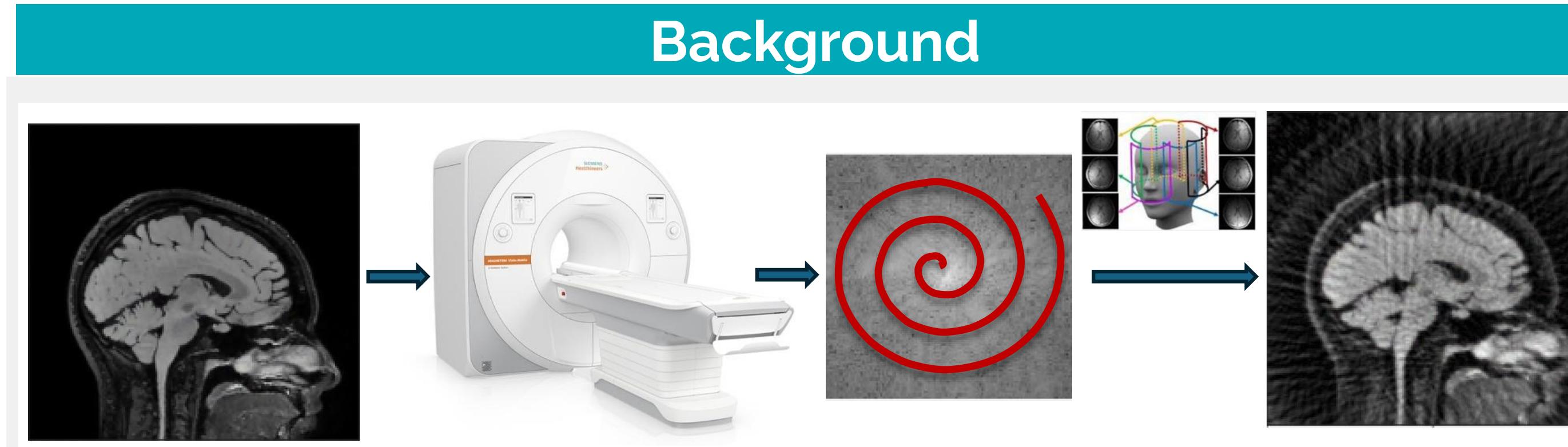


A Convergent Generalized Krylov Subspace Method for Compressed Sensing MRI Reconstruction with Gradient-Driven Denoisers



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Model-Based compressive sensing (CS) MRI reconstruction:

$$\mathbf{x}_* = \arg \min_{\mathbf{x} \in \mathbb{C}^N} h(\mathbf{x}) + f(\mathbf{x}), \quad h(\mathbf{x}) \equiv \frac{1}{2} \|\mathbf{Ax} - \mathbf{y}\|_2^2, \quad f(\cdot) : \text{regularizer} \quad (1)$$

- \mathbf{y} : k-space data, forward model: $\mathbf{A} \in \mathbb{C}^{M \times N} = \{\mathbf{DF}_{\mathbf{S}_c}\}_c$.
- \mathbf{D} : downsampling mask; \mathbf{F} : (nonuniform) FFT; $\{\mathbf{S}_c\}_c$: sensitivity maps.

Proximal method for (1) at k th iteration: $\mathbf{x}_{k+1} = \text{prox}_{\alpha f}(\mathbf{x}_k - \alpha \nabla f(\mathbf{x}_k))$

$$\text{prox}_{\alpha f}(\cdot) \equiv \arg \min_{\mathbf{x} \in \mathbb{C}^N} \frac{1}{2} \|\mathbf{x} - \cdot\|_2^2 + \alpha f(\mathbf{x}) \Rightarrow \mathbf{D}_{\sigma>0}(\cdot) \equiv \text{prox}_{\alpha f}(\cdot).$$

$\mathbf{D}_{\sigma}(\cdot)$: an abstract denoiser and $\sigma > 0$: noise level \leftrightarrow the strength of $f(\cdot)$.

$$\text{PnP [4]: } \mathbf{x}_{k+1} = \mathbf{D}_{\sigma}(\mathbf{x}_k - \alpha \nabla f(\mathbf{x}_k)).$$

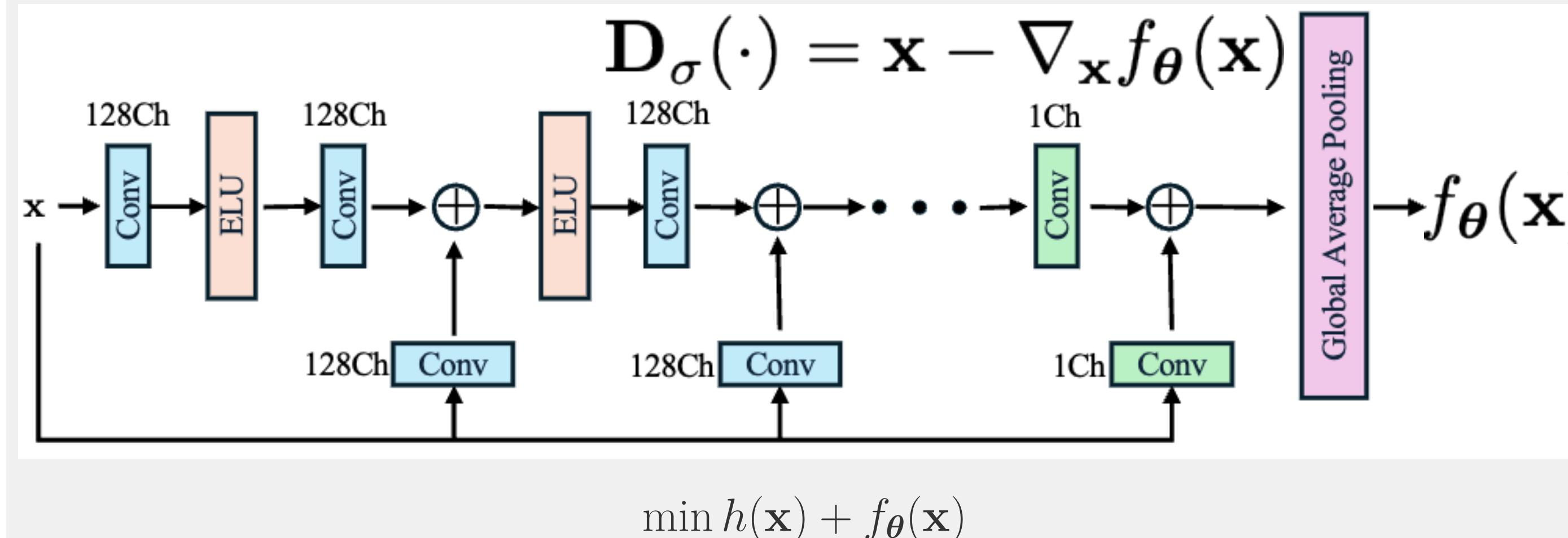
Convergence PnP \rightarrow Assumption: $\mathbf{D}_{\sigma}(\cdot)$ is nonexpansive ✕

Goal: (1) Bridge the gap between practice and theory—learned priors;

(2) Solve the minimization problem efficiently.

(3) Convergence is guaranteed.

Gradient-Driven Denoisers



Observations: 1. h and f_θ are differentiable; 2. f_θ is nonconvex.

Solvers: Gradient based methods, e.g., (accelerated) proximal gradient

Complex quasi-Newton proximal method (CQNPM) [5]:

$$\mathbf{x}_{k+1} = \underset{\mathbf{x} \in \mathcal{C}}{\text{argmin}} \langle \nabla f(\mathbf{x}_k), \mathbf{x} - \mathbf{x}_k \rangle + \frac{1}{2\alpha_k} \|\mathbf{x} - \mathbf{x}_k\|_{\mathbf{B}_k}^2 + \underbrace{\frac{1}{2} \|\mathbf{Ax} - \mathbf{y}\|_2^2}_{h(\mathbf{x})}, \quad \mathbf{B}_k \succ 0 \quad (2)$$

Messages: 1. CQNPM > first-order methods. 2. Convergence is guaranteed:

I) f_θ satisfies Polyak–Łojasiewicz inequality; II) $\nabla_{\mathbf{x}} f_\theta$ is Lipschitz continuous.

NB: computing \mathbf{Ax} is quite expensive in multi-coil non-Cartesian sampling MRI.

Q: how to solve (2) efficiently?

Generalized Krylov Subspace Method (GKSM)

Given a subspace basis $\mathbf{V}_k \in \mathbb{C}^{N \times k}$ satisfying $\mathbf{V}_k^H \mathbf{V}_k = \mathbf{I}_k$:

$$\beta_k = \arg \min_{\mathbf{x} = \mathbf{V}_k \beta} \underbrace{\frac{1}{2} \|\mathbf{Ax} - \mathbf{y}\|_2^2 + \bar{f}(\mathbf{x}, \mathbf{x}_k, \mathbf{B}_k, \alpha_k)}_{\bar{F}(\mathbf{x}, \mathbf{x}_k)}, \quad (3)$$

where $\bar{f}(\mathbf{x}, \mathbf{x}_k, \mathbf{B}_k, \alpha_k) \equiv \langle \nabla f(\mathbf{x}_k), \mathbf{x} \rangle + \frac{1}{2\alpha_k} \|\mathbf{x} - \mathbf{x}_k\|_{\mathbf{B}_k}^2$.

$$\beta_k = \arg \min_{\beta \in \mathbb{C}^k} \left\| \begin{bmatrix} \mathbf{A} \mathbf{V}_k \beta \\ \bar{\mathbf{B}}_k^{\frac{1}{2}} \mathbf{V}_k \beta \end{bmatrix} - \begin{bmatrix} \mathbf{y} \\ \bar{\mathbf{B}}_k^{\frac{1}{2}} \mathbf{w}_k \end{bmatrix} \right\|_2^2, \quad (4)$$

where $\mathbf{w}_k = \mathbf{x}_k - \alpha_k \mathbf{B}_k^{-1} \nabla f(\mathbf{x}_k)$ and $\bar{\mathbf{B}}_k = \bar{\mathbf{B}}_k^{\frac{1}{2}} \bar{\mathbf{B}}_k^{\frac{1}{2}}$ with $\bar{\mathbf{B}}_k = \frac{1}{\alpha_k} \mathbf{B}_k$. Then

$$\beta_k = (\mathbf{V}_k^H \mathbf{A}^H \mathbf{A} \mathbf{V}_k + \mathbf{V}_k^H \bar{\mathbf{B}}_k \mathbf{V}_k)^{-1} \mathbf{V}_k^H (\mathbf{A}^H \mathbf{y} + \bar{\mathbf{B}}_k \mathbf{w}_k) \quad \mathcal{C} = \mathbb{C}^N \quad (5)$$

$$\beta_k = \arg \min_{(\mathbf{V}_k \beta) \in \mathcal{C}} \left\| \begin{bmatrix} \mathbf{A} \mathbf{V}_k \beta \\ \bar{\mathbf{B}}_k^{\frac{1}{2}} \mathbf{V}_k \beta \end{bmatrix} - \begin{bmatrix} \mathbf{y} \\ \bar{\mathbf{B}}_k^{\frac{1}{2}} \mathbf{w}_k \end{bmatrix} \right\|_2^2 \quad \mathcal{C} \neq \mathbb{C}^N \quad (6)$$

$$\mathbf{x}_{k+1} = \arg \min_{\mathbf{z} \in \mathcal{C}} \left\| \begin{bmatrix} \mathbf{A} \mathbf{V}_k \mathbf{V}_k^H \mathbf{z} \\ \bar{\mathbf{B}}_k^{\frac{1}{2}} \mathbf{V}_k \mathbf{V}_k^H \mathbf{z} \end{bmatrix} - \begin{bmatrix} \mathbf{y} \\ \bar{\mathbf{B}}_k^{\frac{1}{2}} \mathbf{w}_k \end{bmatrix} \right\|_2^2 \quad (7)$$

Algorithm 1 Generalized Krylov Subspace Method (GKSM)

Initialization: \mathbf{x}_1 , stepsize $\alpha_k > 0$, $\mathbf{V}_1 = \frac{\mathbf{A}^H \mathbf{y}}{\|\mathbf{A}^H \mathbf{y}\|}$, \mathbf{AV}_1 , maximal number of subspace iterations K , and maximal number of total iterations **Max_Iter**

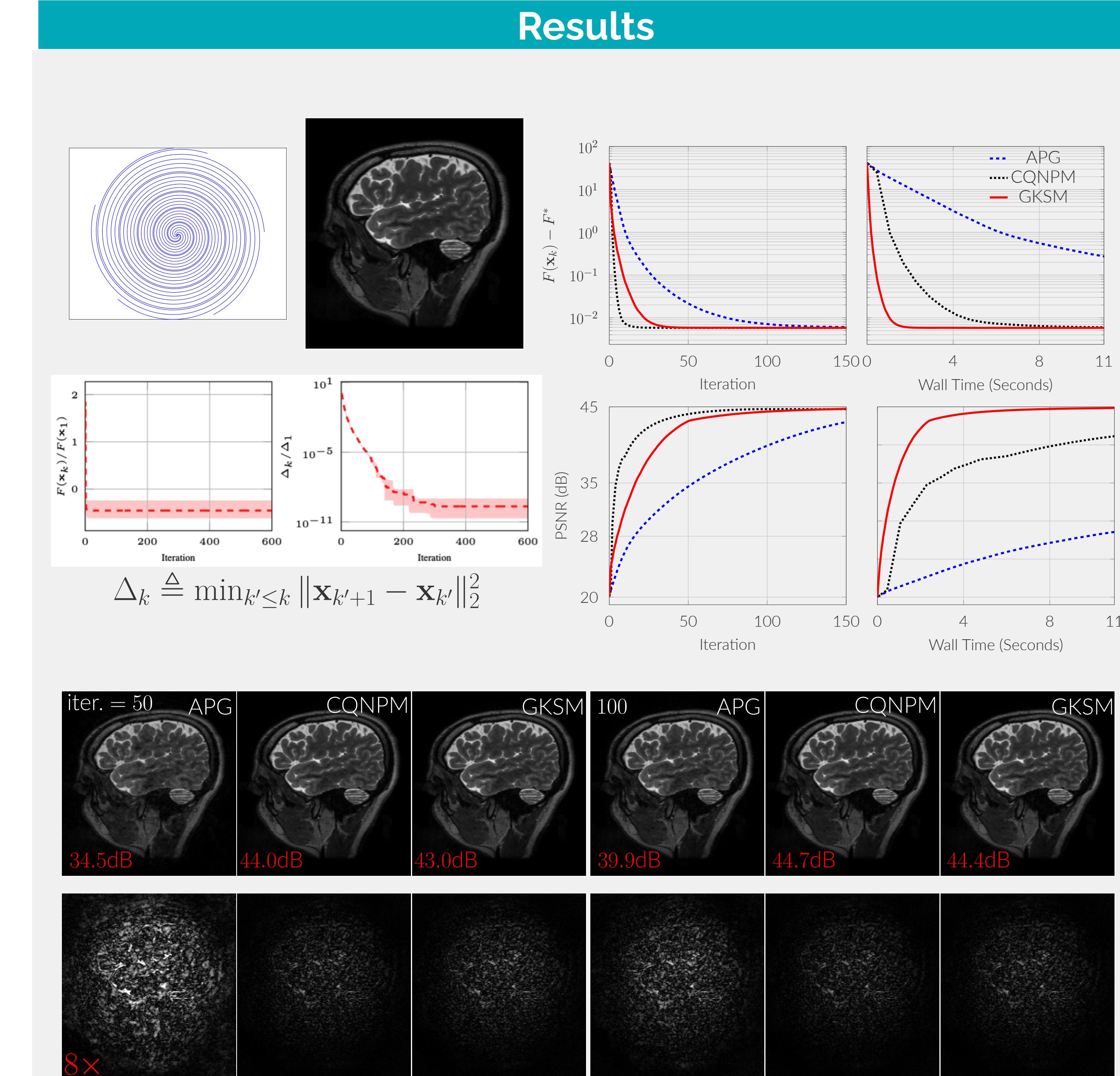
Iteration:

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1: for  $k = 1, 2, \dots, \text{Max\_Iter}$  do
2:   Compute  $\nabla f(\mathbf{x}_k)$ 
3:   Set  $\mathbf{B}_k \succ 0$ 
4:   Compute  $\beta_k$  with (5) or (6)
5:   Compute  $\mathbf{x}_{k+1} \leftarrow \mathbf{V}_k \beta_k$ 
6:   if  $k \leq K$  then
7:     Compute  $\mathbf{r}_k \leftarrow \nabla_{\mathbf{x}} \bar{F}(\mathbf{x}_{k+1}, \mathbf{x}_k)$ .
8:      $\tilde{\mathbf{r}}_k \leftarrow (\mathbf{I} - \mathbf{V}_k \mathbf{V}_k^H) \mathbf{r}_k$ 
9:     if  $\|\tilde{\mathbf{r}}_k\| \neq 0$  then
10:       $\mathbf{v}_{k+1} \leftarrow \tilde{\mathbf{r}}_k / \|\tilde{\mathbf{r}}_k\|$ 
11:       $\mathbf{V}_{k+1} \leftarrow [\mathbf{V}_k \mathbf{v}_{k+1}]$ 
12:       $\mathbf{AV}_{k+1} \leftarrow [\mathbf{AV}_k \mathbf{Av}_{k+1}]$ 
13:    else
14:       $\mathbf{V}_{k+1} \leftarrow \mathbf{V}_k$ 
15:       $\mathbf{AV}_{k+1} \leftarrow \mathbf{AV}_k$ 
16:    end if
17:  else
18:     $\mathbf{V}_{k+1} \leftarrow \mathbf{I}_N$ 
19:  end if
20: end for

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NB: Convergence is guaranteed under the Kurdyka–Łojasiewicz inequality.



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References

- [1] Shreyas Chaudhari, Srinivasa Pranav, and José MF Moura. Gradient networks. *IEEE Transactions on Signal Processing*, 73:324 – 339, 2024.
- [2] Regev Cohen, Yochai Blau, Daniel Freedman, and Ehud Rivlin. It has potential: Gradient-driven denoisers for convergent solutions to inverse problems. *NeurIPS*, 34:18152–18164, 2021.
- [3] Tao Hong, Umberto Villa, and Jeffrey A Fessler. A convergent generalized Krylov subspace method for compressed sensing MRI reconstruction with gradient-driven denoisers. *arXiv preprint arXiv:2508.11219*, 2025.
- [4] Tao Hong, Xiaojian Xu, Jason Hu, and Jeffrey A Fessler. Provable preconditioned plug-and-play approach for compressed sensing MRI reconstruction. *IEEE Transactions on Computational Imaging*, 10:372 – 384, 2024.
- [5] Tao Hong, Zhaoyi Xu, Se Young Chun, Luis Hernandez-Garcia, and Jeffrey A Fessler. Convergent complex quasi-Newton proximal methods for gradient-driven denoisers in compressed sensing MRI reconstruction. *IEEE Transactions on Computational Imaging*, 2025, to appear.
- [6] Samuel Hurault, Arthur Leclaire, and Nicolas Papadakis. Gradient step denoiser for convergent plug-and-play. *ICLR*, 2022.