

Lecture 12

Resolution for predicate logic

Unification, resolution

Print version of the lecture in *Introduction to Logic for Computer Science*

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These lecture notes are very minor variants of the ones made by Prof James Worrell and Prof Christoph Haase for their 'Logic and Proof' course at Oxford.

A serious drawback of the ground resolution procedure is that it requires looking ahead to predict which ground instances of clauses will be needed in a proof. In this lecture, we introduce the predicate-logic version of resolution, which allows us to perform substitution “by need”. This relies on the notion of unification, which we introduce next.

1 Unification

A *substitution* is a function θ from the set of σ -terms back to itself such that (writing function application on the right) $c\theta = c$ for each constant symbol c and $f(t_1, \dots, t_k)\theta = f(t_1\theta, \dots, t_k\theta)$ for each k -ary function symbol f . It is clear that the composition of two such substitutions (as functions) is also a substitution. We have previously considered substitutions of the form $[t/x]$ for a σ -term t and a variable x .

We write composition of substitutions diagrammatically, that is, $\theta \cdot \theta'$ denotes the substitution obtained by applying θ first and then θ' . (This convention matches the fact that for substitutions, we write function application on the right.) In particular, $[t_1/x_1] \cdots [t_k/x_k]$ denotes the substitution obtained by sequentially applying the substitutions $[t_1/x_1], \dots, [t_k/x_k]$ left-to-right.

Given a set of literals $D = \{L_1, \dots, L_k\}$ and a substitution θ , define $D\theta := \{L_1\theta, \dots, L_k\theta\}$. We say that θ *unifies* D if $D\theta = \{L\}$ for some literal L . For example, the substitution $\theta = [f(a)/x][a/y]$ unifies $\{P(x), P(f(y))\}$, as does the substitution $\theta' = [f(y)/x]$. In this example, we regard θ' as a *more general unifier* because $\theta = \theta' \cdot [a/y]$, that is, θ factors through θ' .

We say that θ is a *most general unifier* of a set of literals D if θ is a unifier of D and any other unifier θ' factors through θ , i.e., we have $\theta' = \theta \cdot \theta''$ for some substitution θ'' . Note that both the substitutions $[x/y]$ and $[y/x]$ are most general unifiers of $\{P(x), P(y)\}$. In fact, most-general unifiers are only unique up to renaming variables.

We will show that a set of literals either has no unifier or it has a most general unifier. Examples of sets of literals that cannot be unified are $\{P(f(x)), P(g(x))\}$ and $\{P(f(x)), P(x)\}$. The problem in the second case is that we cannot unify a variable x and term t if x occurs in t .

Theorem 1 (Unification Theorem). *A unifiable set of literals D has a most general unifier.*

Proof. We claim that the following algorithm determines whether a set of literals has a unifier and, if so, outputs a most general unifier.

Unification Algorithm

Input: Set of literals D

Output: Either a most general unifier of D or “fail”

$\theta :=$ identity substitution

while θ is not a unifier of D **do**

begin

 pick two distinct literals in $D\theta$ and find the left-most positions at which they differ

if one of the corresponding sub-terms is a variable x

 and the other term t does not contain x

then $\theta := \theta \cdot [t/x]$ **else** output “fail” and halt

end

return θ

We argue termination as follows. In any iteration of the while loop that does not cause the program to halt, a variable x is replaced everywhere in $D\theta$ by a term t that does not contain x . Thus, the number of different variables occurring in $D\theta$ decreases by one in each iteration, and the loop must terminate.

The loop invariant is that for any unifier θ' of D , we have $\theta' = \theta \cdot \theta'$. Clearly, the invariant is established by the initial assignment of the identity substitution to θ . To see that the invariant is maintained by an iteration of the loop, suppose we find an occurrence of variable x in a literal in $D\theta$ such that a different term t occurs in the same position in another literal in $D\theta$. From the invariant, we know that θ' is a unifier of $D\theta$, and thus $t\theta' = x\theta'$. It immediately follows that $\theta' = [t/x] \cdot \theta'$. Thus, the loop invariant is maintained by the assignment $\theta := \theta \cdot [t/x]$.

The termination condition of the while loop is that θ is a unifier of D . In conjunction with the loop invariant, this implies that the final value of θ is a most general unifier of D . Finally, the invariant implies that if θ' is a unifier of D , then it is also a unifier of $D\theta$. But the algorithm only outputs “fail” if $D\theta$ has no unifier, in which case D has no unifier. \square

Example 2. Consider an execution of the unification algorithm on input $D = \{P(x, y), P(f(z), x)\}$. Scanning left-to-right, the leftmost discrepancy is underlined in $\{P(\underline{x}, y), P(\underline{f}(z), x)\}$. Applying the substitution $[f(z)/x]$ to D yields the set $D' = \{P(f(z), y), P(\underline{f}(z), \underline{f}(z))\}$, where the underlined positions again indicate the leftmost discrepancy. Applying the substitution $[f(z)/y]$ to D' yields the singleton set $\{P(f(z), f(z))\}$. Thus $[f(z)/x][f(z)/y]$ is a most general unifier of the set D .

2 Resolution

First-order resolution operates on a set of clauses, that is, a set of sets of literals. Given a formula $\forall x_1 \dots \forall x_n F$ in Skolem form, we perform resolution on the clauses in the matrix F with the goal of deriving the empty clause. Although quantifiers do not explicitly appear in resolution proofs, we can see the variables in such a proof as being implicitly universally quantified. This is made more formal when we formulate the Resolution Lemma in the next section.

For any set of literals D , let \bar{D} denote the set of complementary literals. For example, if $D = \{\neg P(x), R(x, y)\}$ then $\bar{D} = \{P(x), \neg R(x, y)\}$.

Definition 3 (Resolution). Let C_1 and C_2 be clauses *with no variable in common*. We say that a clause R is a *resolvent* of C_1 and C_2 if there are sets of literals $D_1 \subseteq C_1$ and $D_2 \subseteq C_2$ such that $D_1 \cup \bar{D}_2$ has a most general unifier θ , and

$$R = (C_1\theta \setminus \{L\}) \cup (C_2\theta \setminus \{\bar{L}\}), \quad (1)$$

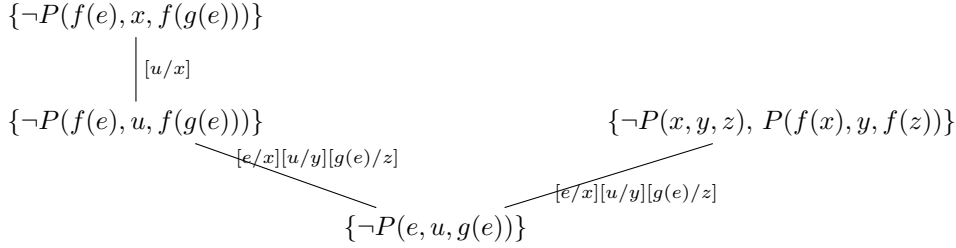


Figure 1: First-order resolution example

where $\{L\} = D_1\theta$ and $\{\bar{L}\} = D_2\theta$. More generally, if C_1 and C_2 are arbitrary clauses, we say that R is a resolvent of C_1 and C_2 if there are variable renamings θ_1 and θ_2 such that $C_1\theta_1$ and $C_2\theta_2$ have no variable in common, and R is a resolvent of $C_1\theta_1$ and $C_2\theta_2$ according to the definition above.

Example 4. Consider a signature with constant symbol e , unary function symbols f and g , and a ternary predicate symbol P . We compute a resolvent of the clauses $C_1 = \{\neg P(f(e), x, f(g(e)))\}$ and $C_2 = \{\neg P(x, y, z), P(f(x), y, f(z))\}$ as follows (see Figure 1). First apply the substitution $[u/x]$ to C_1 , obtaining a clause C'_1 that has no variable in common in C_2 . Now unify complementary literals under the substitution $[e/x][u/y][g(e)/z]$, and obtain the resolvent $\{\neg P(e, u, g(e))\}$.

A *predicate-logic resolution derivation* of a clause C from a set of clauses F is a sequence of clauses C_1, \dots, C_m , with $C_m = C$ such that each C_i is either a clause of F (possibly with the variables renamed) or follows by a resolution step from two preceding clauses C_j, C_k , with $j, k < i$. We write $\text{Res}^*(F)$ for the set of clauses C such that there is a derivation of C from F .

Example 5. Consider the following sentences over a signature with ternary predicate symbol A , constant symbol e , and unary function symbol s . The idea is that A represents the ternary addition relation, e the zero element, and s the successor function.

$$\begin{aligned} F_1 &: \forall x A(e, x, x) \\ F_2 &: \forall x \forall y \forall z (\neg A(x, y, z) \vee A(s(x), y, s(z))) \\ F_3 &: \forall x \exists y A(s(s(e)), x, y) \end{aligned}$$

We use first-order resolution to show that $F_1 \wedge F_2 \models F_3$, that is, we show that $F_1 \wedge F_2 \wedge \neg F_3$ is unsatisfiable. We proceed in two steps.

Step (i): separately Skolemise each formula. Formula $\neg F_3$ is equivalent to

$$\exists y \forall z \neg A(s(s(e)), y, z).$$

Skolemising the above formula, we obtain the formula $G_3 := \forall z \neg A(s(s(e)), c, z)$, where c is a new constant symbol. Now $F_1 \wedge F_2 \wedge G_3$ is equisatisfiable with $F_1 \wedge F_2 \wedge \neg F_3$, and so it suffices to give a resolution refutation of $F_1 \wedge F_2 \wedge G_3$.¹ **Important:** When Skolemising more than one formula separately, different formulas may not introduce the same constant symbols. Always make sure to choose fresh constant symbols for every formula you are Skolemising.

¹Formally, the notion of a resolution proof assumes a single Skolem-form formula. So strictly speaking, the proof below is a resolution refutation of the formula $\forall x \forall y \forall z (A(e, x, x) \wedge ((\neg A(x, y, z) \vee A(s(x), y, s(z))) \wedge \neg A(s(s(e)), c, z)))$, which is logically equivalent to $F_1 \wedge F_2 \wedge G_3$.

Step (ii). *derive the empty clause using resolution.* The proof is as follows. Note that in order to always ensure that we resolve clauses with disjoint variables, we arrange it so that the variables in line k of the proof are subscripted with k . In particular, we add a variable renaming at the end of each unifying substitution so that the variables in the output formula have the right subscript for the next line of the proof.

- | | | |
|----|---|---|
| 1. | $\{\neg A(s(e)), c, z_1\}$ | clause of G_3 |
| 2. | $\{\neg A(x_2, y_2, z_2), A(s(x_2), y_2, s(z_2))\}$ | clause of F_2 |
| 3. | $\{\neg A(s(e), c, z_3)\}$ | 1,2 Res. Sub $[s(e)/x_2][c/y_2][s(z_2)/z_1][z_3/z_2]$ |
| 4. | $\{\neg A(e, c, z_4)\}$ | 2,3 Res. Sub $[e/x_2][c/y_2][s(z_2)/z_3][z_4/z_2]$ |
| 5. | $\{A(e, y_5, y_5)\}$ | clause of F_1 |
| 6. | \square | 4,5 Res. Sub $[c/y_5][c/z_4]$ |

Given a formula H with free variables x_1, x_2, \dots, x_n , its *universal closure* $\forall^* H$ is the sentence $\forall x_1 \forall x_2 \dots \forall x_n H$. The following lemma is key to the soundness of resolution.

Lemma 6 (Resolution Lemma). *Let $F = \forall x_1 \dots \forall x_n G$ be a closed formula in Skolem form, with G quantifier-free and in CNF. Let R be a resolvent of two clauses in G . Then $F \equiv \forall^*(G \cup \{R\})$.*

Proof. Clearly $\forall^*(G \cup \{R\}) \models F$. The non-trivial direction is to show that $F \models \forall^* R$. For this, since F is closed, it suffices to show that $F \models R$. (Check that you understand why this is so!)

To this end, suppose that R is a resolvent of clauses $C_1, C_2 \in G$, with $R = (C_1 \theta \setminus \{L\}) \cup (C_2 \theta' \setminus \{\bar{L}\})$ for some substitutions θ, θ' and complementary literals $L \in C_1 \theta$ and $\bar{L} \in C_2 \theta'$.

Let \mathcal{A} be an assignment that satisfies $F = \forall^* G$. Since $C_1, C_2 \in G$, by the Translation Lemma $\mathcal{A} \models C_1 \theta$ and $\mathcal{A} \models C_2 \theta'$. Moreover, since \mathcal{A} satisfies at most one of the complementary literals L and \bar{L} , it follows that \mathcal{A} satisfies at least one of $C_1 \theta \setminus \{L\}$ and $C_2 \theta' \setminus \{\bar{L}\}$. We conclude that \mathcal{A} satisfies R , as required. \square

Corollary 7 (Soundness). *Let $F = \forall x_1 \dots \forall x_n G$ be a closed formula in Skolem form. Let clause C be obtained from G by a resolution derivation. Then $F \equiv \forall^*(G \cup C)$.*

Proof. Induction on the length of the resolution derivation, using the Resolution Lemma for the induction step. \square

A Refutation Completeness

In this appendix we prove the refutation completeness of predicate-logic resolution proofs by showing that ground resolution proofs lift to predicate-logic resolution proofs. The proofs here are more technical and can be regarded as optional.

Lemma 8 (Lifting Lemma). *Let C_1 and C_2 be clauses with respective ground instances G_1 and G_2 . Suppose that R is a propositional resolvent of G_1 and G_2 . Then C_1 and C_2 have a predicate-logic resolvent R' such that R is a ground instance of R' .*

Proof. The situation of the lemma is shown in Figure 2. We can write the ground resolvent R in the form $R = (G_1 \setminus \{L\}) \cup (G_2 \setminus \{\bar{L}\})$, for complementary literals $L \in G_1$ and $\bar{L} \in G_2$.

Let C'_1 and C'_2 be variable-disjoint renamings of C_1 and C_2 , cf. Figure 2. Then G_1 and G_2 are also ground instances of C'_1 and C'_2 . Thus we can write $G_1 = C'_1 \theta'$ and $G_2 = C'_2 \theta'$ for some ground substitution θ' . Let $D_1 \subseteq C'_1$ be the set of literals mapped to the literal L by θ' and let $D_2 \subseteq C'_2$ be the set of literals mapped to the



Figure 2: Ground resolution step on the left, and its predicate-logic lifting on the right.

literal \bar{L} by θ' . Then θ' is a unifier of $D_1 \cup \overline{D_2}$. Writing θ for the most general unifier of $D_1 \cup D_2$, we have that

$$R' := (C'_1\theta \setminus D_1\theta) \cup (C'_2\theta \setminus D_2\theta) \quad (2)$$

is a predicate-logic resolvent of C_1 and C_2 .

By the definition of most general unifier, there exists some θ'' such that $\theta' = \theta \cdot \theta''$. Thus we have

$$G_1 = C'_1\theta' = C'_1(\theta \cdot \theta'') = (C'_1\theta)\theta'' \quad \text{and} \quad G_2 = C'_2\theta' = C'_2(\theta \cdot \theta'') = (C'_2\theta)\theta''.$$

Now from (2) we have that

$$\begin{aligned} R'\theta'' &= ((C'_1\theta \setminus D_1\theta) \cup (C'_2\theta \setminus D_2\theta))\theta'' \\ &= ((C'_1\theta)\theta'' \setminus (D_1\theta)\theta'') \cup ((C'_2\theta)\theta'' \setminus (D_2\theta)\theta'') \\ &= (C'_1(\theta \cdot \theta'') \setminus D_1(\theta \cdot \theta'')) \cup (C'_2(\theta \cdot \theta'') \setminus D_2(\theta \cdot \theta'')) \\ &= (G_1 \setminus \{L\}) \cup (G_2 \setminus \{\bar{L}\}). \end{aligned}$$

(Note that the last equality uses the fact that D_1 is precisely the set of literals in C'_1 that map to L under θ' and similarly D_2 is precisely the set of literals in C'_2 that map to \bar{L} under θ' .) We conclude that R is a ground instance of R' under the substitution θ'' . □

Corollary 9 (Completeness). *Let F be a closed formula in Skolem form with its matrix F' in CNF. If F is unsatisfiable then there is a predicate-logic resolution proof of \square from F' .*

Proof. Suppose F is unsatisfiable. By the completeness of ground resolution there is a proof C'_1, C'_2, \dots, C'_n , where $C'_n = \square$ and each C'_i is either a ground instance of a clause in F' or is a resolvent of two clauses C'_j, C'_k for $j, k < i$. We inductively define a corresponding predicate-logic resolution proof C_1, C_2, \dots, C_n , such that C'_i is a ground instance of C_i . For each i , if C'_i is a ground instance of a clause $C \in F'$ then define $C_i = C$. On the other hand, suppose that C'_i is a resolvent of two ground clauses C'_j, C'_k , with $j, k < i$. By induction we have constructed clauses C_j and C_k such that C'_j is a ground instance of C_j and C'_k is a ground instance of C_k . By the Lifting Lemma we can find a clause C_i which is a resolvent of C_j and C_k such that C'_i is a ground instance of C_i . □