# Lecture 13

# Compactness for predicate logic

## The compactness theorem, non-standard models of arithmetic

Print version of the lecture in *Introduction to Logic for Computer Science*

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These lecture notes are very minor variants of the ones made by Prof James Worrell and Prof Christoph Haase for their 'Logic and Proof' course at Oxford.

In this lecture, we are going to prove a statement analogous to the one we already saw for propositional logic and show that first-order logic has the compactness property. In fact, the proof relies on compactness for propositional logic and incorporates many of the concepts we have encountered in this course so far. The compactness theorem for first-order logic in particular allows us to construct models with specific properties. A particular example that we will focus on is the construction of non-standard models of classical arithmetic.

## 1 The compactness theorem

In this section, we prove the compactness theorem for first-order logic:

**Theorem 1.** *Let $\mathcal{S}$ be a countably infinite set of first-order formulas. Then, $\mathcal{S}$ is satisfiable if and only if every finite subset of $\mathcal{S}$ is satisfiable.*

We first give the proof on a high level and then justify the individual steps. One direction is of course trivial: If $\mathcal{S}$ is satisfiable, then any model $\mathcal{A}$ of $\mathcal{S}$ will also be a model of any finite subset of $\mathcal{S}$. Hence, we only need to focus on the reverse direction. Let $\mathcal{T}$ be obtained from skolemising every formula in $\mathcal{S}$, and let $\mathcal{E}$ be the Herbrand expansion of $\mathcal{T}$. Then

$$
\begin{aligned}
& \text{all finite subsets of } \mathcal{S} \text{ are satisfiable} && (1)\\
\Rightarrow\ & \text{all finite subsets of } \mathcal{T} \text{ are satisfiable} && (2)\\
\Rightarrow\ & \text{all finite subsets of } \mathcal{E} \text{ are satisfiable (viewed as propositional formulas)} && (3)\\
\Rightarrow\ & \mathcal{E} \text{ is satisfiable (viewed as propositional formulas)} && (4)\\
\Rightarrow\ & \mathcal{T} \text{ is satisfiable} && (5)\\
\Rightarrow\ & \mathcal{S} \text{ is satisfiable.} && (6)
\end{aligned}
$$

Even though the high-level proof seems convincing, there are some subtleties that need to be taken care of. For instance, even the first step of the proof is not immediate. When skolemising an infinite set of formulas, we may not have any fresh function symbols available since all function symbols could have already been used in $\mathcal{S}$. Since there are countably many function symbols, we can assume them to be of the form $f_1, f_2, f_3, \ldots$ In order to make sure we have infinitely many fresh function symbols available, we replace in $\mathcal{S}$ every function symbol $f_i$ by $f_{2i}$.

This ensures that all infinitely many function symbols $f_{2i+1}$, $i \geq 0$ can be used for skolemising $\mathcal{S}$.

Next, we look at the implication $(1) \Rightarrow (2)$. We have shown in Lecture 11 in Theorem 1 that a conjunction of first-order formulas $F_1 \wedge \cdots \wedge F_n$ is satisfiable if and only if $G_1 \wedge \cdots \wedge G_n$ is satisfiable, where $G_i$ is the Skolem form of $F_i$ and fresh function symbols are used for every $G_i$. Now take a finite set $\mathcal{T}' = \{G_1, \ldots, G_n\} \subseteq \mathcal{T}$ emerging from formulas $\mathcal{S}' = \{F_1, \ldots, F_n\} \subseteq \mathcal{S}$. By assumption, there is some $\mathcal{A}$ such that $\mathcal{A} \models \mathcal{S}'$, and hence $\mathcal{A} \models \bigwedge_{1 \leq i \leq n} F_i$. But then Theorem 1 from Lecture 11 yields $\mathcal{B}$ such that $\mathcal{B} \models \bigwedge_{1 \leq i \leq n} G_i$, i.e., $\mathcal{B} \models \mathcal{T}'$.

Now for the implication $(2) \Rightarrow (3)$, let $\mathcal{E}' \subseteq \mathcal{E}$ be a finite. Then, $\mathcal{E}'$ is contained in the Herbrand expansion $E(\mathcal{T}')$ for some finite $\mathcal{T}' \subseteq T$ (it is in general not the case that $\mathcal{E}' = E(\mathcal{T}')$). But since by assumption $\mathcal{T}'$ is satisfiable, it follows from Theorem 7 in Lecture 10 that $E(\mathcal{T}')$ has a (propositional) model $\mathcal{B}$, which is also a model for $\mathcal{E}'$.

The implication $(3) \Rightarrow (4)$ follows from the compactness theorem for propositional logic. In order to show $(4) \Rightarrow (5)$, we can straight-forwardly adapt the proof of Theorem 7 in Lecture 10 in order to obtain a Herbrand model of $\mathcal{T}$. Finally, for the implication $(5) \Rightarrow (6)$, we have shown in Proposition 8 in Lecture 9 that skolemising a single formula preserves satisfiability. In fact, it is not difficult to see that the construction used in the proof of Proposition 8 carries over to the setting of infinite sets of formulas, i.e., given $\mathcal{B}$ such that $\mathcal{B} \models \mathcal{T}$, we can construct an $\mathcal{A}$ from $\mathcal{B}$ such that $\mathcal{A} \models \mathcal{S}$.

## 2 Non-standard models of arithmetic

We will now take a look at some applications of the compactness theorem. A particularly useful application scenario is that the compactness theorem allows us to enforce models with an infinite domain, as illustrated in the following example.

**Proposition 2.** Let $F$ be a $\sigma$-sentence over some signature $\sigma$ such that $F$ has a model $\mathcal{A}_n$ with $|U_{\mathcal{A}_n}| = n$ for every $n > 1$. Then, $F$ has a model with an infinite universe.

*Proof.* Let $\gamma$ be an extension of $\sigma$ by a fresh binary predicate symbol $R$. For every $n > 1$, we define the following $\gamma$-sentence:

$$G_n = \forall x \, \neg R(x, x) \wedge \exists x_1 \ldots \exists x_n \bigwedge_{1 \leq i < j \leq n} R(x_i, x_j).$$

Now for any $\gamma$-structure $\mathcal{B}$, we have that

$$\mathcal{B} \models G_n \text{ implies } |U_{\mathcal{B}}| \geq n \tag{7}$$

Moreover, $F_n := F \wedge G_n$ is satisfiable for every $n > 1$, since by assumption $F$ has a $\sigma$-model $\mathcal{A}_n$ such that $|U_{\mathcal{A}_n}| = n$, and $\mathcal{A}_n$ can be extended to a $\gamma$-model $\mathcal{B}_n$ of $F_n$ by setting

$$R_{\mathcal{B}_n} := (U_{\mathcal{A}_n} \times U_{\mathcal{A}_n}) \setminus \{(u, u) : u \in U_{\mathcal{A}_n}\}.$$

Now define

$$\mathcal{S} := \bigcup_{n > 1} \{F_n\}.$$

We claim that every finite subset of $\mathcal{S}$ is satisfiable: Suppose $\mathcal{T} = \{F_{i_1}, \ldots, F_{i_k}\} \subseteq \mathcal{S}$, and let $n = \max\{i_1, \ldots, i_k\}$. Then, $\mathcal{B}_n$ as defined above is a model for all $F_{i_j} \in T$.

Consequently, by an application of the compactness theorem for first-order logic, $\mathcal{S}$ has a model $\mathcal{B}$. We claim that $U_{\mathcal{B}}$ is infinite. To the contrary, assume that $|U_{\mathcal{B}}| = n$ for some finite $n \in \mathbb{N}$. Then $\mathcal{B} \not\models F_{n+1}$ due to (7), which contradicts $\mathcal{B} \models \mathcal{S}$. □

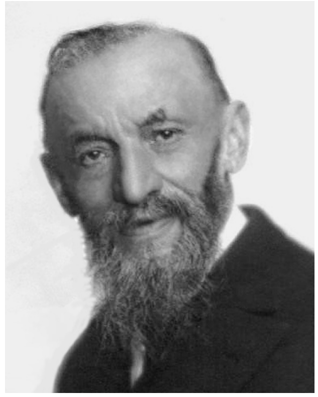

Figure 1: Giuseppe Peano (1858 - 1932)

What is even more interesting is that compactness enables us to construct non-standard models of elementary arithmetic. Doing so requires us to show that first-order logic with equality is compact.

**Theorem 3.** *First-order logic with equality is compact.*

Though not difficult to show, we will omit the proof of this theorem. In order to construct non-standard models of arithmetic, we first need to clarify what a standard model of arithmetic should look like. Perhaps the easiest is to agree on the signature of arithmetic, which we take as $\sigma = \langle 0, s, +, \cdot, = \rangle$. Then, the standard model of arithmetic is the $\sigma$-structure whose domain is the natural numbers $\mathbb{N}$, and which interprets the constants, functions and relation symbols as we would expect. A natural question that now arises is whether we can find a potentially infinite set of first-order formulas over $\sigma$ whose only model is the standard model of arithmetic up to isomorphism. In other words, can we axiomatise classical arithmetic in first-order logic? The precise notion of axiomatisation will be made clear in the next lecture.

Which axioms allow for deriving all mathematical theorems proved in human history was a hot debate in the second half of the 19th century. Building up on the work of Richard Dedekind, Guiseppe Peano published in 1889 the following set of axioms which are nowadays known as *Peano axioms*:

$$\begin{array}{ll} \forall x\, \neg(s(x) = 0), & \forall x\, \forall y\, (x + s(y) = s(x + y)), \\ \forall x\, \forall y\, (s(x) = s(y) \to x = y), & \forall x\, (x \cdot 0 = 0), \\ \forall x\, (x + 0 = x), & \forall x\, \forall y\, (x \cdot s(y) = (x \cdot y) + x). \end{array}$$

Though those axioms certainly seem necessary, an important ingredient for obtaining classical arithmetic is missing: induction. Suppose for now that we could quantify over sets $Y$, then we could express induction as follows:

$$\forall Y \Big( (0 \in Y \wedge \forall x (x \in Y \to s(x) \in Y)) \to \forall x\, (x \in Y) \Big).$$

In order to "emulate" quantification over sets, instead we express induction for all possible predicates and introduce the following induction scheme for all formulas $\phi(x, y_1, \dots, y_k)$ with free variables $x, y_1, \dots, y_k$:

$$\forall y_1 \dots \forall y_k \Bigg( \Big( \phi(0, y_1, \dots, y_k) \wedge \forall x\, (\phi(x, y_1 \dots, y_k) \to \phi(s(x), y_1, \dots, y_k)) \Big) \\ \to \forall x\, \phi(x, y_1, \dots, y_k) \Bigg).$$

Let $\mathcal{S}_{PA}$ be the union of all formulas above. Then, "classical arithmetic" is a model of $\mathcal{S}_{PA}$.

Are the Peano axioms a good set of axioms? So far, it is fair to say that the answer is "yes". It is possible yet tedious to prove numerous non-trivial mathematical statements using only those axioms. Some serious mathematicians even claim that Andrew Wiles' famous proof of Fermat's Last Theorem[1] can be carried out in Peano arithmetic though nobody has been willing to sacrifice their life for doing so. It should, however, be noted that it is not difficult to construct mathematical statements which cannot be proven using Peano axioms, and such statements naturally appear in Ramsey theory.

After this diversion, let us get back to our original goal of constructing non-standard models of arithmetic. To this end, we proceed similar to the example above. Let $c$ be a fresh constant symbol, and set

$$\mathcal{C} = \{\neg(c = s^i(0)) : i \in \mathbb{N}\}.$$

Now every finite subset of $\mathcal{S}_{PA} \cup \mathcal{C}$ is satisfiable. Hence, by compactness $\mathcal{S}_{PA} \cup \mathcal{C}$ has a model $\mathcal{A}$. But $c_{\mathcal{A}} \neq s^i_{\mathcal{A}}(0_{\mathcal{A}})$ for all $i \in \mathbb{N}$. Hence, $\mathcal{A}$ fulfills all Peano axioms but $\mathcal{A}$ is not isomorphic to the standard "classical" model of arithmetic since its universe contains an element which is not a natural number. Thus, $\mathcal{A}$ is a non-standard model of arithmetic.

Of course, one could now argue that maybe the Peano axioms are insufficient, and there is a better set of axioms that allows for axiomatising classical arithmetic in first-order logic. However, the following theorem, which we only state here, entails that no such axiomatisation can exist.

**Theorem 4** (Upward Löwenheim-Skolem theorem). *If $\mathcal{S}$ has an infinite model $\mathcal{A}$, then for any cardinal $\kappa$, it has a model $\mathcal{B}$ with a universe of cardinality $\kappa$ that extends $\mathcal{A}$.*

**Corollary 5.** *Classical arithmetic is not first-order axiomatisable.*

[1]Fermat's Last Theorem states that the Diophantine equation $x^n + y^n = z^n$ has no integral solutions for $n > 2$. Fermat claimed to have an elegant proof that unfortunately did not fit on the margin of his notebook. It took over 200 years and many failed attempts until Andrew Wiles published a proof in 1995 which now fits into a book of more than 500 pages.