Introduction to Logic for Computer Science Spring 2021

Assignment 1 (Deadline: 6:00pm on 31 March 2021)

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Submit your solutions in KLMS. (Reminder: We adopt a very strict policy for handling dishonest behaviours. If a student is found to copy answers from fellow students or other sources in his or her homework submission, she or he will get F.)

Notation: symbols F, G, H denote propositional formulas, and p, q denote propositional variables.

1. Let F, G and H be formulas and let $\mathcal{S}$ be a set of formulas. Let $p_1, \ldots, p_n$ be propositional variables. Which of the following statements are true? Justify your answer. **(10 points)**

   (a) If $F \to G$ is satisfiable and F is satisfiable, then G is satisfiable.

   (b) $p_1 \to (p_2 \to (p_3 \to \ldots (p_n \to p_1) \ldots))$ is valid.

   (c) $\mathcal{S} \models F$ and $\mathcal{S} \models \neg F$ cannot both hold.

   (d) If $\mathcal{S} \models F \vee G$, $\mathcal{S} \cup \{F\} \models H$ and $\mathcal{S} \cup \{G\} \models H$, then $\mathcal{S} \models H$.

2. Suppose that F and G are formulas such that $F \models G$.

   (a) Show that if F and G have no variable in common, then either F is unsatisfiable or G is valid. **(10 points)**

   (b) Now let F and G be arbitrary formulas. Show that there is a formula H, mentioning only propositional variables common to F and G, such that $F \models H$ and $H \models G$. **(10 points)**

3. A **clique** in an undirected graph is a set of vertices S such that there is an edge between every pair of distinct vertices in S. Given a finite graph G and integer k, describe how to obtain a propositional formula φ such that φ is satisfiable if and only if G has a clique of size k. The formula φ should be computable in time polynomial in the number of vertices of G. **(20 points)**

4. Fix a non-empty set U. A U**-assignment** $\mathcal{A}$ his a function from the collection of propositional variables to 2^U, the power set of U, that is, $\mathcal{A}$ maps each propositional variable to a subset of U. Such an assignment is extended to all formulas as follows:

   - $\hat{\mathcal{A}}(\mathit{false}) = \emptyset$ and $\hat{\mathcal{A}}(\mathit{true}) = U$;
   - $\hat{\mathcal{A}}(p) = \mathcal{A}(p)$;
   - $\hat{\mathcal{A}}(F \wedge G) = \hat{\mathcal{A}}(F) \cap \hat{\mathcal{A}}(G)$;
   - $\hat{\mathcal{A}}(F \vee G) = \hat{\mathcal{A}}(F) \cup \hat{\mathcal{A}}(G)$;
   - $\hat{\mathcal{A}}(\neg F) = U \setminus \hat{\mathcal{A}}(F)$.

   Say that a formula F is U-**valid** if $\hat{\mathcal{A}}(F) = U$ for all U-assignments $\mathcal{A}$.

   (a) Show that if F is U-valid, then F is valid with respect to the standard semantics defined in the lecture notes. **(10 points)**

(b) Show that if F is valid, then F is U-valid. **(10 points)**

5. (a) Write down a **DNF**-formula equivalent to $(p_1 \vee q_1) \wedge (p_2 \vee q_2) \wedge \cdots \wedge (p_n \vee q_n)$. Here the p_i and q_i are propositional variables. **(10 points)**

   (b) Prove that any **DNF**-formula equivalent to the above formula must have at least 2^n clauses. **(20 points)**