A Convergent Generalized Krylov Subspace Method for Compressed Sensing MRI Reconstruction with Gradient-Driven Denoisers

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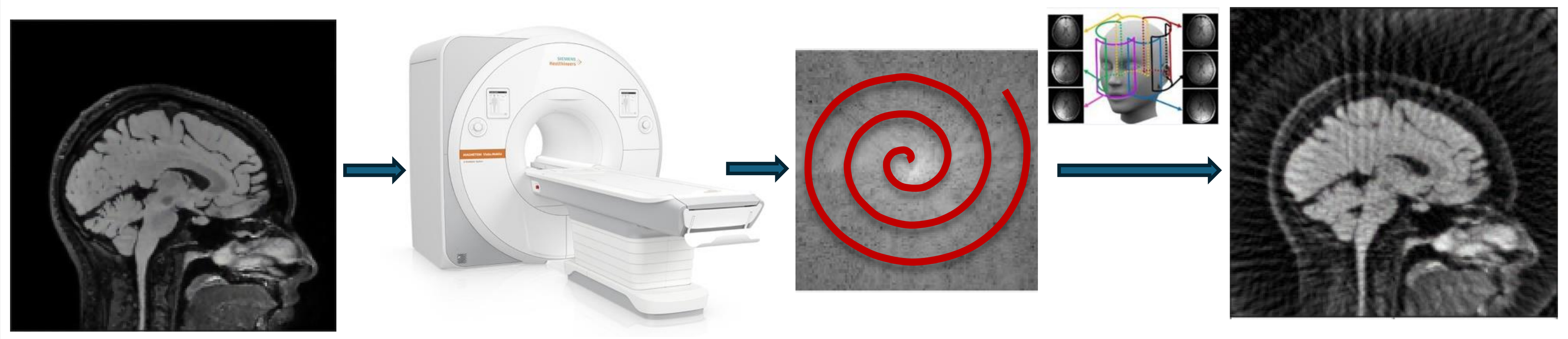
Background



Model-Based compressive sensing (CS) MRI reconstruction:

$$\mathbf{x}_* = \arg\min_{\mathbf{x}\in\mathbb{C}^N} h(\mathbf{x}) + f(\mathbf{x}),\ \ h(\mathbf{x}) \equiv \frac{1}{2}\|\mathbf{A}\mathbf{x}-\mathbf{y}\|_2^2,\ f(\cdot):\ \text{regularizer} \tag{1}$$

- $\mathbf{y}$: k-space data, forward model: $\mathbf{A}\in\mathbb{C}^{M\times N} = \{\mathbf{DFS}_c\}_c$.
- $\mathbf{D}$: downsampling mask; $\mathbf{F}$: (nonuniform) FFT; $\{\mathbf{S}_c\}_c$: sensitivity maps.

Proximal method for (1) at kth iteration: $\mathbf{x}_{k+1} = \mathrm{prox}_{\alpha f}(\mathbf{x}_k - \alpha\nabla h(\mathbf{x}_k))$

$$\mathrm{prox}_{\alpha f}(\cdot) \equiv \arg\min_{\boldsymbol{x}\in\mathbb{C}^N} \frac{1}{2}\|\mathbf{x}-\cdot\|_2^2 + \alpha f(\mathbf{x}) \Rightarrow \mathbf{D}_{\sigma>0}(\cdot) \equiv \mathrm{prox}_{\alpha f}(\cdot).$$

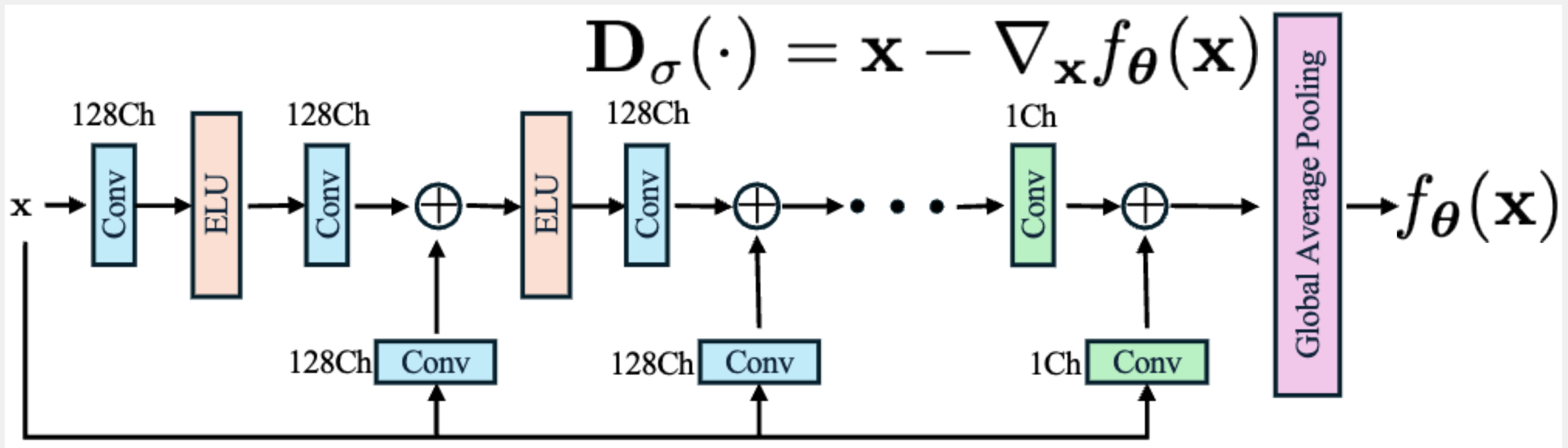
$\mathbf{D}_\sigma(\cdot)$: an abstract denoiser and $\sigma>0$: noise level $\leftrightarrow$ the strength of $f(\cdot)$.

PnP [4]: $\mathbf{x}_{k+1} = \mathbf{D}_\sigma(\mathbf{x}_k - \alpha\nabla h(\mathbf{x}_k))$.

Convergence PnP $\rightarrow$ Assumption: $\mathbf{D}_\sigma(\cdot)$ is nonexpansive ✘

Goal: (1) Bridge the gap between practice and theory—learned priors;
(2) Solve the minimization problem efficiently.
(3) Convergence is guaranteed.

Gradient-Driven Denoisers



$$\min_{\mathbf{x}\in\mathcal{C}} h(\mathbf{x}) + f_{\boldsymbol{\theta}}(\mathbf{x})$$

Observations: 1. h and $f_{\boldsymbol{\theta}}$ are differentiable; 2. $f_{\boldsymbol{\theta}}$ is nonconvex.
Solvers: Gradient based methods, e.g., (accelerated) proximal gradient
Complex quasi-Newton proximal method (CQNPM) [5]:

$$\mathbf{x}_{k+1} = \operatorname*{argmin}_{\mathbf{x}\in\mathcal{C}} \langle\nabla f(\mathbf{x}_k), \mathbf{x}-\mathbf{x}_k\rangle + \frac{1}{2\alpha_k}\|\mathbf{x}-\mathbf{x}_k\|_{\mathbf{B}_k}^2 + \underbrace{\frac{1}{2}\|\mathbf{A}\mathbf{x}-\mathbf{y}\|_2^2}_{h(\mathbf{x})},\ \mathbf{B}_k\succ 0 \tag{2}$$

Messages: 1. CQNPM > first-order methods. 2. Convergence is guaranteed: I) $f_{\boldsymbol{\theta}}$ satisfies Polyak-Łojasiewicz inequality; II) $\nabla_{\mathbf{x}} f_{\boldsymbol{\theta}}$ is Lipschitz continuous.
NB: computing $\mathbf{A}\mathbf{x}$ is quite expensive in multi-coil non-Cartesian sampling MRI.

Q: how to solve (2) efficiently?

Generalized Krylov Subspace Method (GKSM)

Given a subspace basis $\mathbf{V}_k\in\mathbb{C}^{N\times k}$ satisfying $\mathbf{V}_k^{\mathrm{H}}\mathbf{V}_k = \mathbf{I}_k$:

$$\boldsymbol{\beta}_k = \arg\min_{\mathbf{x}=\mathbf{V}_k\boldsymbol{\beta}} \underbrace{\frac{1}{2}\|\mathbf{A}\mathbf{x}-\mathbf{y}\|_2^2 + \bar{f}(\mathbf{x},\mathbf{x}_k,\mathbf{B}_k,\alpha_k)}_{\bar{F}(\mathbf{x},\mathbf{x}_k)}, \tag{3}$$

where $\bar{f}(\mathbf{x},\mathbf{x}_k,\mathbf{B}_k,\alpha_k) \equiv \langle\nabla f(\mathbf{x}_k),\mathbf{x}\rangle + \frac{1}{2\alpha_k}\|\mathbf{x}-\mathbf{x}_k\|_{\mathbf{B}_k}^2$.

$$\boldsymbol{\beta}_k = \arg\min_{\boldsymbol{\beta}\in\mathbb{C}^k} \left\| \begin{bmatrix}\mathbf{A}\mathbf{V}_k\boldsymbol{\beta}\\ \bar{\mathbf{B}}_k^{\frac{1}{2}}\mathbf{V}_k\boldsymbol{\beta}\end{bmatrix} - \begin{bmatrix}\mathbf{y}\\ \bar{\mathbf{B}}_k^{\frac{1}{2}}\mathbf{w}_k\end{bmatrix} \right\|_2^2, \tag{4}$$

where $\mathbf{w}_k = \mathbf{x}_k - \alpha_k\mathbf{B}_k^{-1}\nabla f(\mathbf{x}_k)$ and $\bar{\mathbf{B}}_k = \bar{\mathbf{B}}_k^{\frac{1}{2}}\bar{\mathbf{B}}_k^{\frac{1}{2}}$ with $\bar{\mathbf{B}}_k = \frac{1}{\alpha_k}\mathbf{B}_k$. Then

$$\boldsymbol{\beta}_k = (\mathbf{V}_k^{\mathrm{H}}\mathbf{A}^{\mathrm{H}}\mathbf{A}\mathbf{V}_k + \mathbf{V}_k^{\mathrm{H}}\bar{\mathbf{B}}_k\mathbf{V}_k)^{-1}\mathbf{V}_k^{\mathrm{H}}(\mathbf{A}^{\mathrm{H}}\mathbf{y} + \bar{\mathbf{B}}_k\mathbf{w}_k) \quad \mathcal{C}=\mathbb{C}^N \tag{5}$$

$$\boldsymbol{\beta}_k = \arg\min_{(\mathbf{V}_k\boldsymbol{\beta})\in\mathcal{C}} \left\| \begin{bmatrix}\mathbf{A}\mathbf{V}_k\boldsymbol{\beta}\\ \bar{\mathbf{B}}_k^{\frac{1}{2}}\mathbf{V}_k\boldsymbol{\beta}\end{bmatrix} - \begin{bmatrix}\mathbf{y}\\ \bar{\mathbf{B}}_k^{\frac{1}{2}}\mathbf{w}_k\end{bmatrix} \right\|_2^2 \quad \mathcal{C}\neq\mathbb{C}^N \tag{6}$$

$$\mathbf{x}_{k+1} = \arg\min_{\mathbf{z}\in\mathcal{C}} \left\| \begin{bmatrix}\mathbf{A}\mathbf{V}_k\mathbf{V}_k^{\mathrm{H}}\mathbf{z}\\ \bar{\mathbf{B}}_k^{\frac{1}{2}}\mathbf{V}_k\mathbf{V}_k^{\mathrm{H}}\mathbf{z}\end{bmatrix} - \begin{bmatrix}\mathbf{y}\\ \bar{\mathbf{B}}_k^{\frac{1}{2}}\mathbf{w}_k\end{bmatrix} \right\|_2^2 \tag{7}$$

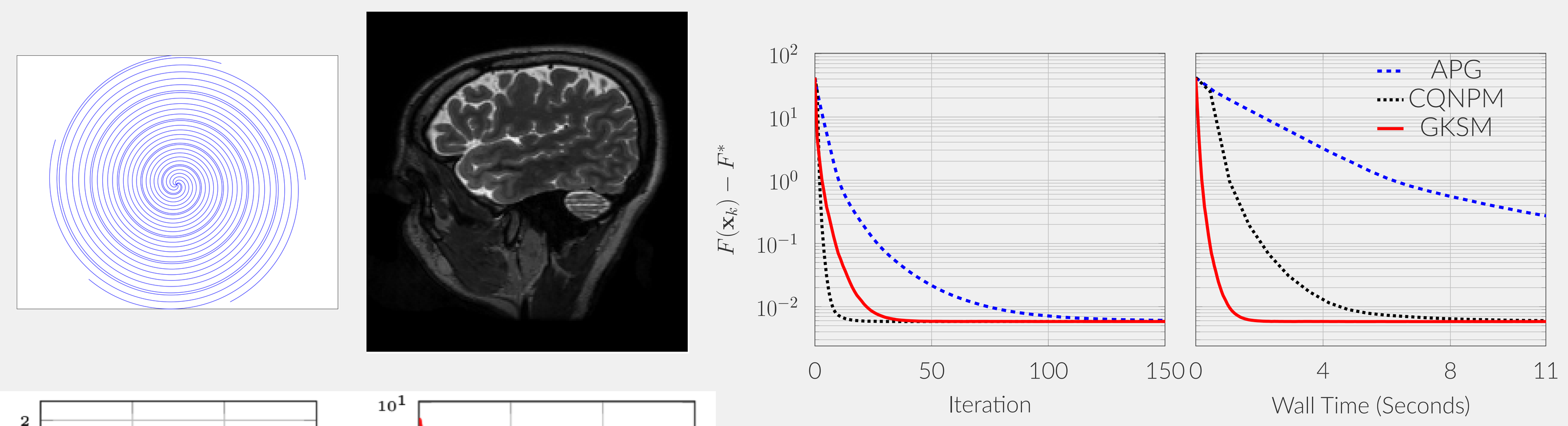
Algorithm 1 Generalized Krylov Subspace Method (GKSM)

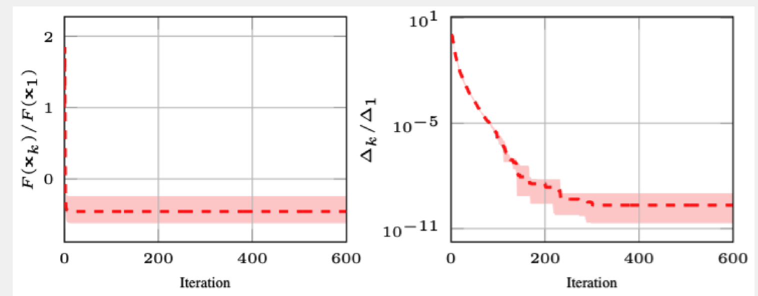
Initialization: $\mathbf{x}_1$, stepsize $\alpha_k>0$, $\mathbf{V}_1 = \frac{\mathbf{A}^{\mathrm{H}}\mathbf{y}}{\|\mathbf{A}^{\mathrm{H}}\mathbf{y}\|}$, $\mathbf{A}\mathbf{V}_1$, maximal number of subspace iterations K, and maximal number of total iterations `Max_Iter`
Iteration:

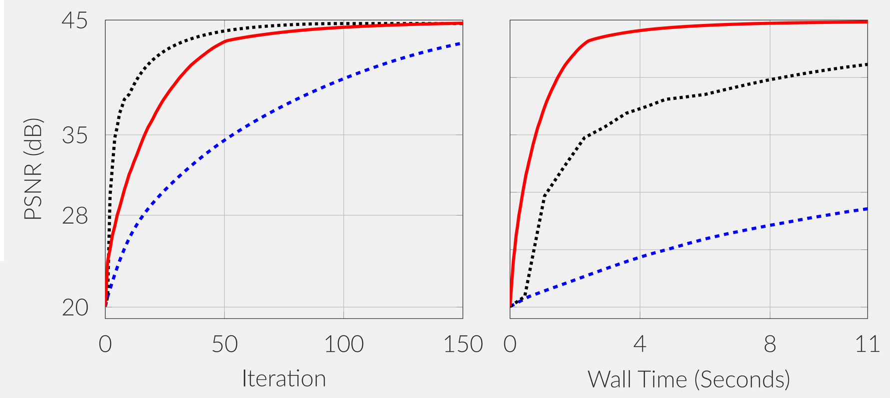
for $k = 1,2,\ldots,$ `Max_Iter` do
  Compute $\nabla f(\mathbf{x}_k)$
  Set $\mathbf{B}_k\succ 0$
  Compute $\boldsymbol{\beta}_k$ with (5) or (6)
  Compute $\mathbf{x}_{k+1}\leftarrow\mathbf{V}_k\boldsymbol{\beta}_k$
  if $k\le K$ then
    Compute $\mathbf{r}_k\leftarrow\nabla_{\mathbf{x}}\bar{F}(\mathbf{x}_{k+1},\mathbf{x}_k)$.
    $\tilde{\mathbf{r}}_k\leftarrow(\mathbf{I}-\mathbf{V}_k\mathbf{V}_k^{\mathrm{H}})\mathbf{r}_k$
    if $\|\tilde{\mathbf{r}}_k\|\neq 0$ then
      $\mathbf{v}_{k+1}\leftarrow\tilde{\mathbf{r}}_k/\|\tilde{\mathbf{r}}_k\|$
      $\mathbf{V}_{k+1}\leftarrow[\mathbf{V}_k\ \mathbf{v}_{k+1}]$
      $\mathbf{A}\mathbf{V}_{k+1}\leftarrow[\mathbf{A}\mathbf{V}_k\ \mathbf{A}\mathbf{v}_{k+1}]$
    else
      $\mathbf{V}_{k+1}\leftarrow\mathbf{V}_k$
      $\mathbf{A}\mathbf{V}_{k+1}\leftarrow\mathbf{A}\mathbf{V}_k$
    end if
  else
    $\mathbf{V}_{k+1}\leftarrow\mathbf{I}_N$
  end if
end for

NB: Convergence is guaranteed under the Kurdyka–Łojasiewicz inequality.

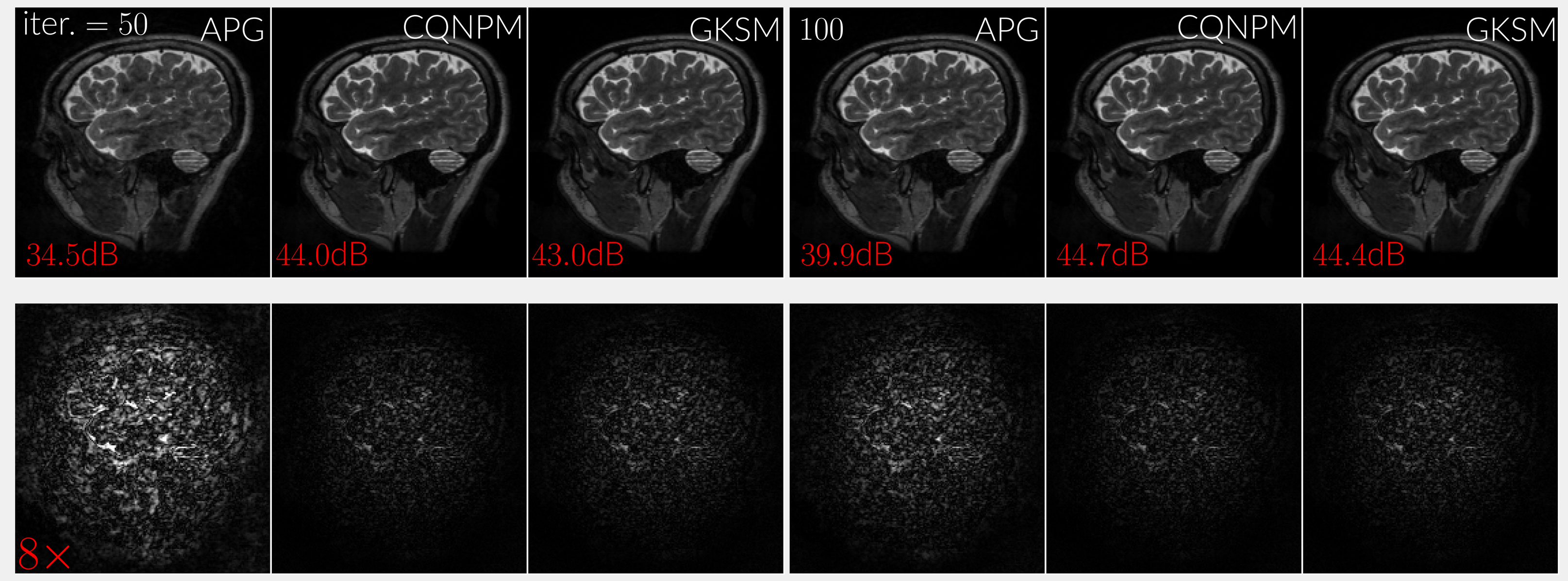
Results







$\Delta_k \triangleq \min_{k'\le k}\|\mathbf{x}_{k'+1}-\mathbf{x}_{k'}\|_2^2$



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