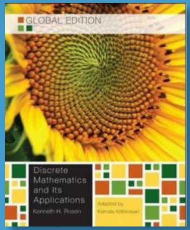
# Induction and recursion

Section 5

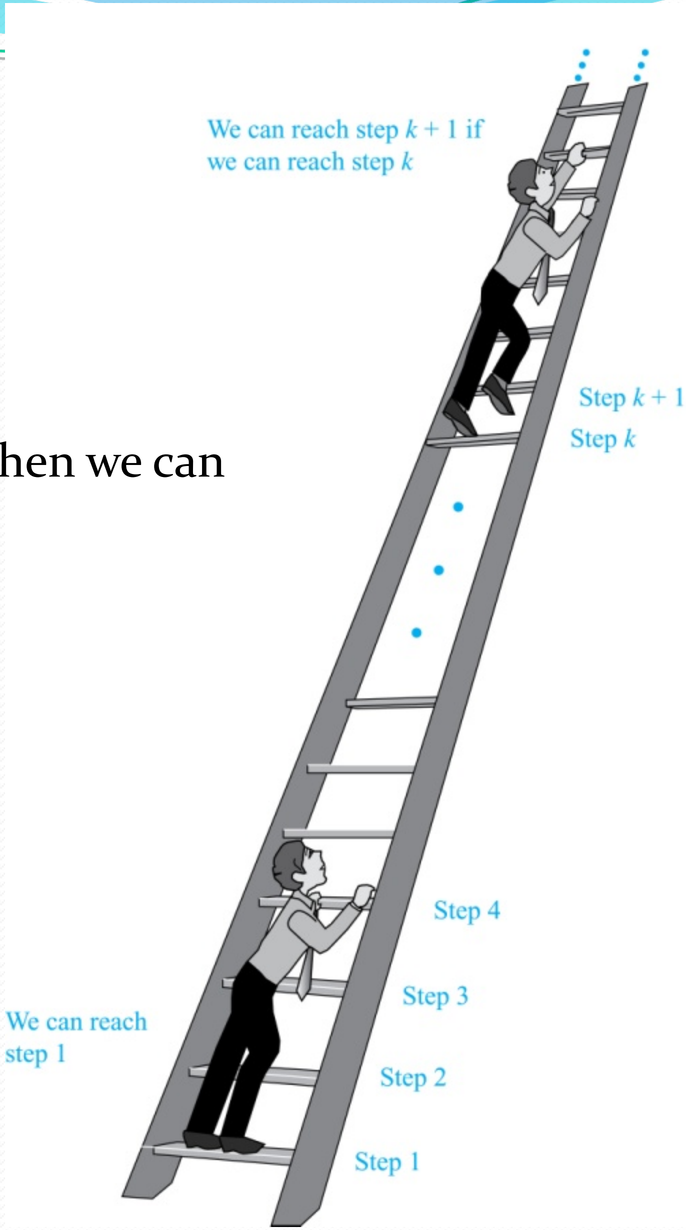



- Taken from the instructor's resource of *Discrete Mathematics and Its Applications*, 7/e
- Edited by Shin Hong hongshin@handong.edu

# Climbing an Infinite Ladder

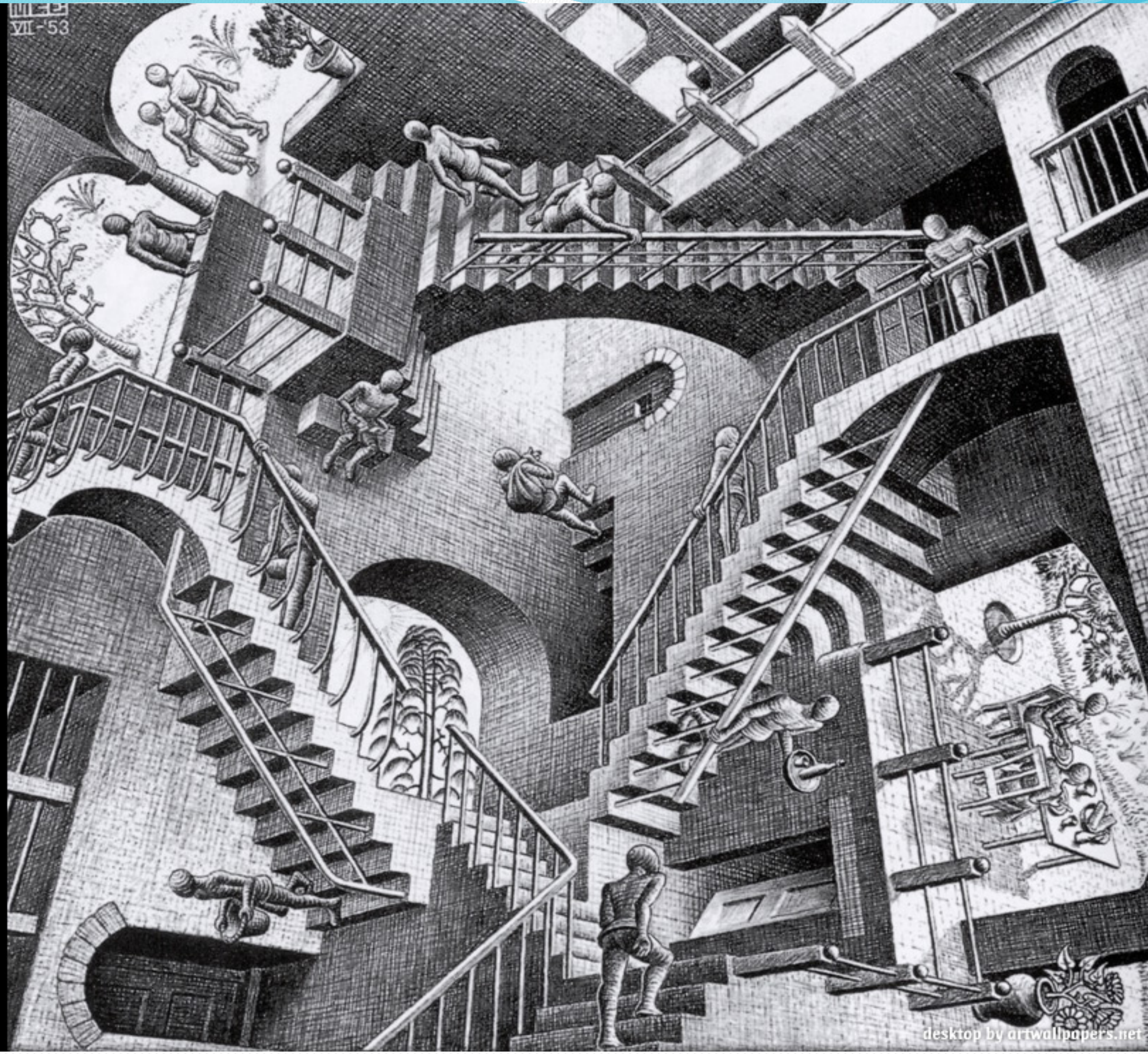
Suppose we have an infinite ladder:

1. We can reach the first rung of the ladder.
2. If we can reach a particular rung of the ladder, then we can reach the next rung.




From (1), we can reach the first rung. Then by applying (2), we can reach the second rung. Applying (2) again, the third rung. And so on. We can apply (2) any number of times to reach any particular rung, no matter how high up.

This example motivates proof by mathematical induction.




M.C Escher - Relativity

# Principle of Mathematical Induction

*Principle of Mathematical Induction*

To prove $P$, define $P$ with subproblems such that $P= P(1) \wedge P(2) \wedge \ldots$

And then to prove that $P(n)$ is true for all positive integers $n$, we complete these steps:

- *Basis Step*: Show that $P(1)$ is true.
- *Inductive Step*: Show that $P(k) \to P(k+1)$ is true for all positive integers $k$.

To complete the inductive step, assuming the *inductive hypothesis* that $P(k)$ holds for an arbitrary integer $k$, show that must $P(k+1)$ be true.

**Climbing an Infinite Ladder Example**:

- BASIS STEP: By (1), we can reach rung 1.
- INDUCTIVE STEP: Assume the inductive hypothesis that we can reach rung $k$. Then by (2), we can reach rung $k + 1$.

Hence, $P(k) \to P(k+1)$ is true for all positive integers $k$. We can reach every rung on the ladder.

# Important Points About Using Mathematical Induction

- Mathematical induction can be expressed as the rule of inference

$$(P(1) \land \forall k\,(P(k) \rightarrow P(k+1))) \rightarrow \forall n\, P(n),$$

  where the domain is the set of positive integers.
- In a proof by mathematical induction, we don't assume that $P(k)$ is true for all positive integers! We show that if we assume that $P(k)$ is true, then $P(k+1)$ must also be true.
- Proofs by mathematical induction do not always start at the integer 1.

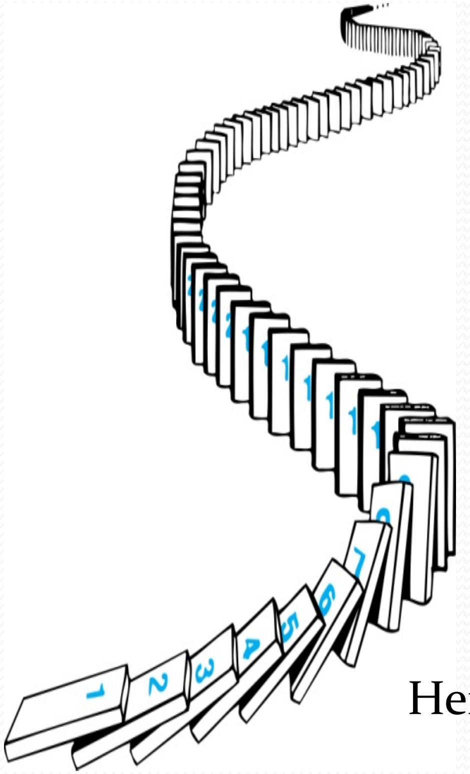
# Validity of Mathematical Induction

- Mathematical induction is valid because of the well ordering property, which states that every nonempty subset of the set of positive integers has a least element:
  - Suppose that $P(1)$ holds and $P(k) \rightarrow P(k + 1)$ is true for all positive integers $k$.
  - Assume there is at least one positive integer $n$ for which P($n$) is false. Then the set $S$ of positive integers for which P($n$) is false is nonempty.
  - By the well-ordering property, $S$ has a least element, say $m$.
  - We know that $m$ cannot be 1 since $P(1)$ holds.
  - Since $m$ is positive and greater than 1, $m - 1$ must be a positive integer. Since $m - 1 < m$, it is not in S, so $P(m - 1)$ must be true.
  - But then, since the conditional $P(k) \rightarrow P(k + 1)$ for every positive integer $k$ holds, $P(m)$ must also be true. This contradicts $P(m)$ being false.
  - Hence, $P(n)$ must be true for every positive integer $n$.

# Remembering How Mathematical Induction Works

Consider an infinite sequence of dominoes, labeled 1,2,3, …, where each domino is standing.

Let $P(n)$ be the proposition that the $n$th domino is knocked over.




We know that the first domino is knocked down, i.e., $P(1)$ is true .

We also know that if whenever the $k$-th domino is knocked over, it knocks over the $(k + 1)$st domino, i.e, $P(k) \rightarrow P(k + 1)$ is true for all positive integers $k$.

Hence, all dominos are knocked over.

$P(n)$ is true for all positive integers $n$.

# Proving a Summation Formula by Mathematical Induction

**Example**: Show that: $$\sum_{i=1}^{n} = \frac{n(n+1)}{2}$$

Note: Once we have this conjecture, mathematical induction can be used to prove it correct.

**Solution**:

- BASIS STEP: $P(1)$ is true since $1(1 + 1)/2 = 1$.
- INDUCTIVE STEP: Assume true for $P(k)$.

The inductive hypothesis is $$\sum_{i=1}^{k} = \frac{k(k+1)}{2}$$

Under this assumption,

$$\begin{aligned} 1+2+\ldots+k+(k+1) &= \frac{k(k+1)}{2}+(k+1) \\ &= \frac{k(k+1)+2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

# Conjecturing and Proving Correct a Summation Formula

**Example**: Conjecture and prove correct a formula for the sum of the first $n$ positive odd integers. Then prove your conjecture.

**Solution**: We have: $1 = 1$, $1 + 3 = 4$, $1 + 3 + 5 = 9$, $1 + 3 + 5 + 7 = 16$, $1 + 3 + 5 + 7 + 9 = 25$.

- We can conjecture that the sum of the first $n$ positive odd integers is $n^2$,

$$1 + 3 + 5 + \cdots + (2n - 1) + (2n + 1) = n^2.$$

- We prove the conjecture is proved correct with mathematical induction.
- BASIS STEP: $P(1)$ is true since $1^2 = 1$.
- INDUCTIVE STEP: $P(k) \rightarrow P(k + 1)$ for every positive integer $k$.
  Assume the inductive hypothesis holds and then show that $P(k)$ holds has well.

**Inductive Hypothesis**: $1 + 3 + 5 + \cdots + (2k - 1) = k^2$

- So, assuming $P(k)$, it follows that:

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) &= [1 + 3 + 5 + \cdots + (2k - 1)] + (2k + 1) \\ &= k^2 + (2k + 1) \quad (\textit{by the inductive hypothesis}) \\ &= k^2 + 2k + 1 \\ &= (k + 1)^2 \end{aligned}$$

- Hence, we have shown that $P(k + 1)$ follows from $P(k)$. Therefore the sum of the first $n$ positive odd integers is $n^2$.

# Proving Inequalities

**Example**: Use mathematical induction to prove that $n < 2^n$ for all positive integers $n$.

**Solution**: Let $P(n)$ be the proposition that $n < 2^n$.

- BASIS STEP: $P(1)$ is true since $1 < 2^1 = 2$.
- INDUCTIVE STEP: Assume $P(k)$ holds, i.e., $k < 2^k$, for an arbitrary positive integer $k$.
- Must show that $P(k + 1)$ holds. Since by the inductive hypothesis, $k < 2^k$, it follows that:

$$k + 1 < 2^k + 1 \leq 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$$

Therefore $n < 2^n$ holds for all positive integers $n$. ◀

# Proving Inequalities

**Example**: Use mathematical induction to prove that $2^n < n!$, for every integer $n \geq 4$.

**Solution**: Let $P(n)$ be the proposition that $2^n < n!$.

- BASIS STEP: $P(4)$ is true since $2^4 = 16 < 4! = 24$.
- INDUCTIVE STEP: Assume $P(k)$ holds, i.e., $2^k < k!$ for an arbitrary integer $k \geq 4$. To show that $P(k + 1)$ holds:

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &< 2 \cdot k! \qquad \textit{(by the inductive hypothesis)} \\ &< (k+1)k! \\ &= (k+1)! \end{aligned}$$

Therefore, $2^n < n!$ holds, for every integer $n \geq 4$. ◀

Note that here the basis step is $P(4)$, since $P(0)$, $P(1)$, $P(2)$, and $P(3)$ are all false.

# Number of Subsets of a Finite Set

**Example**: Use mathematical induction to show that if $S$ is a finite set with n elements, where $n$ is a nonnegative integer, then $S$ has $2^n$ subsets.

(*Chapter* 6 *uses combinatorial methods to prove this result*.)

**Solution**: Let $P(n)$ be the proposition that a set with $n$ elements has $2^n$ subsets.

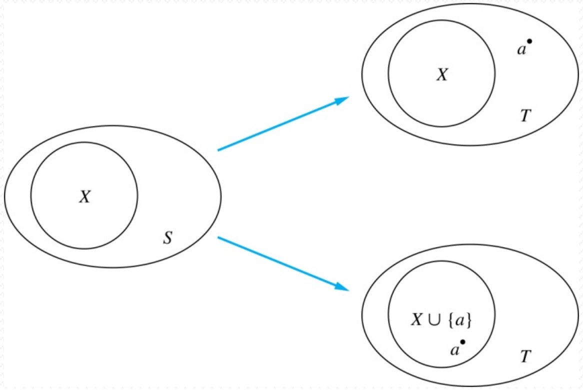
- Basis Step: $P(0)$ is true, because the empty set has only itself as a subset and $2^0 = 1$.
- Inductive Step: Assume $P(k)$ is true for an arbitrary nonnegative integer $k$.

*continued* →

# Number of Subsets of a Finite Set

**Inductive Hypothesis**: For an arbitrary nonnegative integer $k$, every set with $k$ elements has $2^k$ subsets.

- Let $T$ be a set with $k + 1$ elements. Then $T = S \cup \{a\}$, where $a \in T$ and $S = T - \{a\}$. Hence $|S| = k$.
- For each subset $X$ of $S$, there are exactly two subsets of $T$, i.e., $X$ and $X \cup \{a\}$.




- By the inductive hypothesis $S$ has $2^k$ subsets. Since there are two subsets of T for each subset of $S$, the number of subsets of $T$ is $2 \cdot 2^k = 2^{k+1}$ .

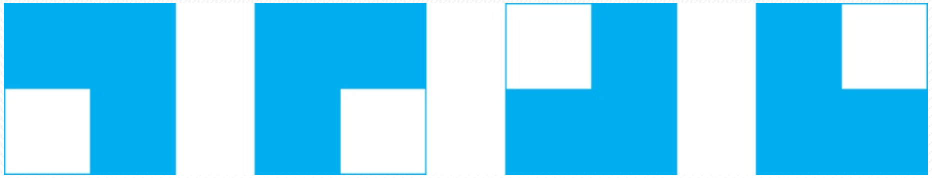
# Tiling Checkerboards

**Example**: Show that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using right triominoes.

A right triomino is an L-shaped tile which covers three squares at a time.

**Solution**: Let $P(n)$ be the proposition that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using right triominoes. Use mathematical induction to prove that $P(n)$ is true for all positive integers $n$.

- BASIS STEP: P(1) is true, because each of the four $2 \times 2$ checkerboards with one square removed can be tiled using one right triomino.



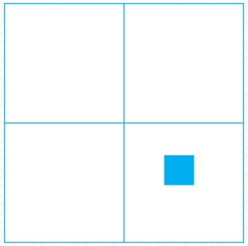
- INDUCTIVE STEP: Assume that $P(k)$ is true for every $2^k \times 2^k$ checkerboard, for some positive integer $k$.

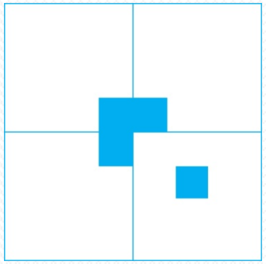
*continued* →

# Tiling Checkerboards

**Inductive Hypothesis**: Every $2^k \times 2^k$ checkerboard, for some positive integer $k$, with one square removed can be tiled using right triominoes.

- Consider a $2^{k+1} \times 2^{k+1}$ checkerboard with one square removed. Split this checkerboard into four checkerboards of size $2^k \times 2^k$, by dividing it in half in both directions.





- Remove a square from one of the four $2^k \times 2^k$ checkerboards. By the inductive hypothesis, this board can be tiled. Also by the inductive hypothesis, the other three boards can be tiled with the square from the corner of the center of the original board removed. We can then cover the three adjacent squares with a triominoe.
- Hence, the entire $2^{k+1} \times 2^{k+1}$ checkerboard with one square removed can be tiled using right triominoes.

# Guidelines: Mathematical Induction Proofs

*Template for Proofs by Mathematical Induction*

1. Express the statement that is to be proved in the form "for all $n \geq b$, $P(n)$" for a fixed integer $b$.
2. Write out the words "Basis Step." Then show that $P(b)$ is true, taking care that the correct value of $b$ is used. This completes the first part of the proof.
3. Write out the words "Inductive Step."
4. State, and clearly identify, the inductive hypothesis, in the form "assume that $P(k)$ is true for an arbitrary fixed integer $k \geq b$."
5. State what needs to be proved under the assumption that the inductive hypothesis is true. That is, write out what $P(k+1)$ says.
6. Prove the statement $P(k+1)$ making use the assumption $P(k)$. Be sure that your proof is valid for all integers $k$ with $k \geq b$, taking care that the proof works for small values of $k$, including $k = b$.
7. Clearly identify the conclusion of the inductive step, such as by saying "this completes the inductive step."
8. After completing the basis step and the inductive step, state the conclusion, namely that by mathematical induction, $P(n)$ is true for all integers $n$ with $n \geq b$.

# Strong Induction

- *Strong Induction*: To prove that $P(n)$ is true for all positive integers $n$, where $P(n)$ is a propositional function, complete two steps:
  - *Basis Step*: Verify that the proposition $P(1)$ is true.
  - *Inductive Step*: Show the conditional statement $[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$ holds for all positive integers $k$.

Strong Induction is sometimes called the *second principle of mathematical induction* or *complete induction*.

# Strong Induction and the Infinite Ladder

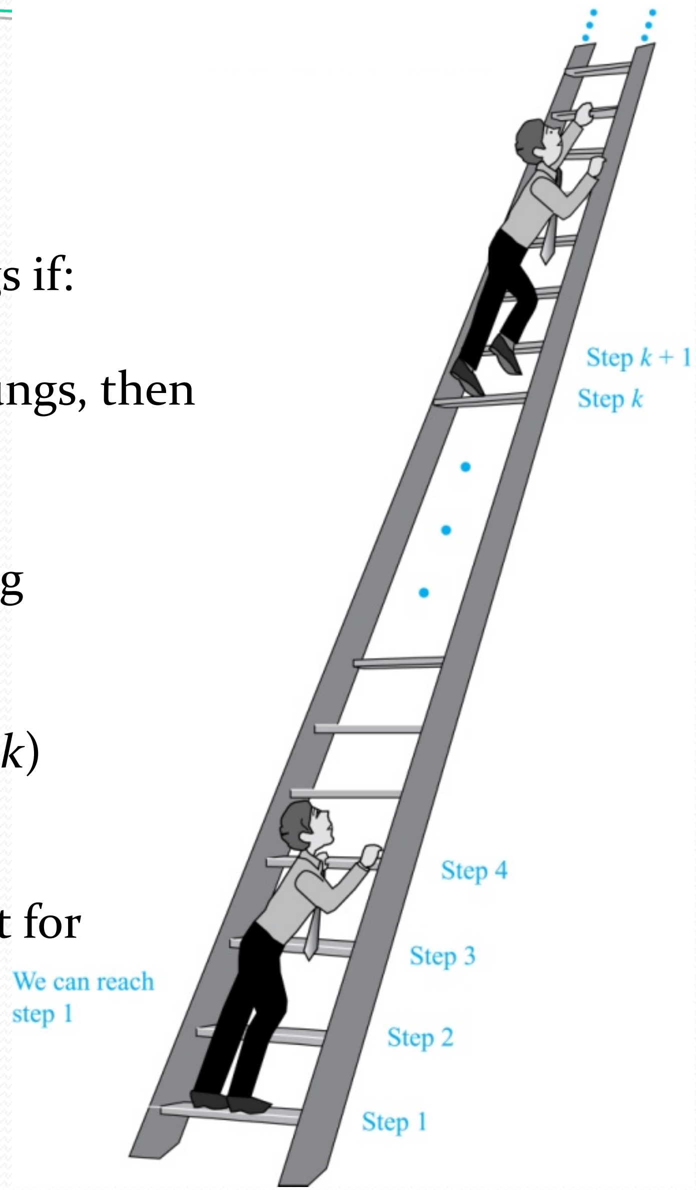
Strong induction tells us that we can reach all rungs if:

1. We can reach the first rung of the ladder.
2. For every integer $k$, if we can reach the first $k$ rungs, then we can reach the $(k + 1)$st rung.

To conclude that we can reach every rung by strong induction:

- BASIS STEP: $P(1)$ holds
- INDUCTIVE STEP: Assume $P(1) \wedge P(2) \wedge \cdots \wedge P(k)$ holds for an arbitrary integer $k$, and show that $P(k + 1)$ must also hold.

We will have then shown by strong induction that for every positive integer $n$, $P(n)$ holds, i.e., we can reach the $n$th rung of the ladder.



# Proof using Strong Induction

**Example**: Suppose we can reach the first and second rungs of an infinite ladder, and we know that if we can reach a rung, then we can reach two rungs higher. Prove that we can reach every rung.

(Try this with mathematical induction.)

**Solution**: Prove the result using strong induction.

- BASIS STEP: We can reach the first step.
- INDUCTIVE STEP: The inductive hypothesis is that we can reach the first $k$ rungs, for any $k \geq 2$. We can reach the $(k+1)$st rung since we can reach the $(k-1)$st rung by the inductive hypothesis.
- Hence, we can reach all rungs of the ladder. ◀

# Which Form of Induction Should Be Used?

- We can always use strong induction instead of mathematical induction. But there is no reason to use it if it is simpler to use mathematical induction. (*See page* 335 *of text*.)
- In fact, the principles of mathematical induction, strong induction, and the well-ordering property are all equivalent. (*Exercises* 41-43)
- Sometimes it is clear how to proceed using one of the three methods, but not the other two.

# Proof using Strong Induction

**Example**: Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

**Solution**: Let $P(n)$ be the proposition that postage of $n$ cents can be formed using 4-cent and 5-cent stamps.

- BASIS STEP: $P(12)$, $P(13)$, $P(14)$, and $P(15)$ hold.
  - $P(12)$ uses three 4-cent stamps.
  - $P(13)$ uses two 4-cent stamps and one 5-cent stamp.
  - $P(14)$ uses one 4-cent stamp and two 5-cent stamps.
  - $P(15)$ uses three 5-cent stamps.
- INDUCTIVE STEP: The inductive hypothesis states that $P(j)$ holds for $12 \leq j \leq k$, where $k \geq 15$. Assuming the inductive hypothesis, it can be shown that $P(k + 1)$ holds.
- Using the inductive hypothesis, $P(k - 3)$ holds since $k - 3 \geq 12$. To form postage of $k + 1$ cents, add a 4-cent stamp to the postage for $k - 3$ cents.

Hence, $P(n)$ holds for all $n \geq 12$. ◀

# Proof of Same Example using Mathematical Induction

**Example**: Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

**Solution**: Let $P(n)$ be the proposition that postage of $n$ cents can be formed using 4-cent and 5-cent stamps.

- BASIS STEP: Postage of 12 cents can be formed using three 4-cent stamps.
- INDUCTIVE STEP: The inductive hypothesis $P(k)$ for any positive integer $k$ is that postage of $k$ cents can be formed using 4-cent and 5-cent stamps. To show $P(k + 1)$ where $k \geq 12$ , we consider two cases:
  - If at least one 4-cent stamp has been used, then a 4-cent stamp can be replaced with a 5-cent stamp to yield a total of $k + 1$ cents.
  - Otherwise, no 4-cent stamp have been used and at least three 5-cent stamps were used. Three 5-cent stamps can be replaced by four 4-cent stamps to yield a total of $k + 1$ cents.

Hence, $P(n)$ holds for all $n \geq 12$. ◀

# Ex. Triangulation of Polygons

- A polygon is a closed geometric figure consisting of a sequence of line segements $s_1$ to $s_n$ called *sides*
  - An end point of a side is called a vertex
- A polygon is simple when no two nonconsecutive sides intersect.
  - A simple polygon divides the plane into interior and exterior
- A polygon is called *convex* if every line connecting two points in the interior lies entirely in the interior.
- A diagonal is a line connecting two nonconsecutive verticies
  - An interior diagonal lies in the interior entirely

# Ex. Triangulation of Polygons

- Triangulation is a process to divide a polygon into triangles by adding nonintersecting diagonals
- Theorem. A simple polygon with $n$ sides for $n \geq 3$ can be triangulated into $n$-2 triangles
  - Lemma. A simple polygon with at least 4 sides has an interior diagonal
  - Proof. T(n): a simple polygon with $n$ sides can be triangulated into n-2 triangles
    - Basis: T(3) holds, obviously.
    - Induction step: for T(i+1)
      - By Induction hypotheses, T(j) holds for 3 <= j <= i
      - By the lemma, a simple polygon with i+1 sides has an interior diagonal that divides the polygon into another two simple polygons Q and R.
      - Each of Q and R can be triangulated since the number of sides in Q or R is less than i+1.

# Recursively Defined Functions

**Definition**: A *recursive* or *inductive definition* of a function consists of two steps.

- BASIS STEP: Specify the value of the function at zero.
- RECURSIVE STEP: Give a rule for finding its value at an integer from its values at smaller integers.

- A function $f(n)$ is the same as a sequence $a_0, a_1, \ldots$ , where $a_i$, where $f(i) = a_i$. This was done using recurrence relations in Section 2.4.

# Recursively Defined Functions

**Example**: Suppose $f$ is defined by:

$f(0) = 3,$

$f(n + 1) = 2f(n) + 3$

Find $f(1), f(2), f(3), f(4)$

**Solution**:

- $f(1) = 2f(0) + 3 = 2\cdot3 + 3 = 9$
- $f(2) = 2f(1)+ 3 = 2\cdot9 + 3 = 21$
- $f(3) = 2f(2) + 3 = 2\cdot21 + 3 = 45$
- $f(4) = 2f(3) + 3 = 2\cdot45 + 3 = 93$

**Example:** Give a recursive definition of the factorial function $n!$:

**Solution**:

$f(0) = 1$

$f(n + 1) = (n + 1)\cdot f(n)$

# Recursively Defined Functions

**Example**: Give a recursive definition of:

$$\sum_{k=0}^{n} a_k.$$

**Solution**: The first part of the definition is

$$\sum_{k=0}^{0} a_k = a_0.$$

The second part is

$$\sum_{k=0}^{n+1} a_k = \left(\sum_{k=0}^{n} a_k\right) + a_{n+1}.$$

Fibonacci
(1170- 1250)

# Fibonacci Numbers

**Example** : The Fibonacci numbers are defined as follows:

$f_0 = 0$

$f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$

Find $f_2, f_3, f_4, f_5$.

- $f_2 = f_1 + f_0 = 1 + 0 = 1$
- $f_3 = f_2 + f_1 = 1 + 1 = 2$
- $f_4 = f_3 + f_2 = 2 + 1 = 3$
- $f_5 = f_4 + f_3 = 3 + 2 = 5$

In Chapter 8, we will use the Fibonacci numbers to model population growth of rabbits. This was an application described by Fibonacci himself.

Next, we use strong induction to prove a result about the Fibonacci numbers.

# Fibonacci Numbers

**Example 4**: Show that whenever $n \geq 3$, $f_n > \alpha^{n-2}$, where $\alpha = (1 + \sqrt{5})/2$.

**Solution**: Let $P(n)$ be the statement $f_n > \alpha^{n-2}$. Use strong induction to show that $P(n)$ is true whenever $n \geq 3$.

- BASIS STEP: $P(3)$ holds since $\alpha < 2 = f_3$

  $P(4)$ holds since $\alpha^2 = (3 + \sqrt{5})/2 < 3 = f_4$ .
- INDUCTIVE STEP: Assume that $P(j)$ holds, i.e., $f_j > \alpha^{j-2}$ for all integers $j$ with $3 \leq j \leq k$, where $k \geq 4$. Show that $P(k + 1)$ holds, i.e., $f_{k+1} > \alpha^{k-1}$ .
  - Since $\alpha^2 = \alpha + 1$ (because $\alpha$ is a solution of $x^2 - x - 1 = 0$),

$$\alpha^{k-1} = \alpha^2 \cdot \alpha^{k-3} = (\alpha + 1) \cdot \alpha^{k-3} = \alpha \cdot \alpha^{k-3} + 1 \cdot \alpha^{k-3} = \alpha^{k-2} + \alpha^{k-3}$$

  - By the inductive hypothesis, because $k \geq 4$ we have

$$f_{k-1} > \alpha^{k-3}, \qquad f_{k-1} > \alpha^{k-2}.$$

  - Therefore, it follows that

$$f_{k+1} = f_k + f_{k-1} > \alpha^{k-2} + \alpha^{k-3} = \alpha^{k-1}.$$

  - Hence, $P(k + 1)$ is true.

Why does this equality hold?

# Recursively Defined Sets and Structures

*Recursive definitions* of sets have two parts:

- The *basis step* specifies an initial collection of elements.
- The *recursive step* gives the rules for forming new elements in the set from those already known to be in the set.

- Sometimes the recursive definition has an *exclusion rule*, which specifies that the set contains nothing other than those elements specified in the basis step and generated by applications of the rules in the recursive step.
- We will always assume that the exclusion rule holds, even if it is not explicitly mentioned.
- We will later develop a form of induction, called *structural induction*, to prove results about recursively defined sets.

# Recursively Defined Sets and Structures

**Example** : Subset of Integers $S$:

BASIS STEP: $3 \in S$.

RECURSIVE STEP: If $x \in S$ and $y \in S$, then $x + y$ is in $S$.

- Initially 3 is in $S$, then $3 + 3 = 6$, then $3 + 6 = 9$, etc.

**Example**: The natural numbers **N**.

BASIS STEP: $0 \in \mathbf{N}$.

RECURSIVE STEP: If $n$ is in **N**, then $n + 1$ is in **N**.

- Initially 0 is in $S$, then $0 + 1 = 1$, then $1 + 1 = 2$, etc.

# Strings

**Definition**: The set $\Sigma^*$ of *strings* over the alphabet $\Sigma$:

BASIS STEP: $\lambda \in \Sigma^*$ ($\lambda$ is the empty string)

RECURSIVE STEP: If $w$ is in $\Sigma^*$ and $x$ is in $\Sigma$, then $wx \in \Sigma^*$.

**Example**: If $\Sigma = \{0,1\}$, the strings in in $\Sigma^*$ are the set of all bit strings, $\lambda$,0,1, 00,01,10, 11, etc.

**Example**: If $\Sigma = \{a,b\}$, show that $aab$ is in $\Sigma^*$.

- Since $\lambda \in \Sigma^*$ and $a \in \Sigma$, $a \in \Sigma^*$.
- Since $a \in \Sigma^*$ and $a \in \Sigma$, $aa \in \Sigma^*$.
- Since $aa \in \Sigma^*$ and $b \in \Sigma$, $aab \in \Sigma^*$.

# String Concatenation

**Definition**: Two strings can be combined via the operation of *concatenation*. Let Σ be a set of symbols and Σ* be the set of strings formed from the symbols in Σ. We can define the concatenation of two strings, denoted by ·, recursively as follows.

BASIS STEP: If $w \in \Sigma^*$, then $w \cdot \lambda = w$.

RECURSIVE STEP: If $w_1 \in \Sigma^*$ and $w_2 \in \Sigma^*$ and $x \in \Sigma$, then $w \cdot (w_2 x) = (w_1 \cdot w_2)x$.

- Often $w_1 \cdot w_2$ is written as $w_1 w_2$.
- If $w_1 = abra$ and $w_2 = cadabra$, the concatenation $w_1 w_2 = abracadabra$.

# Length of a String

**Example**: Give a recursive definition of $l(w)$, the length of the string $w$.

**Solution**: The length of a string can be recursively defined by:

$l(\lambda) = 0$;

$l(wx) = l(w) + 1$ if $w \in \Sigma^*$ and $x \in \Sigma$.

# Balanced Parentheses

**Example**: Give a recursive definition of the set of balanced parentheses $P$.

**Solution**:

BASIS STEP: $() \in P$

RECURSIVE STEP:

- $(w) \in P$ for $w \in P$
- $w\,w' \in P$ for $w \in P$ and $w' \in P$

- Show that (() ()) is in $P$.
- Why is ))(() not in $P$?

# Well-Formed Formulae in Propositional Logic

**Definition**: The set of *well-formed formulae* in propositional logic involving **T**, **F**, propositional variables, and operators from the set $\{\neg,\wedge,\vee,\to,\leftrightarrow\}$.

BASIS STEP: **T**,**F**, and $s$, where $s$ is a propositional variable, are well-formed formulae.

RECURSIVE STEP: If $E$ and $F$ are well formed formulae, then $(\neg E)$, $(E \wedge F)$, $(E \vee F)$, $(E \to F)$, $(E \leftrightarrow F)$, are well-formed formulae.

**Examples**: $((p \vee q) \to (q \wedge \mathbf{F}))$ is a well-formed formula.

$pq \wedge$ is not a well formed formula.