Lecture 11

Applications of Herbrand's theorem

Ground resolution proofs, semi-decidability of validity, undecidability of validity

Introduction to Logic for Computer Science

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These slides are minor variants of those made by Prof Worrell and Prof Haase for their logic course at Oxford.

Recap

Theorem (Herbrand's theorem)

Let $F = \forall x_1 \forall x_2 \ldots \forall x_n\, F^*$ be a Skolem formula. Then F is satisfiable if and only if F has a Herbrand model.

Generalisation of the ground resolution theorem

Theorem

Let $F_1, \ldots, F_n$ be closed rectified formulas in prenex form with Skolem forms $G_1, \ldots, G_n$. Assume each G_i is obtained using different Skolem functions. Then

$$F_1 \wedge F_2 \wedge \cdots \wedge F_n \text{ is satisfiable}$$
$$\text{iff } G_1 \wedge G_2 \wedge \cdots \wedge G_n \text{ is satisfiable.}$$

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Theorem (Ground resolution theorem)

Let $G_1, \ldots, G_n$ be closed formulas in Skolem form whose respective matrices $G_1^*, G_2^*, \ldots, G_n^*$ are in CNF. Then $G_1 \wedge G_2 \wedge \cdots \wedge G_n$ is unsatisfiable if and only if there is a propositional resolution proof of $\Box$ starting from the set of ground instances of clauses from $G_1^*, \ldots, G_n^*$.

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Ex: Prove both theorems.

An example

An example

Example

Consider the following hypothetical scenario:

(a) Everyone at Oriel[a] is lazy, a rower or drunk.

(b) All rowers are lazy.

(c) Someone at Oriel is not drunk.

(d) Someone at Oriel is lazy.

Show that (a), (b) and (c) together entail (d).

[a] Oriel is one of the oldest Oxford colleges. Oxford colleges are like houses in the Harry Potter movie.

An example

Translation into first-order logic:

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\begin{aligned}
F_1 &:= \forall x(O(x) \to (L(x) \vee R(x) \vee D(x))),\\
F_2 &:= \forall x(R(x) \to L(x)),\\
F_3 &:= \exists x(O(x) \wedge \neg D(x)),\\
F_4 &:= \neg\exists x(O(x) \wedge L(x)).
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Ex: Translate $F_1 \wedge F_2 \wedge F_3 \wedge F_4$ into CNF Skolem form. Then, prove that the result is unsat using ground resolution.

Transformation into CNF Skolem form:

$$\begin{aligned}
G_1 &:= \forall x(\neg O(x) \vee L(x) \vee R(x) \vee D(x)),\\
G_2 &:= \forall x(\neg R(x) \vee L(x)),\\
G_3 &:= O(a) \wedge \neg D(a),\\
G_4 &:= \forall x(\neg O(x) \vee \neg L(x)).
\end{aligned}$$

Ground resolution proof for the example:

$$\dfrac{\dfrac{\dfrac{\dfrac{\{\neg R(a), L(a)\} \qquad \{\neg O(a), L(a), R(a), D(a)\}}{\{L(a), \neg O(a), D(a)\}} \qquad \{\neg O(a), \neg L(a)\}}{\{\neg O(a), D(a)\}} \qquad \{\neg D(a)\}}{\{\neg O(a)\}} \qquad \{O(a)\}}{\square}$$

Another example

Show that the following formula is valid:

$$F = \forall x \exists y (P(x) \to Q(y)) \to \exists y \forall x (P(x) \to Q(y)).$$

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F is valid if and only if $\neg F$ is unsatisfiable:

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Express $\neg F$ as $F_1 \land F_2$:

$$\begin{aligned} \neg F &\equiv F_1 \land F_2, \\ F_1 &= \forall x \exists y (P(x) \to Q(y)), \\ F_2 &= \neg \exists y \forall x (P(x) \to Q(y)). \end{aligned}$$

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Ex: Prove that $F_1 \wedge F_2$ is unsat via Skolemisation and resolution.

Hint: In this case, Skolemisation does not introduce any constants, so that you won't have any ground terms. To overcome this, introduce a constant symbol *a*. Justify why this introduction is ok.

Skolemise:

$$F_1 = \forall x(\neg P(x) \vee Q(f(x))) \qquad F_2 = \forall y(P(g(y)) \wedge \neg Q(y))$$

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No constant symbols. No ground terms. Introduce a constant *a*.

Ground resolution proof:

$$\dfrac{\dfrac{\{P(g(a))\} \qquad \{\neg P(g(a)), Q(f(g(a)))\}}{\{Q(f(g(a)))\}} \qquad \{\neg Q(f(g(a)))\}}{\square}$$

Semi-decidability of validity

Theorem

Validity of first-order logic is semi-decidable.

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Semi-Decision Procedure for Validity

Input: Closed formula F

Output: Either that F is valid or compute forever

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Compute a Skolem-form formula $G$ equisatisfiable with $\neg F$
  such that $G$'s quantifier-free part is in CNF
Let $G_1, G_2, \ldots$ be an enumeration of the Herbrand expansion $E(G)$
for $n = 1$ to $\infty$ do
begin
    if $\Box \in \mathrm{Res}^*(G_1 \cup \ldots \cup G_n)$ then stop and output "$F$ is valid"
end
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Ex: Can we do better? Can we design an *algorithm* for validity?

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Ex: Can we do better? Can we design an *algorithm* for validity?

Answer: No.

How to show undecidability?

Principle:

- Take an undecidable problem P.
- Provide a computable function f that translates an instance I of P into the validity problem for first order logic $f(I)$.
- Ensure that the answer for I is yes iff $f(I)$ is valid.
- “Validity for first-order logic is at least as difficult as P and hence undecidable.”

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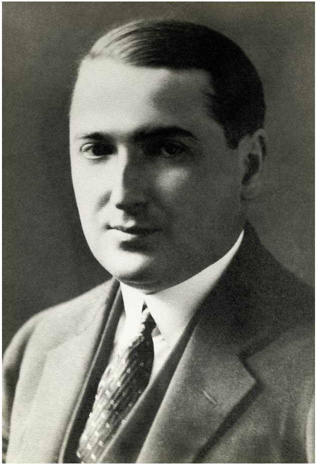
We choose P to be the **Post Correspondence Problem (PCP).**

Emil Post (1897 – 1954)



The post correspondence problem

In PCP, given a set of **tiles** $(x_i, y_i) \in \{0,1\}^* \times \{0,1\}^*$, e.g.:

$$\left\{ \begin{bmatrix} 1 \\ 101 \end{bmatrix}, \begin{bmatrix} 10 \\ 00 \end{bmatrix}, \begin{bmatrix} 011 \\ 11 \end{bmatrix} \right\}.$$

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A solution is a sequence of tiles such that the top string equals the bottom string:

$$\begin{bmatrix} 1 \\ 101 \end{bmatrix} \begin{bmatrix} 011 \\ 11 \end{bmatrix} \begin{bmatrix} 10 \\ 00 \end{bmatrix} \begin{bmatrix} 011 \\ 11 \end{bmatrix}.$$

The post correspondence problem

Definition (Post Correspondence Problem (PCP))

An **instance of PCP** is a finite set

$$P = \{(x_1, y_1), \ldots, (x_k, y_k)\} \subseteq \{0,1\}^* \times \{0,1\}^*.$$

A **solution of** P is a sequence of indices $i_1, i_2, \ldots, i_n$ such that $i_j \in \{1, \ldots, k\}$ for all $1 \leq j \leq n$, and

$$x_{i_1} x_{i_2} \cdots x_{i_n} = y_{i_1} y_{i_2} \cdots y_{i_n}.$$

Theorem

The PCP is undecidable.

Reduction to first-order logic

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Encode strings using terms.

- Introduce constant symbol e.
- Introduce unary function symbols f_0 and f_1.
- Write e.g. $f_{10110}(e)$ instead of $f_1(f_0(f_1(f_1(f_0(e)))))$.

Reduction to first-order logic

$$\left\{ \left[\begin{array}{c} 1 \\ 101 \end{array} \right], \left[\begin{array}{c} 10 \\ 00 \end{array} \right], \left[\begin{array}{c} 011 \\ 11 \end{array} \right] \right\}.$$

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Introduce binary predicate symbol $P(x, y)$.

- Write a formula which expresses that $P(x, y)$ hold iff the pair of strings (x, y) can be built using a sequence of given tiles.

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Ex1: Find such a formula for the three tiles from above.

Ex2: Using a formula, express the existence of a solution.

$$\left\{ \begin{bmatrix} 1 \\ 101 \end{bmatrix}, \begin{bmatrix} 10 \\ 00 \end{bmatrix}, \begin{bmatrix} 011 \\ 11 \end{bmatrix} \right\}.$$

$$\begin{aligned}
F &= F_1 \wedge F_2 \to F_3, \\
F_1 &= P(f_1(e), f_{101}(e)) \wedge P(f_{10}(e), f_{00}(e)) \wedge P(f_{011}(e), f_{11}(e)), \\
F_2 &= \forall u \forall v\, (P(u,v) \to P(f_1(u), f_{101}(v))) \\
&\qquad\qquad \wedge (P(u,v) \to P(f_{10}(u), f_{00}(v)) \\
&\qquad\qquad \wedge (P(u,v) \to P(f_{011}(u), f_{11}(v))). \\
F_3 &= \exists u\, P(u,u).
\end{aligned}$$

Reduction to first-order logic

Given instance P of PCP

$$P = \{(x_1, y_1), \ldots, (x_k, y_k)\} \subseteq \{0, 1\}^* \times \{0, 1\}^*.$$

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$$\begin{aligned}
F_1 &= \bigwedge_{i=1}^{k} P(f_{x_i}(e), f_{y_i}(e)), \\
F_2 &= \forall u \forall v \bigwedge_{i=1}^{k} (P(u,v) \rightarrow P(f_{x_i}(u), f_{y_i}(v))), \\
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Proposition

P has a solution if and only if $F_1 \wedge F_2 \to F_3$ is valid.

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Ex: Prove the proposition.

$$F_1 = \bigwedge_{i=1}^{k} P(f_{x_i}(e), f_{y_i}(e))$$

$$F_3 = \exists u \, P(u, u)$$

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- If $F_1 \wedge F_2 \to F_3$ is valid, consider the Herbrand structure $\mathcal{H}$ with

$$P_{\mathcal{H}} = \{(f_u(e), f_v(e)) : \exists i_1 \ldots \exists i_t \,.\, u = x_{i_1} \ldots x_{i_t} \text{ and } v = y_{i_1} \ldots y_{i_t}\}.$$

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  Now $\mathcal{H} \models F_1 \wedge F_2$. So, $\mathcal{H} \models F_3$. But then P has a solution.

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  Now $\mathcal{H} \models F_1 \wedge F_2$. So, $\mathcal{H} \models F_3$. But then P has a solution.

- If P has a solution, consider $\mathcal{A}$ that satisfies $F_1 \wedge F_2$. Show by induction on t that for every sequence of tiles $i_1 \ldots i_t$,

  $$\mathcal{A} \models P(f_u(e), f_v(e)), \text{ where } u = x_{i_1} \ldots x_{i_t} \text{ and } v = y_{i_1} \ldots y_{i_t}.$$

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  But since P has a solution, $\mathcal{A} \models P(f_u(e), f_u(e))$ for some string u. Thus, $\mathcal{A} \models F_3$.

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Validity in first-order logic is undecidable.

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Ex: Prove these theorems. Hint: Use the proposition and the undecidability of the PCP problem P.

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Satisfiability in first-order logic is not semi-decidable.

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Theorem

Satisfiability in first-order logic is not semi-decidable.

Ex: Prove it. Hint: Use the semi-decidability and the undecidability of validity in first-order logic.