Introduction to Logic for Computer Science Spring 2025

Assignment 2 (Deadline: 6:00pm on 9 May 2025)

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Submit your solutions in KLMS. (Reminder: We adopt a very strict policy for handling dishonest behaviours. If a student is found to copy answers from fellow students or other sources in his or her homework submission, she or he will get F.

1. Using resolution, show that there is a polynomial-time algorithm to decide satisfiability of those CNF formulas F in which each propositional variable occurs at most twice. Explain why your answer is correct and briefly explain why it meets the required time bound.

   (20 points)

2. *Positive resolution* is a restriction of ordinary resolution, which is defined as follows: derive a resolvent from C_1 and C_2 only if C_1 is a positive clause, i.e., it consists only of positive literals. By adapting the completeness proof from lectures, show that if a propositional formula F is an unsatisfiable CNF formula then one can derive the empty clause from F using only positive resolution. **(20 points)**

3. Given an undirected graph $G = (V, E)$, a set of vertices $S \subseteq V$ is a *clique* if every pair of distinct vertices $u, v \in S$ is connected by an edge and S is an *independent set* if no pair of distinct vertices $u, v \in S$ is connected by an edge. Now consider the following two statements:

   (A) Every infinite graph either has an infinite clique or an infinite independent set.

   (B) For all k there exists n such that any graph with n vertices has a clique of size k or an independent set of size k.

   Using the Compactness Theorem, prove that (A) implies (B).[1] **(10 points)**

4. Let σ be a signature with finitely many relation and constant symbols, but no function symbols.

   (a) Given σ-formulas $G_1, \ldots, G_n$ and a propositional formula F that mentions variables $p_1, \ldots, p_n$, let $F[G_1/p_1, \ldots, G_n/p_n]$ denote the σ-formula obtained by substituting G_i for all occurrences of p_i in F. Give a formal definition of $F[G_1/p_1, \ldots, G_n/p_n]$.

   (5 points)

   (b) Given propositional formulas $F \equiv F'$, both over variables $p_1, \ldots, p_n$, and σ-formulas $G_1 \equiv G'_1, \ldots, G_n \equiv G'_n$, show that $F[G_1/p_1, \ldots, G_n/p_n] \equiv F'[G'_1/p_1, \ldots, G'_n/p_n]$.

   (5 points)

   (c) Fix $n \in \mathbb{N}$. Show that up to logical equivalence there are only finitely many quantifier-free σ-formulas that use first-order variables $x_1, \ldots, x_n$.

   (10 points)

[1] As an optional exercise, beyond the scope of the course, you can try to prove (A). This result is Theorem 1 in `https://www.dpmms.cam.ac.uk/~par31/notes/ramsey.pdf`. Combining this with your answer, we obtain a proof of (B).

(d) Fix $n, k \in \mathbb{N}$. Show that up to logical equivalence there are only finitely many σ-formulas of quantifier depth at most k that use first-order variables $x_1, \ldots, x_n$.

(10 points)

5. Fix a signature σ. Consider a relation $\sim$ on σ-assignments that satisfies the following two properties:

(P1) If $\mathcal{A} \sim \mathcal{B}$, then for every atomic formula F we have $\mathcal{A} \models F$ iff $\mathcal{B} \models F$.

(P2) If $\mathcal{A} \sim \mathcal{B}$, then for each variable x we have (i) for each $a \in U_{\mathcal{A}}$ there exists $b \in U_{\mathcal{B}}$ such that $\mathcal{A}_{[x \mapsto a]} \sim \mathcal{B}_{[x \mapsto b]}$, and (ii) for all $b \in U_{\mathcal{B}}$ there exists $a \in U_{\mathcal{A}}$ such that $\mathcal{A}_{[x \mapsto a]} \sim \mathcal{B}_{[x \mapsto b]}$.

Prove that if $\mathcal{A} \sim \mathcal{B}$ then for any formula F, $\mathcal{A} \models F$ if and only if $\mathcal{B} \models F$. You may assume that F is built from atomic formulas using the connectives $\wedge$ and $\neg$ and the quantifier $\exists$.

(20 points)