : i \in \mathbb{N} \}.$$

Then, every finite subset of  $S_{PA} \cup C$  is satisfiable. (Ex2: Why?)

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Answer3: Because  $c_{\mathcal{A}} \neq s_{\mathcal{A}}^{i}(0_{\mathcal{A}})$  for all  $i \in \mathbb{N}$ .

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Answer: No. Because the following theorem says that we can make an arbitrary big model.

### Theorem (Upward Löwenheim-Skolem theorem)

If a set of formulas  $\mathcal S$  over a finite signature  $\sigma$  has an infinite model  $\mathcal A$ , then for any cardinal  $\kappa$ , it has a model  $\mathcal B$  with a universe of cardinality  $\kappa$ .

### Downward Löwenheim-Skolem theorem

Let  $\sigma$  be the signature

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### **Corollary**

S has a model with a countable universe but not isomorphic to the classic arithmetic.