Introduction to Logic for Computer Science Spring 2024

Assignment 1 (Deadline: 6:00pm on 29 March 2024)

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Submit your solutions in KLMS. (Reminder: We adopt a very strict policy for handling dishonest behaviours. If a student is found to copy answers from fellow students or other sources in his or her homework submission, she or he will get F.)

Notation: symbols F, G, H denote propositional formulas, and p, q denote propositional variables.

1. Suppose that F and G are formulas such that $F \models G$.
   (a) Show that if F and G have no variable in common, then either F is unsatisfiable or G is valid. **(10 points)**
   (b) Now let F and G be arbitrary formulas. Show that there is a formula H, mentioning only propositional variables common to F and G, such that $F \models H$ and $H \models G$. **(20 points)**

2. A **clique** in an undirected graph is a set of vertices S such that there is an edge between every pair of distinct vertices in S. Given a finite graph G and integer k, describe how to obtain a propositional formula φ such that φ is satisfiable if and only if G has a clique of size k. The formula φ should be computable in time polynomial in the number of vertices of G. **(20 points)**

3. Fix a non-empty set U. A U**-assignment** $\mathcal{A}$ his a function from the collection of propositional variables to 2^U, the power set of U, that is, $\mathcal{A}$ maps each propositional variable to a subset of U. Such an assignment is extended to all formulas as follows:
   - $\hat{\mathcal{A}}(\mathit{false}) = \emptyset$ and $\hat{\mathcal{A}}(\mathit{true}) = U$;
   - $\hat{\mathcal{A}}(p) = \mathcal{A}(p)$;
   - $\hat{\mathcal{A}}(F \wedge G) = \hat{\mathcal{A}}(F) \cap \hat{\mathcal{A}}(G)$;
   - $\hat{\mathcal{A}}(F \vee G) = \hat{\mathcal{A}}(F) \cup \hat{\mathcal{A}}(G)$;
   - $\hat{\mathcal{A}}(\neg F) = U \setminus \hat{\mathcal{A}}(F)$.

   Say that a formula F is U-**valid** if $\hat{\mathcal{A}}(F) = U$ for all U-assignments $\mathcal{A}$.

   (a) Show that if F is U-valid, then F is valid with respect to the standard semantics defined in the lecture notes. **(10 points)**
   (b) Show that if F is valid, then F is U-valid. **(10 points)**

4. This question is about **DNF** formulas.
   (a) Write down a **DNF**-formula equivalent to $(p_1 \vee q_1) \wedge (p_2 \vee q_2) \wedge \cdots \wedge (p_n \vee q_n)$. Here the p_i and q_i are propositional variables. **(10 points)**
   (b) Prove that any **DNF**-formula equivalent to the above formula must have at least 2^n clauses. **(20 points)**