Introduction to Logic for Computer Science Spring 2021

Assignment 3 (Deadline: 6:00pm on 14 May 2021)

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Submit your solutions in KLMS. (Reminder: We adopt a very strict policy for handling dishonest behaviours. If a student is found to copy answers from fellow students or other sources in his or her homework submission, she or he will get F.)

1. Given an undirected graph $G = (V, E)$, a set of vertices $S \subseteq V$ is a *clique* if every pair of distinct vertices $u, v \in S$ is connected by an edge and S is an *independent set* if no pair of distinct vertices $u, v \in S$ is connected by an edge. Now consider the following two statements:

   (A) Every infinite graph either has an infinite clique or an infinite independent set.

   (B) For all k there exists n such that any graph with n vertices has a clique of size k or an independent set of size k.

   Using the Compactness Theorem, prove that (A) implies (B).[1] **(20 points)**

2. Translate the following formula to rectified form, then to prenex form, and finally to Skolem form:
$$\forall z\, \exists y\, (Q(x, g(y), z) \vee \neg \forall x\, P(x)) \wedge \neg \forall z\, \exists x\, \neg R(f(x, z), z).$$
   (10 points)

3. Let σ be a signature with finitely many relation and constant symbols, but no function symbols.

   (a) Given σ-formulas $G_1, \ldots, G_n$ and a propositional formula F that mentions variables $p_1, \ldots, p_n$, let $F[G_1/p_1, \ldots, G_n/p_n]$ denote the σ-formula obtained by substituting G_i for all occurrences of p_i in F. Give a formal definition of $F[G_1/p_1, \ldots, G_n/p_n]$.
   (5 points)

   (b) Given propositional formulas $F \equiv F'$, both over variables $p_1, \ldots, p_n$, and σ-formulas $G_1 \equiv G'_1, \ldots, G_n \equiv G'_n$, show that $F[G_1/p_1, \ldots, G_n/p_n] \equiv F'[G'_1/p_1, \ldots, G'_n/p_n]$.
   (5 points)

   (c) Fix $n \in \mathbb{N}$. Show that up to logical equivalence there are only finitely many quantifier-free σ-formulas that use first-order variables $x_1, \ldots, x_n$.
   (10 points)

   (d) Fix $n, k \in \mathbb{N}$. Show that up to logical equivalence there are only finitely many σ-formulas of quantifier depth at most k that use first-order variables $x_1, \ldots, x_n$.
   (10 points)

4. Fix a signature σ. Consider a relation $\sim$ on σ-assignments that satisfies the following two properties:

[1] As an optional exercise, beyond the scope of the course, you can try to prove (A). This result is Theorem 1 in `https://www.dpmms.cam.ac.uk/~par31/notes/ramsey.pdf`. Combining this with your answer, we obtain a proof of (B).

(P1) If $\mathcal{A} \sim \mathcal{B}$, then for every atomic formula F we have $\mathcal{A} \models F$ iff $\mathcal{B} \models F$.

(P2) If $\mathcal{A} \sim \mathcal{B}$, then for each variable x we have (i) for each $a \in U_{\mathcal{A}}$ there exists $b \in U_{\mathcal{B}}$ such that $\mathcal{A}_{[x \mapsto a]} \sim \mathcal{B}_{[x \mapsto b]}$, and (ii) for all $b \in U_{\mathcal{B}}$ there exists $a \in U_{\mathcal{A}}$ such that $\mathcal{A}_{[x \mapsto a]} \sim \mathcal{B}_{[x \mapsto b]}$.

Prove that if $\mathcal{A} \sim \mathcal{B}$ then for any formula F, $\mathcal{A} \models F$ if and only if $\mathcal{B} \models F$. You may assume that F is built from atomic formulas using the connectives $\wedge$ and $\neg$ and the quantifier $\exists$.

(20 points)

5. In this question we work with first-order logic without equality.

   (a) Consider a signature σ containing only a binary relation symbol R. For each positive integer n show that there is a satisfiable σ-formula F_n such that every model $\mathcal{A}$ of F_n has at least n elements. **(5 points)**

   (b) Consider a signature σ containing only unary predicate symbols $P_1, \ldots, P_k$. Using Question 4, or otherwise, show that any satisfiable σ-formula has a model with at most 2^k elements. **(15 points)**