Tutorial 8: Integration

1. Find $\int \ln(x^2+2)\,dx$.

Let $u = \ln(x^2+2)$, $dv = dx$

$u = \ln(x^2+2)$	$dv = dx$
$du = \dfrac{2x}{x^2+2}dx$	$v = x$

So,

$$\begin{aligned}\int \ln(x^2+2)\,dx &= x\ln(x^2+2) - 2\int \frac{x^2}{x^2+2}dx \\ &= x\ln(x^2+2) - 2\int \left(1 - \frac{2}{x^2+2}\right)dx \\ &= x\ln(x^2+2) - 2x + \frac{4}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C \\ &= \mathbf{x(\ln(x^2+2) - 2) + 2\sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C}\end{aligned}$$

2. Find $\int x^2 \ln x\,dx$.

Let $u = \ln x$, $dv = x^2\,dx$

$u = \ln x$	$dv = x^2dx$
$du = \dfrac{dx}{x}$	$v = \dfrac{x^3}{3}$

So,

$$\begin{aligned}\int x^2 \ln x\,dx &= \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\frac{dx}{x} \\ &= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2dx \\ &= \mathbf{\frac{x^3}{3}\ln x - \frac{1}{9}x^3 + C}\end{aligned}$$

3. Find $\int x^3 e^{x^2} dx$.

Let $u = x^2$ and $dv = xe^{x^2} dx$.

$$u = x^2$$
$$du = 2x\,dx$$

$$dv = xe^{x^2}$$

Let $w = x^2 \implies \frac{dw}{dx} = 2x \equiv dx = \frac{dw}{2x}$

$$\begin{aligned} v &= \int xe^{x^2} dx \\ &= \int xe^{w} \frac{dw}{2x} \\ &= \frac{1}{2}\int e^{w} dw \\ &= \frac{1}{2}e^{w} \\ &= \frac{1}{2}e^{x^2} \end{aligned}$$

So,

$$\begin{aligned} \int x^3 e^{x^2}\, dx &= \frac{1}{2}x^2 e^{x^2} - \int xe^{x^2} dx \\ &= \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + C \\ &= \mathbf{\frac{1}{2}e^{x^2}\left(x^2 - 1\right) + C} \end{aligned}$$

4. Find $\int \frac{(x+1)}{x^3+x^2-6x} dx$.

Factoring the denomenator $x^3+x^2-6x = x(x^2+x-6) = x(x-2)(x+3)$.

The integrand is now $\frac{(x+1)}{x(x-2)(x+3)}$.

Representing the integrand such that:

$$\frac{(x+1)}{x(x-2)(x+3)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+3} \tag{1}$$

Multiplying equation (1) with $x(x-2)(x+3)$,

$$x+1 = A(x-2)(x+3) + Bx(x+3) + Cx(x-2) \tag{2}$$

Let $x = 0$ in (2), $1 = A(-2)(3) + B(0)(3) + C(0)(-2) \Rightarrow 1 = -6A$. So, $A = -\frac{1}{6}$.

Let $x = 2$ in (2), $3 = A(0)(5) + B(2)(5) + C(2)(0) \Rightarrow 3 = 10B$. So, $B = \frac{3}{10}$.

Let $x = -3$ in (2), $-2 = A(-5)(0) + B(-3)(0) + C(-3)(-5) \Rightarrow -2 = 15C$. So, $C = -\frac{2}{15}$.

Therefore,

$$\int \frac{(x+1)}{x^3+x^2-6x} dx = \int \left(-\frac{1}{6}\frac{1}{x} + \frac{3}{10}\frac{1}{x-2} - \frac{2}{15}\frac{1}{x+3}\right) dx$$

$$= -\frac{1}{6}\ln|x| + \frac{3}{10}\ln|x-2| - \frac{2}{15}\ln|x+3| + C$$

5. Find $\int \frac{x^3+x^2+x+2}{x^4+3x^2+2}dx$.

Factoring the denomenator $x^4+3x^2+2 = \left(x^2+1\right)\left(x^2+2\right)$.

The integrand is now $\frac{x^3+x^2+x+2}{\left(x^2+1\right)\left(x^2+2\right)}$.

Representing the integrand such that:

$$\frac{x^3+x^2+x+2}{\left(x^2+1\right)\left(x^2+2\right)} = \frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+2} \tag{1}$$

Multiplying equation (1) with $\left(x^2+1\right)\left(x^2+2\right)$,

$$\begin{aligned} x^3+x^2+x+2 &= (Ax+B)\left(x^2+2\right)+(Cx+D)\left(x^2+1\right) \\ &= Ax^3+2Ax+Bx^2+2B+Cx^3+Cx+Dx^2+D \\ &= (A+C)x^3+(B+D)x^2+(2A+C)x+(2B+D) \end{aligned} \tag{2}$$

Comparing LHS and RHS of equation (2),

$$\begin{aligned} A+C &= 1 \\ B+D &= 1 \\ 2A+C &= 1 \\ 2B+D &= 2 \end{aligned} \tag{3}$$

Solving equation (3) simultaneously to obtain $A = 0,\ B = 1,\ C = 1,\ D = 0$

Therefore,

$$\begin{aligned} \int \frac{x^3+x^2+x+2}{\left(x^2+1\right)\left(x^2+2\right)}dx &= \int \left(\frac{1}{x^2+1}+\frac{x}{x^2+2}\right)dx \\ &= \tan^{-1}x+\frac{1}{2}\ln\left(x^2+2\right)+C \end{aligned}$$

6. Find $\int \tan^3(3x)\sec^4(3x)\,dx$.

$$\begin{aligned}\int \tan^3(3x)\sec^4(3x)\,dx &= \int \tan^3(3x)\,\left(1+\tan^2(3x)\right)\sec^2(3x)\,dx\\ &= \int \tan^3(3x)\sec^2(3x)\,dx + \int \tan^5(3x)\sec^2(3x)\,dx\\ &= \frac{1}{3}\frac{1}{4}\tan^4(3x) + \frac{1}{3}\frac{1}{6}\tan^6(3x) + C\\ &= \mathbf{\frac{1}{12}\tan^4(3x) + \frac{1}{18}\tan^6(3x) + C}\end{aligned}$$

7. Find $\int \sin^4(x)\cos^7(x)\,dx$.

Using trigonometry identity, $\sin^2 x + \cos^2 x = 1 \implies \cos^2 x = 1 - \sin^2 x$

$$\begin{aligned}\int \sin^4(x)\cos^7(x)\,dx &= \int \sin^4(x)\cos^6(x)\cos(x)\,dx\\ &= \int \sin^4(x)\left(1-sin^2(x)\right)^3\cos(x)\,dx\end{aligned}$$

Let $u = \sin x$, $du = \cos(x)\,dx$,

$$\begin{aligned}\int \sin^4(x)\cos^7(x)\,dx &= \int u^4\left(1-u^2\right)^3\,du\\ &= \int u^4\left(1-u^2\right)\left(1-u^2\right)^2 du\\ &= \int u^4\left(1-u^2\right)\left(1-2u^2+u^4\right)du\\ &= \int u^4\left(1-2u^2+u^4-u^2+2u^4-u^6\right)du\\ &= \int u^4\left(1-3u^2+3u^4-u^6\right)du\\ &= \int u^4-3u^6+3u^8-u^{10}du\\ &= \frac{1}{5}u^5-\frac{3}{7}u^7+\frac{3}{9}u^9-\frac{1}{11}u^{11}+C\end{aligned}$$

Substituting back $u = \sin x$,

$$\int \sin^4(x)\cos^7(x)\,dx = \mathbf{\frac{1}{5}\sin^5 x-\frac{3}{7}\sin^7 x+\frac{1}{3}\sin^9 x-\frac{1}{11}\sin^{11} x+C}$$

8. Find $\int \frac{dx}{x^2\sqrt{9-x^2}}$.

Let $x = 3\sin\theta$, therefore $\theta = \sin^{-1}\frac{x}{3}$. Then, $dx = 3\cos\theta\, d\theta$, and

$$\begin{aligned}
\sqrt{9-x^2} &= \sqrt{9-(3\sin\theta)^2} \\
&= \sqrt{9-9\sin^2\theta} \\
&= 3\sqrt{1-\sin^2\theta} \\
&= 3\sqrt{\cos^2\theta} \\
&= 3\cos\theta
\end{aligned}$$

Using defination of inverse sine, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Therefore, $\cos\theta > 0$.

Thus, $\cos\theta = |\cos\theta| = \frac{\sqrt{9-x^2}}{3}$

Hence,

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{9-x^2}} &= \int \frac{3\cos\theta\, d\theta}{9\sin^2\theta(3\cos\theta)} \\
&= \frac{1}{9}\int \csc^2\theta\, d\theta \\
&= -\frac{1}{9}\cot\theta + C \\
&= -\frac{1}{9}\frac{\cos\theta}{\sin\theta} + C \\
&= -\frac{1}{9}\frac{\frac{\sqrt{9-x^2}}{3}}{\frac{x}{3}} + C \\
&= \mathbf{-\frac{1}{9}\frac{\sqrt{9-x^2}}{x} + C}
\end{aligned}$$