Lecture 14

Decidable Theories

Logical theories, quantifier elimination, unbounded dense linear orders, linear arithmetic over the rationals, Presburger arithmetic

Introduction to Logic for Computer Science

Prof Hongseok Yang
KAIST

These slides are minor variants of those made by Prof Worrell and Prof Haase for their logic course at Oxford.

Logical theories

A **theory** $\mathcal{T}$ is a set of sentences closed under semantic entailment:

$$\mathcal{T} \models F \text{ implies } F \in \mathcal{T}.$$

A theory is **complete** if either $F \in \mathcal{T}$ or $\neg F \in \mathcal{T}$ for any F.

Logical theories

A **theory** $\mathcal{T}$ is a set of sentences closed under semantic entailment:

$$\mathcal{T} \models F \text{ implies } F \in \mathcal{T}.$$

A theory is **complete** if either $F \in \mathcal{T}$ or $\neg F \in \mathcal{T}$ for any F.

Two recipes for generating theories:

- Pick a σ-structure $\mathcal{A}$. Define

  $$\mathrm{Th}(\mathcal{A}) = \{F \,:\, \mathcal{A} \models F \text{ and } F \text{ is a sentence}\}.$$

  $\mathrm{Th}(\mathcal{A})$ is called the **theory of** $\mathcal{A}$.

- Pick a set of **axioms** $\mathcal{S}$ (i.e., a set of setences). Define

  $$\mathcal{T} = \{F \,:\, \mathcal{S} \models F \text{ and } F \text{ is a sentence}\}.$$

Logical theories

A **theory** $\mathcal{T}$ is a set of sentences closed under semantic entailment:

$$\mathcal{T} \models F \text{ implies } F \in \mathcal{T}.$$

A theory is **complete** if either $F \in \mathcal{T}$ or $\neg F \in \mathcal{T}$ for any F.

Two recipes for generating theories:

- Pick a σ-structure $\mathcal{A}$. Define

  $$\mathrm{Th}(\mathcal{A}) = \{F : \mathcal{A} \models F \text{ and } F \text{ is a sentence}\}.$$

  $\mathrm{Th}(\mathcal{A})$ is called the **theory of** $\mathcal{A}$.

- Pick a set of **axioms** $\mathcal{S}$ (i.e., a set of setences). Define

  $$\mathcal{T} = \{F : \mathcal{S} \models F \text{ and } F \text{ is a sentence}\}.$$

Ex1: Prove that both recipes give theories.

Ex2: Which one always generates a complete theory?

Ex3: Give an example of an incomplete theory.

Example (Structure-based Theory)

The theory of **linear arithmetic over the rationals** is

$$\mathcal{T}_{LAR} = \mathrm{Th}(\mathbb{Q}, 1, +, \{c\,\cdot\,\}_{c \in \mathbb{Q}}, <, =).$$

It tells the truth of the following sentences:

- The system of linear inequalities $A\boldsymbol{x} \leq \boldsymbol{b}$ has no solution.
- Every solution of $A\boldsymbol{x} \leq \boldsymbol{b}$ is also a solution of $C\boldsymbol{x} \leq \boldsymbol{d}$.

Example (Structure-based Theory)

The theory of **linear arithmetic over the rationals** is

$$\mathcal{T}_{LAR} = \mathrm{Th}(\mathbb{Q}, 1, +, \{c\,\cdot\,\}_{c\in\mathbb{Q}}, <, =).$$

It tells the truth of the following sentences:

- The system of linear inequalities $A\boldsymbol{x} \leq \boldsymbol{b}$ has no solution.
- Every solution of $A\boldsymbol{x} \leq \boldsymbol{b}$ is also a solution of $C\boldsymbol{x} \leq \boldsymbol{d}$.

Example (Axiom-based Theory)

The theory $\mathcal{T}_{UDLO}$ of **unbounded dense linear orders** over the signature $(<, =)$ is the set of sentences entailed by the following set of axioms:

$$\begin{array}{ll}
F_1 & \forall x\,\forall y\,(x < y \to \neg(x = y \lor y < x)) \\
F_2 & \forall x\,\forall y\,\forall z\,(x < y \land y < z \to x < z) \\
F_3 & \forall x\,\forall y\,(x < y \lor y < x \lor x = y) \\
F_4 & \forall x\,\forall y\,(x < y \to \exists z\,(x < z \land z < y)) \\
F_5 & \forall x\,\exists y\,\exists z\,(y < x < z)\,.
\end{array}$$

Decidable theories

A theory $\mathcal{T}$ is **decidable** if there is an algorithm for checking $F \in \mathcal{T}$ for every sentence F.

Decidable theories

A theory $\mathcal{T}$ is **decidable** if there is an algorithm for checking $F \in \mathcal{T}$ for every sentence F.

Ex: Do you think $\mathcal{T}_{LAR}$ is decidable? What about $\mathcal{T}_{UDLO}$?

Decidable theories

A theory $\mathcal{T}$ is **decidable** if there is an algorithm for checking $F \in \mathcal{T}$ for every sentence F.

Ex: Do you think $\mathcal{T}_{LAR}$ is decidable? What about $\mathcal{T}_{UDLO}$?

Ans: Both are decidable. Proved by **quantifier-elimination**.

Decidable theories

A theory $\mathcal{T}$ is **decidable** if there is an algorithm for checking $F \in \mathcal{T}$ for every sentence F.

Ex: Do you think $\mathcal{T}_{LAR}$ is decidable? What about $\mathcal{T}_{UDLO}$?

Ans: Both are decidable. Proved by **quantifier-elimination**.

A theory $\mathcal{T}$ **admits quantifier-elimination** if for any $\exists x\, F$ with F quantifier-free, there is a quantifier-free formula G such that

$$\mathcal{T} \models \exists x\, F \leftrightarrow G.$$

$\mathcal{T}$ has a **quantifier-elimination procedure** if G is computable.

Decidable theories

A theory $\mathcal{T}$ is **decidable** if there is an algorithm for checking $F \in \mathcal{T}$ for every sentence F.

Ex: Do you think $\mathcal{T}_{LAR}$ is decidable? What about $\mathcal{T}_{UDLO}$?

Ans: Both are decidable. Proved by **quantifier-elimination**.

A theory $\mathcal{T}$ **admits quantifier-elimination** if for any $\exists x\, F$ with F quantifier-free, there is a quantifier-free formula G such that

$$\mathcal{T} \models \exists x\, F \leftrightarrow G.$$

$\mathcal{T}$ has a **quantifier-elimination procedure** if G is computable.

Theorem

A theory $\mathcal{T}$ is decidable if $\mathcal{T}$ has (i) a quantifier-elimination (QE) procedure, and (ii) a procedure for deciding $F \in \mathcal{T}$ for quantifier-free (QF) sentences F.

Decidable theories

A theory $\mathcal{T}$ is **decidable** if there is an algorithm for checking $F \in \mathcal{T}$ for every sentence F.

Ex: Do you think $\mathcal{T}_{LAR}$ is decidable? What about $\mathcal{T}_{UDLO}$?

Ans: Both are decidable. Proved by **quantifier-elimination**.

A theory $\mathcal{T}$ **admits quantifier-elimination** if for any $\exists x\, F$ with F quantifier-free, there is a quantifier-free formula G such that

$$\mathcal{T} \models \exists x\, F \leftrightarrow G.$$

$\mathcal{T}$ has a **quantifier-elimination procedure** if G is computable.

Theorem

A theory $\mathcal{T}$ is decidable if $\mathcal{T}$ has (i) a quantifier-elimination (QE) procedure, and (ii) a procedure for deciding $F \in \mathcal{T}$ for quantifier-free (QF) sentences F.

Ex: Prove the theorem.

Theorem

The theory $\mathcal{T}_{UDLO}$ of unbounded dense linear orders is decidable.

Theorem

The theory $\mathcal{T}_{UDLO}$ of unbounded dense linear orders is decidable.

$\mathcal{T}_{UDLO}$ has (i) a QE proc., and (ii) a decision proc. for QF sentences.

Theorem

The theory $\mathcal{T}_{UDLO}$ of unbounded dense linear orders is decidable.

$\mathcal{T}_{UDLO}$ has (i) a QE proc., and (ii) a decision proc. for QF sentences.

Ex1: Design a decision procedure for QF sentences.

Theorem

The theory $\mathcal{T}_{UDLO}$ of unbounded dense linear orders is decidable.

$\mathcal{T}_{UDLO}$ has (i) a QE proc., and (ii) a decision proc. for QF sentences.

Ex1: Design a decision procedure for QF sentences.

Ans1: QF sentences are boolean combinations of **true** and **false**. Compute their truth values using truth tables.

Theorem

The theory $\mathcal{T}_{UDLO}$ of unbounded dense linear orders is decidable.

$\mathcal{T}_{UDLO}$ has (i) a QE proc., and (ii) a decision proc. for QF sentences.

Ex1: Design a decision procedure for QF sentences.

Ans1: QF sentences are boolean combinations of **true** and **false**. Compute their truth values using truth tables.

Ex2: Design a QE procedure.

Theorem

The theory $\mathcal{T}_{UDLO}$ of unbounded dense linear orders is decidable.

$\mathcal{T}_{UDLO}$ has (i) a QE proc., and (ii) a decision proc. for QF sentences.

Ex1: Design a decision procedure for QF sentences.

Ans1: QF sentences are boolean combinations of **true** and **false**. Compute their truth values using truth tables.

Ex2: Design a QE procedure.

Hint: Given $\exists x\, F$ for a quantifier-free F, the proc. works as follows:

1. Transform $\exists x\, F$ to an equivalent $\bigvee_i((\exists x\, G_i) \wedge H_i)$ where H_i is QF and G_i is conjunction of $x < y$ or $y < x$ for some variable $y \neq x$.

2. Transform $\exists x\, G_i$ to an equivalent quantifier-free G_i'.

Find out how to do both steps.

1. Transform $\exists x\, F$ to an equivalent $\bigvee_i((\exists x\, G_i) \wedge H_i)$ where H_i is QF and G_i is conjunction of $x < y$ or $y < x$ for some variable $y \neq x$.

2. Transform $\exists x\, G_i$ to an equivalent quantifier-free G_i'.

How to do the step 2?

1. Transform $\exists x\, F$ to an equivalent $\bigvee_i ((\exists x\, G_i) \wedge H_i)$ where H_i is QF and G_i is conjunction of $x < y$ or $y < x$ for some variable $y \neq x$.

2. Transform $\exists x\, G_i$ to an equivalent quantifier-free G_i'.

How to do the step 2?

Assume that $G_i \equiv (l_1 < x \wedge \ldots \wedge l_m < x \wedge x < u_1 \wedge \ldots \wedge x < u_n)$.

1. Transform $\exists x\, F$ to an equivalent $\bigvee_i ((\exists x\, G_i) \wedge H_i)$ where H_i is QF and G_i is conjunction of $x < y$ or $y < x$ for some variable $y \neq x$.

2. Transform $\exists x\, G_i$ to an equivalent quantifier-free G_i'.

How to do the step 2?

Assume that $G_i \equiv (l_1 < x \wedge \ldots \wedge l_m < x \wedge x < u_1 \wedge \ldots \wedge x < u_n)$.

If $m = 0$ or $n = 0$, then $\mathcal{T}_{UDLO} \models (\exists x\, G_i) \leftrightarrow \textbf{true}$. Thus, $G_i' = \textbf{true}$.

1. Transform $\exists x\, F$ to an equivalent $\bigvee_i ((\exists x\, G_i) \wedge H_i)$ where H_i is QF and G_i is conjunction of $x < y$ or $y < x$ for some variable $y \neq x$.

2. Transform $\exists x\, G_i$ to an equivalent quantifier-free G_i'.

How to do the step 2?

Assume that $G_i \equiv (l_1 < x \wedge \ldots \wedge l_m < x \wedge x < u_1 \wedge \ldots \wedge x < u_n)$.

If $m = 0$ or $n = 0$, then $\mathcal{T}_{UDLO} \models (\exists x\, G_i) \leftrightarrow \textbf{true}$. Thus, $G_i' = \textbf{true}$.

Otherwise,

$$\mathcal{T}_{UDLO} \models (\exists x\, G_i) \leftrightarrow \bigwedge_{i=1}^{m} \bigwedge_{j=1}^{n} l_i < u_j.$$

Thus, $G_i' = \bigwedge_{i=1}^{m} \bigwedge_{j=1}^{n} l_i < u_j$.

Theorem

The theory

$$\mathcal{T}_{LAR} = \mathrm{Th}(\mathbb{Q}, 1, +, \{c\,\cdot\,\}_{c\in\mathbb{Q}}, <, =)$$

of linear arithmetic over the rationals is decidable.

Theorem

The theory

$$\mathcal{T}_{LAR} = \mathrm{Th}(\mathbb{Q}, 1, +, \{c\,\cdot\,\}_{c\in\mathbb{Q}}, <, =)$$

of linear arithmetic over the rationals is decidable.

Ex: Prove the theorem. Hint: The proof is very similar to the one for the decidability of $\mathcal{T}_{UDLO}$, which we have just studied.

Presburger arithmetic

Figure: Mojzesz Presburger (1904 - 1943).

$\mathrm{Th}(\mathbb{N}, 0, 1, +, <, =)$ is commonly known as **Presburger arithmetic**.

Natural numbers, not rationals. Only addition. No multiplication.

Simple number theory in Presburger arithmetic

Example

Every natural number is odd or even:

$$\forall x\, \exists y\, (x = y + y \lor x = y + y + 1)\,.$$

Simple number theory in Presburger arithmetic

Example

Every natural number is odd or even:

$$\forall x \, \exists y \, (x = y + y \vee x = y + y + 1) \, .$$

Ex: Consider the following Chicken McNugget problem.

> *Given $a_1, \ldots, a_n \in \mathbb{N}$, is there some $c \in \mathbb{N}$ such that all numbers greater than c can be represented as a non-negative linear combination of $a_1, \ldots, a_n$?*

Express this problem for given $a_1, \ldots, a_n$ in Presburger arithmetic.

Simple number theory in Presburger arithmetic

Example

Every natural number is odd or even:

$$\forall x \, \exists y \, (x = y + y \vee x = y + y + 1) \, .$$

Ex: Consider the following Chicken McNugget problem.

> *Given $a_1, \ldots, a_n \in \mathbb{N}$, is there some $c \in \mathbb{N}$ such that all numbers greater than c can be represented as a non-negative linear combination of $a_1, \ldots, a_n$?*

Express this problem for given $a_1, \ldots, a_n$ in Presburger arithmetic.

Ans:

$$\exists x \, \forall y \, (x < y \rightarrow (\exists z_1 \ldots \exists z_n \, (y = a_1 \cdot z_1 + \cdots + a_n \cdot z_n))).$$

Here $a_i \cdot z_i$ is an abbreviation for $\underbrace{z_i + \ldots + z_i}_{a_i \text{ copies}}$.

Decidability of Presburger arithmetic

Theorem

Presburger arithmetic $\mathrm{Th}(\mathbb{N}, 0, 1, +, <, =)$ *is decidable.*

Decidability of Presburger arithmetic

Theorem

Presburger arithmetic $\mathrm{Th}(\mathbb{N}, 0, 1, +, <, =)$ *is decidable.*

Can't prove it using QE.

$\mathrm{Th}(\mathbb{N}, 0, 1, +, <, =)$ does not have quantifier elimination. For instance, y cannot be eliminated from $\exists y\,(x = y + y)$.

Decidability of Presburger arithmetic

Theorem

Presburger arithmetic $\mathrm{Th}(\mathbb{N}, 0, 1, +, <, =)$ *is decidable.*

Can't prove it using QE.

$\mathrm{Th}(\mathbb{N}, 0, 1, +, <, =)$ does not have quantifier elimination. For instance, y cannot be eliminated from $\exists y\,(x = y + y)$.

Solution: extend the signature with unary divisibility relations $c \mid \cdot$ for all $c > 0$ such that

$$c \mid n \text{ iff there is } k \in \mathbb{N} \text{ such that } n = k \cdot c.$$

Theorem

$\mathrm{Th}(\mathbb{N}, 0, 1, +, <, =, \{c \mid \cdot\}_{c>0})$ *has a QE procedure.*

Decidability of Presburger arithmetic

Theorem

Presburger arithmetic $\mathrm{Th}(\mathbb{N}, 0, 1, +, <, =)$ *is decidable.*

Can't prove it using QE.

$\mathrm{Th}(\mathbb{N}, 0, 1, +, <, =)$ does not have quantifier elimination. For instance, y cannot be eliminated from $\exists y\,(x = y + y)$.

Solution: extend the signature with unary divisibility relations $c \mid \cdot$ for all $c > 0$ such that

$$c \mid n \text{ iff there is } k \in \mathbb{N} \text{ such that } n = k \cdot c.$$

Theorem

$\mathrm{Th}(\mathbb{N}, 0, 1, +, <, =, \{c \mid \cdot\}_{c>0})$ *has a QE procedure.*

Ex: Use the theorem and prove the decid. of Presburger arithmetic.

The QE procedure for $\mathrm{Th}(\mathbb{N}, 0, 1, +, <, =, \{c \mid \cdot\}_{c>0})$ works in two steps.

The QE procedure for $\mathrm{Th}(\mathbb{N}, 0, 1, +, <, =, \{c \mid \cdot\}_{c>0})$ works in two steps.

1. Transform $\exists x\, F$ to an equivalent $\bigvee_l((\exists x\, F_l) \wedge F'_l)$ where F'_l is QF and F_l has the form

$$F_l = \Big(\bigwedge_{i \in L} q_i(\vec{y}) < a_i \cdot x \wedge \bigwedge_{j \in U} a_j \cdot x < p_j(\vec{y}) \wedge \bigwedge_{k \in D} c_k \mid a_k \cdot x + r_k(\vec{y}) \Big).$$

2. Transform $\exists x\, F_l$ to an equivalent quantifier-free G_l.

The QE procedure for $\mathrm{Th}(\mathbb{N}, 0, 1, +, <, =, \{c \mid \cdot\}_{c>0})$ works in two steps.

1. Transform $\exists x\, F$ to an equivalent $\bigvee_l ((\exists x\, F_l) \wedge F'_l)$ where F'_l is QF and F_l has the form

$$F_l = \Big(\bigwedge_{i \in L} q_i(\vec{y}) < a_i \cdot x \wedge \bigwedge_{j \in U} a_j \cdot x < p_j(\vec{y}) \wedge \bigwedge_{k \in D} c_k \mid a_k \cdot x + r_k(\vec{y}) \Big).$$

2. Transform $\exists x\, F_l$ to an equivalent quantifier-free G_l.

Step 1 is similar to what we did before. Will focus on step 2.

$$F_l = \Big(\bigwedge_{i \in L} q_i(\vec{y}) < a_i \cdot x \wedge \bigwedge_{j \in U} a_j \cdot x < p_j(\vec{y}) \wedge \bigwedge_{k \in D} c_k \mid a_k \cdot x + r_k(\vec{y}) \Big).$$

Goal: Eliminate quantifiers from $\exists x\, F_l$.

$$F_l = \Big(\bigwedge_{i \in L} q_i(\vec{y}) < a_i \cdot x \wedge \bigwedge_{j \in U} a_j \cdot x < p_j(\vec{y}) \wedge \bigwedge_{k \in D} c_k \mid a_k \cdot x + r_k(\vec{y}) \Big).$$

Goal: Eliminate quantifiers from $\exists x\, F_l$.

Let $b = \mathrm{lcm}\{a_i \mid i \in L \cup U \cup D\}$.

$$F_l = \Big(\bigwedge_{i \in L} q_i(\vec{y}) < a_i \cdot x \wedge \bigwedge_{j \in U} a_j \cdot x < p_j(\vec{y}) \wedge \bigwedge_{k \in D} c_k \mid a_k \cdot x + r_k(\vec{y}) \Big).$$

Goal: Eliminate quantifiers from $\exists x\, F_l$.

Let $b = \mathrm{lcm}\{a_i \mid i \in L \cup U \cup D\}$.

Ex1: Show that $\exists x\, F_l$ is equivalent to $\exists x\, H$ where

$$\begin{aligned} H = {} & \bigwedge_{i \in L} \frac{b}{a_i} \cdot q_i(\vec{y}) < x \wedge \bigwedge_{j \in U} x < \frac{b}{a_j} \cdot p_j(\vec{y}) \\ & \wedge \bigwedge_{k \in D} \left(\frac{b}{a_k} \cdot c_k \right) \Bigm| \left(x + \frac{b}{a_k} \cdot r_k(\vec{y}) \right) \wedge b \mid x. \end{aligned}$$

$$F_I = \Big(\bigwedge_{i \in L} q_i(\vec{y}) < a_i \cdot x \wedge \bigwedge_{j \in U} a_j \cdot x < p_j(\vec{y}) \wedge \bigwedge_{k \in D} c_k \mid a_k \cdot x + r_k(\vec{y}) \Big).$$

Goal: Eliminate quantifiers from $\exists x \, F_I$.

Let $b = \mathrm{lcm}\{a_i \mid i \in L \cup U \cup D\}$.

Ex1: Show that $\exists x \, F_I$ is equivalent to $\exists x \, H$ where

$$H = \bigwedge_{i \in L} \frac{b}{a_i} \cdot q_i(\vec{y}) < x \wedge \bigwedge_{j \in U} x < \frac{b}{a_j} \cdot p_j(\vec{y})$$
$$\wedge \bigwedge_{k \in D} \left(\frac{b}{a_k} \cdot c_k \right) \Bigm| \left(x + \frac{b}{a_k} \cdot r_k(\vec{y}) \right) \wedge b \mid x.$$

Define $c = \mathrm{lcm}\big(\{b\} \cup \{b \cdot c_k / a_k : k \in D\}\big)$.

$$F_l = \Big(\bigwedge_{i \in L} q_i(\vec{y}) < a_i \cdot x \wedge \bigwedge_{j \in U} a_j \cdot x < p_j(\vec{y}) \wedge \bigwedge_{k \in D} c_k \mid a_k \cdot x + r_k(\vec{y}) \Big).$$

Goal: Eliminate quantifiers from $\exists x \, F_l$.

Let $b = \mathrm{lcm}\{a_i \mid i \in L \cup U \cup D\}$.

Ex1: Show that $\exists x \, F_l$ is equivalent to $\exists x \, H$ where

$$H = \bigwedge_{i \in L} \frac{b}{a_i} \cdot q_i(\vec{y}) < x \wedge \bigwedge_{j \in U} x < \frac{b}{a_j} \cdot p_j(\vec{y})$$
$$\wedge \bigwedge_{k \in D} \left(\frac{b}{a_k} \cdot c_k \right) \Bigm| \left(x + \frac{b}{a_k} \cdot r_k(\vec{y}) \right) \wedge b \mid x.$$

Define $c = \mathrm{lcm}\big(\{b\} \cup \{b \cdot c_k / a_k : k \in D\}\big)$.

Ex2: Show that $\exists x \, H$ is equivalent to

$$\begin{cases} \bigvee_{0 \le m < c} H[m/x] & \text{if } L = \emptyset, \\ \bigvee_{i \in L} \bigvee_{1 \le m \le c} H[((b/a_i) \cdot q_i(\vec{y}) + m)/x] & \text{otherwise.} \end{cases}$$

Time complexity of the decidability algorithm for Presburger arithmetic

Theorem (Oppen)

Presburger arithmetic is decidable in time $2^{2^{2^{O(n)}}}$.

Good article on Presburger arithmetic

A suvival guide to Presburger arithmetic.

Written by Christoph Hasse. Published in ACM SIGLOG News 2018.

https://dl.acm.org/citation.cfm?id=3242964.