

# Lecture 12

## Resolution for predicate logic

### Unification, resolution

Print version of the lecture in *Introduction to Logic for Computer Science*

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These lecture notes are very minor variants of the ones made by Prof James Worrell and Prof Christoph Haase for their 'Logic and Proof' course at Oxford.

A serious drawback of the ground resolution procedure is that it requires looking ahead to predict which ground instances of clauses will be needed in a proof. In this lecture, we introduce the predicate-logic version of resolution, which allows us to perform substitution “by need”. This relies on the notion of unification, which we introduce next.

## 1 Unification

A *substitution* is a function  $\theta$  from the set of  $\sigma$ -terms back to itself such that (writing function application on the right)  $c\theta = c$  for each constant symbol  $c$  and  $f(t_1, \dots, t_k)\theta = f(t_1\theta, \dots, t_k\theta)$  for each  $k$ -ary function symbol  $f$ . It is clear that the composition of two such substitutions (as functions) is also a substitution. We have previously considered substitutions of the form  $[t/x]$  for a  $\sigma$ -term  $t$  and a variable  $x$ .

We write composition of substitutions diagrammatically, that is,  $\theta \cdot \theta'$  denotes the substitution obtained by applying  $\theta$  first and then  $\theta'$ . (This convention matches the fact that for substitutions, we write function application on the right.) In particular,  $[t_1/x_1] \cdots [t_k/x_k]$  denotes the substitution obtained by sequentially applying the substitutions  $[t_1/x_1], \dots, [t_k/x_k]$  left-to-right.

Given a set of literals  $D = \{L_1, \dots, L_k\}$  and a substitution  $\theta$ , define  $D\theta := \{L_1\theta, \dots, L_k\theta\}$ . We say that  $\theta$  *unifies*  $D$  if  $D\theta = \{L\}$  for some literal  $L$ . For example, the substitution  $\theta = [f(a)/x][a/y]$  unifies  $\{P(x), P(f(y))\}$ , as does the substitution  $\theta' = [f(y)/x]$ . In this example, we regard  $\theta'$  as a *more general unifier* because  $\theta = \theta' \cdot [a/y]$ , that is,  $\theta$  factors through  $\theta'$ .

We say that  $\theta$  is a *most general unifier* of a set of literals  $D$  if  $\theta$  is a unifier of  $D$  and any other unifier  $\theta'$  factors through  $\theta$ , i.e., we have  $\theta' = \theta \cdot \theta''$  for some substitution  $\theta''$ . Note that both the substitutions  $[x/y]$  and  $[y/x]$  are most general unifiers of  $\{P(x), P(y)\}$ . In fact, most-general unifiers are only unique up to renaming variables.

We will show that a set of literals either has no unifier or it has a most general unifier. Examples of sets of literals that cannot be unified are  $\{P(f(x)), P(g(x))\}$  and  $\{P(f(x)), P(x)\}$ . The problem in the second case is that we cannot unify a variable  $x$  and term  $t$  if  $x$  occurs in  $t$ .

**Theorem 1** (Unification Theorem). *A unifiable set of literals  $D$  has a most general unifier.*

*Proof.* We claim that the following algorithm determines whether a set of literals has a unifier and, if so, outputs a most general unifier.

### Unification Algorithm

**Input:** Set of literals  $D$

**Output:** Either a most general unifier of  $D$  or “fail”

$\theta :=$  identity substitution

**while**  $\theta$  is not a unifier of  $D$  **do**

**begin**

pick two distinct literals in  $D\theta$  and find the left-most positions at which they differ

**if** one of the corresponding sub-terms is a variable  $x$

and the other term  $t$  does not contain  $x$

**then**  $\theta := \theta \cdot [t/x]$  **else** output “fail” and halt

**end**

**return**  $\theta$

We argue termination as follows. In any iteration of the while loop that does not cause the program to halt, a variable  $x$  is replaced everywhere in  $D\theta$  by a term  $t$  that does not contain  $x$ . Thus, the number of different variables occurring in  $D\theta$  decreases by one in each iteration, and the loop must terminate.

The loop invariant is that for any unifier  $\theta'$  of  $D$ , we have  $\theta' = \theta \cdot \theta'$ . Clearly, the invariant is established by the initial assignment of the identity substitution to  $\theta$ . To see that the invariant is maintained by an iteration of the loop, suppose we find an occurrence of variable  $x$  in a literal in  $D\theta$  such that a different term  $t$  occurs in the same position in another literal in  $D\theta$ . From the invariant, we know that  $\theta'$  is a unifier of  $D\theta$ , and thus  $t\theta' = x\theta'$ . It immediately follows that  $\theta' = [t/x] \cdot \theta'$ . Thus, the loop invariant is maintained by the assignment  $\theta := \theta \cdot [t/x]$ .

The termination condition of the while loop is that  $\theta$  is a unifier of  $D$ . In conjunction with the loop invariant, this implies that the final value of  $\theta$  is a most general unifier of  $D$ . Finally, the invariant implies that if  $\theta'$  is a unifier of  $D$ , then it is also a unifier of  $D\theta$ . But the algorithm only outputs “fail” if  $D\theta$  has no unifier, in which case  $D$  has no unifier.  $\square$

*Example 2.* Consider an execution of the unification algorithm on input  $D = \{P(x, y), P(f(z), x)\}$ . Scanning left-to-right, the leftmost discrepancy is underlined in  $\{P(\underline{x}, y), P(\underline{f}(z), x)\}$ . Applying the substitution  $[f(z)/x]$  to  $D$  yields the set  $D' = \{P(f(z), y), P(\underline{f}(z), \underline{f}(z))\}$ , where the underlined positions again indicate the leftmost discrepancy. Applying the substitution  $[f(z)/y]$  to  $D'$  yields the singleton set  $\{P(f(z), f(z))\}$ . Thus  $[f(z)/x][f(z)/y]$  is a most general unifier of the set  $D$ .

## 2 Resolution

First-order resolution operates on a set of clauses, that is, a set of sets of literals. Given a formula  $\forall x_1 \dots \forall x_n F$  in Skolem form, we perform resolution on the clauses in the matrix  $F$  with the goal of deriving the empty clause. Although quantifiers do not explicitly appear in resolution proofs, we can see the variables in such a proof as being implicitly universally quantified. This is made more formal when we formulate the Resolution Lemma in the next section.

For any set of literals  $D$ , let  $\bar{D}$  denote the set of complementary literals. For example, if  $D = \{\neg P(x), R(x, y)\}$  then  $\bar{D} = \{P(x), \neg R(x, y)\}$ .

**Definition 3** (Resolution). Let  $C_1$  and  $C_2$  be clauses *with no variable in common*. We say that a clause  $R$  is a *resolvent* of  $C_1$  and  $C_2$  if there are sets of literals  $D_1 \subseteq C_1$  and  $D_2 \subseteq C_2$  such that  $D_1 \cup \bar{D}_2$  has a most general unifier  $\theta$ , and

$$R = (C_1\theta \setminus \{L\}) \cup (C_2\theta \setminus \{\bar{L}\}), \quad (1)$$

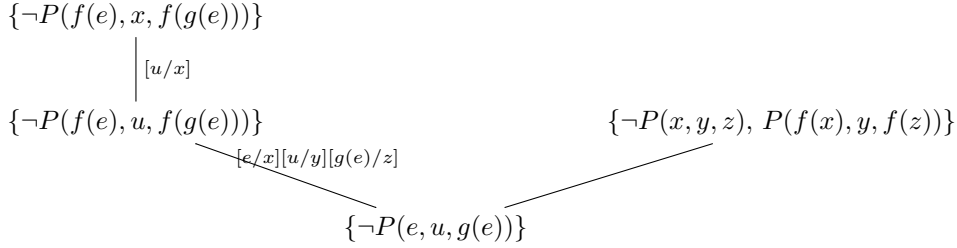


Figure 1: First-order resolution example

where  $\{L\} = D_1\theta$  and  $\{\bar{L}\} = D_2\theta$ . More generally, if  $C_1$  and  $C_2$  are arbitrary clauses, we say that  $R$  is a resolvent of  $C_1$  and  $C_2$  if there are variable renamings  $\theta_1$  and  $\theta_2$  such that  $C_1\theta_1$  and  $C_2\theta_2$  have no variable in common, and  $R$  is a resolvent of  $C_1\theta_1$  and  $C_2\theta_2$  according to the definition above.

*Example 4.* Consider a signature with constant symbol  $e$ , unary function symbols  $f$  and  $g$ , and a ternary predicate symbol  $P$ . We compute a resolvent of the clauses  $C_1 = \{\neg P(f(e), x, f(g(e)))\}$  and  $C_2 = \{\neg P(x, y, z), P(f(x), y, f(z))\}$  as follows (see Figure 1). First apply the substitution  $[u/x]$  to  $C_1$ , obtaining a clause  $C'_1$  that has no variable in common in  $C_2$ . Now unify complementary literals under the substitution  $[e/x][u/y][g(e)/z]$ , obtaining the clause  $\{\neg P(e, u, g(e))\}$ .

A *predicate-logic resolution derivation* of a clause  $C$  from a set of clauses  $F$  is a sequence of clauses  $C_1, \dots, C_m$ , with  $C_m = C$  such that each  $C_i$  is either a clause of  $F$  (possibly with the variables renamed) or follows by a resolution step from two preceding clauses  $C_j, C_k$ , with  $j, k < i$ . We write  $\text{Res}^*(F)$  for the set of clauses  $C$  such that there is a derivation of  $C$  from  $F$ .

*Example 5.* Consider the following sentences over a signature with ternary predicate symbol  $A$ , constant symbol  $e$ , and unary function symbol  $s$ . The idea is that  $A$  represents the ternary addition relation,  $e$  the zero element, and  $s$  the successor function.

$$\begin{aligned} F_1 &: \forall x A(e, x, x) \\ F_2 &: \forall x \forall y \forall z (\neg A(x, y, z) \vee A(s(x), y, s(z))) \\ F_3 &: \forall x \exists y A(s(s(e)), x, y) \end{aligned}$$

We use first-order resolution to show that  $F_1 \wedge F_2 \models F_3$ , that is, we show that  $F_1 \wedge F_2 \wedge \neg F_3$  is unsatisfiable. We proceed in two steps.

**Step (i): separately Skolemise each formula.** Formula  $\neg F_3$  is equivalent to

$$\exists y \forall z \neg A(s(s(e)), y, z).$$

Skolemising, we obtain the formula  $G_3 := \forall z \neg A(s(s(e)), c, z)$ , where  $c$  is a new constant symbol. Now  $F_1 \wedge F_2 \wedge G_3$  is equisatisfiable with  $F_1 \wedge F_2 \wedge \neg F_3$  and so it suffices to give a resolution refutation of  $F_1 \wedge F_2 \wedge G_3$ .<sup>1</sup> **Important:** When Skolemising more than one formula separately, different formulas may not introduce the same constant symbols. Always make sure to choose fresh constant symbols for every formula you are Skolemising.

<sup>1</sup>Formally the notion of a resolution proof assumes a single Skolem-form formula. So strictly speaking the proof below is a resolution refutation of the formula  $\forall x \forall y \forall z (A(e, x, x) \wedge ((\neg A(x, y, z) \vee A(s(x), y, s(z))) \wedge \neg A(s(s(e)), c, z)))$ , which is logically equivalent to  $F_1 \wedge F_2 \wedge G_3$ .

Step (ii). *derive the empty clause using resolution.* The proof is as follows. Note that in order to always ensure that we resolve clauses with disjoint variables, we arrange it so that the variables in line  $k$  of the proof are subscripted with  $k$ . In particular, we add a variable renaming at the end of each unifying substitution so that the variables in the output formula have the right subscript for the next line of the proof.

- |    |   |   |
|----|---|---|
| 1. | $\{\neg A(s(e)), c, z_1\}$                          | clause of $G_3$                                       |
| 2. | $\{\neg A(x_2, y_2, z_2), A(s(x_2), y_2, s(z_2))\}$ | clause of $F_2$                                       |
| 3. | $\{\neg A(s(e), c, z_3)\}$                          | 1,2 Res. Sub $[s(e)/x_2][c/y_2][s(z_2)/z_1][z_3/z_2]$ |
| 4. | $\{\neg A(e, c, z_4)\}$                             | 2,3 Res. Sub $[e/x_2][c/y_2][s(z_2)/z_3][z_4/z_2]$    |
| 5. | $\{A(e, y_5, y_5)\}$                                | clause of $F_1$                                       |
| 6. | $\square$   | 4,5 Res. Sub $[c/y_5][c/z_4]$                         |

Given a formula  $H$  with free variables  $x_1, x_2, \dots, x_n$ , its *universal closure*  $\forall^* H$  is the sentence  $\forall x_1 \forall x_2 \dots \forall x_n H$ . The following lemma is key to the soundness of resolution.

**Lemma 6** (Resolution Lemma). *Let  $F = \forall x_1 \dots \forall x_n G$  be a closed formula in Skolem form, with  $G$  quantifier-free and in CNF. Let  $R$  be a resolvent of two clauses in  $G$ . Then  $F \equiv \forall^*(G \cup \{R\})$ .*

*Proof.* Clearly  $\forall^*(G \cup \{R\}) \models F$ . The non-trivial direction is to show that  $F \models \forall^* R$ . For this, since  $F$  is closed, it suffices to show that  $F \models R$ . (Check that you understand why this is so!)

To this end, suppose that  $R$  is a resolvent of clauses  $C_1, C_2 \in G$ , with  $R = (C_1 \theta \setminus \{L\}) \cup (C_2 \theta' \setminus \{\bar{L}\})$  for some substitutions  $\theta, \theta'$  and complementary literals  $L \in C_1 \theta$  and  $\bar{L} \in C_2 \theta'$ .

Let  $\mathcal{A}$  be an assignment that satisfies  $F = \forall^* G$ . Since  $C_1, C_2 \in G$ , by the Translation Lemma  $\mathcal{A} \models C_1 \theta$  and  $\mathcal{A} \models C_2 \theta'$ . Moreover, since  $\mathcal{A}$  satisfies at most one of the complementary literals  $L$  and  $\bar{L}$ , it follows that  $\mathcal{A}$  satisfies at least one of  $C_1 \theta \setminus \{L\}$  and  $C_2 \theta' \setminus \{\bar{L}\}$ . We conclude that  $\mathcal{A}$  satisfies  $R$ , as required.  $\square$

**Corollary 7** (Soundness). *Let  $F = \forall x_1 \dots \forall x_n G$  be a closed formula in Skolem form. Let clause  $C$  be obtained from  $G$  by a resolution derivation. Then  $F \equiv \forall^*(G \cup C)$ .*

*Proof.* Induction on the length of the resolution derivation, using the Resolution Lemma for the induction step.  $\square$

## A Refutation Completeness

In this appendix we prove the refutation completeness of predicate-logic resolution proofs by showing that ground resolution proofs lift to predicate-logic resolution proofs. The proofs here are more technical and can be regarded as optional.

**Lemma 8** (Lifting Lemma). *Let  $C_1$  and  $C_2$  be clauses with respective ground instances  $G_1$  and  $G_2$ . Suppose that  $R$  is a propositional resolvent of  $G_1$  and  $G_2$ . Then  $C_1$  and  $C_2$  have a predicate-logic resolvent  $R'$  such that  $R$  is a ground instance of  $R'$ .*

*Proof.* The situation of the lemma is shown in Figure 2. We can write the ground resolvent  $R$  in the form  $R = (G_1 \setminus \{L\}) \cup (G_2 \setminus \{\bar{L}\})$ , for complementary literals  $L \in G_1$  and  $\bar{L} \in G_2$ .

Let  $C'_1$  and  $C'_2$  be variable-disjoint renamings of  $C_1$  and  $C_2$ , cf. Figure 2. Then  $G_1$  and  $G_2$  are also ground instances of  $C'_1$  and  $C'_2$ . Thus we can write  $G_1 = C'_1 \theta'$  and  $G_2 = C'_2 \theta'$  for some ground substitution  $\theta'$ . Let  $D_1 \subseteq C'_1$  be the set of literals mapped to the literal  $L$  by  $\theta'$  and let  $D_2 \subseteq C'_2$  be the set of literals mapped to the



Figure 2: Ground resolution step on the left, and its predicate-logic lifting on the right.

literal  $\bar{L}$  by  $\theta'$ . Then  $\theta'$  is a unifier of  $D_1 \cup \overline{D_2}$ . Writing  $\theta$  for the most general unifier of  $D_1 \cup D_2$ , we have that

$$R' := (C'_1\theta \setminus D_1\theta) \cup (C'_2\theta \setminus D_2\theta) \quad (2)$$

is a predicate-logic resolvent of  $C_1$  and  $C_2$ .

By the definition of most general unifier, there exists some  $\theta''$  such that  $\theta' = \theta \cdot \theta''$ . Thus we have

$$G_1 = C'_1\theta' = C'_1(\theta \cdot \theta'') = (C'_1\theta)\theta'' \quad \text{and} \quad G_2 = C'_2\theta' = C'_2(\theta \cdot \theta'') = (C'_2\theta)\theta''.$$

Now from (2) we have that

$$\begin{aligned} R'\theta'' &= ((C'_1\theta \setminus D_1\theta) \cup (C'_2\theta \setminus D_2\theta))\theta'' \\ &= ((C'_1\theta)\theta'' \setminus (D_1\theta)\theta'') \cup ((C'_2\theta)\theta'' \setminus (D_2\theta)\theta'') \\ &= (C'_1(\theta \cdot \theta'') \setminus D_1(\theta \cdot \theta'')) \cup (C'_2(\theta \cdot \theta'') \setminus D_2(\theta \cdot \theta'')) \\ &= (G_1 \setminus \{L\}) \cup (G_2 \setminus \{\bar{L}\}). \end{aligned}$$

(Note that the last equality uses the fact that  $D_1$  is precisely the set of literals in  $C'_1$  that map to  $L$  under  $\theta'$  and similarly  $D_2$  is precisely the set of literals in  $C'_2$  that map to  $\bar{L}$  under  $\theta'$ .) We conclude that  $R$  is a ground instance of  $R'$  under the substitution  $\theta''$ . □

**Corollary 9** (Completeness). *Let  $F$  be a closed formula in Skolem form with its matrix  $F'$  in CNF. If  $F$  is unsatisfiable then there is a predicate-logic resolution proof of  $\square$  from  $F'$ .*

*Proof.* Suppose  $F$  is unsatisfiable. By the completeness of ground resolution there is a proof  $C'_1, C'_2, \dots, C'_n$ , where  $C'_n = \square$  and each  $C'_i$  is either a ground instance of a clause in  $F'$  or is a resolvent of two clauses  $C'_j, C'_k$  for  $j, k < i$ . We inductively define a corresponding predicate-logic resolution proof  $C_1, C_2, \dots, C_n$ , such that  $C'_i$  is a ground instance of  $C_i$ . For each  $i$ , if  $C'_i$  is a ground instance of a clause  $C \in F'$  then define  $C_i = C$ . On the other hand, suppose that  $C'_i$  is a resolvent of two ground clauses  $C'_j, C'_k$ , with  $j, k < i$ . By induction we have constructed clauses  $C_j$  and  $C_k$  such that  $C'_j$  is a ground instance of  $C_j$  and  $C'_k$  is a ground instance of  $C_k$ . By the Lifting Lemma we can find a clause  $C_i$  which is a resolvent of  $C_j$  and  $C_k$  such that  $C'_i$  is a ground instance of  $C_i$ . □