Lecture 9

Normal forms for first-order logic

Equivalences, prenex form, Skolem form

Introduction to Logic for Computer Science

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These slides are minor variants of those made by Prof Worrell and Prof Haase for their logic course at Oxford.

Recap

Syntax of first-order formulas:

- Signature σ (constant, function and predicate symbols).
- σ-terms.
- Formulas (predicate symbols, logical connectives of propositional logic, and additional $\forall x$ and $\exists x$).

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- σ-structure $\mathcal{A}$ with universe $U_{\mathcal{A}}$ and interpretations of constants, functions, predicates, and variables.
- $\mathcal{A} \models F$ defined by structural induction on F.

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- $\mathcal{A} \models F$ defined by structural induction on F.

Relevance lemma: "If $\mathcal{A}$ and $\mathcal{A}'$ only differ on variables other than free variables in F, then $\mathcal{A} \models F$ if and only if $\mathcal{A}' \models F$."

Normal forms

$$\neg(\exists x\, P(x,y) \vee \forall z\, Q(z)) \wedge \exists w\, Q(w)$$

vs

$$\forall x\, \exists z\, \exists w\, ((\neg P(x,y) \wedge \neg Q(z)) \wedge Q(w)).$$

Normal forms

$$\neg(\exists x\, P(x,y) \vee \forall z\, Q(z)) \wedge \exists w\, Q(w)$$

vs

$$\forall x\, \exists z\, \exists w\, ((\neg P(x,y) \wedge \neg Q(z)) \wedge Q(w)).$$

Today:

- Establish elementary equivalences.
- Prenex form: all quantifiers first.
- Skolem form: prenex form with no existential quantifiers.

Equivalences

Definition

Two first-order logic formulas F and G over the signature σ are **logically equivalent** (written $F \equiv G$) if $\mathcal{A} \models F$ iff $\mathcal{A} \models G$ for all σ-structures $\mathcal{A}$.

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Proposition

Let F and G be arbitrary formulas. Then, the following hold.

(A) $\neg \forall x F \equiv \exists x \neg F$ and $\neg \exists x F \equiv \forall x \neg F$.

(B) If x does not occur free in G then:

$$
\begin{aligned}
(\forall x F \wedge G) &\equiv \forall x (F \wedge G), & (\forall x F \vee G) &\equiv \forall x (F \vee G),\\
(\exists x F \wedge G) &\equiv \exists x (F \wedge G), & (\exists x F \vee G) &\equiv \exists x (F \vee G).
\end{aligned}
$$

(C) $(\forall x F \wedge \forall x G) \equiv \forall x (F \wedge G)$ and $(\exists x F \vee \exists x G) \equiv \exists x (F \vee G)$.

(D) $\forall x \forall y F \equiv \forall y \forall x F$ and $\exists x \exists y F \equiv \exists y \exists x F$.

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(A) $\neg\forall xF \equiv \exists x\neg F$ and $\neg\exists xF \equiv \forall x\neg F$.

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$$(\forall xF \wedge G) \equiv \forall x(F \wedge G), \qquad (\forall xF \vee G) \equiv \forall x(F \vee G),$$
$$(\exists xF \wedge G) \equiv \exists x(F \wedge G), \qquad (\exists xF \vee G) \equiv \exists x(F \vee G).$$

(C) $(\forall xF \wedge \forall xG) \equiv \forall x(F \wedge G)$ and $(\exists xF \vee \exists xG) \equiv \exists x(F \vee G)$.

(D) $\forall x\forall yF \equiv \forall y\forall xF$ and $\exists x\exists yF \equiv \exists y\exists xF$.

Ex: Prove the highlighted cases of (B) and (C).

Prenex form

Definition

A formula is in **prenex form** if it can be written

$$Q_1 y_1\, Q_2 y_2 \ldots Q_n y_n\, F,$$

where $Q_i \in \{\exists, \forall\}$, $n \geq 0$, and F contains no quantifiers. In this case F is called the **matrix** of the formula.

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Ex: Transform $\neg\Big(\forall x\, \exists y\, P(x,y,z) \to \forall x\big(\neg\exists y\, Q(y,z) \to R(x)\big)\Big)$.

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Ex: Define $F[t/x]$ formally using structural induction.

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Lemma (Translation Lemma)

If t is a term and F is a formula such that no variable in t occurs bound in F, then $\mathcal{A} \models F[t/x]$ iff $\mathcal{A}_{[x \mapsto \mathcal{A}(t)]} \models F$.

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Proof: By structural induction.

Ex1: Prove the case that $F = \forall y G$.

Ex2: Why do we need the variable condition in the lemma? Find a counter-example $(F, \mathcal{A})$.

Rectified formulas

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Proposition

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Ex1: Use the proposition and show that for every formula F, there is a rectified formula G such that $F \equiv G$.

Ex2: Prove the proposition. Hint: Use the Translation lemma and the Relevance lemma.

Proposition

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Proof.

Proof for $\forall$:

$$
\begin{array}{ll}
 & \mathcal{A} \models \forall y\,(G[y/x]) \\
\text{iff} & \mathcal{A}_{[y\mapsto a]} \models G[y/x] \text{ for all } a \in U_{\mathcal{A}} \\
\text{iff} & \mathcal{A}_{[y\mapsto a][x\mapsto \mathcal{A}_{[y\mapsto a]}(y)]} \models G \text{ for all } a \in U_{\mathcal{A}} \text{ (Translation Lemma)} \\
\text{iff} & \mathcal{A}_{[y\mapsto a][x\mapsto a]} \models G \text{ for all } a \in U_{\mathcal{A}} \\
\text{iff} & \mathcal{A}_{[x\mapsto a][y\mapsto a]} \models G \text{ for all } a \in U_{\mathcal{A}} \\
\text{iff} & \mathcal{A}_{[x\mapsto a]} \models G \text{ for all } a \in U_{\mathcal{A}} \text{ (Relevance Lemma)} \\
\text{iff} & \mathcal{A} \models \forall x\, G\,. \quad \square
\end{array}
$$

Prenex form again

Theorem

Every formula is equivalent to a formula in rectified prenex form.

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Proof (sketch).

- Push negation symbols inwards.
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Ex: Apply the construction to the following formulas:

- $\neg(\exists x\, P(x, y) \vee \forall y\, Q(y)) \wedge \exists x\, Q(x)$.
- $\neg(\forall x\, \exists y\, P(x, y, z) \rightarrow \forall x(\neg\exists y\, Q(y, z) \rightarrow R(x)))$.

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Proposition (Skolemisation)

Let $F = \forall y_1 \forall y_2 \ldots \forall y_n \exists z\, G$ be a rectified formula. Given a function symbol f of arity n that does not occur in F, write

$$F' = \forall y_1 \forall y_2 \ldots \forall y_n\, G[f(y_1, \ldots, y_n)/z]\,.$$

Then F and F' are equisatisfiable.

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Ex3: Use the prop. and find a poly-time conversion.

Ex4: Prove the proposition.

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Proved by the construction we discussed.

Ex: Apply the construction to following formulas.

- $\neg(\exists x\, P(x, y) \vee \forall y\, Q(y)) \wedge \exists x\, Q(x)$.
- $\neg(\forall x\, \exists y\, P(x, y, z) \to \forall x(\neg\exists y\, Q(y, z) \to R(x)))$.
- $\forall x\, \exists y\, \forall z\, \exists w\, (\neg P(a, w) \vee Q(f(x), y))$.