Lecture 15
Automatic Structures

Introduction to Logic for Computer Science

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These slides are minor variants of those made by Prof Worrell and Prof Haase for their logic course at Oxford.

Overview

Today:

- Study another approach for proving the decidability of a theory.
- Structures whose universe and relations are regular languages.
- Automata-based decision procedures for the theories of those structures.

Relational structures

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To get the variant, we replace every $f_{\mathcal{A}}\colon U_{\mathcal{A}}^{k} \to U_{\mathcal{A}}$ with relation

$$F_{\mathcal{A}} = \{(a_1, \ldots, a_k, b) \in U_{\mathcal{A}}^{k+1} \ : \ f_{\mathcal{A}}(a_1, \ldots, a_k) = b\},$$

and each constant $c_{\mathcal{A}}$ by unary relation

$$C_{\mathcal{A}} = \{a \in U_{\mathcal{A}} \ : \ a = c_{\mathcal{A}}\}.$$

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Ex: Construct a relational variant of the structure $(\mathbb{N}, 0, 1, +, \times)$.

Automatic structure infomally

Key concept of this lecture.

Informally refers to a structure $\mathcal{A} = (U_{\mathcal{A}}, R_1, \ldots, R_m)$ where $U_{\mathcal{A}}$ and the R_i can be represented by finite automata.

Theorem (Khoussainov & Nerode, Simpler Version)

$\mathrm{Th}(\mathcal{A})$ *is decidable for every automatic structure* $\mathcal{A}$.

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Some terminology from formal language theory

An **alphabet** Σ is a finite set.

A **word** w over Σ is a finite sequence of elements in Σ.

A **language** L over Σ is a set of words. That is, $L \subseteq \Sigma^*$.

A language L over Σ is **regular** if it can be recognised by a finite state automaton.

Word convolutions

Given an alphabet Σ, we want to represent relations R on Σ^*, by words over another alphabet. **Word convolutions** let us do it.

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For words $w_1, w_2, \ldots, w_n \in \Sigma^*$,

- let $w_i = a_{(i,1)} a_{(i,2)} \cdots a_{(i,\ell_i)}$, so that $|w_i| = \ell_i$;
- let $\ell = \max\{\ell_1, \ldots, \ell_n\}$;
- set $a_{(i,j)} := \#$ for all $\ell_i < j \leq \ell$ and $1 \leq i \leq n$.

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The **convolution** of $w_1, \ldots, w_n$ is

$$
\begin{aligned}
w_1 \otimes w_2 \otimes \cdots \otimes w_n \in (\Sigma_\#^n)^* \\
:= (a_{(1,1)}, \ldots, a_{(n,1)})(a_{(1,2)}, \ldots, a_{(n,2)}) \cdots (a_{(1,\ell)}, \ldots, a_{(n,\ell)}).
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Ex: Compute $abba \otimes abaabba \otimes ac$.

Automatic relations

Definition

A relation $R \subseteq (\Sigma^*)^n$ is **automatic** if the following language over $\Sigma_\#^n$

$$L_R := \{w_1 \otimes w_2 \otimes \cdots \otimes w_n : (w_1, \ldots, w_n) \in R\}$$

is regular (i.e. it can be recognised by a finite automaton over $\Sigma_\#^n$).

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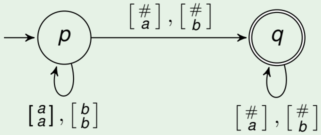
Example

$R = \{(u, v) \in (\Sigma^*)^2 : u \text{ is a prefix of } v\}$ with $\Sigma = \{a, b\}$ is automatic:



Automatic structures

Definition

A relational structure $\mathcal{A} = (U_\mathcal{A}, R_1, \ldots, R_m)$ is **automatic** if there are a finite alphabet Σ and regular languages $L, L_1, \ldots, L_m$ such that

- $L = U_\mathcal{A}$;
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A structure $\mathcal{A}$ has an **automatic presentation** if $\mathcal{A}$ is isomorphic to an automatic structure.

Note: Structures $\mathcal{A} = (U_\mathcal{A}, R_1, \ldots, R_m)$ and $\mathcal{B} = (U_\mathcal{B}, S_1, \ldots, S_m)$ are **isomorphic** if there is a bijection $f : U_\mathcal{A} \to U_\mathcal{B}$ such that

$$(a_1, \ldots, a_k) \in R_i \iff (f(a_1), \ldots, f(a_k)) \in S_i$$

for all $1 \leq i \leq m$ and all $a_1, \ldots, a_k \in U_\mathcal{A}$.

Theorem (Khoussainov & Nerode)

$\mathrm{Th}(\mathcal{A})$ *is decidable for every structure* $\mathcal{A}$ *with an automatic presentation.*

Reminder of Presburger arithmetic

$\mathrm{Th}(\mathbb{N}, 0, 1, +, =)$ is **Presburger arithmetic.**[1]

Theorem

Presburger arithmetic $\mathrm{Th}(\mathbb{N}, 0, 1, +, =)$ *is decidable.*

Last time we proved the theorem using quantifier elimination.

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Last time we proved the theorem using quantifier elimination.

We will give another proof here that uses Khoussainov & Nerode.

Theorem

The structure $(\mathbb{N}, 0, 1, +, =)$ *is automatic.*

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Presburger arithmetic has an automatic presentation

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Set $N := (\{0,1\}^*1) \cup \{0\} \subseteq \{0,1\}^*$.

For $w = b_0 b_1 \cdots b_m \in N$, define $\mathrm{val}\colon N \to \mathbb{N}$ by

$$\mathrm{val}(w) := \sum_{i=0}^{m} 2^i \cdot b_i.$$

Set $A := \big\{(a,b,c) \in N^3 : \mathrm{val}(a) + \mathrm{val}(b) = \mathrm{val}(c)\big\} \subseteq N^3$.

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Then, $(\mathbb{N}, +)$ is isomorphic to (N, A) by mapping $n \in \mathbb{N}$ to its unique minimal binary expansion $\mathrm{val}^{-1}(n)$.

Proposition

The structure (N, A) is automatic.

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The structure (N, A) is automatic.

Ex: Prove the proposition.

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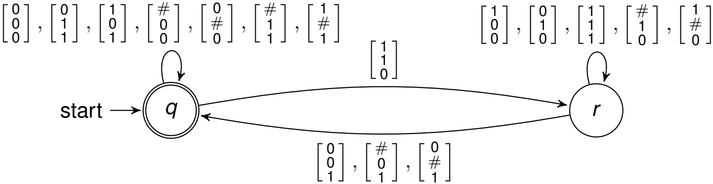
L_A is contained in the language of the following DFA:



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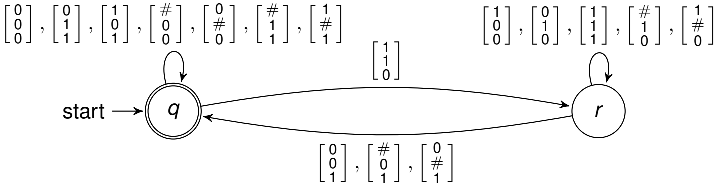

Now intersect it with $\{a \otimes b \otimes c : a, b, c \in N\}$ to obtain a finite automaton for L_A.

Unbounded dense linear orders have automatic presentations

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Any countable structure $\mathcal{A} = (Q, <)$ that is a model of the unbounded dense linear order axioms has an automatic presentation.

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- Show that any two countable unbounded dense linear orders are isomorphic.
- Show the statement for a suitable countable unbounded dense linear order.

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Theorem (Cantor)

Any two countable unbounded dense linear orders are isomorphic.

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Theorem (Cantor)

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An automatic unbounded dense linear order

Let $L = \{0,1\}^* \cdot 1$ and $<$ such that $x < y$ iff either

- $y = xu$ for some $u \in \{0,1\}^*$, or
- $x = z0u$ and $y = z1v$ for some $u, v, z \in \{0,1\}^*$.

Ex1: Prove that $(L, <)$ is automatic.

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Ex1: Prove that $(L, <)$ is automatic.

Proposition

The structure $(L, <)$ is an automatic unbounded dense linear order (in short, UDLO).

Ex2: Prove the proposition. You need to show that $(L, <)$ is an UDLO.

$(L, <)$ is an UDLO.

No smallest element: for $u1 \in L$, have ??? $< u1$.

No largest element: for $u1 \in L$, have $u1 <$???.

Density: Let $x, y \in L$ with $x < y$. Should find z with $x < z < y$.

- Case $x = u1, y = u1v1$: Then, $x < z =$??? $< y$.
- Case $x = u0v1, y = u1w$: Then, $x < z =$??? $< y$.

$(L, <)$ is an UDLO.

No smallest element: for $u1 \in L$, have $u01 < u1$.

No largest element: for $u1 \in L$, have $u1 < u11$.

Density: Let $x, y \in L$ with $x < y$. Should find z with $x < z < y$.

- Case $x = u1, y = u1v1$: Then, $x < z = u10^{|v|+1}1 < y$.
- Case $x = u0v1, y = u1w$: Then, $x < z = u01^{|v|+2} < y$.

Structures with automatic presentations are decidable

Theorem (Khoussainov & Nerode)

$\mathrm{Th}(\mathcal{A})$ *is decidable for every structure* $\mathcal{A}$ *with an automatic presentation.*

Proposition

Let $\mathcal{A} = (L, R_1, \ldots, R_m)$ be an automatic σ-structure. Let F be a σ-formula with at most free variables $x_1, \ldots, x_n$. There is an effectively constructible regular language $L_F \subseteq (\Sigma_\#^*)^n$ such that

$$L_F = \left\{ w_1 \otimes \cdots \otimes w_n \ : \ \mathcal{A}_{[x_1 \mapsto w_1] \cdots [x_n \mapsto w_n]} \models F \right\}.$$

Ex1: Why does the proposition imply the theorem?

Ex2: Prove the proposition by induction.

Proof of the proposition: case $F = R_i(x_{i_1}, \ldots, x_{i_k})$ with $1 \le i_1, \ldots, i_k \le n$

Define homomorphism $h\colon (\Sigma_\#^n)^* \to (\Sigma_\#^k)^*$ such that for $a_1, \ldots, a_n \in \Sigma_\#$:

$$h(a_1, \ldots, a_n) = \begin{cases} \epsilon & \text{if } a_{i_1} = \cdots = a_{i_k} = \# \\ (a_{i_1}, \ldots, a_{i_k}) & \text{otherwise.} \end{cases}$$

By assumption, $L_{R_i} \subseteq (\Sigma_\#^k)^*$ is regular. Using the closure under inverse homomorphisms, obtain the regularity of

$$L_F = h^{-1}(L_{R_i}) \cap \{w_1 \otimes \cdots \otimes w_n \ : \ w_1, \ldots, w_n \in L\}.$$

Proof of the proposition: case $F = G \wedge H$, $F = G \vee H$, or $F = \neg G$

Induction hypothesis yields regular languages $L_G, L_H \subseteq (\Sigma_{\#}^{n})^*$.

The statement in the proposition follows from the closure of regular languages under intersection, union and complement.

Example: for $F = \neg G$, we have

$$L_F = \{w_1 \otimes \cdots \otimes w_n : w_1, \ldots, w_n \in L\} \setminus L_G.$$

Then, L_F is regular because of the closure of regular languages under complement.

Proof of the proposition: case $F = \exists x_{n+1}\, G$ with $x_1, \ldots, x_n, x_{n+1}$ free in G

Induction hypothesis yields regular languages L_G for G.

Define homomorphism $h\colon (\Sigma_{\#}^{n+1})^* \to (\Sigma_{\#}^{n})^*$ such that for $a_1, \ldots, a_n \in \Sigma_{\#}$,

$$h(a_1, \ldots, a_n, a_{n+1}) = \begin{cases} \epsilon & \text{if } a_1 = \cdots = a_n = \# \\ (a_1, \ldots, a_n) & \text{otherwise.} \end{cases}$$

Then, $L_F = h(L_G)$, which is regular because the homomorphic image of a regular language is regular.

Intractability

Theorem

There exists an automatic structure $\mathcal{A}$ with non-elementary complexity, i.e., no algorithm decides $F \in \mathrm{Th}(\mathcal{A})$ in time $2^{2^{\cdot^{\cdot^{\cdot^{2^n}}}}}$.

Proof idea.

This can be shown for the structure $\mathcal{A} = (\{0,1\}^*, S_1, S_2, \leq)$, where

- $S_0 = \{(w, w0) : w \in \{0,1\}^*\}$,
- $S_1 = \{(w, w1) : w \in \{0,1\}^*\}$,
- $\leq \ = \{(w, u) : w, u \in \{0,1\}^*\}$. $\square$

Proving Lagrange-style theorems automatically

Theorem (Lagrange, 1770)

Every natural number can be written as the sum of four integer squares.

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Call $n \in \mathbb{N}$ a **binary palindrome** if the string representing its binary presentation is a palindrome, e.g.,

$$27 = \mathrm{val}(11011).$$

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$$27 = \mathrm{val}(11011).$$

Theorem (Rajasekaran, Shallit, Smith, 2017)

Every natural number can be written as the sum of four binary palindromes.

Proof idea.

Translate the statement into a suitably constructed nested-word automaton accepting all numbers that are the sum of four binary palindromes. Show that the automaton accepts all numbers. $\square$