Tutorial 1: Basic Functions & Derivatives

1. **Find the limit of** $\lim\limits_{x \to 5} \dfrac{x^2 - 25}{x^2 + x - 30}$.

$$\lim_{x \to 5} \frac{\cancel{(x-5)}(x+5)}{\cancel{(x-5)}(x+6)}$$

$$= \lim_{x \to 5} \frac{x+5}{x+6}$$

$$= \frac{5+5}{5+6} = \frac{10}{11} \ \#$$

2. **Find the limit of** $\lim_{x \to 9} \frac{\sqrt{x}-3}{x-9}$.

$(a-b)(a+b) = a^2 - b^2$

$$\lim_{x \to 9} \frac{\sqrt{x}-3}{x-9} \times \frac{\sqrt{x}+3}{\sqrt{x}+3}$$

$$= \lim_{x \to 9} \frac{(\sqrt{x})^2 - 3^2}{(x-9)(\sqrt{x}+3)}$$

$$= \lim_{x \to 9} \frac{\cancel{x-9}}{\cancel{(x-9)}(\sqrt{x}+3)}$$

$$= \lim_{x \to 9} \frac{1}{\sqrt{x}+3}$$

$$= \frac{1}{\sqrt{9}+3} = \frac{1}{3+3} = \frac{1}{6} \#$$

3. **Find the limit of** $\lim_{x \to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x}$.

$$\lim_{x \to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x} \times \frac{\sqrt{2+x}+\sqrt{2}}{\sqrt{2+x}\cancel{x}+\sqrt{2}}$$

$$= \lim_{x \to 0} \frac{(\sqrt{2+x})^2-(\sqrt{2})^2}{x(\sqrt{2+x}+\sqrt{2})}$$

$$= \lim_{x \to 0} \frac{2+x-2}{x(\sqrt{2+x}+\sqrt{2})}$$

$$= \lim_{x \to 0} \frac{\cancel{x}}{\cancel{x}(\sqrt{2+x}+\sqrt{2})}$$

$$= \lim_{x \to 0} \frac{1}{\sqrt{2+x}+\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}+\sqrt{2}} = \frac{1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{2}}{4}$$ #

4. **Find the limit of** $\lim_{\theta \to \frac{\pi}{2}} \frac{\tan\theta}{\sec\theta}$.

$$\lim_{\theta \to \frac{\pi}{2}} \frac{\left(\frac{\sin\theta}{\cancel{\cos\theta}}\right)}{\left(\frac{1}{\cancel{\cos\theta}}\right)} = \lim_{\theta \to \frac{\pi}{2}} \sin\theta$$

$$= \sin\left(\frac{\pi}{2}\right)$$

$$= \sin 90^\circ$$

$$= 1 \ \#$$

5. **Find the limit of** $\lim_{\theta \to 0} \frac{\cos\theta - 1}{\sin\theta}$.

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

$$\lim_{\theta \to 0} \frac{\theta}{\sin\theta} = 1$$

$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\sin\theta} \times \frac{\theta}{\theta}$$

$$= \lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} \times \frac{\theta}{\sin\theta}$$

$$\frac{\cos 0 - 1}{0}$$

$$= \boxed{\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta}} \times \lim_{\theta \to 0} \frac{\theta}{\sin\theta}$$

$$\frac{0}{0}$$

$$= \lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} \times 1$$

$$= 0 \times 1 = 0$$ #

6. If $2x \le g(x) \le x^2 - x + 2$ for all x, evaluate $\lim_{x \to 1} g(x)$.

$$\lim_{x \to 1} 2x \le \lim_{x \to 1} g(x) \le \lim_{x \to 1} (x^2 - x + 2)$$

$$2 \le \lim_{x \to 1} g(x) \le 1^2 - 1 + 2 = 2$$

$$2 \le \lim_{x \to 1} g(x) \le 2$$

$$\lim_{x \to 1} g(x) = 2 \ \#$$

7. **Solve** y' **if** $y = \sqrt{3x^2 - 2x + 3}$.

let $u = 3x^2 - 2x + 3$

$\frac{du}{dx} = 6x - 2$

$\Rightarrow y = \sqrt{u} = u^{1/2}$

$\frac{dy}{du} = \frac{1}{2} u^{-1/2}$

$$y' = \frac{dy}{dx}$$

$$= \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{1}{2} u^{-1/2} \times (6x - 2)$$

$$= \frac{1}{2} (3x^2 - 2x + 3)^{-1/2} \times (6x - 2)$$

$$= \frac{6x - 2}{2(3x^2 - 2x + 3)^{1/2}} \quad 6x - 2 \rightarrow 2(3x - 1)$$

$$= \frac{3x - 1}{\sqrt{3x^2 - 2x + 3}} \quad \#$$

8. Solve y' if $y = 5\sqrt[3]{x^2 + \sqrt{x^3}}$.

$$\text{let } u = x^2 + \sqrt{x^3} = x^2 + (x^3)^{1/2}$$

$$= x^2 + x^{3/2}$$

$$\frac{du}{dx} = 2x + \frac{3}{2}x^{1/2}$$

$$y = 5\sqrt[3]{u} = 5u^{1/3}$$

$$\frac{dy}{du} = 5\left(\frac{1}{3}\right)u^{-2/3}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{5}{3}u^{-2/3}\left(2x + \frac{3}{2}x^{1/2}\right)$$

$$= \frac{5}{3}\left(x^2 + \sqrt{x^3}\right)^{-2/3}\left(2x + \frac{3}{2}x^{1/2}\right)$$

9. Solve y if $y = \ln \cos x^2$.

let $u = \cos x^2$ $\quad \frac{du}{dx} = (-\sin x^2)(2x)$

$y = \ln u$ $\quad \frac{dy}{du} = \frac{1}{u}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{1}{u} \times (2x(-\sin x^2))$$

$$= \frac{1}{\cos x^2}(-\sin x^2)(2x)$$

$$= \frac{-2x \boxed{\sin x^2}}{\boxed{\cos x^2}}$$

$$= -2x \tan x^2 \ \#$$

10. Differentiate $y = \log(\underbrace{4 + \cos x})$.

$$\frac{d}{dx}\left(\log_b x\right) = \frac{1}{\ln(b)\cdot x} = \frac{1}{x}\log e$$

$$\frac{dy}{dx} = \frac{1}{4+\cos x} \cdot \log e \cdot -\sin x$$

$$= \frac{-(\log e)(\sin x)}{4 + \cos x} \quad \#$$

11. **Find y' for** $10e^{2xy} = e^{15y} + e^{13x}$.

y'_x

$u = 2x,\ v = y$

$$10e^{2xy}\left(2x \cdot y' + y(2)\right) = e^{15y}(15y') + e^{13x}(13)$$

$$20xy'e^{2xy} + 20ye^{2xy} = 15y'e^{15y} + 13e^{13x}$$

$$y'\left[20xe^{2xy} - 15e^{15y}\right] = 13e^{13x} - 20ye^{2xy}$$

$$y' = \frac{13e^{13x} - 20ye^{2xy}}{20xe^{2xy} - 15e^{15y}} \quad \#$$

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$$

12. Find $f'(x)$ if $f(x) = \underbrace{2x}_{u}\underbrace{(\arctan 5x)^2}_{v} + 6\tan(\cos 6x)$.

$$\frac{d}{dx}(\tan^{-1}5x)^2 = 2(\tan^{-1}5x)\left(\frac{1}{1+(5x)^2}\right)(5)$$

$$= \frac{10\tan^{-1}5x}{1+25x^2}$$

$$\frac{d}{dx}\left(\underbrace{2x}_{u}\underbrace{(\tan^{-1}5x)^2}_{v}\right) = 2x\left(\frac{10\tan^{-1}5x}{1+25x^2}\right) + (\tan^{-1}5x)^2(2)$$

$$= \frac{20x\tan^{-1}5x}{1+25x^2} + 2(\tan^{-1}5x)^2$$

$$\frac{d}{dx}\left(6\tan(\cos 6x)\right) = 6\sec^2(\cos 6x)(-\sin 6x)\cdot 6$$

$$= -36(\sec^2(\cos 6x))\sin 6x$$

$$\Rightarrow \frac{20x\tan^{-1}5x}{1+25x^2} + 2(\tan^{-1}5x)^2 - 36\sec^2(\cos 6x)\sin 6x \ \#$$

13. **Solve** y' **if** $y = 4x \sinh^{-1}\left(\frac{x}{6}\right) + \tanh^{-1}\left(\frac{x}{6}\right) + \tanh^{-1}(\cos 10x)$.

u v

$$\frac{d}{dx}\left(\sinh^{-1}(x)\right) = \frac{1}{\sqrt{1}}$$

$$\frac{d}{dx}\left(4x \sinh^{-1}\left(\frac{x}{6}\right)\right) = 4x\left[\frac{1}{\sqrt{1+\left(\frac{x}{6}\right)^2}} \cdot \frac{1}{6}\right] + \sinh$$

$$= \frac{2x}{3\sqrt{1+\frac{x^2}{36}}} + 4\sinh^{-1}\left(\frac{x}{6}\right)$$

$$\frac{d}{dx}\left(\tanh^{-1}\left[\frac{x}{6}\right]\right) = \frac{1}{1-\left(\frac{x}{6}\right)^2}\left(\frac{1}{6}\right)$$

$$= \frac{\frac{1}{6}}{1-\frac{x^2}{36}}$$

6

$$\frac{d}{dx}\tanh^{-1}(\cos 10x) = \frac{1}{1-(\cos 10x)^2}(-\sin 10$$

$$= \frac{-10\sin 10x}{1-(\cos 10x)^2}$$

$$= \frac{-10\sin 10x}{\sin^2 10x}$$

$$= \frac{-10}{\sin 10x}$$

$=$

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14. Differentiate $y = \dfrac{1}{\sin^{-1}x}$.

$$\frac{d}{dx}\left(\sin^{-1}x\right) = \frac{1}{\sqrt{1-x^2}}$$

$$y = \left(\sin^{-1}x\right)^{-1}$$

$$\frac{dy}{dx} = -\left(\sin^{-1}x\right)^{-2}\frac{1}{\sqrt{1-x^2}}$$

$$= \frac{1}{\left(\sin^{-1}x\right)^2\sqrt{1-x^2}}$$

15. Differentiate $y = (x^3 - 1)^{100}$.

$$\frac{dy}{dx} = 100(x^3-1)^{99}(3x^2)$$

$$= 300x^2(x^3-1)^{99} \#$$