A Convergent Generalized Krylov Subspace Method for CS-MRI Reconstruction with Gradient-Driven Denoisers

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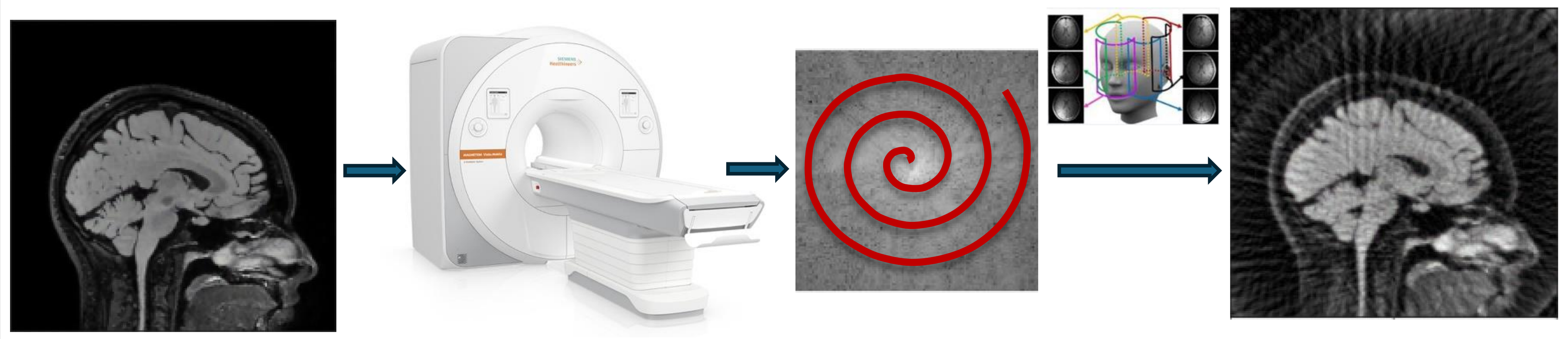
Background



Model-Based compressed sensing (CS) MRI reconstruction:

$$\mathbf{x}_* = \arg\min_{\mathbf{x}\in\mathbb{C}^N} h(\mathbf{x}) + f(\mathbf{x}),\ h(\mathbf{x}) \equiv \frac{1}{2}\|\mathbf{A}\mathbf{x}-\mathbf{y}\|_2^2,\ f(\cdot):\ \text{regularizer} \qquad (1)$$

- $\mathbf{y}$: k-space data, forward model: $\mathbf{A}\in\mathbb{C}^{M\times N} = \{\mathbf{D}\mathbf{F}\mathbf{S}_c\}_c$.
- $\mathbf{D}$: downsampling mask; $\mathbf{F}$: (nonuniform) FFT; $\{\mathbf{S}_c\}_c$: sensitivity maps.

Proximal method for (1) at kth iteration: $\mathbf{x}_{k+1} = \mathrm{prox}_{\alpha f}(\mathbf{x}_k - \alpha\nabla h(\mathbf{x}_k))$

$$\mathrm{prox}_{\alpha f}(\cdot) \equiv \arg\min_{\mathbf{x}\in\mathbb{C}^N} \frac{1}{2}\|\mathbf{x}-\cdot\|_2^2 + \alpha f(\mathbf{x}) \Rightarrow \mathbf{D}_{\sigma>0}(\cdot) \equiv \mathrm{prox}_{\alpha f}(\cdot).$$

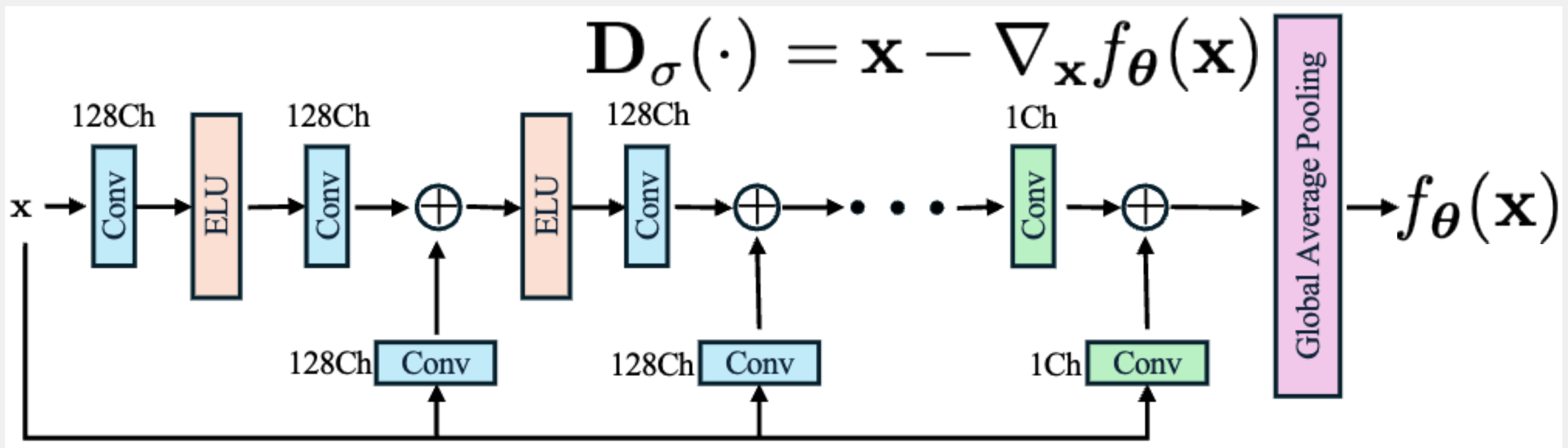
$\mathbf{D}_\sigma(\cdot)$: an abstract denoiser and $\sigma > 0$: noise level $\leftrightarrow$ the strength of $f(\cdot)$.

$$\text{PnP [4]:}\quad \mathbf{x}_{k+1} = \mathbf{D}_\sigma(\mathbf{x}_k - \alpha\nabla h(\mathbf{x}_k)).$$

Convergence PnP $\rightarrow$ Assumption: $\mathbf{D}_\sigma(\cdot)$ is nonexpansive ✘

Goal: (1) Bridge the gap between practice and theory—learned priors;
(2) Solve the minimization problem efficiently.
(3) Convergence is guaranteed.

Gradient-Driven Denoisers



$$\min_{\mathbf{x}\in\mathcal{C}} h(\mathbf{x}) + f_{\boldsymbol{\theta}}(\mathbf{x}) \qquad f_{\boldsymbol{\theta}} \rightarrow f$$

Observations: 1. h and f are differentiable; 2. f is nonconvex.
Solvers: Gradient based methods, e.g., (accelerated) proximal gradient;
Complex quasi-Newton proximal method (CQNPM) [5]: $(\mathbf{B}_k \succ 0)$

$$\mathbf{x}_{k+1} = \operatorname*{argmin}_{\mathbf{x}\in\mathcal{C}} \langle\nabla f(\mathbf{x}_k), \mathbf{x}-\mathbf{x}_k\rangle + \frac{1}{2\alpha_k}\|\mathbf{x}-\mathbf{x}_k\|_{\mathbf{B}_k}^2 + \underbrace{\frac{1}{2}\|\mathbf{A}\mathbf{x}-\mathbf{y}\|_2^2}_{h(\mathbf{x})}, \qquad (2)$$

Messages: 1. CQNPM > first-order methods. 2. Convergence is guaranteed: I) f satisfies Polyak-Łojasiewicz inequality; II) $\nabla_{\mathbf{x}} f$ is Lipschitz continuous.

NB: computing $\mathbf{A}\mathbf{x}$ is quite expensive in multi-coil non-Cartesian sampling MRI.

Q: how to solve (2) efficiently?

Generalized Krylov Subspace Method (GKSM)

Given a subspace basis $\mathbf{V}_k \in \mathbb{C}^{N\times k}$ satisfying $\mathbf{V}_k^{\mathrm{H}}\mathbf{V}_k = \mathbf{I}_k$:

$$\boldsymbol{\beta}_k = \arg\min_{\mathbf{x}=\mathbf{V}_k\boldsymbol{\beta}} \underbrace{\frac{1}{2}\|\mathbf{A}\mathbf{x}-\mathbf{y}\|_2^2 + \bar{f}(\mathbf{x},\mathbf{x}_k,\mathbf{B}_k,\alpha_k)}_{\bar{F}(\mathbf{x},\mathbf{x}_k)}, \qquad (3)$$

where $\bar{f}(\mathbf{x},\mathbf{x}_k,\mathbf{B}_k,\alpha_k) \equiv \langle\nabla f(\mathbf{x}_k),\mathbf{x}\rangle + \frac{1}{2\alpha_k}\|\mathbf{x}-\mathbf{x}_k\|_{\bar{\mathbf{B}}_k}^2$.

$$\boldsymbol{\beta}_k = \arg\min_{\boldsymbol{\beta}\in\mathbb{C}^k} \left\| \begin{bmatrix}\mathbf{A}\mathbf{V}_k\boldsymbol{\beta}\\ \bar{\mathbf{B}}_k^{\frac{1}{2}}\mathbf{V}_k\boldsymbol{\beta}\end{bmatrix} - \begin{bmatrix}\mathbf{y}\\ \bar{\mathbf{B}}_k^{\frac{1}{2}}\mathbf{w}_k\end{bmatrix} \right\|_2^2, \qquad (4)$$

where $\mathbf{w}_k = \mathbf{x}_k - \alpha_k\mathbf{B}_k^{-1}\nabla f(\mathbf{x}_k)$ and $\bar{\mathbf{B}}_k = \bar{\mathbf{B}}_k^{\frac{1}{2}}\bar{\mathbf{B}}_k^{\frac{1}{2}}$ with $\bar{\mathbf{B}}_k = \frac{1}{\alpha_k}\mathbf{B}_k$. Then

$$\boldsymbol{\beta}_k = (\mathbf{V}_k^{\mathrm{H}}\mathbf{A}^{\mathrm{H}}\mathbf{A}\mathbf{V}_k + \mathbf{V}_k^{\mathrm{H}}\bar{\mathbf{B}}_k\mathbf{V}_k)^{-1}\mathbf{V}_k^{\mathrm{H}}(\mathbf{A}^{\mathrm{H}}\mathbf{y} + \bar{\mathbf{B}}_k\mathbf{w}_k) \qquad \mathcal{C} = \mathbb{C}^N \qquad (5)$$

$$\boldsymbol{\beta}_k = \arg\min_{(\mathbf{V}_k\boldsymbol{\beta})\in\mathcal{C}} \left\| \begin{bmatrix}\mathbf{A}\mathbf{V}_k\boldsymbol{\beta}\\ \bar{\mathbf{B}}_k^{\frac{1}{2}}\mathbf{V}_k\boldsymbol{\beta}\end{bmatrix} - \begin{bmatrix}\mathbf{y}\\ \bar{\mathbf{B}}_k^{\frac{1}{2}}\mathbf{w}_k\end{bmatrix} \right\|_2^2 \qquad \mathcal{C} \neq \mathbb{C}^N \qquad (6)$$

$$\mathbf{x}_{k+1} = \arg\min_{\mathbf{z}\in\mathcal{C}} \left\| \begin{bmatrix}\mathbf{A}\mathbf{V}_k\mathbf{V}_k^{\mathrm{H}}\mathbf{z}\\ \bar{\mathbf{B}}_k^{\frac{1}{2}}\mathbf{V}_k\mathbf{V}_k^{\mathrm{H}}\mathbf{z}\end{bmatrix} - \begin{bmatrix}\mathbf{y}\\ \bar{\mathbf{B}}_k^{\frac{1}{2}}\mathbf{w}_k\end{bmatrix} \right\|_2^2 \qquad (7)$$

Algorithm 1 Generalized Krylov Subspace Method (GKSM)

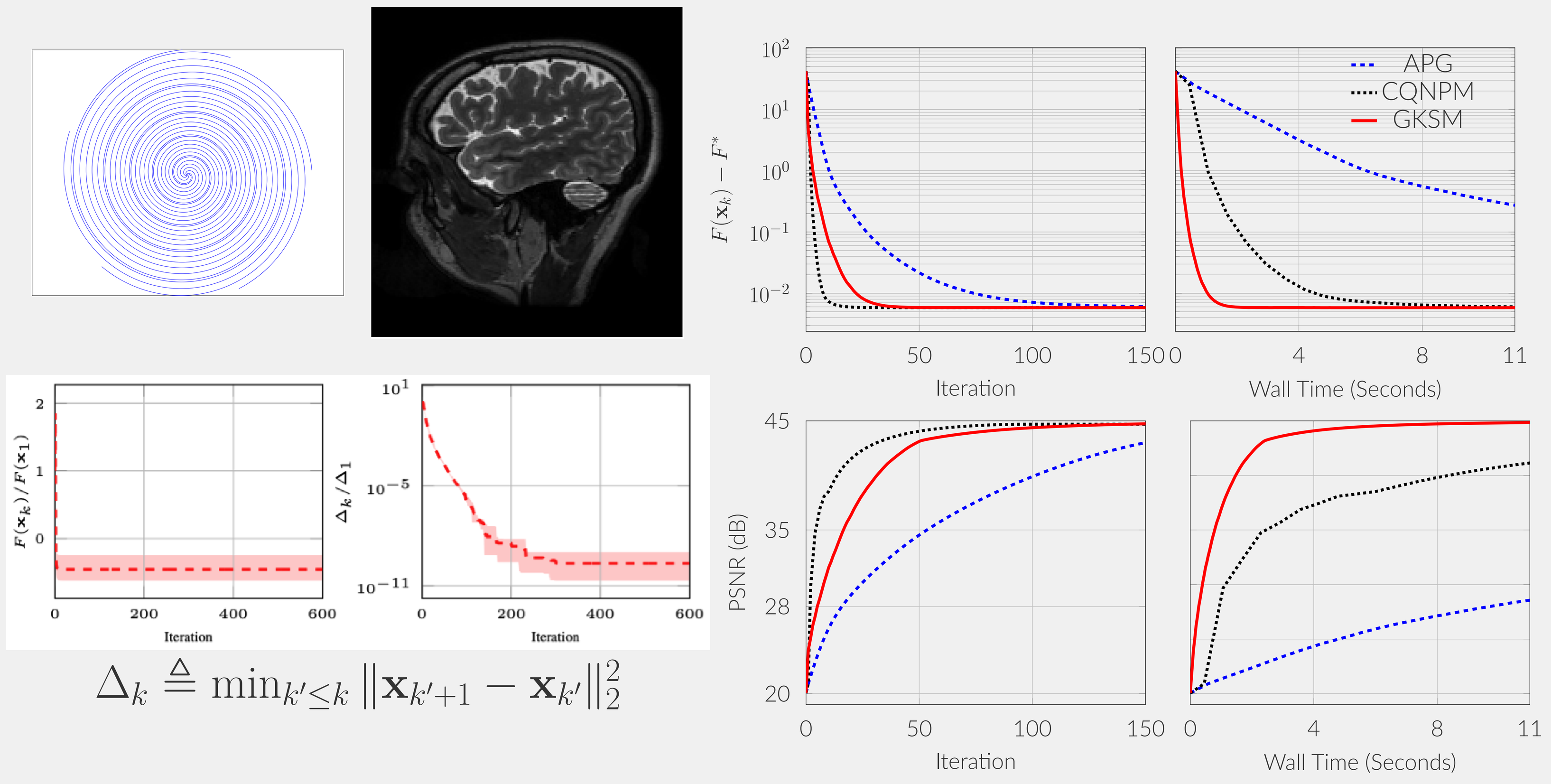
Initialization: $\mathbf{x}_1$, stepsize $\alpha_k > 0$, $\mathbf{V}_1 = \frac{\mathbf{A}^{\mathrm{H}}\mathbf{y}}{\|\mathbf{A}^{\mathrm{H}}\mathbf{y}\|}$, $\mathbf{A}\mathbf{V}_1$, maximal number of subspace iterations K, and maximal number of total iterations `Max_Iter`

Iteration:

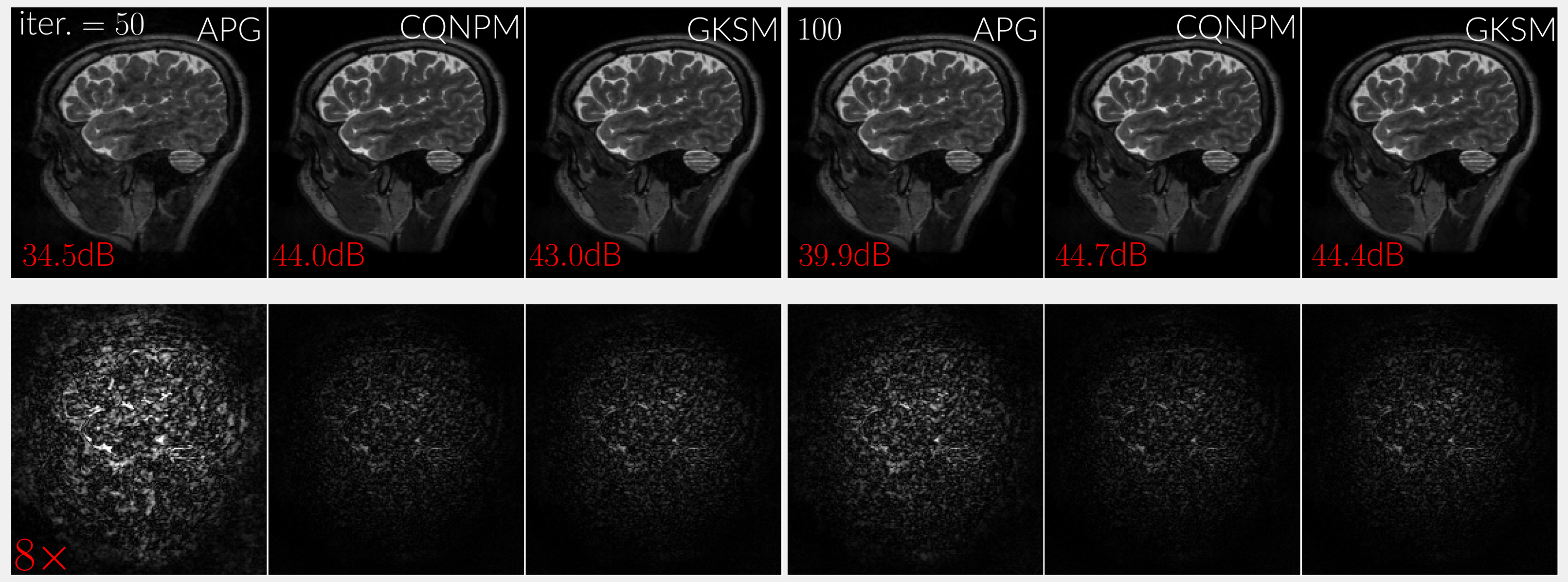
```
for k = 1, 2, ..., Max_Iter do
    Compute ∇f(x_k)
    Set B_k ≻ 0
    Compute β_k with (5) or (6)
    Compute x_{k+1} ← V_k β_k
    if k ≤ K then
        Compute r_k ← ∇_x F̄(x_{k+1}, x_k).
        r̃_k ← (I − V_k V_k^H) r_k
        if ‖r̃_k‖ ≠ 0 then
            v_{k+1} ← r̃_k / ‖r̃_k‖
            V_{k+1} ← [V_k  v_{k+1}]
            AV_{k+1} ← [AV_k  Av_{k+1}]
        else
            V_{k+1} ← V_k
            AV_{k+1} ← AV_k
        end if
    else
        V_{k+1} ← I_N
    end if
end for
```

NB: Convergence is guaranteed under the Kurdyka–Łojasiewicz inequality and the Lipschitz continuous of ∇f.

Results



$\Delta_k \triangleq \min_{k'\leq k} \|\mathbf{x}_{k'+1} - \mathbf{x}_{k'}\|_2^2$



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References

[1] Shreyas Chaudhari, Srinivasa Pranav, and José MF Moura. Gradient networks. *IEEE Transactions on Signal Processing*, 73:324 – 339, 2024.

[2] Regev Cohen, Yochai Blau, Daniel Freedman, and Ehud Rivlin. It has potential: Gradient-driven denoisers for convergent solutions to inverse problems. *NeurIPS*, 34:18152–18164, 2021.

[3] Tao Hong, Umberto Villa, and Jeffrey A Fessler. A convergent generalized Krylov subspace method for compressed sensing MRI reconstruction with gradient-driven denoisers. *arXiv preprint arXiv:2508.11219*, 2025.

[4] Tao Hong, Xiaojian Xu, Jason Hu, and Jeffrey A Fessler. Provable preconditioned plug-and-play approach for compressed sensing MRI reconstruction. *IEEE Transactions on Computational Imaging*, 10:372 – 384, 2024.

[5] Tao Hong, Zhaoyi Xu, Se Young Chun, Luis Hernandez-Garcia, and Jeffrey A Fessler. Convergent complex quasi-Newton proximal methods for gradient-driven denoisers in compressed sensing MRI reconstruction. *IEEE Transactions on Computational Imaging*, 11:1534 – 1547, October 2025.

[6] Samuel Hurault, Arthur Leclaire, and Nicolas Papadakis. Gradient step denoiser for convergent plug-and-play. *ICLR*, 2022.