Introduction to Logic for Computer Science Spring 2025

Assignment 3 (Deadline: 6:00pm on 30 May 2025)

Prof Hongseok Yang

Submit your solutions in KLMS. (Reminder: We adopt a very strict policy for handling dishonest behaviours. If a student is found to copy answers from fellow students or other sources in his or her homework submission, she or he will get F.)

1. In this question we work with first-order logic without equality.
   (a) Consider a signature σ containing only a binary relation symbol R. For each positive integer n, show that there is a satisfiable σ-formula F_n such that every model $\mathcal{A}$ of F_n has at least n elements. **(10 points)**
   (b) Consider a signature σ containing only unary predicate symbols $P_1, \ldots, P_k$. Show that any satisfiable σ-formula has a model with at most 2^k elements. **(10 points)**

2. Express the following by formulas of first-order logic, using predicate $H(x)$ for "x is happy", $R(x)$ for "x is rich", $G(x)$ for "x is a graduate", and $C(x, y)$ for "y is a child of x".
   (a) Any person is happy if all their children are rich.
   (b) All graduates are rich.
   (c) Someone is a graduate if she or he is a child of a graduate.
   (d) All graduates are happy.

   Use first-order resolution (i.e. unification-based resolution) to show that (d) is entailed by (a), (b), and (c). Indicate the substitutions in each resolution step.

   (20 points)

3. Give an example of a finite set of clauses F in first-order logic such that $Res^*(F)$ is infinite.

   (20 points)

4. Let σ be a fixed signature with at least one constant symbol. Professor Long claims that given closed formulas F and G over σ in Skolem form, if every Herbrand model of F over σ is a Herbrand model of G, then $F \models G$. Is Professor Long correct? Justify your answer.

   (20 points)

5. A closed formula (i.e., formula without free variables) is in the class $\exists^*\forall^*$ if it has the form $\exists x_1 \ldots \exists x_m \forall y_1 \ldots \forall y_n F$, where F is quantifier-free and $m, n \geq 0$.
   (a) Prove that if a closed $\exists^*\forall^*$-formula over a signature with no function symbols has a model, then it has a finite model.

   (6 points)

   (b) Suggest an algorithm for deciding whether a given closed $\exists^*\forall^*$-formula over a signature with no function symbols has a model.

   (6 points)

(c) A closed formula is in the class $\exists^*$ if it has the form $\exists x_1 \ldots \exists x_m\, F$, where F is quantifier-free and $m \geq 0$. Prove that the validity problem for the class of closed $\exists^*$-formulas that may mention function symbols is undecidable.

(8 points)