# Lecture 10

## Herbrand's theorem and ground resolution

*Introduction to Logic for Computer Science*

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These slides are minor variants of those made by Prof Worrell and Prof Haase for their logic course at Oxford.

# Recap

Let $F$ be a quantifier-free formula.

Prenex form: $Q_1 x_1 \, Q_2 x_2 \, \cdots Q_n x_n \, F$, where $Q_i \in \{\forall, \exists\}$.

Skolem form: $\forall x_1 \, \forall x_2 \, \cdots \forall x_n \, F$.

Every first-order formula can be translated into an equi-satisfiable formula in Skolem form in poly. time.

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**Example**

Let $\sigma = \langle c, d,\ f, g,\ P, Q\rangle$ with unary $f$ and binary $g$, the set of ground terms is

$$\{c, d, f(c), f(d), g(c, c), g(c, d), g(d, c), g(d, d), f(f(c)), f(f(d)), \ldots\}.$$

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Ex: Find a signature $\sigma$ such that its set of ground terms is isomorphic to $\mathbb{N}$.

# Herbrand structures

**Definition**

Let $\sigma$ be a signature with at least one constant symbol. A $\sigma$-structure $\mathcal{H}$ is a **Herbrand structure** if the following hold:

- The universe $U_{\mathcal{H}}$ is the set of ground terms over $\sigma$.
- For every constant symbol $c$, we have $c_{\mathcal{H}} = c$.
- For every function symbol $f$, $f_{\mathcal{H}}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$.

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**Translation Lemma:** For all ground $t$, $\mathcal{H} \models F[t/x]$ iff $\mathcal{H}_{[x \mapsto t]} \models F$.

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Ex2: Prove the lemma.

# Jaques Herbrand (1908 – 1931)

## Herbrand's theorem

**Theorem**

*Let $F = \forall x_1 \forall x_2 \ldots \forall x_n F^*$ be a formula in Skolem form. Then, $F$ is satisfiable if and only if $F$ has a Herbrand model.*

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Suppose $\mathcal{A} \models F$. Define a Herbrand model $\mathcal{H}$ by setting the interpretation of each predicate symbol $P$ as follows:

$$(t_1, \ldots, t_k) \in P_{\mathcal{H}} \text{ iff } \mathcal{A} \models P(t_1, \ldots, t_k).$$

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Now show that for all closed $G$ in Skolem form, if $\mathcal{A} \models G$, then $\mathcal{H} \models G$. Use induction on the number of $\forall$ in $G$.

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**Proof.**

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But then $\mathcal{H} \models \forall x\, G'$. $\square$

## Ground resolution

**Herbrand expansion** of $F = \forall x_1 \forall x_2 \ldots \forall x_n F^*$:

$$E(F) := \{F^*[t_1/x_1][t_2/x_2]\ldots[t_n/x_n] : t_1, \ldots, t_n \text{ ground terms}\}.$$

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Ex1: What does this imply in terms of developing a validity checker for first-order formulas?

Ex2: Prove the theorem.

Hint: Prove that Herbrand' theorem implies the following theorem. Then, use the Compactness theorem for propositional logic.

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**Proof.**

By Herbrand's theorem, $F$ is satisfiable if and only if $F$ has a Herbrand model. Now

$$\begin{aligned}\mathcal{H} \models F \text{ iff } & \mathcal{H}_{[x_1 \mapsto t_1]\cdots[x_n \mapsto t_n]} \models F^* \text{ for all ground terms } t_1, \ldots, t_n \\ \text{iff } & \mathcal{H} \models F^*[t_1/x_1]\ldots[t_n/x_n] \text{ for all ground } t_i \text{ (by Trans. Lemma)} \\ \text{iff } & \mathcal{H} \models E(F) \\ \text{iff } & E(F) \text{ as a set of prop. formulas is satisfied by } \mathcal{H}.\end{aligned}$$

In the last line, we view $\mathcal{H}$ as an assignment to prop. variables of the form $P(t_1, \ldots, t_k)$. □

**Theorem (Ground Resolution)**

*A closed formula $F$ in Skolem form is unsatisfiable if and only if there is a propositional resolution proof of $\Box$ from $E(F)$.*

**Proof.**

By the Compactness theorem, $E(F)$ is unsatisfiable if and only if some finite subset of $E(F)$ is unsatisfiable. The latter happens if and only if $\Box$ can be derived from $E(F)$ using resolution. $\Box$