Lecture 9

Normal forms for first-order logic

Equivalences, prenex form, Skolem form

Introduction to Logic for Computer Science

Prof Hongseok Yang
KAIST

These slides are minor variants of those made by Prof Worrell and Dr Haase for their logic course at Oxford.

Recap

Syntax of first-order formulas:

- Signature σ (constant, function and predicate symbols).
- σ-terms.
- Formulas (predicate symbols, logical connectives of propositional logic, and additional $\forall x$ and $\exists x$).

Recap

Syntax of first-order formulas:

- Signature σ (constant, function and predicate symbols).
- σ-terms.
- Formulas (predicate symbols, logical connectives of propositional logic, and additional $\forall x$ and $\exists x$).

Semantics of first-order formulas:

- σ-structure $\mathcal{A}$ with universe $U_{\mathcal{A}}$ and interpretations of constants, functions, predicates, and variables.
- $\mathcal{A} \models F$ defined by structural induction on F.

Recap

Syntax of first-order formulas:

- Signature σ (constant, function and predicate symbols).
- σ-terms.
- Formulas (predicate symbols, logical connectives of propositional logic, and additional $\forall x$ and $\exists x$).

Semantics of first-order formulas:

- σ-structure $\mathcal{A}$ with universe $U_{\mathcal{A}}$ and interpretations of constants, functions, predicates, and variables.
- $\mathcal{A} \models F$ defined by structural induction on F.

Relevance lemma: "If $\mathcal{A}$ and $\mathcal{A}'$ only differ on variables other than free variables in F, then $\mathcal{A} \models F$ if and only if $\mathcal{A}' \models F$."

Normal forms

$$\neg(\exists x\, P(x,y) \vee \forall z\, Q(z)) \wedge \exists w\, Q(w)$$

vs

$$\forall x\, \exists z\, \exists w\, ((\neg P(x,y) \wedge \neg Q(z)) \wedge Q(w)).$$

Normal forms

$$\neg(\exists x\, P(x,y) \vee \forall z\, Q(z)) \wedge \exists w\, Q(w)$$

vs

$$\forall x\, \exists z\, \exists w\, ((\neg P(x,y) \wedge \neg Q(z)) \wedge Q(w)).$$

Today:

- Establish elementary equivalences.
- Prenex form: all quantifiers first.
- Skolem form: prenex form with no existential quantifiers.

Equivalences

Definition

Two first-order logic formulas F and G over the signature σ are **logically equivalent** (written $F \equiv G$) if $\mathcal{A} \models F$ iff $\mathcal{A} \models G$ for all σ-structures $\mathcal{A}$.

Equivalences

Definition

Two first-order logic formulas F and G over the signature σ are **logically equivalent** (written $F \equiv G$) if $\mathcal{A} \models F$ iff $\mathcal{A} \models G$ for all σ-structures $\mathcal{A}$.

Proposition

Let F and G be arbitrary formulas. Then, the following hold.

(A) $\neg \forall x F \equiv \exists x \neg F$ and $\neg \exists x F \equiv \forall x \neg F$.

(B) If x does not occur free in G then:

$$
\begin{aligned}
(\forall x F \wedge G) &\equiv \forall x (F \wedge G), & (\forall x F \vee G) &\equiv \forall x (F \vee G),\\
(\exists x F \wedge G) &\equiv \exists x (F \wedge G), & (\exists x F \vee G) &\equiv \exists x (F \vee G).
\end{aligned}
$$

(C) $(\forall x F \wedge \forall x G) \equiv \forall x (F \wedge G)$ and $(\exists x F \vee \exists x G) \equiv \exists x (F \vee G)$.

(D) $\forall x \forall y F \equiv \forall y \forall x F$ and $\exists x \exists y F \equiv \exists y \exists x F$.

Equivalences

Definition

Two first-order logic formulas F and G over the signature σ are **logically equivalent** (written $F \equiv G$) if $\mathcal{A} \models F$ iff $\mathcal{A} \models G$ for all σ-structures $\mathcal{A}$.

Proposition

Let F and G be arbitrary formulas. Then, the following hold.

(A) $\neg\forall xF \equiv \exists x\neg F$ and $\neg\exists xF \equiv \forall x\neg F$.

(B) If x does not occur free in G then:

$$(\forall xF \wedge G) \equiv \forall x(F \wedge G), \qquad (\forall xF \vee G) \equiv \forall x(F \vee G),$$
$$(\exists xF \wedge G) \equiv \exists x(F \wedge G), \qquad (\exists xF \vee G) \equiv \exists x(F \vee G).$$

(C) $(\forall xF \wedge \forall xG) \equiv \forall x(F \wedge G)$ and $(\exists xF \vee \exists xG) \equiv \exists x(F \vee G)$.

(D) $\forall x\forall yF \equiv \forall y\forall xF$ and $\exists x\exists yF \equiv \exists y\exists xF$.

Ex: Prove the highlighted cases of (B) and (C).

Prenex form

Definition

A formula is in **prenex form** if it can be written

$$Q_1 y_1\, Q_2 y_2 \ldots Q_n y_n\, F,$$

where $Q_i \in \{\exists, \forall\}$, $n \geq 0$, and F contains no quantifiers. In this case F is called the **matrix** of the formula.

Prenex form

Definition

A formula is in **prenex form** if it can be written

$$Q_1 y_1 \; Q_2 y_2 \ldots Q_n y_n \; F,$$

where $Q_i \in \{\exists, \forall\}$, $n \geq 0$, and F contains no quantifiers. In this case F is called the **matrix** of the formula.

Example:

$$\neg((\exists x \, P(x, y)) \vee \forall z \, Q(z)) \wedge \exists w \, Q(w)$$

Prenex form

Definition

A formula is in **prenex form** if it can be written

$$Q_1 y_1\, Q_2 y_2 \ldots Q_n y_n\, F,$$

where $Q_i \in \{\exists, \forall\}$, $n \geq 0$, and F contains no quantifiers. In this case F is called the **matrix** of the formula.

Example:

$$\begin{aligned} & \neg((\exists x\, P(x,y)) \vee \forall z\, Q(z)) \wedge \exists w\, Q(w) \\ \equiv \quad & ((\neg\exists x\, P(x,y)) \wedge \neg\forall z\, Q(z)) \wedge \exists w\, Q(w) \end{aligned}$$

Prenex form

Definition

A formula is in **prenex form** if it can be written

$$Q_1 y_1\, Q_2 y_2 \ldots Q_n y_n\, F,$$

where $Q_i \in \{\exists, \forall\}$, $n \geq 0$, and F contains no quantifiers. In this case F is called the **matrix** of the formula.

Example:

$$\begin{aligned}
& \neg((\exists x\, P(x,y)) \vee \forall z\, Q(z)) \wedge \exists w\, Q(w) \\
\equiv \quad & ((\neg\exists x\, P(x,y)) \wedge \neg\forall z\, Q(z)) \wedge \exists w\, Q(w) \\
\equiv \quad & ((\forall x\, \neg P(x,y)) \wedge \exists z\, \neg Q(z)) \wedge \exists w\, Q(w)
\end{aligned}$$

Prenex form

Definition

A formula is in **prenex form** if it can be written

$$Q_1 y_1 \, Q_2 y_2 \ldots Q_n y_n \, F,$$

where $Q_i \in \{\exists, \forall\}$, $n \geq 0$, and F contains no quantifiers. In this case F is called the **matrix** of the formula.

Example:

$$\begin{aligned}
& \neg((\exists x \, P(x,y)) \vee \forall z \, Q(z)) \wedge \exists w \, Q(w) \\
\equiv \quad & ((\neg \exists x \, P(x,y)) \wedge \neg \forall z \, Q(z)) \wedge \exists w \, Q(w) \\
\equiv \quad & ((\forall x \, \neg P(x,y)) \wedge \exists z \, \neg Q(z)) \wedge \exists w \, Q(w) \\
\equiv \quad & \forall x \, (\neg P(x,y) \wedge \exists z \, \neg Q(z)) \wedge \exists w \, Q(w)
\end{aligned}$$

Prenex form

Definition

A formula is in **prenex form** if it can be written

$$Q_1 y_1\, Q_2 y_2 \dots Q_n y_n\, F,$$

where $Q_i \in \{\exists, \forall\}$, $n \geq 0$, and F contains no quantifiers. In this case F is called the **matrix** of the formula.

Example:

$$\begin{aligned}
& \neg((\exists x\, P(x,y)) \vee \forall z\, Q(z)) \wedge \exists w\, Q(w) \\
\equiv\;& ((\neg\exists x\, P(x,y)) \wedge \neg\forall z\, Q(z)) \wedge \exists w\, Q(w) \\
\equiv\;& ((\forall x\, \neg P(x,y)) \wedge \exists z\, \neg Q(z)) \wedge \exists w\, Q(w) \\
\equiv\;& \forall x\, (\neg P(x,y) \wedge \exists z\, \neg Q(z)) \wedge \exists w\, Q(w) \\
\equiv\;& \forall x\, \exists z\, (\neg P(x,y) \wedge \neg Q(z)) \wedge \exists w\, Q(w)
\end{aligned}$$

Prenex form

Definition

A formula is in **prenex form** if it can be written

$$Q_1 y_1\, Q_2 y_2 \dots Q_n y_n\, F,$$

where $Q_i \in \{\exists, \forall\}$, $n \geq 0$, and F contains no quantifiers. In this case F is called the **matrix** of the formula.

Example:

$$\begin{aligned}
& \neg((\exists x\, P(x,y)) \vee \forall z\, Q(z)) \wedge \exists w\, Q(w) \\
\equiv\ & ((\neg\exists x\, P(x,y)) \wedge \neg\forall z\, Q(z)) \wedge \exists w\, Q(w) \\
\equiv\ & ((\forall x\, \neg P(x,y)) \wedge \exists z\, \neg Q(z)) \wedge \exists w\, Q(w) \\
\equiv\ & \forall x\, (\neg P(x,y) \wedge \exists z\, \neg Q(z)) \wedge \exists w\, Q(w) \\
\equiv\ & \forall x\, \exists z\, (\neg P(x,y) \wedge \neg Q(z)) \wedge \exists w\, Q(w) \\
\equiv\ & \forall x\, \exists z\, \exists w\, ((\neg P(x,y) \wedge \neg Q(z)) \wedge Q(w))\,.
\end{aligned}$$

Prenex form

Definition

A formula is in **prenex form** if it can be written

$$Q_1 y_1\, Q_2 y_2 \ldots Q_n y_n\, F,$$

where $Q_i \in \{\exists, \forall\}$, $n \geq 0$, and F contains no quantifiers. In this case F is called the **matrix** of the formula.

Example:

$$\begin{aligned}
& \neg((\exists x\, P(x,y)) \vee \forall z\, Q(z)) \wedge \exists w\, Q(w) \\
\equiv\; & ((\neg\exists x\, P(x,y)) \wedge \neg\forall z\, Q(z)) \wedge \exists w\, Q(w) \\
\equiv\; & ((\forall x\, \neg P(x,y)) \wedge \exists z\, \neg Q(z)) \wedge \exists w\, Q(w) \\
\equiv\; & \forall x\, (\neg P(x,y) \wedge \exists z\, \neg Q(z)) \wedge \exists w\, Q(w) \\
\equiv\; & \forall x\, \exists z\, (\neg P(x,y) \wedge \neg Q(z)) \wedge \exists w\, Q(w) \\
\equiv\; & \forall x\, \exists z\, \exists w\, ((\neg P(x,y) \wedge \neg Q(z)) \wedge Q(w))\,.
\end{aligned}$$

Ex: Transform $\neg\Big(\forall x\, \exists y\, P(x,y,z) \rightarrow \forall x \big(\neg\exists y\, Q(y,z) \rightarrow R(x)\big)\Big)$.

Prenex form

Definition

A formula is in **prenex form** if it can be written

$$Q_1 y_1 \, Q_2 y_2 \ldots Q_n y_n \, F,$$

where $Q_i \in \{\exists, \forall\}$, $n \geq 0$, and F contains no quantifiers. In this case F is called the **matrix** of the formula.

Example:

$$\begin{aligned}
& \neg((\exists x\, P(x,y)) \vee \forall z\, Q(z)) \wedge \exists w\, Q(w) \\
\equiv \quad & ((\neg\exists x\, P(x,y)) \wedge \neg\forall z\, Q(z)) \wedge \exists w\, Q(w) \\
\equiv \quad & ((\forall x\, \neg P(x,y)) \wedge \exists z\, \neg Q(z)) \wedge \exists w\, Q(w) \\
\equiv \quad & \forall x\, (\neg P(x,y) \wedge \exists z\, \neg Q(z)) \wedge \exists w\, Q(w) \\
\equiv \quad & \forall x\, \exists z\, (\neg P(x,y) \wedge \neg Q(z)) \wedge \exists w\, Q(w) \\
\equiv \quad & \forall x\, \exists z\, \exists w\, ((\neg P(x,y) \wedge \neg Q(z)) \wedge Q(w)) \, .
\end{aligned}$$

Ex: Transform $\neg\Big((\forall x\, \exists y\, P(x,y,z)) \rightarrow \forall x\big((\neg\exists y\, Q(y,z)) \rightarrow R(x)\big)\Big)$.

Translation lemma

Denote by $F[t/x]$ the formula obtained by replacing every *free* occurrence of x in F with t.

Translation lemma

Denote by $F[t/x]$ the formula obtained by replacing every *free* occurrence of x in F with t.

Example

$$(\forall x\, P(x, y) \wedge Q(x))[t/x] = \forall x\, P(x, y) \wedge Q(t)\,.$$

Translation lemma

Denote by $F[t/x]$ the formula obtained by replacing every *free* occurrence of x in F with t.

Example

$$(\forall x\, P(x,y) \wedge Q(x))[t/x] = \forall x\, P(x,y) \wedge Q(t)\,.$$

Ex: Define $F[t/x]$ formally using structural induction.

Translation lemma

Denote by $F[t/x]$ the formula obtained by replacing every *free* occurrence of x in F with t.

Example

$$(\forall x\, P(x,y) \wedge Q(x))[t/x] = \forall x\, P(x,y) \wedge Q(t)\,.$$

Ex: Define $F[t/x]$ formally using structural induction.

Lemma (Translation Lemma)

If t is a term and F is a formula such that no variable in t occurs bound in F, then $\mathcal{A} \models F[t/x]$ iff $\mathcal{A}_{[x \mapsto \mathcal{A}(t)]} \models F$.

Translation lemma

Denote by $F[t/x]$ the formula obtained by replacing every *free* occurrence of x in F with t.

Example

$$(\forall x\, P(x,y) \wedge Q(x))[t/x] = \forall x\, P(x,y) \wedge Q(t)\,.$$

Ex: Define $F[t/x]$ formally using structural induction.

Lemma (Translation Lemma)

If t is a term and F is a formula such that no variable in t occurs bound in F, then $\mathcal{A} \models F[t/x]$ iff $\mathcal{A}_{[x \mapsto \mathcal{A}(t)]} \models F$.

Proof: By structural induction.

Ex1: Prove the case that $F = \forall y G$.

Ex2: Why do we need the variable condition in the lemma? Find a counter-example $(F, \mathcal{A})$.

Rectified formulas

Definition

A formula is **rectified** if (i) no variable occurs both bound and free and (ii) all quantifiers in the formula refer to different variables.

Rectified formulas

Definition

A formula is **rectified** if (i) no variable occurs both bound and free and (ii) all quantifiers in the formula refer to different variables.

Proposition

Let $F = Qx\, G$ be a formula where $Q \in \{\forall, \exists\}$. Let y be a variable that does not occur free or bound in G. Then $F \equiv Qy\, (G[y/x])$.

Rectified formulas

Definition

A formula is **rectified** if (i) no variable occurs both bound and free and (ii) all quantifiers in the formula refer to different variables.

Proposition

Let $F = Qx\, G$ be a formula where $Q \in \{\forall, \exists\}$. Let y be a variable that does not occur free or bound in G. Then $F \equiv Qy\,(G[y/x])$.

Ex1: Use the proposition and show that for every formula F, there is a rectified formula G such that $F \equiv G$.

Ex2: Prove the proposition. Hint: Use the Translation lemma and the Relevance lemma.

Proposition

Let $F = Qx\,G$ be a formula where $Q \in \{\forall, \exists\}$. Let y be a variable that does not occur in G. Then $F \equiv Qy\,(G[y/x])$.

Proof.

Proof for $\forall$:

$$
\begin{array}{ll}
 & \mathcal{A} \models \forall y\,(G[y/x]) \\
\text{iff} & \mathcal{A}_{[y\mapsto a]} \models G[y/x] \text{ for all } a \in U_{\mathcal{A}} \\
\text{iff} & \mathcal{A}_{[y\mapsto a][x\mapsto \mathcal{A}_{[y\mapsto a]}(y)]} \models G \text{ for all } a \in U_{\mathcal{A}} \text{ (Translation Lemma)} \\
\text{iff} & \mathcal{A}_{[y\mapsto a][x\mapsto a]} \models G \text{ for all } a \in U_{\mathcal{A}} \\
\text{iff} & \mathcal{A}_{[x\mapsto a][y\mapsto a]} \models G \text{ for all } a \in U_{\mathcal{A}} \\
\text{iff} & \mathcal{A}_{[x\mapsto a]} \models G \text{ for all } a \in U_{\mathcal{A}} \text{ (Relevance Lemma)} \\
\text{iff} & \mathcal{A} \models \forall x\,G\,. \quad \square
\end{array}
$$

Prenex form again

Theorem

Every formula is equivalent to a formula in rectified prenex form.

Prenex form again

Theorem

Every formula is equivalent to a formula in rectified prenex form.

Proof (sketch).

- Push negation symbols inwards.
- Rectify formula.
- Move all quantifiers outside. □

Prenex form again

Theorem

Every formula is equivalent to a formula in rectified prenex form.

Proof (sketch).

- Push negation symbols inwards.
- Rectify formula.
- Move all quantifiers outside. $\square$

Ex: Apply the construction to the following formulas:

- $\neg(\exists x\, P(x, y) \vee \forall y\, Q(y)) \wedge \exists x\, Q(x)$.
- $\neg(\forall x\, \exists y\, P(x, y, z) \rightarrow \forall x(\neg\exists y\, Q(y, z) \rightarrow R(x)))$.

Skolem form

Definition

A formula is in **Skolem form** if it is in rectified prenex form and no existential quantifier occurs in it.

Skolem form

Definition

A formula is in **Skolem form** if it is in rectified prenex form and no existential quantifier occurs in it.

Ex1: Is every formula equivalent to a formula in Skolem form?

Ex2: Can every formula be converted to an equisatisfiable formula in Skolem form in poly. time?

Skolem form

Definition

A formula is in **Skolem form** if it is in rectified prenex form and no existential quantifier occurs in it.

Ex1: Is every formula equivalent to a formula in Skolem form?

Ex2: Can every formula be converted to an equisatisfiable formula in Skolem form in poly. time?

Proposition (Skolemisation)

Let $F = \forall y_1 \forall y_2 \ldots \forall y_n \exists z\, G$ be a rectified formula. Given a function symbol f of arity n that does not occur in F, write

$$F' = \forall y_1 \forall y_2 \ldots \forall y_n\, G[f(y_1, \ldots, y_n)/z]\,.$$

Then F and F' are equisatisfiable.

Skolem form

Definition

A formula is in **Skolem form** if it is in rectified prenex form and no existential quantifier occurs in it.

Ex1: Is every formula equivalent to a formula in Skolem form?

Ex2: Can every formula be converted to an equisatisfiable formula in Skolem form in poly. time?

Proposition (Skolemisation)

Let $F = \forall y_1 \forall y_2 \ldots \forall y_n \exists z\, G$ be a rectified formula. Given a function symbol f of arity n that does not occur in F, write

$$F' = \forall y_1 \forall y_2 \ldots \forall y_n\, G[f(y_1, \ldots, y_n)/z]\,.$$

Then F and F' are equisatisfiable.

Ex3: Use the prop. and find a poly-time conversion.

Ex4: Prove the proposition.

Skolem form

Theorem

Every formula of first-order logic can be converted to an equisatisfiable formula in Skolem form in polynomial time.

Skolem form

Theorem

Every formula of first-order logic can be converted to an equisatisfiable formula in Skolem form in polynomial time.

Proved by the construction we discussed.

Ex: Apply the construction to following formulas.

- $\neg(\exists x\, P(x, y) \vee \forall y\, Q(y)) \wedge \exists x\, Q(x)$.
- $\neg(\forall x\, \exists y\, P(x, y, z) \rightarrow \forall x(\neg\exists y\, Q(y, z) \rightarrow R(x)))$.
- $\forall x\, \exists y\, \forall z\, \exists w\, (\neg P(a, w) \vee Q(f(x), y))$.