Laboratory 7: Wave dispersion

In this laboratory, we will explore wave propagation in lossless media with different dispersion relations.

The instantaneous electric field amplitude of a pulse of electromagnetic waves travelling along the z-axis at a carrier frequency ω_0 is represented by the following sum of ($2N + 1$) travelling wave components at different frequencies:

$$E(z,t) = \sum_{n'=-N}^{+N} E_0 \cos(\omega_{n'} t - \beta_{n'} z)$$

where $\omega_{n'} = \omega_0 + n'\delta\omega$ is the frequency of each forward travelling wave, with a corresponding phase constant $\beta_{n'} = \beta(\omega_{n'})$ that is a function of frequency. E_0 is a constant amplitude for each component and the bandwidth is $N\delta\omega$. Note that the Fourier-like sum above could be expressed with complex exponentials by applying Euler's theorem.

1. Pulse propagation in vacuum

Consider vacuum, where the dispersion relation is given by $\beta = \omega/c$. We will further take the following parameters: E_0 = 1 V/m, $\omega_0 = 2\pi \times 10^9$ rad/s, N = 20, $\delta\omega = \omega_0/60$. The simulation will take place over a time interval 0 ns < t < 33 ns in steps of 0.3 ns, with field displayed on the spatial interval -2 m < z < 10 m with 0.01 m resolution.

A MATLAB code implementing the above simulation is given below, producing an animation of the field $E(z,t)$ versus position z as time t evolves:

```
close all;
clear all;

eps0 = 8.854e-12;
mu0 = 4*pi*1e-7;
c = 1/sqrt(mu0*eps0);

t = (0:0.3:33)*1e-9;
z = -2:0.01:10;

E0 = 1;
omega0 = 2*pi*1e9;
domega = omega0/60;
for n = 1:41;
    omega(n) = omega0+(n-21)*domega;
end;

% vacuum dispersion
beta = omega/c;

for m=1:length(t);
    E=zeros(1,length(z));
    for n = 1:41;
```

```
        Etemp=E0*cos( omega(n)*t(m) - beta(n)*z );
        E=E+Etemp;
    end;
   plot(z, E);
   axis( [ -2 +10 -50 +50 ] );
   xlabel("z [m]");
   ylabel("E(z,t)[V/m]");
   title("instantaneous field, \beta = \omega / c");
   M(m)=getframe;
end;

movie2gif(M,"vacuum.gif")
```

Copy the above code into a script and execute. Notice how the phase velocity v_p and group velocity v_g are equal.

Observe the distance Δz that the pulse travels in the elapsed simulation time Δt = 33 ns. What is the group velocity $v_g = \Delta z/\Delta t$? Use the mid-point of the pulse to approximate Δz.
How does v_g compare with c, the speed of light?

№ 1: Show your results to the teaching assistant.

2. Pulse propagation in a medium with constant refractive index *n* = 1.5

Consider now the propagation of the pulse through a lossless medium with a constant refractive index n = 1.5. The modified dispersion relation is $\beta = \omega/c \times 1.5$.

Use the code of exercise 1 with a phase constant $\beta = \omega/c \times 1.5$, and calculate $E(z,t)$ to simulate pulse propagation.

slower

Observe the distance Δz that the pulse travels in the elapsed simulation time Δt = 33 ns. What is the group velocity $v_g = \Delta z/\Delta t$? Use the mid-point of the pulse to approximate Δz.
Is the phase velocity v_p equal to the group velocity v_g?
Why does the pulse appear compressed versus the distance z as compared to the pulse in vacuum?

compress because of a smaller wavelength

№ 2: Show your results to the teaching assistant.

3. Pulse propagation in a medium with a dispersive refractive index $n(\omega)$ = 1.5 (1 + ω / Ω)

Consider now the propagation of the pulse through a lossless medium with a frequency dependent refractive index $n(\omega) = 1.5\,(\,1\,+\,\omega/\Omega\,)$ where the frequency $\Omega = 2\pi \times 8 \times 10^9$ rad/s. The refractive index increases with frequency. The modified dispersion relation is $\beta = \omega/c \times 1.5\,(\,1\,+\,\omega/\Omega\,)$.

Use the code of exercise 1 with a phase constant $\beta = \omega/c \times 1.5\,(\,1\,+\,\omega/\Omega\,)$ and calculate $E(z,t)$ to simulate pulse propagation.

Observe the distance Δz that the pulse travels in the elapsed simulation time Δt = 33 ns. What is the group velocity $v_g = \Delta z/\Delta t$? Use the mid-point of the pulse to approximate Δz.

Is the phase velocity v_p equal to the group velocity v_g?
Observe the pulse very carefully, and you will see that the pulse broadens, and the apparent value of β (the spatial periodicity) is different at the leading edge and trailing edge of the pulse. How can this be explained in terms of $n(\omega)$?

№ 3: Show your results to the teaching assistant.

Optional: Is v_g larger or smaller than v_p?

4. Pulse propagation in a medium with a dispersive refractive index $n(\omega) = 1.5\ (\ 1 - \omega\ /\ \Omega\ )$

Consider now the propagation of the pulse through a lossless medium with a frequency dependent refractive index $n(\omega) = 1.5\ (\ 1\ -\ \omega/\Omega\ )$ where the frequency $\Omega = 2\pi \times 8 \times 10^9$ rad/s. The refractive index increases with frequency. The modified dispersion relation is $\beta = \omega/c\ \times\ 1.5\ (\ 1\ -\ \omega/\Omega\ )$.

Use the code of exercise 1 with a phase constant $\beta = \omega/c\ \times\ 1.5\ (\ 1\ -\ \omega/\Omega\ )$ and calculate $E(z,t)$ to simulate pulse propagation.

Observe the distance Δz that the pulse travels in the elapsed simulation time Δt = 33 ns. What is the group velocity $v_g = \Delta z/\Delta t$?
Is the phase velocity v_p equal to the group velocity v_g?
Observe the pulse very carefully, and you will see that the pulse broadens, and the apparent value of β (the spatial periodicity) is different at the leading edge and trailing edge of the pulse. How can this be explained in terms of $n(\omega)$?

№ 4: Show your results to the teaching assistant.

Optional: Is v_g larger or smaller than v_p?

A cautionary note: The causality of physical laws imposes strict bounds on physically realizable dispersion relations and the associated frequency dependent attenuation (which we have neglected in this exercise).