Lecture 13

Compactness for first-order logic

The compactness theorem, non-standard models of arithmetic

Introduction to Logic for Computer Science

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These slides are minor variants of those made by Prof Worrell and Prof Haase for their logic course at Oxford.

The compactness theorem

Theorem

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Hint: Consider the Herbrand expansion $\mathcal{E}$ of $\mathcal{T}$. Use the compactness theorem for prop. logic and the Herbrand theorem.

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Trick: Rename f_i to f_{2i} to ensure that infinitely many unused function symbols f_{2i+1} are available.

Ex: Explain why it is ok to rename. You need to prove the following.

- Every finite subset of $\mathcal{S}$ before renaming is satisfiable if and only if every finite subset of $\mathcal{S}$ after renaming is satisfiable.
- The whole of $\mathcal{S}$ before renaming is satisfiable if and only if the whole of $\mathcal{S}$ after renaming is satisfiable.

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(5) $\Rightarrow$ (6). Ex: Why does this step hold?

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Proof: Introduce a fresh binary predicate R. For $n > 1$, define

$$G_n = \forall x \neg R(x, x) \wedge \exists x_1 \ldots \exists x_n \bigwedge_{1 \leq i < j \leq n} R(x_i, x_j).$$

Then, $\mathcal{B} \models G_n$ implies $|U_{\mathcal{B}}| \geq n$.

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Every finite subset of $\mathcal{S}$ is satisfiable. (Ex2: Why?) By compactness, $\mathcal{S}$ has a model $\mathcal{B}$. Then, $|U_{\mathcal{B}}|$ is infinite, and $\mathcal{B} \models F$. (Ex3: Why?)

Peano axioms

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Figure: Giuseppe Peano (1858 - 1932)

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Let $\sigma = \langle 0, s, +, \cdot, = \rangle$. Peano axioms axiomatise elementary facts about arithmetic in first-order logic with equality:

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&\forall x\, \neg(s(x) = 0), && \forall x \forall y\, (s(x) = s(y) \rightarrow x = y),\\
&\forall x\, (x + 0 = x), && \forall x \forall y\, (x + s(y) = s(x + y)),\\
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$$\forall Y \big((0 \in Y \wedge \forall x (x \in Y \to s(x) \in Y)) \to \forall x\, (x \in Y)\big).$$

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But, cannot quantify over sets in first-order logic. We instead resort to infinitely many axioms, one for each formula $\phi(x, y_1, \ldots, y_K)$:

$$\forall y_1 \ldots y_k \left(\Big(\phi(0, y_1, \ldots, y_k) \wedge \forall x \Big(\phi(x, y_1, \ldots, y_k) \to \phi(s(x), y_1, \ldots, y_k) \Big) \Big) \right.$$
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Let $\mathcal{S}_{PA}$ be the union of all formulas above. Then, "classical arithmetic" is a model of $\mathcal{S}_{PA}$.

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Theorem (without proof)

The compactness theorem holds for first-order logic ***with equality****.*

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Answer1: Introduce a new constant symbol c, and set

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Then, every finite subset of $\mathcal{S}_{PA} \cup \mathcal{C}$ is satisfiable. (Ex2: Why?)

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Thus, $\mathcal{S}_{PA} \cup \mathcal{C}$ has a model $\mathcal{A}$ by the compactness theorem.

The model $\mathcal{A}$ is not isomorphic to the "classical" model of arithmetic. (Ex3: Why?)

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Answer3: Because $c_{\mathcal{A}} \neq s^i_{\mathcal{A}}(0_{\mathcal{A}})$ for all $i \in \mathbb{N}$.

Upward Löwenheim-Skolem theorem

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Answer: No. Because the following theorem says that we can make an arbitrary big model.

Theorem (Upward Löwenheim-Skolem theorem)

If a set of sentences $\mathcal{S}$ over a finite signature σ has an infinite model $\mathcal{A}$, then for any infinite cardinal κ, it has a model $\mathcal{B}$ with a universe of cardinality κ.

Downward Löwenheim-Skolem theorem

Let σ be the signature

$$\langle 0, 1, +, \cdot, <, = \rangle.$$

Let $\mathcal{S}$ be the set of first-order sentences over σ that holds for $\mathbb{R}$.

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Corollary

$\mathcal{S}$ has a model with a countable universe but not isomorphic to $\mathcal{R}$.