Lecture 1

History of mathematical logic in computer science

Introduction to Logic for Computer Science

Prof Hongseok Yang
KAIST

These slides are minor variants of those made by Prof Worrell and Dr Haase for their logic course at Oxford.

Agenda

1. A historical perspective on logic
2. Recent applications of computational logic in math, CS and philosophy
3. Overview of the course
4. Practicalities

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What is logic?

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"To discover truths is the task of all sciences; it falls to logic to discern the laws of truth. ... I assign to logic the task of discovering the laws of truth, not of assertion or thought."

—Gottlob Frege, "Der Gedanke. Eine logische Untersuchung".

What is logic?

Logic is the study of the principles of correct reasoning.

Aristotle (384–322 BC)

Syllogism: *"...a discourse in which certain (specific) things having been supposed, something different from the things supposed results of necessity because these things are so."*

$$\begin{array}{l} \text{All } \textit{beings} \text{ are } \textit{mortal} \\ \text{All } \textit{humans} \text{ are } \textit{beings} \\ \hline \text{All } \textit{humans} \text{ are } \textit{mortal} \end{array}$$

$$\begin{array}{l} \text{All } B \text{ are } M \\ \text{All } H \text{ are } B \\ \hline \text{All } H \text{ are } M \end{array}$$

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Aristotle only gave a compendium of valid arguments.

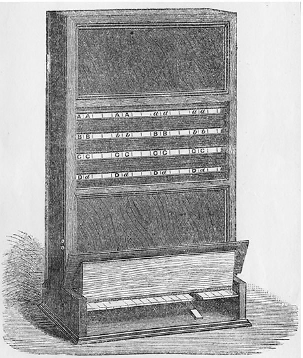
Gottfried Wilhelm Leibniz (1646–1716)

- Envisioned a calculus that allows for testing *any* argument for validity.

"It is obvious that if we could find characters or signs suited for expressing all our thoughts as clearly and as exactly as arithmetic expresses numbers or geometry expresses lines [...] all investigations which depend on reasoning would be carried out by transposing these characters and by a species of calculus."

George Boole (1815–1864)

- Equational rules for *propositional logic* in 1854.
- Built into the logic piano by William Stanley Jevons.

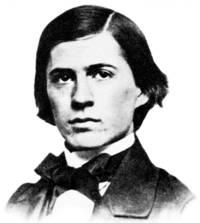


Gottlob Frege (1848–1925) and Charles Sanders Pierce (1839–1914)

- Generalisation of propositional logic to *predicate logic*.

“There is just one point where I have encountered a difficulty . . . ”

— Bertrand Russell to Frege, 1902.



“Beyond doubt . . . he was one of the most original minds of the later nineteenth century, and certainly the greatest American thinker ever.”

— Bertrand Russell about Pierce, 1959.

Bertrand Russell (1872–1970) and Alfred North Whitehead (1861–1947)

- Principia Mathematica: attempt to base all of mathematics (set theory, analysis, geometry) rigorously on predicate logic.

324 QUANTITY [PART VI

$*311{\cdot}511.\ \vdash : \text{Infin ax} \,.\, \xi \in C`\Theta \,.\, Y \in C`H \,.\supset.\, (\exists X) \,.\, X \in \xi \,.\, Y +_g X \sim\in \xi$
[$*311{\cdot}51$. Transp]

$*311{\cdot}52.\ \vdash : \text{Infin ax} \,.\, \xi, \eta \in C`\Theta \,.\supset.\, \xi \Theta (\xi +_p \eta)$

Dem.

$\vdash . *311{\cdot}511 . \supset \vdash :. \text{Hp} . \supset : Y \in C`H . \supset . (\exists X) . X \in \xi . X +_g Y \sim\in \xi :$
[$*311{\cdot}11$] $\supset : (\exists X, Y) . X +_g Y \in (\xi +_p \eta) - \xi :$
[$*310{\cdot}11 . *311{\cdot}27$] $\supset : \xi \Theta (\xi +_p \eta) :. \supset \vdash .$ Prop

$*311{\cdot}53.\ \vdash : \text{Infin ax} \,.\, \xi, \eta \in C`\Theta_n \,.\supset.\, \xi \Theta_n (\xi +_p \eta)$ [$*311{\cdot}52{\cdot}33$]

$*311{\cdot}56.\ \vdash :. \text{Infin ax} \,.\, \xi \in C`\Theta_g \,.\supset : \xi = \xi +_p \eta \,.\equiv.\, \eta = \iota`0_q$ [$*311{\cdot}1{\cdot}43{\cdot}52{\cdot}53$]

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$*311{\cdot}58.\ \vdash : \text{Infin ax} \,.\, \mu \in C`\Theta \,.\supset.\, \mu = H``\mu$ [$*304{\cdot}3 . *270{\cdot}31$]

$*311{\cdot}6.\ \vdash : \text{Infin ax} \,.\, \mu \Theta \nu \,.\, X, Y \in \nu - \mu \,.\, XHY \,.\, M \in \mu \,.\supset.\, M +_g (Y -_s X) \in \nu$

Dem.

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[$*308{\cdot}42{\cdot}72$] $\supset . \{M +_g (Y -_s X)\} HY$ (1)
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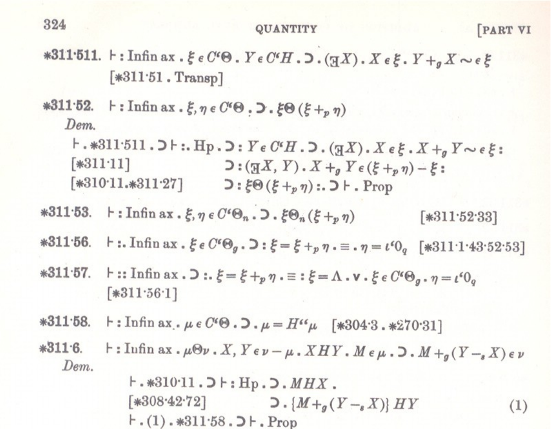

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(proves that 1+1=2 on page 379)

Kurt Gödel (1906–1978)

- No sufficiently strong logical system can be both consistent and complete.

"Kurt Gödel's achievement in modern logic is singular and monumental - indeed it is more than a monument, it is a landmark which will remain visible far in space and time. [. . .] The subject of logic has certainly completely changed its nature and possibilities with Gödel's achievement."

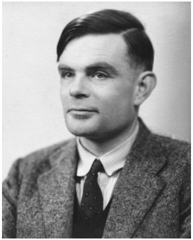
— John von Neumann

Alonso Church (1903–1995) and Alan Turing (1912–1954)

- There is no algorithm that decides whether a given logical argument is valid or not.
- λ-*calculus* and *Turing machines* lay the foundations for theoretical computer science.

Alonzo Church, A note on the Entscheidungsproblem", *Journal of Symbolic Logic*, 1 (1936), pp 40-41.

Alan Turing, On computable numbers, with an application to the Entscheidungsproblem, *Proceedings of the London Mathematical Society*, Series 2, 42 (1936-7), pp 230-265.



Claude Shannon (1916–2001) and John Robinson (1930–2016)

Using electrical switches to compute Boolean functions:

Claude Shannon. A Symbolic Analysis of Relay and Switching Circuits, *MSc Thesis*, 1937.

Resolution and *unification*: automated reasoning and logic programming:

John Robinson. A Machine-Oriented Logic Based on the Resolution Principle, *Journal of the ACM*, 1965.

Logic is fundamental to computer science

When I was a student, even the topologists regarded mathematical logicians as living in outer space. Today the connections between logic and computers are a matter of engineering practice at every level of computer organization.

—Martin Davis, "Influences of Mathematical Logic on Computer Science".

Logic is everywhere

- Hardware design.
- Database theory.
- Automated verification.
- Knowledge representation.
- Programming-language theory.
- Complexity theory.
- Constraint satisfaction problems.

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- . . . we won't be focussing on any of the above. Our focus is on foundations of logic.
- But we study the logic from the computational perspective.

Finding a needle amongst $1,566 \times 10^{349}$ needles

Erdős discrepancy conjecture

For any $C > 0$ and any infinite sequence $x_1 x_2 x_3 \cdots$ of $+1$'s and -1's, there exist $d, k \in \mathbb{N}$ such that

$$\Big| \sum_{1 \leq i \leq k} x_{id} \Big| > C.$$

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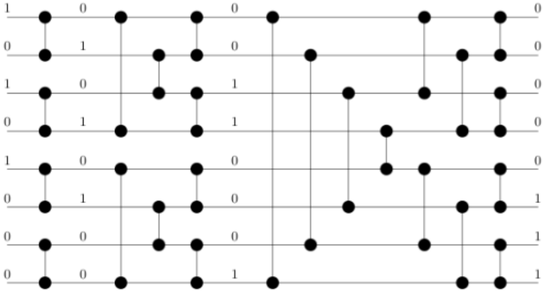
$$\left| \sum_{1 \leq i \leq k} x_{id} \right| > C.$$

- First shown to hold when $\sum_{1 \leq i \leq k} x_{id} \leq 1$ for all d, k by A.R.D. Mathias in 1993. This implies the case $C = 1$.
- Investigated as a Polymath project in 2009-10:

  "Given how long a finite sequence can be, it seems unlikely that we could answer this question [for $C = 2$] just by a clever search of all possibilities on a computer."
- B. Konev and A. Lisitsa solve the case $C = 2$ in 2014 using SAT solver.
- 13Gb of proof that no sequence of length 1161 with discrepancy 2 exists.
- Conjecture proved for all $C > 0$ by T. Tao in 2015.

Optimal sorting networks



- Consist of wires and comparators, input flows from left to right.
- Depth is the maximum number of comparators an input can encounter.
- D. Knuth and R. Floyd found optimal depths for $n = 1, \ldots, 8$ in 1960s.
- D. Bundala and J. Závodný found optimal depths for $n = 11, \ldots, 16$ in 2014 using SAT solvers.

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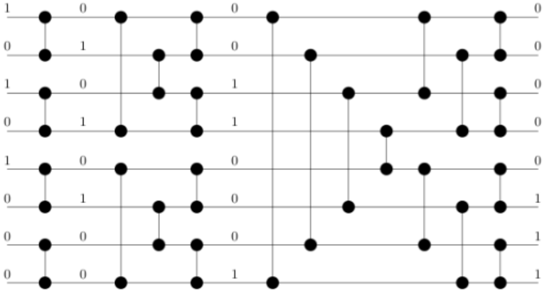

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n	5	6	7	8	9	10	11	12	13	14	15	16
d	5	5	6	6	7	7	8	8	9	9	9	9

Leibniz's ontological proof

Leibniz's proof for the existence of god

1. Definition: God is a being having all perfections.
2. Definition: A perfection is a simple and absolute property.
3. Existence is a perfection.
4. If existence is part of the essence of a thing, then it is a necessary being.
5. If it is possible for a necessary being to exist, then a necessary being does exist.
6. It is possible for a being to have all perfections.
7. Therefore, a necessary being (God) does exist.

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7. Therefore, a necessary being (God) does exist.

- Leibniz's "algebra of concepts" formalised in the theorem prover Isabelle/HOL by Bentert, Benzmüller, Streit and Paleo in 2016.
- Showed validity and invalidity of Leibniz's proof in the algebra of concepts depending on the interpretation of some informal statements.

Focus

- Foundations.
- Computational aspects in particular.
- Computability and algorithms for satisfiability/validity problems.
- Based on the course by Prof Worrell and Dr Haase at Oxford.

Topics

- Propositional logic.
  - The SAT problem.
  - Polynomial-time algorithms for Horn, 2-SAT, and XOR formulas.
  - Resolution.
  - DPLL with clause learning.
  - Compactness theorem.
- First-order logic.
  - Prenex normal form and Skolemisation.
  - Herbrand models and ground resolution.
  - Unification and resolution for predicate logic.
  - Undecidability of satisfiability.
  - Logical theories and quantifier elimination.
  - Automatic structures.
  - Ehrenfeucht-Fraisse games.

Topics - Basics

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Topics – Algorithms or semi-algorithms

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Topics – Computability and complexity

- Propositional logic.
  - The SAT problem.
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  - Resolution.
  - DPLL with clause learning.
  - Compactness theorem.
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  - Ehrenfeucht-Fraisse games.

Course Materials and Information

- KLMS is the main source. Check it regularly.
- Lecture notes, slides, homework sheets all there.
- Also, announcements and Q&A.

Teaching staffs

- Lecturer: Prof Hongseok Yang (email: hongseok00@gmail.com).
- TA1: Mr Hyoungjin Lim (email: lmkmkr@kaist.ac.kr).
- TA2: Mr Jungmin Park (email: qkrclrl701@kaist.ac.kr).
- Yang's office hours: 6pm - 7pm on Mon. To be held online using ZOOM.
- TAs' office hours: To be announced at KLMS.

Evaluation

- Final exam (35%).
- Homework (30%) – 4-5 problem sheets.
- Programming project (15%) – DPLL-based SAT solver.
- Critical review on binary decision diagram (20%).

Critical review

- On binary decision diagram.
- At most 3 pages excluding bibliography.
- Submission deadline: 23:59 of 27 May 2020 (Wednesday).
- Submit it in KLMS.
- Difficult but comprehensive reference: Chapter 7.1.4 of Knuth's "The Art of Computer Programming" (vol. 4A or fasc. 1 of vol. 4).
- All quoted sentences from existing literature should be marked so explicitly with pointers to their sources. If not, zero mark.

Honour code

- We adopt a very strict policy for handling the violation of the standard honour code.
- If a student is found to cheat in an exam or copy answers or code from friends' or other souces, the student will get F.

Recommended Textbook

- Lecture notes in KLMS.
- *'The Calculus of Computation: Decision Procedures with Applications to Verification'*, A. Bradley and Z. Manna.
- *'Logic in Computer Science: Modelling and Reasoning about Systems'*, M. Huth and M. Ryan.

Further Literature

- *'Logicomix: An Epic Search for Truth'*, A. Doxiadis and C. Papadimitriou.
- *'Handbook of Practical Logic and Automated Reasoning'*, J. Harrison.
- *'Mathematical Logic for Computer Science'*, M. Ben-Ari.
- *'Logic for Computer Scientists'*, U. Schöning.
- *'Gödel, Escher, Bach: an Eternal Golden Braid'*, D. Hofstadter.

Questions to check your background knowledge

- Let X be a non-negative real-valued random variable, and $c > 0$ be a positive real number. Prove that

$$P(X > c) \leq \frac{\mathbb{E}(X)}{c}.$$

- What is the time complexity of solving a system of linear equations $Ax = b$ using Gaussian elimination?
- Explain the following notions: NP, NP-complete, regular, recursive, and recursively enumerable.
- Consider a subset $L \subseteq \Sigma^*$ over a finite Σ. Show that if both L and L^c are recursively enumerable, then L is recursive.
- Assume that $L, L' \subseteq \Sigma^*$ over a finite Σ are regular. Let $f : \Sigma \to \Sigma'$ be a map to a finte Σ'. Show that $L \cap L$ is regular. Also, prove that $f(L) = \{f(x_1)\dots f(x_n) \mid x_1 \dots x_n \in L\}$ is regular.
- Prove: the program (`y=1; while (x>0) y=y*x; x=x-1;`) computes the factorial of `x`.