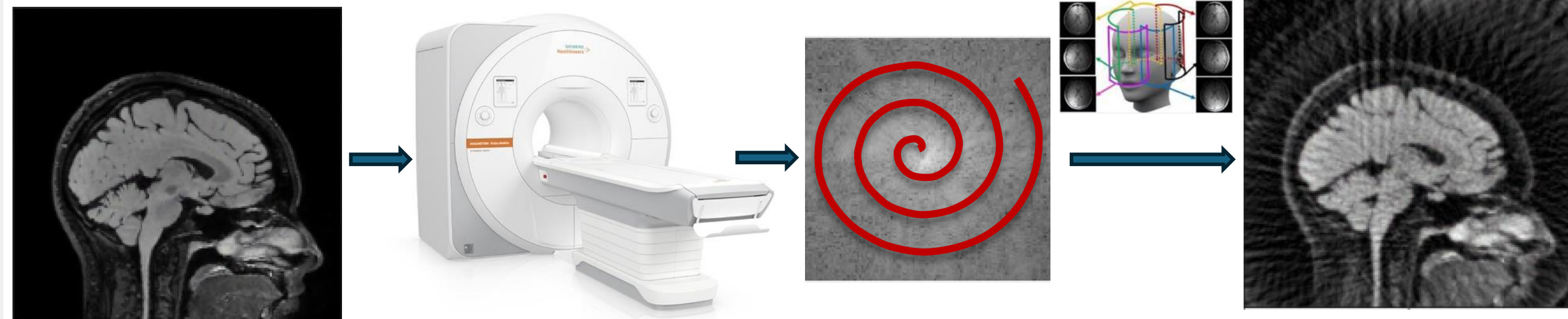


A Convergent Generalized Krylov Subspace Method for Compressed Sensing MRI Reconstruction with Gradient-Driven Denoisers

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Background



Model-Based compressed sensing (CS) MRI reconstruction:

$$\mathbf{x}_* = \arg \min_{\mathbf{x} \in \mathbb{C}^N} h(\mathbf{x}) + f(\mathbf{x}), \quad h(\mathbf{x}) \equiv \frac{1}{2} \|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2, \quad f(\cdot) : \text{regularizer} \quad (1)$$

- \mathbf{y} : k-space data, forward model: $\mathbf{A} \in \mathbb{C}^{M \times N} = \{\mathbf{DFS}_c\}_c$.
 - \mathbf{D} : downsampling mask; \mathbf{F} : (nonuniform) FFT; $\{\mathbf{S}_c\}_c$: sensitivity maps.
- Proximal method for (1) at k th iteration: $\mathbf{x}_{k+1} = \text{prox}_{\alpha f}(\mathbf{x}_k - \alpha \nabla h(\mathbf{x}_k))$

$$\text{prox}_{\alpha f}(\cdot) \equiv \arg \min_{\mathbf{x} \in \mathbb{C}^N} \frac{1}{2} \|\mathbf{x} - \cdot\|_2^2 + \alpha f(\mathbf{x}) \Rightarrow \mathbf{D}_{\sigma > 0}(\cdot) \equiv \text{prox}_{\alpha f}(\cdot).$$

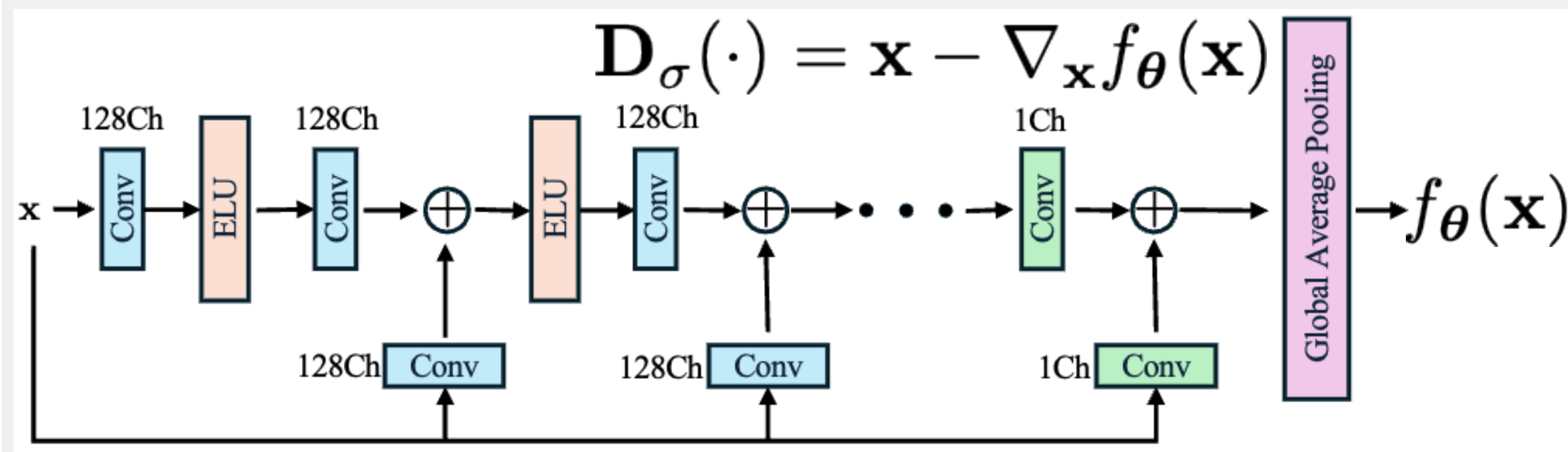
$\mathbf{D}_\sigma(\cdot)$: an abstract denoiser and $\sigma > 0$: noise level \leftrightarrow the strength of $f(\cdot)$.

$$\text{PnP [4]: } \mathbf{x}_{k+1} = \mathbf{D}_\sigma(\mathbf{x}_k - \alpha \nabla h(\mathbf{x}_k)).$$

Convergence PnP \rightarrow Assumption: $\mathbf{D}_\sigma(\cdot)$ is nonexpansive \times

- Goal:** (1) Bridge the gap between practice and theory—learned priors;
(2) Solve the minimization problem efficiently.
(3) Convergence is guaranteed.

Gradient-Driven Denoisers



$$\min_{\mathbf{x} \in \mathcal{C}} h(\mathbf{x}) + f_\theta(\mathbf{x})$$

Observations: 1. h and f_θ are differentiable; 2. f_θ is nonconvex.

Solvers: Gradient based methods, e.g., (accelerated) proximal gradient;
Complex quasi-Newton proximal method (CQNPM) [5]: ($\mathbf{B}_k \succ 0$, $f_\theta \rightarrow f$)

$$\mathbf{x}_{k+1} = \arg \min_{\mathbf{x} \in \mathcal{C}} \langle \nabla f(\mathbf{x}_k), \mathbf{x} - \mathbf{x}_k \rangle + \frac{1}{2\alpha_k} \|\mathbf{x} - \mathbf{x}_k\|_{\mathbf{B}_k}^2 + \underbrace{\frac{1}{2} \|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2}_{h(\mathbf{x})} \quad (2)$$

Messages: 1. CQNPM $>$ first-order methods. 2. Convergence is guaranteed:
I) f satisfies Polyak-Łojasiewicz inequality; II) $\nabla_{\mathbf{x}} f$ is Lipschitz continuous.

NB: computing $\mathbf{A}\mathbf{x}$ is quite expensive in multi-coil non-Cartesian sampling MRI.

Q: how to solve (2) efficiently?

Generalized Krylov Subspace Method (GKSM)

Given a subspace basis $\mathbf{V}_k \in \mathbb{C}^{N \times k}$ satisfying $\mathbf{V}_k^H \mathbf{V}_k = \mathbf{I}_k$:

$$\beta_k = \arg \min_{\mathbf{x} = \mathbf{V}_k \beta} \frac{1}{2} \underbrace{\|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2 + \bar{f}(\mathbf{x}, \mathbf{x}_k, \mathbf{B}_k, \alpha_k)}_{\bar{F}(\mathbf{x}, \mathbf{x}_k)}, \quad (3)$$

where $\bar{f}(\mathbf{x}, \mathbf{x}_k, \mathbf{B}_k, \alpha_k) \equiv \langle \nabla f(\mathbf{x}_k), \mathbf{x} \rangle + \frac{1}{2\alpha_k} \|\mathbf{x} - \mathbf{x}_k\|_{\mathbf{B}_k}^2$.

$$\beta_k = \arg \min_{\beta \in \mathbb{C}^k} \left\| \begin{bmatrix} \mathbf{A}\mathbf{V}_k \beta \\ \bar{\mathbf{B}}_k^{\frac{1}{2}} \mathbf{V}_k \beta \end{bmatrix} - \begin{bmatrix} \mathbf{y} \\ \bar{\mathbf{B}}_k^{\frac{1}{2}} \mathbf{w}_k \end{bmatrix} \right\|_2, \quad (4)$$

where $\mathbf{w}_k = \mathbf{x}_k - \alpha_k \mathbf{B}_k^{-1} \nabla f(\mathbf{x}_k)$ and $\bar{\mathbf{B}}_k = \bar{\mathbf{B}}_k^{\frac{1}{2}} \bar{\mathbf{B}}_k^{\frac{1}{2}}$ with $\bar{\mathbf{B}}_k = \frac{1}{\alpha_k} \mathbf{B}_k$. Then

$$\beta_k = (\mathbf{V}_k^H \mathbf{A}^H \mathbf{A} \mathbf{V}_k + \mathbf{V}_k^H \bar{\mathbf{B}}_k \mathbf{V}_k)^{-1} \mathbf{V}_k^H (\mathbf{A}^H \mathbf{y} + \bar{\mathbf{B}}_k \mathbf{w}_k) \quad \mathcal{C} = \mathbb{C}^N \quad (5)$$

$$\beta_k = \arg \min_{(\mathbf{V}_k \beta) \in \mathcal{C}} \left\| \begin{bmatrix} \mathbf{A}\mathbf{V}_k \beta \\ \bar{\mathbf{B}}_k^{\frac{1}{2}} \mathbf{V}_k \beta \end{bmatrix} - \begin{bmatrix} \mathbf{y} \\ \bar{\mathbf{B}}_k^{\frac{1}{2}} \mathbf{w}_k \end{bmatrix} \right\|_2 \quad \mathcal{C} \neq \mathbb{C}^N \quad (6)$$

$$\mathbf{x}_{k+1} = \arg \min_{\mathbf{z} \in \mathcal{C}} \left\| \begin{bmatrix} \mathbf{A}\mathbf{V}_k \mathbf{V}_k^H \mathbf{z} \\ \bar{\mathbf{B}}_k^{\frac{1}{2}} \mathbf{V}_k \mathbf{V}_k^H \mathbf{z} \end{bmatrix} - \begin{bmatrix} \mathbf{y} \\ \bar{\mathbf{B}}_k^{\frac{1}{2}} \mathbf{w}_k \end{bmatrix} \right\|_2 \quad (7)$$

Algorithm 1 Generalized Krylov Subspace Method (GKSM)

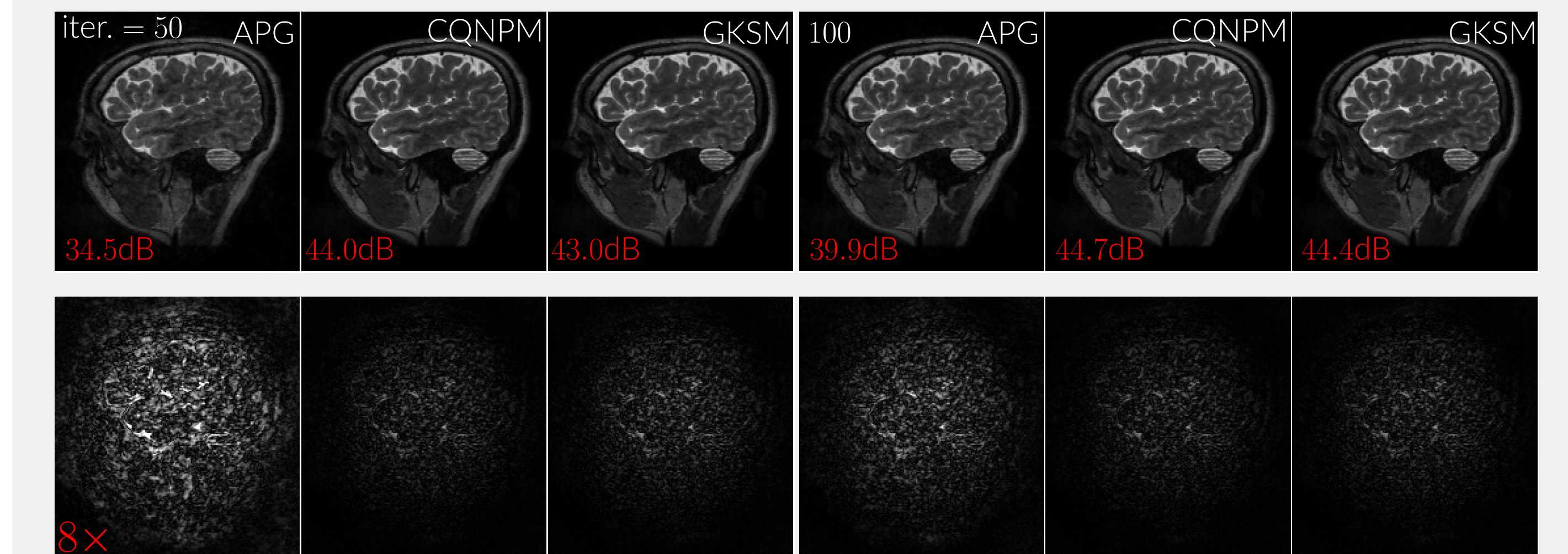
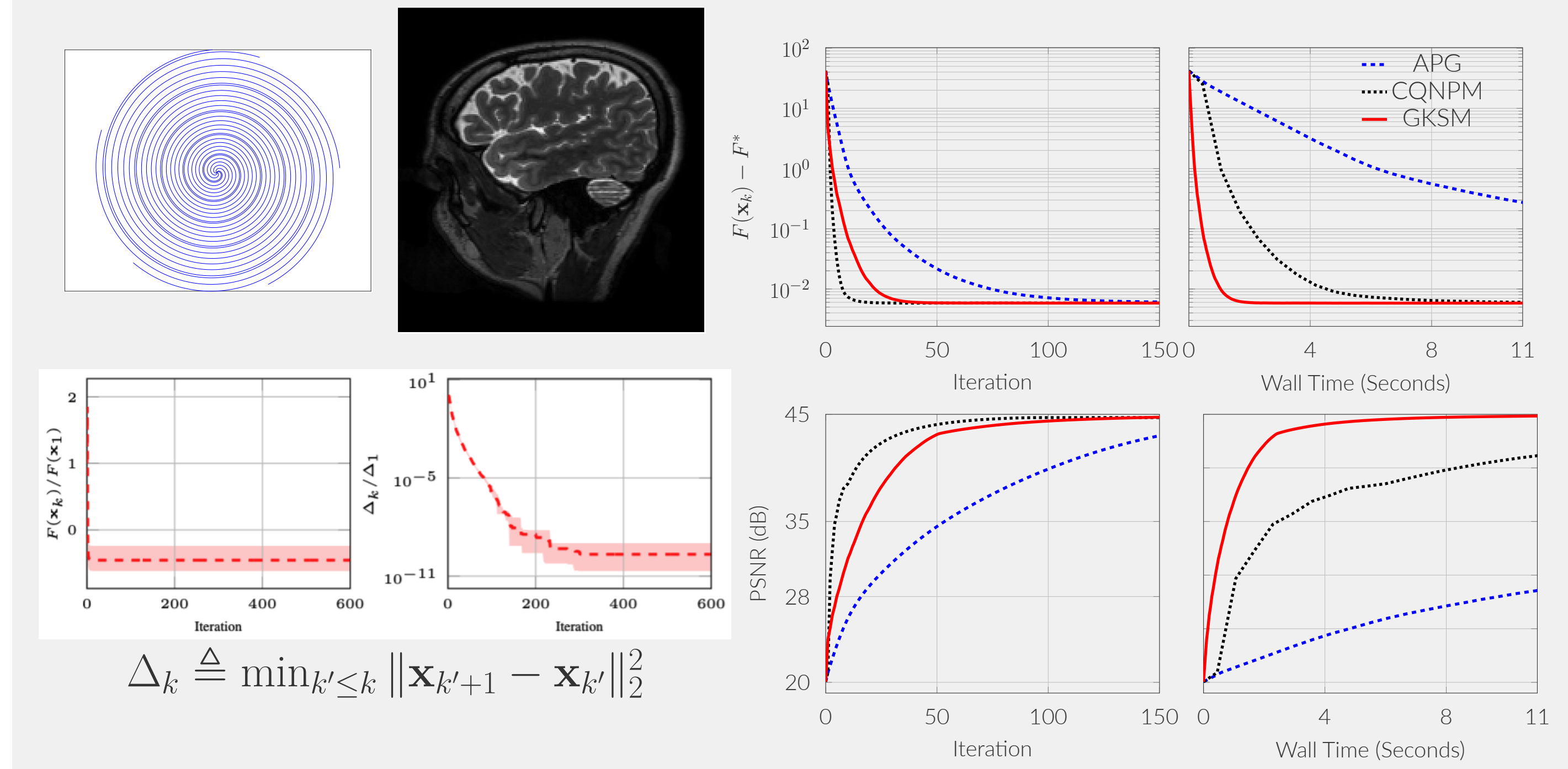
Initialization: \mathbf{x}_1 , stepsize $\alpha_k > 0$, $\mathbf{V}_1 = \frac{\mathbf{A}^H \mathbf{y}}{\|\mathbf{A}^H \mathbf{y}\|}$, $\mathbf{A}\mathbf{V}_1$, maximal number of subspace iterations K , and maximal number of total iterations Max_Iter

Iteration:

- 1: **for** $k = 1, 2, \dots, \text{Max_Iter}$ **do**
- 2: Compute $\nabla f(\mathbf{x}_k)$
- 3: Set $\mathbf{B}_k \succ 0$
- 4: Compute β_k with (5) or (6)
- 5: Compute $\mathbf{x}_{k+1} \leftarrow \mathbf{V}_k \beta_k$
- 6: **if** $k \leq K$ **then**
- 7: Compute $\mathbf{r}_k \leftarrow \nabla_{\mathbf{x}} \bar{F}(\mathbf{x}_{k+1}, \mathbf{x}_k)$.
- 8: $\tilde{\mathbf{r}}_k \leftarrow (\mathbf{I} - \mathbf{V}_k \mathbf{V}_k^H) \mathbf{r}_k$
- 9: **if** $\|\tilde{\mathbf{r}}_k\| \neq 0$ **then**
- 10: $\mathbf{v}_{k+1} \leftarrow \tilde{\mathbf{r}}_k / \|\tilde{\mathbf{r}}_k\|$
- 11: $\mathbf{V}_{k+1} \leftarrow [\mathbf{V}_k \ \mathbf{v}_{k+1}]$
- 12: $\mathbf{A}\mathbf{V}_{k+1} \leftarrow [\mathbf{A}\mathbf{V}_k \ \mathbf{A}\mathbf{v}_{k+1}]$
- 13: **else**
- 14: $\mathbf{V}_{k+1} \leftarrow \mathbf{V}_k$
- 15: $\mathbf{A}\mathbf{V}_{k+1} \leftarrow \mathbf{A}\mathbf{V}_k$
- 16: **end if**
- 17: **else**
- 18: $\mathbf{V}_{k+1} \leftarrow \mathbf{I}_N$
- 19: **end if**
- 20: **end for**

NB: Convergence is guaranteed under the Kurdyka-Łojasiewicz inequality and the Lipschitz continuous of ∇f .

Results



Acknowledgements

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References

- [1] Shreyas Chaudhari, Srinivasa Pranav, and José MF Moura. Gradient networks. *IEEE Transactions on Signal Processing*, 73:324 – 339, 2024.
- [2] Regev Cohen, Yochai Blau, Daniel Freedman, and Ehud Rivlin. It has potential: Gradient-driven denoisers for convergent solutions to inverse problems. *NeurIPS*, 34:18152–18164, 2021.
- [3] Tao Hong, Umberto Villa, and Jeffrey A Fessler. A convergent generalized Krylov subspace method for compressed sensing MRI reconstruction with gradient-driven denoisers. *arXiv preprint arXiv:2508.11219*, 2025.
- [4] Tao Hong, Xiaojian Xu, Jason Hu, and Jeffrey A Fessler. Provable preconditioned plug-and-play approach for compressed sensing MRI reconstruction. *IEEE Transactions on Computational Imaging*, 10:372 – 384, 2024.
- [5] Tao Hong, Zhaoyi Xu, Se Young Chun, Luis Hernandez-Garcia, and Jeffrey A Fessler. Convergent complex quasi-Newton proximal methods for gradient-driven denoisers in compressed sensing MRI reconstruction. *IEEE Transactions on Computational Imaging*, 11:1534 – 1547, October 2025.
- [6] Samuel Hurault, Arthur Leclaire, and Nicolas Papadakis. Gradient step denoiser for convergent plug-and-play. *ICLR*, 2022.