Lecture 10

Herbrand's theorem and ground resolution

Print version of the lecture in *Introduction to Logic for Computer Science*

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These lecture notes are very minor variants of the ones made by Prof James Worrell and Prof Christoph Haase for their 'Logic and Proof' course at Oxford.

In this lecture, we introduce Herbrand structures and state Herbrand's theorem. We then prove the Ground Resolution Theorem, which justifies the use of the ground resolution deduction technique in first-order logic. The Ground Resolution Theorem is one of the central results of the course. Its proof combines Herbrand's Theorem with the Resolution Theorem and Compactness Theorem for propositional logic. Throughout this lecture, we work in first-order logic *without equality*. There are versions of resolution that handle equality but we do not consider them in this course.

1 Herbrand's Theorem

Definition 1. Let σ be a signature with at least one constant symbol. A σ-structure $\mathcal{H}$ is called a *Herbrand structure* if the following hold:

1. The universe $U_{\mathcal{H}}$ is the set of ground terms over σ.
2. For every constant symbol c in σ, we have $c_{\mathcal{H}} = c$.
3. For every k-ary function symbol f in σ and for all ground terms $t_1, t_2 \ldots, t_n \in U_{\mathcal{H}}$, we have $f_{\mathcal{H}}(t_1, \ldots, t_k) = f(t_1, \ldots, t_k)$.

Thus, a Herbrand structure is built from syntax, with terms and function symbols being interpreted "as themselves".

Example 2. Consider a signature with a constant symbol a, unary function symbol f, and unary predicate symbol P. Then, a Herbrand structure $\mathcal{H}$ has $U_{\mathcal{H}} = \{a, f(a), f(f(a)), \ldots\}$, $a_{\mathcal{H}} = a$ and $f_{\mathcal{H}}(f^n(a)) = f^{n+1}(a)$. Note that $P_{\mathcal{H}}$ can be an arbitrary subset of $U_{\mathcal{H}}$.

The following proposition expresses a key property of Herbrand structures: the interpretation of a ground term in a Herbrand structure is the term itself.

Proposition 3. Let $\mathcal{H}$ be a Herbrand structure and t a ground term. Then, $\mathcal{H}(t) = t$.

Proof. The proof is by structural induction over terms. The base case is that t is constant symbol c. Then, $\mathcal{H}(c) = c$ by definition of a Herbrand structure. The induction step is that t has the form $f(t_1, \ldots, t_k)$ for a k-ary function symbol f and ground terms $t_1, \ldots, t_k$. Then

$$\begin{aligned}
\mathcal{H}(f(t_1, \ldots, t_k)) &= f_{\mathcal{H}}(\mathcal{H}(t_1), \ldots, \mathcal{H}(t_k)) \\
&= f_{\mathcal{H}}(t_1, \ldots, t_k) \quad \text{induction hyp.} \\
&= f(t_1, \ldots, t_k) \quad \text{defn. of } f_{\mathcal{H}}.
\end{aligned}$$

□

Building on Proposition 3, we show that the Translation Lemma has a particularly simple form for Herbrand structures.

Lemma 4 (Translation Lemma for Herbrand structures)**.** *Let $\mathcal{H}$ be a Herbrand structure, F a formula, and t a ground term. Then, $\mathcal{H} \models F[t/x]$ if and only if $\mathcal{H}_{[x \mapsto t]} \models F$.*

Proof. By the version of the Translation Lemma proved previously, we have $\mathcal{H} \models F[t/x]$ if and only if $\mathcal{H}_{[x \mapsto \mathcal{H}(t)]} \models F$. But by Proposition 3, we have $\mathcal{H}(t) = t$ and the result follows. □

We now come to what can be regarded as the central result of the course. The value of this result is that it cuts down the "search space" of potential models for a given formula.

Theorem 5 (Herbrand's Theorem)**.** *Let $F = \forall x_1 \ldots \forall x_n \, F^*$ be a closed formula in Skolem form. Then F is satisfiable if and only if it has a Herbrand model.*

Proof. If F has a Herbrand model, then it is clearly satisfiable.

Conversely, suppose that F is satisfied by some structure $\mathcal{A}$. Then, we show that F has a Herbrand model $\mathcal{H}$. To define $\mathcal{H}$, it suffices to define the interpretation of the predicate symbols since the universe and the interpretation of the constants and function symbols are already determined. The idea is to define $\mathcal{H}$ to mimic $\mathcal{A}$. To this end, given a k-ary predicate symbol P, we define $(t_1, \ldots, t_k) \in P_{\mathcal{H}}$ if and only if $\mathcal{A} \models P(t_1, \ldots, t_k)$.

We claim that for all closed formulas $G = \forall y_1 \ldots \forall y_n G^*$ in Skolem form, if $\mathcal{A} \models G$ then $\mathcal{H} \models G$. It follows from this that $\mathcal{H} \models F$. The proof of the claim is by induction on the number of quantifiers n.

The base case is that $n = 0$. Since G is closed, it is a Boolean combination of atomic formulas $P(t_1, \ldots, t_k)$, where $t_1, \ldots, t_k$ are ground terms. But, by construction, $\mathcal{A}$ and $\mathcal{H}$ assign the same truth value to each such atom. Thus, $\mathcal{A} \models G$ implies $\mathcal{H} \models G$.

The induction step is as follows. Suppose $\mathcal{A} \models \forall y \, G$. We cannot directly apply the induction hypothesis to G since y might appear free in G, in which case it is not closed. However by the Translation Lemma, we have that $\mathcal{A} \models G[t/y]$ iff $\mathcal{A}_{[y \mapsto \mathcal{A}(t)]} \models G$, and thus $\mathcal{A} \models G[t/y]$ for all ground terms t. Since $G[t/y]$ is closed, we can apply the induction hypothesis to it and conclude that $\mathcal{H} \models G[t/y]$ for all ground terms t. But now by the Translation Lemma for Herbrand structures, we have $\mathcal{H}_{[y \mapsto t]} \models G$ for all $t \in U_{\mathcal{H}}$, i.e., $\mathcal{H} \models \forall y \, G$. □

Example 6. Is the following first-order formula satisfiable?

$$F = \exists x_1 \, \exists x_2 \, \exists x_3 \, (\neg(\neg P(x_1) \rightarrow P(x_2)) \wedge \neg(\neg P(x_1) \rightarrow \neg P(x_3)))\,.$$

One way to simplify the problem is to skolemise, that is, eliminate the existential quantifiers by introducing new constants a, b, and c. Doing this we obtain an equisatisfiable formula

$$G = \neg(\neg P(a) \rightarrow P(b)) \wedge \neg(\neg P(a) \rightarrow \neg P(c))\,.$$

Now by Herbrand's Theorem, G is satisfiable if and only if it has a Herbrand model.

A Herbrand model $\mathcal{H}$ of G has as universe the set of ground terms $U_{\mathcal{H}} = \{a, b, c\}$. The constants are interpreted "as themselves", i.e., we have $a_{\mathcal{H}} = a$, $b_{\mathcal{H}} = b$, and $c_{\mathcal{H}} = c$. Thus, to specify $\mathcal{H}$, it remains to say how to interpret the predicate symbol P. We can represent the possibilities in the following truth table, each line of which represents a Herbrand structure.

$P(a)$	$P(b)$	$P(c)$	G
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	0

We conclude that G is satisfiable and therefore F is satisfiable.

Generalising the above example, we get a method to decide satisfiability of any formula $F = \exists x_1 \dots \exists x_n \, F^*$ for which the matrix F^* does not contain a function symbol. The key property of such a formula is that if it satisfiable, then it has a finite model.

The technique in the above example breaks down on a slightly more complex formula. Consider the formula $F = \forall x_1 \, \exists x_2 \, F^*$. The Skolem form is $\forall x_1 \, F^*[f(x_1)/x_2]$, where f is fresh unary function symbol. The presence of f ensures that each Herbrand structure is also infinite. More generally, it can be the case that a formula F is only satisfied by infinite structures (can you give an example of such a formula, without using equality?).

2 Ground Resolution

In general, a satisfiable formula may not have a finite model. Intuitively, it might not be possible to provide a finite *witness* that a formula is satisfiable. By contrast we will show that if a formula F is unsatisfiable, then there is always a ground resolution proof of $\Box$ from F. Such a proof could be considered a finite witness of unsatisfiability.

Fix a signature σ. Let $F = \forall x_1 \dots \forall x_n \, F^*$ be a closed formula in Skolem form with matrix F^*. Then, the *Herbrand expansion* $E(F)$ is defined as

$$E(F) := \{F^*[t_1/x_1] \dots [t_n/x_n] \mid t_1, \dots, t_n \text{ ground } \sigma\text{-terms}\}.$$

That is, the formulas in $E(F)$ are obtained by substituting ground terms for the variables in F^* in all possible ways.

Each formula in $E(F)$ is a Boolean combination of atomic formulas $P(t_1, \dots, t_k)$, for P a k-ary predicate symbol and $t_1, \dots, t_k$ ground terms. In particular, $E(F)$ has a Herbrand model if and only if it is "propositionally satisfiable", that is, there is some truth assignment to the set of closed atomic formulas that makes all formulas in $E(F)$ evaluate to true (cf. Example 6).

Theorem 7. *A closed formula F in Skolem form is satisfiable if and only if $E(F)$ is satisfiable when considered as a set of propositional formulas.*

Proof. By Herbrand's Theorem, a formula in Skolem form is satisfiable if and only if it has a Herbrand model. Thus, it suffices to show that F has a Herbrand model if and only if $E(F)$ is satisfiable considered as a set of propositional formulas. Let

F have the form $\forall x_1 \dots \forall x_n\, F^*$. Given a Herbrand structure $\mathcal{H}$, we have

$$\begin{array}{llll} \mathcal{H} \models F & \text{iff} & \mathcal{H}_{[x_1 \mapsto t_1]\dots[x_n \mapsto t_n]} \models F^* & \text{for all (ground terms) } t_1, \dots, t_n \in \mathcal{U}_{\mathcal{H}} \\ & \text{iff} & \mathcal{H} \models F^*[t_1/x_1]\dots[t_n/x_n] & \text{Translation Lemma for Herbrand structures} \\ & \text{iff} & \mathcal{H} \models E(F)\,. & \end{array}$$

Observe that $\mathcal{H} \models E(F)$ for some Herbrand structure $\mathcal{H}$ iff $E(F)$ is satisfiable as a set of propositional formulas. □

As a corollary of Theorem 7 we can prove the Ground Resolution Theorem.

Theorem 8 (Ground Resolution)**.** *A closed formula F in Skolem form is unsatisfiable if and only if there is a propositional resolution proof of $\Box$ from $E(F)$.*

Proof. By the compactness theorem of propositional logic, $E(F)$ is unsatisfiable if and only if some finite subset of $E(F)$ is unsatisfiable. By the soundness and completeness of propositional resolution, this holds if and only if we can derive $\Box$ from $E(F)$ using resolution. □

In summary, we have the following situation. Given a first-order formula F in Skolem form, if F is unsatisfiable, then by systematically generating all resolvents of ground instances of clauses in F, we are guaranteed to eventually generate $\Box$. However, if F is satisfiable, then this process of generating resolvents can go on forever.