Tutorial 13: Power Series Solutions

1. Find the radius of convergence and interval of convergence for the given power series.

a. $\sum_{n=1}^{\infty} \frac{2^n}{n} x^n$

b. $\sum_{n=1}^{\infty} \frac{(-1)^n}{4^n}(x+3)^n$

c. $\sum_{n=0}^{\infty} \frac{(100)^n}{n!}(x+7)^n$

d. $\sum_{n=0}^{\infty} n!(2x+1)^n$

(a)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}x^{n+1}/(n+1)}{2^n x^n/n}\right| = \lim_{n\to\infty}\frac{2n}{n+1}|x| = 2|x|$$

The series is absolutely convergent for $2|x| < 1$ or $|x| < \frac{1}{2}$. At $x = -\frac{1}{2}$, the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges by the alternating series test. At $x = \frac{1}{2}$, the series $\sum_{n=1}^{\infty}\frac{1}{n}$ is the harmonic series which diverges. Thus, the given series converges on $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

(b)

$$\begin{aligned} L &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)(x+3)^{n+1}}{4^{n+1}}\frac{4^n}{(-1)^n(n)(x+3)^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{-(n+1)(x+3)}{4n}\right| \\ &= |x+3|\lim_{n\to\infty}\frac{n+1}{4n} \\ &= \frac{1}{4}|x+3| \end{aligned}$$

$\frac{1}{4}|x+3| < 1 \implies |x+3| < 4$ series converges

$\frac{1}{4}|x+3| > 1 \implies |x+3| > 4$ series diverges

The radius of convergence for this power series is $R = 4$.

$$\begin{aligned} -4 &< x+3 < 4 \\ -7 &< x < 1 \end{aligned}$$

(c)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{100^{n+1}(x+7)^{n+1}/(n+1)!}{100^n(x+7)^n/n!}\right| = \lim_{n\to\infty}\frac{100}{n+1}|x+7| = 0$$

The series is absolutely convergent on $(-\infty, \infty)$.

(d)

$$\begin{aligned} L &= \lim_{n\to\infty}\left|\frac{(n+1)!(2x+1)^{n+1}}{n!(2x+1)^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{(n+1)n!(2x+1)}{n!}\right| \\ &= |2x+1| \lim_{n\to\infty}(n+1) \end{aligned}$$

At this point we need to be careful. The limit is infinite, but there is that term with the x's in front of the limit. We will have $L = \infty > 1$ provided $x \neq -\frac{1}{2}$ will only converge if $x = -\frac{1}{2}$.

The radius of convergence is $R = 0$ and the interval of convergence is $x = -\frac{1}{2}$.

2. Rewrite the given power series by shifting the index, so that its general term involves x^k.

a. $\sum_{n=3}^{\infty}(2n-1)c_n x^{n-3}$

b. $\sum_{n=3}^{\infty}\frac{3^n}{(2n)!}x^{n-2}$

c. $\sum_{n=3}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$

(a)

$$\sum_{n=3}^{\infty}(2n-1)c_n x^{n-3} = \sum_{k=0}^{\infty}(2(k+3)-1)c_{k+3}x^k = \sum_{\mathbf{k=0}}^{\infty}\mathbf{(2k+5)c_{k+3}x^k}$$

(b)

$$\sum_{n=3}^{\infty}\frac{3^n}{(2n)!}x^{n-2} = \sum_{k=1}^{\infty}\frac{3^{k+2}}{(2(k+2))!}x^k = \sum_{\mathbf{k=1}}^{\infty}\mathbf{\frac{3^{k+2}}{(2k+1)!}x^k}$$

(c)

$$\sum_{n=3}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} = \sum_{k=7}^{\infty}\frac{(-1)^{\frac{k-1}{2}}}{\left(2\left(\frac{k-1}{2}\right)+1\right)!}x^k = \sum_{k=7}^{\infty}\frac{(-1)^{\frac{k-1}{2}}}{k!}x^k = \sum_{\mathbf{k=7}}^{\infty}\mathbf{\frac{\sqrt{(-1)^{k-1}}}{k!}x^k}$$

3. Rewrite the given expression as a single power series whose general term involves x^k.

a. $\displaystyle\sum_{n=2}^{\infty} n(n-1)C_n x^n + 2\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 3\sum_{n=1}^{\infty} nc_n x^n$

b. $\displaystyle 3x^2 \sum_{n=-2}^{\infty} n(n-1)x^{n-2} + x\sum_{n=1}^{\infty} nx^n$

c. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{(2n)!}x^{n-1} + 3x^3 \sum_{n=-1}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$

(a)

$$\sum_{n=2}^{\infty} n(n-1)c_n x^n + 2\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 3\sum_{n=1}^{\infty} nc_n x^n$$

$$= 2\cdot 2\cdot 1 c_2 x^0 + 2\cdot 3\cdot 2c_3 x^1 + 3\cdot 1\cdot c_1 x' + \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^n}_{k=n} + 2\underbrace{\sum_{n=4}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + 3\underbrace{\sum_{n=2}^{\infty} nc_n x^n}_{k=n}$$

$$= 4c_2 + (12c_3 + (12c_3 + 3c_1)x + \sum_{k=2}^{\infty} k(k-1)c_k x^k + 2\sum_{k=2}^{\infty} (k+2)(k+1)x_{k+2}x^k + 3\sum_{k=2}^{\infty} kc_k x^k$$

$$= 4c_2 + (3c_1 + 12c_3)x + \sum_{k=2}^{\infty} ([k(k-1) + 3k]c_k + 2(k+2)(k+1)x_{k+2})x^k$$

$$= 4c_2 + (3c_1 + 12c_3)x + \sum_{k=2}^{\infty} [k(k+2)c_k + 2(k+1)(k+2)c_{k+2}]x^k$$

(b)

$$3x^2 \sum_{n=-2}^{\infty} n(n-1)x^{n-2} + x\sum_{n=1}^{\infty} nx^n = 3\sum_{n=-2}^{\infty} n(n-1)x^n + \sum_{n=1}^{\infty} nx^{n+1} = 3\sum_{k=-2}^{\infty} k(k-1)x^k + \sum_{k=2}^{\infty} (k-1)x^k \quad (1)$$

Solving first part of Eq(1),

$$\sum_{k=-2}^{\infty} k(k-1)x^k = -2(-2-1)x^{-2} + (-1)(-1-1)x^{-1} + 0 + 1(1-1)x^1 + \sum_{k=2}^{\infty} k(k-1)x^k$$

$$= \frac{6}{x^2} + \frac{2}{x} + \sum_{k=2}^{\infty} k(k-1)x^k$$

Substitute back to Eq (1),

$$\therefore 3\left[\frac{6}{x^2}+\frac{2}{x}+\sum_{k=2}^{\infty}k(k-1)x^k\right]+\sum_{k=2}^{\infty}(k-1)x^k = \frac{18}{x^2}+\frac{6}{x}+\sum_{k=2}^{\infty}[3k(k-1)+(k-1)]x^k$$
$$= \frac{18}{x^2}+\frac{6}{x}+\sum_{k=2}^{\infty}[(k-1)(3k+1)]x^k$$

(c)

$$\sum_{n=1}^{\infty}\frac{3^n}{(2n)!}x^{n-1}+2x^3\sum_{n=-1}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} = \sum_{n=1}^{\infty}\frac{3^n}{(2n)!}x^{n-1}+2\sum_{n=-1}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+4}$$
$$= \sum_{k=0}^{\infty}\frac{3^{k+1}}{(2(k+1))!}x^k+2\sum_{k=2}^{\infty}\frac{(-1)^{\frac{k-4}{2}}}{\left(2\left(\frac{k-4}{2}\right)+1\right)!}x^k \qquad (2)$$
$$= \sum_{k=0}^{\infty}\frac{3^{k+1}}{(2k+2)!}x^k+2\sum_{k=2}^{\infty}\frac{(-1)^{\frac{k-4}{2}}}{(k-3)!}x^k$$

Solving first part of Eq (2),

$$\sum_{k=0}^{\infty}\frac{3^{k+1}}{(2k+2)!}x^k = \frac{3^1}{4!}x^0+\frac{3^2}{4!}x^1+\sum_{k=2}^{\infty}\frac{3^{k+1}}{(2k+2)!}$$
$$= \frac{3}{2}+\frac{3}{8}x+\sum_{k=2}^{\infty}\frac{3^{k+1}}{(2k+2)!}x^k$$

Subtitute back to Eq (2),

$$\therefore \frac{3}{2}+\frac{3}{8}x+\sum_{k=2}^{\infty}\frac{3^{k+1}}{(2k+2)!}x^k+2\sum_{k=2}^{\infty}\frac{(-1)^{\frac{k-4}{2}}}{(k-3)!}x^k = \frac{3}{2}+\frac{3}{8}x+\sum_{k=2}^{\infty}\left[\frac{3^{k+2}}{(2k+2)!}+\frac{2(-1)^{\frac{k-4}{2}}}{(k-3)!}\right]x^k$$
$$= \frac{3}{2}+\frac{3}{8}x+\sum_{k=2}^{\infty}\left[\frac{3^{k+2}}{(2k+2)!}+\frac{2\sqrt{(-1)^{k-4}}}{(k-3)!}\right]x^k$$

4. Find two power series solutions of given differential equation about the ordinary point $x = 0$.

a. $y'' + xy' + y = 0$

b. $(x-1)y'' + y' = 0$

c. $y'' + e^x y' - y = 0$

d. $\left(x^2+1\right)y'' + xy' - y = 0$

(a)

Let $y(x) = \sum_{n=0}^{\infty} c_n x^n \implies y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$ and $y''(x) = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n$.

The differential equation becomes

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n + x\sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty}[(n+2)(n+1)c_{n+2} + nc_n + c_n]x^n = 0$$

$$[\triangle \text{ Since } \sum_{n=1}^{\infty} n c_n x^n = \sum_{n=0}^{\infty} n c_n x^n]$$

Equating coefficients gives,

$$(n+2)(n+1)c_{n+2} + (n+1)c_n = 0$$

Thus, the recursion relation is,

$$c_{n+2} = \frac{-(n+1)c_n}{(n+2)(n+1)} = -\frac{c_n}{n+2}, \quad n = 0, 1, 2, \ldots$$

Then, the even coefficients are given by $c_2 = -\dfrac{c_0}{2}, \quad c_4 = -\dfrac{c_2}{4} = \dfrac{c_0}{2\cdot 4}, \quad c_6 = -\dfrac{c_4}{6} = -\dfrac{c_0}{2\cdot 4\cdot 6}$.

Hence,

$$c_{2n} = (-1)^n \frac{c_0}{2\cdot 4 \cdots 2n} = \frac{(-1)^n c_0}{2^n n!}$$

The odd coefficients are $c_3 = -\dfrac{c_1}{3}, \quad c_5 = -\dfrac{c_3}{5} = \dfrac{c_1}{3\cdot 5}, \quad c_7 = -\dfrac{c_5}{7} = -\dfrac{c_1}{3\cdot 5\cdot 7}$. Hence,

$$c_{2n+1} = (-1)^n \frac{c_1}{3\cdot 5\cdot 7\cdots(2n+1)} = \frac{(-2)^n n! c_1}{(2n+1)!}$$

The solution is,

$$\mathbf{y(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^{2n} + c_1 \sum_{n=0}^{\infty} \frac{(-2)^n n!}{(2n+1)!} x^{2n+1}}$$

(b)

Subtituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation,

$$\begin{aligned}
(x-1)y'' + y' &= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1}}_{k=n-1} - \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + \underbrace{\sum_{n=1}^{\infty} nc_n x^{n-1}}_{k=n-1} \\
&= \sum_{k=1}^{\infty}(k+1)kc_{k+1}x^k - \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=0}^{\infty}(k+1)c_{k+1}x^k \\
&= -2c_2 + c_1 + \sum_{k=1}^{\infty}[(k+1)kc_{k+1} - (k+2)(k+1)c_{k+2} + (k+1)c_{k+1}]x^k = 0
\end{aligned}$$

Thus,

$$\begin{aligned}
-2c_2 + c_1 &= 0 \\
(k+1)^2 c_{k+1} - (k+2)(k+1)c_{k+2} &= 0
\end{aligned}$$

and

$$\begin{aligned}
c_2 &= \frac{1}{2}c_1 \\
c_{k+2} &= \frac{k+1}{k+2}c_{k+1}, \quad k = 1, 2, 3, \ldots
\end{aligned}$$

Choosing $c_0 = 1$ and $c_1 = 0$, we find $c_2 = c_3 = c_4 = \ldots = 0$. For $c_0 = 0$ and $c_1 = 1$, we obtain

$$c_2 = \frac{1}{2}, \quad c_3 = \frac{1}{3}, \quad c_4 = \frac{1}{4}$$

Thus, the two solutions are

$$\mathbf{y_1 = 1 \qquad y_2 = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \ldots}$$

(c)

Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have,

$$\begin{aligned}
y'' - (x+1)y' - y &= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - \underbrace{\sum_{n=1}^{\infty} nc_n x^n}_{k=n} - \underbrace{\sum_{n=1}^{\infty} nc_n x^{n-1}}_{k=n-1} - \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n} \\
&= \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty} kc_k x^k - \sum_{k=0}^{\infty} (k+1)c_{k+1}x^k - \sum_{k=0}^{\infty} c_k x^k \\
&= 2c_2 - c_1 - c_0 + \sum_{k=1}^{\infty} [(k+2)(k+1)c_{k+2} - (k+1)c_{k+1} - (k+1)c_k]x^k = 0
\end{aligned}$$

Thus,

$$\begin{aligned}
2c_2 - c_1 - c_0 &= 0 \\
(k+2)(k+1)c_{k+2} - (k-1)(c_{k+1} + c_k) &= 0
\end{aligned}$$

and,

$$\begin{aligned}
c_2 &= \frac{c_1 + c_0}{2} \\
c_{k+2} &= \frac{c_{k+1} + c_k}{k+2}c_k, \quad k = 2, 3, 4, \ldots
\end{aligned}$$

Choosing $c_0 = 1$ and $c_1 = 0$, we find,

$$c_2 = \frac{1}{2}, \qquad c_3 = \frac{1}{6}, \qquad c_4 = \frac{1}{6}$$

and so on. For $c_0 = 0$ and $c_1 = 1$, we obtain,

$$c_2 = \frac{1}{2}, \qquad c_3 = \frac{1}{2}, \qquad c_4 = \frac{1}{4}$$

and so on. Thus, the two solutions are,

$$y_1 = 1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{6}x^4 + \ldots \qquad \text{and} \qquad y_2 = x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{4}x^4 + \ldots$$

(d)

Let $y(x) = \sum_{n=0}^{\infty} c_n x^n \implies y''(x) = \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2}$ and $xy'' = \sum_{n=0}^{\infty} nc_n x^n$.Thus,

$$\left(x^2+1\right)y'' = \sum_{n=0}^{\infty} n(n-1c_n x^n + \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$$

The differential equation becomes,

$$\sum_{n=0}^{\infty}[(n+2)(n+1)c_{n+2} + [n(n-1)+n-1]c_n]x^n = 0$$

The recursion relation is

$$c_{n+2} = -\frac{(n-1)c_n}{n+2}, \quad n = 0, 1, 2, \ldots$$

Given c_0 and c_1, $c_2 = \frac{c_0}{2}$, $c_4 = -\frac{c_0}{2^2 \cdot 2!}$, $c_6 = -\frac{3c_4}{6} = (-1)^2\frac{3c_0}{2^3 \cdot 3!}, \ldots$

$$c_{2n} = (-1)^{n-1}\frac{1\cdot 3\cdots(2n-3)c_0}{2^n n!} = (-1)^{n-1}\frac{(2n-3)!c_0}{2^n 2^{n-2} n!(n-2)!} = (-1)^{n-1}\frac{(2n-3)!c_0}{2^{2n-2}n!(n-2)!} \text{ for } n = 2, 3, \ldots$$

$c_3 = \frac{0\cdot c_1}{3} = 0 \implies c_{2n+1} = 0$ for $n = 1, 2, \ldots$ Thus the solution is,

$$\mathbf{y(x) = c_0 + c_1 x + c_0\frac{x^2}{2} + c_0\sum_{n=2}^{\infty}\frac{(-1)^{n-1}(2n-3)!}{2^{2n-2}n!(n-2)!}x^{2n}}$$

5. Use the power series method to solve the given initial-value problem.

a. $y'' - xy' - y = 0, \quad y(0) = 1, \quad y'(0) = 0$

b. $y'' + x^2y' + xy = 0, \quad y(0) = 0, \quad y'(0) = 1$

c. $(x+1)y'' - (2-x)y' + y = 0, \quad y(0) = 2, \quad y'(0) = -1$

(a)

Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $-xy'(x) = -x\sum_{n=1}^{\infty} nc_n x^{n-1} = -\sum_{n=1}^{\infty} nc_n x^n = -\sum_{n=0}^{\infty} nc_n x^n$.

$$y''(x) = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n$$

The equation $y'' - xy' - y = 0$ becomes

$$\sum_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} - nc_n - c_n]x^n = 0$$

Thus, the recursion relation is

$$c_{n+2} = \frac{nc_n + c_n}{(n+2)(n+1)} = \frac{c_n(n+1)}{(n+2)(n+1)} = \frac{c_n}{n+2} \text{ for } n = 0, 1, 2, ...$$

One of the condition is, $y(0) = 1$. But $y(0) = \sum_{n=0}^{\infty} c_n(0)^n = c_0 + 0 + 0 + ... = c_0$, so $c_0 = 1$.

Hence, $c_2 = \frac{c_0}{2} = \frac{1}{2}$, $c_4 = \frac{c_2}{4} = \frac{1}{2\cdot 4}$, $c_6 = \frac{c_4}{6} = \frac{1}{2\cdot 4\cdot 6}$,....,$c_{2n} = \frac{1}{2^n n!}$. The other given condition is $y'(0) = 0$.

But $y'(0) = \sum_{n=1}^{\infty} nc_n(0)^{n-1} = c_1 + 0 + 0 + ... = c_1$. So, $c_1 = 0$.

By the recursion relation, $c_3 = \frac{c_1}{3} = 0$, $c_5 = 0$,...,$c_{2n+1} = 0$ for $n = 0, 1, 2, ...$.

Thus, the solution to the initial-value problem is,

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{2n}x^{2n} = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} = \sum_{n=0}^{\infty} \frac{(x^2/2)^n}{n!} = c^{x^2/2}$$

(b)

Assuming that $y(x) = \sum_{n=0}^{\infty} c_n x^n$, we have $xy = x\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+1}$,

$$x^2 y' = x^2 \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} n c_n x^{n+1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} = \sum_{n=-1}^{\infty} (n+3)(n+2) c_{n+3} x^{n+1}$$

$$= 2c_2 + \sum_{n=0}^{\infty} (n+3)(n+2) c_{n+3} x^{n+1}$$

Thus, the equation $y'' + x^2 y' + xy = 0$ becomes

$$2c_2 + \sum_{n=0}^{\infty} [(n+3)(n+2)c_{n+3} + nc_n + c_n] x^{n+1} = 0$$

So, $c_2 = 0$ and the recursion relation is,

$$c_{n+3} = \frac{-nc_n - c_n}{(n+3)(n+2)} = -\frac{(n+1)c_n}{(n+3)(n+2)},\ n = 0, 1, 2, \ldots$$

Also, $c_1 = y'(0) = 1$, so,

$$c_4 = -\frac{2c_1}{4\cdot 3} = -\frac{2}{4\cdot 3}, \quad c_7 = -\frac{5c_4}{7\cdot 6} = (-1)^2 \frac{2\cdot 5}{7\cdot 6\cdot 4\cdot 3} = (-1)^2 \frac{2^2 5^2}{7!}, \ldots$$

$$c_{3n+1} = (-1)^n \frac{2^2 5^2 \cdots (3n-1)^2}{(3n+1)!}$$

Thus, the solution is,

$$\mathrm{y(x)} = \sum_{\mathrm{n=0}}^{\infty} \mathrm{c_m x^n} = \mathrm{x} + \sum_{\mathrm{n=1}}^{\infty} \left[(-1)^{\mathrm{n}} \frac{2^2 5^2 \cdots (3\mathrm{n}-1)^2 \mathrm{x}^{3\mathrm{n}+1}}{(3\mathrm{n}+1)!} \right]$$

(c)

Subtituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation, we have,

$$\begin{aligned}
&(x+1)y'' - (2-x)y' + y \\
=& \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1}}_{k=n-1} + \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - 2\underbrace{\sum_{n=2}^{\infty} nc_n x^{n-1}}_{k=n-1} + \underbrace{\sum_{n=1}^{\infty} nc_n x^{n}}_{k=n} + \underbrace{\sum_{n=0}^{\infty} c_n x^{n}}_{k=n} \\
&= \sum_{k=1}^{\infty}(k+1)kc_{k+1}x^k + \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - 2\sum_{k=0}^{\infty}(k+1)c_{k+1}x^k + \sum_{k=1}^{\infty} kc_k x^k + \sum_{k=0}^{\infty} c_k x^k \\
&= 2c_2 - 2c_1 + c_0 + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} - (k+1)c_{k+1} + (k+1)c_k]x^k = 0
\end{aligned}$$

Thus,

$$\begin{aligned}
2c_2 - 2c_1 + c_0 &= 0 \\
(k+2)(k+1)c_{k+2} - (k+1)c_{k+1} + (k+1)c_k &= 0
\end{aligned}$$

and,

$$\begin{aligned}
c_2 &= c_1 - \frac{1}{2}c_0 \\
c_{k+2} &= \frac{1}{k+2}c_{k+1} - \frac{1}{k+2}c_k, \quad k = 1, 2, 3, \ldots
\end{aligned}$$

Choosing $c_0 = 1$ and $c_1 = 0$, we find

$$c_2 = -\frac{1}{2}, \qquad c_3 = -\frac{1}{6}, \qquad c_4 = \frac{1}{12}$$

and so on. For $c_0 = 0$ and $c_1 = 1$, we obtain,

$$c_2 = 1, \qquad c_3 = 0, \qquad c_4 = -\frac{1}{4}$$

and do on. Thus,

$$y = C_1\left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \ldots\right) + C_2\left(x + x^2 - \frac{1}{4}x^4 + \ldots\right)$$

and

$$y' = C_1\left(-x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots\right) + C_2\left(1 + 2x - x^3 + \ldots\right)$$

The initital condition imply $C_1 = 2$ and $C_2 = -1$, so

$$\begin{aligned}
y &= 2\left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \ldots\right) - \left(x + x^2 - \frac{1}{4}x^4 + \ldots\right) \\
&= 2 - x - 2x^2 - \frac{1}{3}x^3 + \frac{5}{12}x^4 + \ldots
\end{aligned}$$

6. Determine the singular points of the given differential equation. Classify each singular point as regular or irregular.

a. $x^3y'' - 4x^2y' + 3y = 0$

b. $\left(x^2-9\right)^2 y'' + (x+3)y' + 2y = 0$

c. $\left(2x^2 - 5x - 3\right)y'' + (2x+1)y' + \dfrac{6}{(x-3)}y = 0$

d. $\left(x^3 - 2x^2 - 3x\right)^2 y'' + x(x-3)^2 y' - (x+1)y = 0$

(a)

Dividing the equation with x^3

$$y'' + \frac{4x^2}{x^3}y' + \frac{3}{x^3}y = 0$$

$$P(x) = \frac{4}{x} \qquad Q(x) = \frac{3}{x^3}$$

- The factor x appears at most to the first power in the denominator of $P(x)$, but more than the second power in the denominator of $Q(x)$
- $x = 0$ is irregular singular point

(b)

Dividing the eqution with $\left(x^2-9\right)^2 = ((x-3)(x+3))^2$,

$$y'' + \frac{(x+3)}{((x-3)(x+3))^2}y' + \frac{2}{((x-3)(x+3))^2}y = 0$$

$$P(x) = \frac{1}{(x-3)^2(x+3)} \qquad Q(x) = \frac{2}{(x-3)^2(x+3)^2}$$

- The factor $x + 3$ appears at most to the first power in the denominator of $P(x)$ and at most to the second power in the denominator of $Q(x)$
- $x = -3$ is regular singular point
- The factor $x - 3$ appears more than the first power in the denominator of $P(x)$; but at most to the second power in the denominator of $Q(x)$
- $x = 3$ is irregular singular point

(c)

Factor the equation $\left(2x^2 - 5x - 3\right) = (x-3)(2x+1)$,

$$y'' + \frac{(2x+1)}{(x-3)(2x+1)}y' + \frac{6}{(x-3)^2(2x+1)}y = 0$$

$$P(x) = \frac{1}{(x-3)} \qquad Q(x) = \frac{6}{(x-3)^2(2x+1)}$$

- The factor $(x-3)$ appears at most to the first power in $P(x)$ and second power in the denominator of $Q(x)$
- $x = 3$ is regular singular point
- The factor $(2x+1)$ appears at to the first power in the denominator of $Q(x)$
- $x = -\frac{1}{2}$ is regular singular point

(d)

Divide the equation with $\left(x^3 - 2x^2 - 3x\right)^2 = \left(x\left(x^2 - 2x - 3\right)\right)^2 = x^2(x-3)^2(x+1)^2$,

$$y'' + \frac{x(x-3)^2}{x^2(x-3)^2(x+1)^2}y' - \frac{(x+1)}{x^2(x-3)^2(x+1)^2}y = 0$$

$$P(x) = \frac{1}{x(x+1)^2} \qquad Q(x) = \frac{1}{x^2(x-3)^2(x+1)}$$

- The factor x appears at most to the first power in the denominator of $P(x)$ and at most to the second power in the denominator of $Q(x)$
- $x = 0$ is regular singular point
- The factor $x - 3$ appears to the second power in the denominator of $Q(x)$
- $x = 3$ is regular singular point
- The factor $x + 1$ appears to the second power in the denominator of $P(x)$ and to the first power in the denominator of $Q(x)$
- $x = -1$ is irregular singular point

7. Find the indical roots for the given differential equations where $x = 0$ is a regular singular point.

a. $2xy'' - y' + 2y = 0$

b. $3xy'' + (2 - x)y' - y = 0$

c. $9x^2y'' + 9x^2y' + 2y = 0$

d. $x^2y'' + xy' + \left(x^2 - \frac{4}{9}\right)y = 0$

e. $xy'' + (1 - x)y' - y = 0$

(a)

Divide the equation with $2x$,

$$y'' - \frac{1}{2x}y' + \frac{2}{2x}y = y'' - \frac{\frac{1}{2}}{x}y' + \frac{1}{x}y = y'' - \frac{\frac{1}{2}}{x}y' + \frac{x}{x^2}y = 0$$

Hence, $b_0 = -\frac{1}{2}$ and $r_2 = 0$,

$$7r(r-1) + b_0r + c_0 = r^2 - r - \frac{1}{2}r + 0 = r^2 - \frac{3}{2}r = r\left(r - \frac{3}{2}\right) = 0$$

The indical roots are $r_1 = \frac{3}{2}$ and $r_2 = 0$.

(b)

Divide the equation with $3x$,

$$y'' + \frac{(2-x)}{3x}y' - \frac{1}{3x}y = y'' + \frac{\frac{1}{3}(2-x)}{x}y' - \frac{\frac{1}{3}}{x}y = y'' + \frac{\frac{2}{3} - \frac{1}{3}x}{x}y' - \frac{\frac{1}{3}x}{x^2}y = 0$$

Hence, $b_0 = \frac{2}{3}$ and $c_0 = 0$,

$$r(r-1) + b_0r + c_0 = r^2 - r + \frac{2}{3}r + 0 = r^2 - \frac{1}{3}r = r\left(r - \frac{1}{3}\right) = 0$$

The indical roots are $r_1 = \frac{1}{3}$ and $r_2 = 0$.

(c)

Divide the equation with $9x^2$,

$$y'' + \frac{9x^2}{9x^2}y' + \frac{2}{9x^2}y = y'' + y' + \frac{\frac{2}{9}}{x^2}y = y'' + \frac{x}{x}y' + \frac{\frac{2}{9}}{x^2}y = 0$$

Hence, $b_0 = 0$ and $c_0 = \frac{2}{9}$,

$$r(r-1) + b_0 r + c_0 = r^2 - r + \frac{2}{9} = \left(r - \frac{1}{3}\right)\left(r - \frac{2}{3}\right) = 0$$

The indical roots are $r_1 = \frac{2}{3}$ and $r_2 = \frac{1}{3}$.

(d)

Divide the equation with x^2,

$$y'' + \frac{x}{x^2}y' + \frac{\left(x^2 - \frac{4}{9}\right)}{x^2}y = y'' + \frac{1}{x}y' + \frac{-\frac{4}{9} + x^2}{x^2}y = y'' + \frac{x}{x}y' + \frac{\frac{2}{9}}{x^2}y = 0$$

Hence, $b_0 = 1$ and $c_0 = -\frac{4}{9}$,

$$r(r-1) + b_0 r + c_0 = r^2 - r + r - \frac{4}{9} = r^2 - \frac{4}{9} = \left(r + \frac{2}{3}\right)\left(r - \frac{2}{3}\right) = 0$$

The indical roots are $r_1 = \frac{2}{3}$ and $r_2 = -\frac{2}{3}$.

(e)

Divide the equation with x,

$$y'' + \frac{(1-x)}{x}y' - \frac{1}{x}y = y'' + \frac{(1-x)}{x}y' + \frac{x}{x^2}y = 0$$

Hence, $b_0 = 1$ and $c_0 = 0$,

$$r(r-1) + b_0 r + c_0 = r^2 - r + r = r^2 = 0$$

The indical roots are $r_1 = 0$ and $r_2 = 0$.