Lecture 13

Compactness for first-order logic

The compactness theorem, non-standard models of arithmetic

Introduction to Logic for Computer Science

Prof Hongseok Yang
KAIST

These slides are minor variants of those made by Prof Worrell and Prof Haase for their logic course at Oxford.

The compactness theorem

Theorem

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Ex2: Prove the if direction. Assume that we can skolemise $\mathcal{S}$ to $\mathcal{T}$.

Hint: Consider the Herbrand expansion $\mathcal{E}$ of $\mathcal{T}$. Use the compactness theorem for prop. logic and the Herbrand theorem.

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Answer:

(1)	all finite subsets of $\mathcal{S}$ are satisfiable
(2)	$\Rightarrow$ all finite subsets of $\mathcal{T}$ are satisfiable
(3)	$\Rightarrow$ all finite subsets of $\mathcal{E}$ are satisfiable
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Assumed that we can skolemise $\mathcal{S}$ to $\mathcal{T}$.

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Trick: Rename f_i to f_{2i} to ensure that infinitely many unused function symbols f_{2i+1} are available.

Ex: Explain why it is ok to rename. You need to prove the following.

- Every finite subset of $\mathcal{S}$ before renaming is satisfiable if and only if every finite subset of $\mathcal{S}$ after renaming is satisfiable.
- The whole of $\mathcal{S}$ before renaming is satisfiable if and only if the whole of $\mathcal{S}$ after renaming is satisfiable.

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(5) $\Rightarrow$ (6). Ex: Why does this step hold?

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Proof: Introduce a fresh binary predicate R. For $n > 1$, define

$$G_n = \forall x \neg R(x, x) \wedge \exists x_1 \ldots \exists x_n \bigwedge_{1 \leq i < j \leq n} R(x_i, x_j).$$

Then, $\mathcal{B} \models G_n$ implies $|U_{\mathcal{B}}| \geq n$.

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Every finite subset of $\mathcal{S}$ is satisfiable. (Ex2: Why?)

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Every finite subset of $\mathcal{S}$ is satisfiable. (Ex2: Why?) By compactness, $\mathcal{S}$ has a model $\mathcal{B}$. Then, $|U_{\mathcal{B}}|$ is infinite, and $\mathcal{B} \models F$. (Ex3: Why?)

Peano axioms

Let $\sigma = \langle 0, s, +, \cdot, = \rangle$ be the sig. of arithmetic. Can we find a possibly infinite set of σ-formulas whose only model up to isomorphism is the classical arithmetic, i.e. the standard structure on $\mathbb{N}$?

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Figure: Giuseppe Peano (1858 - 1932)

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Let $\sigma = \langle 0, s, +, \cdot, = \rangle$. Peano axioms axiomatise elementary facts about arithmetic in first-order logic with equality:

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\begin{aligned}
&\forall x\, \neg(s(x) = 0), && \forall x \forall y\, (s(x) = s(y) \to x = y),\\
&\forall x\, (x + 0 = x), && \forall x \forall y\, (x + s(y) = s(x + y)),\\
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But, cannot quantify over sets in first-order logic. We instead resort to infinitely many axioms, one for each formula $\phi(x, y_1, \ldots, y_K)$:

$$\forall y_1 \ldots y_k \left(\Big(\phi(0, y_1, \ldots, y_k) \wedge \forall x \Big(\phi(x, y_1, \ldots, y_k) \to \phi(s(x), y_1, \ldots, y_k) \Big) \Big) \to \forall x \phi(x, y_1, \ldots, y_k) \right).$$

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Let $\mathcal{S}_{PA}$ be the union of all formulas above. Then, "classical arithmetic" is a model of $\mathcal{S}_{PA}$.

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Theorem (without proof)

The compactness theorem holds for first-order logic ***with equality****.*

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Answer3: Because $c_{\mathcal{A}} \neq s^i_{\mathcal{A}}(0_{\mathcal{A}})$ for all $i \in \mathbb{N}$.

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Answer: No. Because the following theorem says that we can make an arbitrary big model.

Theorem (Upward Löwenheim-Skolem theorem)

If a set of sentences $\mathcal{S}$ over a finite signature σ has an infinite model $\mathcal{A}$, then for any cardinal κ, it has a model $\mathcal{B}$ with a universe of cardinality κ.

Downward Löwenheim-Skolem theorem

Let σ be the signature

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Let $\mathcal{S}$ be the set of first-order sentences over σ that holds for $\mathbb{R}$.

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Corollary

$\mathcal{S}$ has a model with a countable universe but not isomorphic to $\mathcal{R}$.