Tutorial 2: Functions & Derivatives

1. Find y' for $(x-y)^2 = x+y-1$ by using implicit differentiation.

$$\begin{aligned}
\mathfrak{D}(x-y)^2 &= \mathfrak{D}(x+y-1) \\
\mathfrak{D}(x-y)^2 &= \mathfrak{D}(x)+\mathfrak{D}(y)-\mathfrak{D}(1) \\
2(x-y)\mathfrak{D}(x-y) &= 1+y' \\
2(x-y)(1-y') &= 1+y' \\
2(x-y)-2(x-y)y' &= 1+y' \\
-2(x-y)y'-y' &= 1-2(x-y) \\
y'[-2(x-y)-1] &= 1-2(x-y) \\
y' &= \frac{1-2(x-y)}{-2(x-y)-1} \\
y' &= \mathbf{\frac{2y-2x+1}{2y-2x-1}}
\end{aligned}$$

2. Implicit differentiation to find an equation of the tangent line to the curve $5x^2+10xy^2+5y=20$ at point $(2,2)$.

$$\begin{aligned}
\mathfrak{D}(5x^2+10xy^2+5y) &= \mathfrak{D}(20) \\
10x+10x(2y\cdot y')+y^2\cdot 10+5y' &= 0 \\
10x+20xy\cdot y'+10y^2+5y' &= 0 \\
(20xy+5)y' &= -10y^2-10x \\
y' &= \frac{-10y^2-10x}{20xy+5}
\end{aligned}$$

At point $(2,2)$, $y' = \dfrac{-10(2)^2-10(2)}{20(2)(2)+5} = -\dfrac{12}{17}$

Knowing $y = mx+c$,

$$2 = -\frac{12}{17}(2)+c \rightarrow c = \frac{58}{17}$$

$\therefore$ Tangent Line: $\mathbf{y = -\dfrac{12}{17}x+\dfrac{58}{17}}$

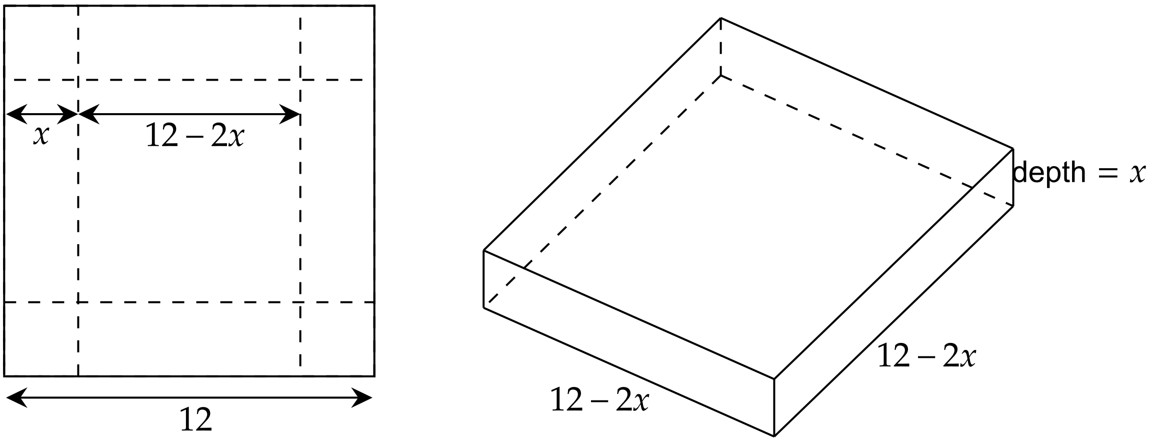
3. $y\cos x = 1+\sin(xy)$. **Find** $\dfrac{dy}{dx}$ **by implicit differentiation.**

$$\begin{aligned}
\mathfrak{D}(y\cos x) &= \mathfrak{D}(1+\sin(xy))\\
y(-\sin x)+(\cos x)y' &= \cos(xy)\cdot(xy'+y)\\
-y\sin x + y'\cos x &= (xy'+y)\cos(xy)\\
y' &= \boldsymbol{\frac{y\cos(xy)+y\sin x}{\cos x - x\cos xy}}
\end{aligned}$$

4. Find $\dfrac{dy}{dx}$ **by implicit differentiation** $4\cos x\sin y = 1$.

$$\begin{aligned}
\mathfrak{D}(4\cos x\sin y) &= \mathfrak{D}(1)\\
(4\cos x)\cdot(\cos y)\cdot y' + (\sin y)\cdot(-4\sin x) &= 0\\
y' &= \frac{4\sin x\sin y}{4\cos x\cos y}\\
&= \boldsymbol{\tan x\tan y}
\end{aligned}$$

5. An open box is to be made from cutting a square from each corner of a 12 in by 12 in piece of metal and then folding up the sides. What size square should be cut from each corner to produce a box of maximum volume?



The volume is to be maximized, $V = lwh = x(12-2x)(12-2x) = 4x^3 - 48x^2 + 144x$.

$$\begin{aligned} V(x) &= 4x^3 - 48x^2 + 144x \\ V'(x) &= 12x^2 - 96x + 144 \end{aligned}$$

The critical point can be calculate such that, $V'(x) = 0$.

$$\begin{aligned} 0 &= 12x^2 - 96x + 144 \\ 0 &= 12(x^2 - 8x + 12) \\ 0 &= 12(x-2)(x-6) \\ \therefore x &= 2, 6 \end{aligned}$$

When $x = 2$, $V(2) = 128$.   When $x = 6$, $V(6) = 0$.

Thus, when $x = 2$, the maximum volume is 128 in^3.

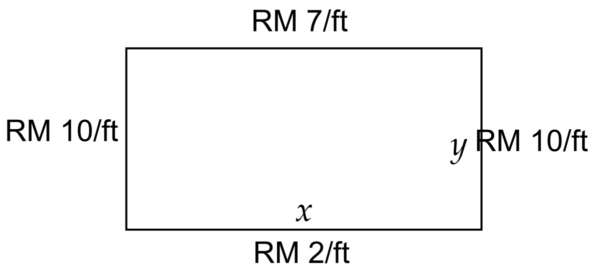
To prove that, using second derivation $V''(x) = 24x - 96$.

When $x = 2$, $V''(2) < 0$ (Maximum Volume)

When $x = 6$, $V''(6) > 0$ (Minimum Volume)

Thus, $x = 2$.

6. We are going to fence in a rectangular field. If we look at the field from above the cost of the vertical sides are RM 10/ft, the cost of the bottom is RM2/ft and the cost of the top is RM 7 /ft. If we have RM 700 determind the dimensions of the field that will maximize the enclosed area.



We have RM 700, therefore

$$700 = 10y + 2x + 10y + 7x = 20y + 9x.$$

In terms of y, $y = 35 - \frac{9}{20}x$.

Since $A = xy$, plugging in yields

$$A(x) = x\left(35 - \frac{9}{20}x\right).$$

The critical point can be calculate such that, $A'(x) = 0$.

$$A'(x) = 35 - \frac{9}{10}(x)$$
$$0 = 35 - \frac{9}{10}(x)$$
$$\therefore x = \frac{350}{9}$$

The second derivation,

$$A''(x) = -\frac{9}{10}$$

This means that the second derivative always negative and so $A(x)$ will always be concave down and so the single critical point. Therefore the value obtained in $A'(x)$ must be the value that yield maximum area.

Substituting $x = \frac{350}{9}$ into the equation of y,

$$y = 35 - \frac{9}{20}\left(\frac{350}{9}\right) = \frac{35}{2}$$

Thus, the final dimension is, $\mathbf{x = \frac{350}{9}, y = \frac{35}{2}}$.