

## Tutorial 13: Power Series Solutions

1. Find the radius of convergence and interval of convergence for the given series.

a. 
$$\sum_{n=1}^{\infty} \frac{2^n}{n} x^n$$

b. 
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4^n} (x+3)^n$$

c. 
$$\sum_{n=0}^{\infty} \frac{(100)^n}{n!} (x+7)^n$$

d. 
$$\sum_{n=0}^{\infty} n! (2x+1)^n$$

(a)

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2^{n+1}x^{n+1}/(n+1)}{2^n x^n / n} \right| = \lim_{n \rightarrow \infty} \frac{2n}{n+1} |x| = 2|x|$$

The series is absolutely convergent for  $2|x| < 1$  or  $|x| < \frac{1}{2}$ . At  $x = -\frac{1}{2}$ , the series  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$  is

by the alternating series test. At  $x = \frac{1}{2}$ , the series  $\sum_{n=1}^{\infty} \frac{1}{n}$  is the harmonic series which diverges.

(c)

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{100^{n+1}(x+7)^{n+1} / (n+1)!}{100^n(x+7)^n / n!} \right| = \lim_{n \rightarrow \infty} \frac{100}{n+1} |x+7| = 0$$

The series is absolutely convergent on  $(-\infty, \infty)$ .

(d)

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)!(2x+1)^{n+1}}{n!(2x+1)^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)n!(2x+1)}{n!} \right| \\ &= |2x+1| \lim_{n \rightarrow \infty} (n+1) \end{aligned}$$

At this point we need to be careful. The limit is infinite, but there is that term with the  $x$ 's in the limit. We will have  $L = \infty > 1$  provided  $x \neq -\frac{1}{2}$  will only converge if  $x = -\frac{1}{2}$ .

The radius of convergence is  $R = 0$  and the interval of convergence is  $x = -\frac{1}{2}$ .

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2. Rewrite the given power series by shifting the index, so that its general term involves  $x^k$ .

a.  $\sum_{n=3}^{\infty} (2n-1)c_n x^{n-3}$

b.  $\sum_{n=3}^{\infty} \frac{3^n}{(2n)!} x^{n-3}$

c.  $\sum_{n=3}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$

(a)

$$\sum_{n=3}^{\infty} (2n-1)c_n x^{n-3} = \sum_{k=0}^{\infty} (2(k+3)-1)c_{k+3} x^k = \sum_{k=0}^{\infty} (2k+5)c_{k+3} x^k$$

(b)

$$\sum_{n=3}^{\infty} \frac{3^n}{(2n)!} x^{n-2} = \sum_{k=1}^{\infty} \frac{3^{k+2}}{(2(k+2))!} x^k = \sum_{k=1}^{\infty} \frac{3^{k+2}}{(2k+4)!} x^k$$

(c)

$$\sum_{n=3}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \sum_{k=7}^{\infty} \frac{(-1)^{\frac{k-1}{2}}}{\left(2\left(\frac{k-1}{2}\right)+1\right)!} x^k = \sum_{k=7}^{\infty} \frac{(-1)^{\frac{k-1}{2}}}{k!} x^k = \sum_{k=7}^{\infty} \frac{\sqrt{(-1)^{k-1}}}{k!} x^k$$

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3. Rewrite the given expression as a single power series whose general term involves  $x^k$ .


a. 
$$\sum_{n=2}^{\infty} n(n-1)C_n x^n + 2 \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 3 \sum_{n=1}^{\infty} n c_n x^n$$

b. 
$$3x^2 \sum_{n=-2}^{\infty} n(n-1)x^{n-2} + x \sum_{n=1}^{\infty} n x^n$$

c. 
$$\sum_{n=1}^{\infty} \frac{3^n}{(2n)!} x^{n-1} + 3x^3 \sum_{n=-1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

(a)

$$\sum_{n=2}^{\infty} n(n-1)c_n x^n + 2 \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 3 \sum_{n=1}^{\infty} n c_n x_n$$

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$$= 2 \cdot 2 \cdot 1 c_2 x^0 + 2 \cdot 3 \cdot 2 c_2 x' + 3 \cdot 1 \cdot c_1 x' + \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^n}_{k=n} + 2 \underbrace{\sum_{n=4}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + 3 \sum_{n=1}^{\infty} n c_n x_n$$



Substitute back to Eq (1),

$$\begin{aligned}\therefore 3 \left[ \frac{6}{x^2} + \frac{2}{x} + \sum_{k=2}^{\infty} k(k-1)x^k \right] + \sum_{k=2}^{\infty} (k-1)x^k &= \frac{18}{x^2} + \frac{6}{x} + \sum_{k=2}^{\infty} [3k(k-1) + (k-1)]x^k \\ &= \frac{18}{x^2} + \frac{6}{x} + \sum_{k=2}^{\infty} [(k-1)(3k+1)]x^k\end{aligned}$$

(c)

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{3^n}{(2n)!} x^{n-1} + 2x^3 \sum_{n=-1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} &= \sum_{n=1}^{\infty} \frac{3^n}{(2n)!} x^{n-1} + 2 \sum_{n=-1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+4} \\ &= \sum_{k=0}^{\infty} \frac{3^{k+1}}{(2(k+1))!} x^k + 2 \sum_{k=2}^{\infty} \frac{(-1)^{\frac{k-4}{2}}}{\left(2\left(\frac{k-4}{2}\right) + 1\right)!} x^k \\ &= \sum_{k=0}^{\infty} \frac{3^{k+1}}{(2k+2)!} x^k + 2 \sum_{k=2}^{\infty} \frac{(-1)^{\frac{k-4}{2}}}{(k-3)!} x^k\end{aligned}$$

Solving first part of Eq (2),

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{3^{k+1}}{(2k+2)!} x^k &= \frac{3^1}{4!} x^0 + \frac{3^2}{4!} x^1 + \sum_{k=2}^{\infty} \frac{3^{k+1}}{(2k+2)!} \\ &= \frac{3}{2} + \frac{3}{8}x + \sum_{k=2}^{\infty} \frac{3^{k+1}}{(2k+2)!} x^k\end{aligned}$$

Subtitute back to Eq (2),

$$\begin{aligned}\therefore \frac{3}{2} + \frac{3}{8}x + \sum_{k=2}^{\infty} \frac{3^{k+1}}{(2k+2)!} x^k + 2 \sum_{k=2}^{\infty} \frac{(-1)^{\frac{k-4}{2}}}{(k-3)!} x^k &= \frac{3}{2} + \frac{3}{8}x + \sum_{k=2}^{\infty} \left[ \frac{3^{k+2}}{(2k+2)!} + \frac{2(-1)^{\frac{k-4}{2}}}{(k-3)!} \right] x^k \\ &= \frac{3}{2} + \frac{3}{8}x + \sum_{k=2}^{\infty} \left[ \frac{3^{k+2}}{(2k+2)!} + \frac{2\sqrt{(-1)^{\frac{k-4}{2}}}}{(k-3)!} \right] x^k\end{aligned}$$

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**4. Find two power series solutions of given differential equation ordinary point  $x = 0$ .**

**a.  $y'' + xy' + y = 0$**

**b.  $(x - 1)y'' + y' = 0$**

**c.  $y'' + e^x y' - y = 0$**

**d.  $(x^2 + 1)y'' + xy' - y = 0$**

(a)

$$\text{Let } y(x) = \sum_{n=0}^{\infty} c_n x^n \implies y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} \text{ and } y''(x) = \sum_{n=2}^{\infty} (n+2)(n+1) c_{n+2} x^n.$$

The differential equation becomes

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$$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + x \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$c_{2n+1} = (-1)^n \frac{c_1}{3 \cdot 5 \cdot 7 \cdots (2n+1)} = \frac{(-2)^n n! c_1}{(2n+1)!}$$

The solution is,

$$y(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^{2n} + c_1 \sum_{n=0}^{\infty} \frac{(-2)^n n!}{(2n+1)!} x^{2n+1}$$

(b)

Substituting  $y = \sum_{n=0}^{\infty} c_n x^n$  into the differential equation,

$$\begin{aligned} (x-1)y'' + y' &= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1}}_{k=n-1} - \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + \underbrace{\sum_{n=1}^{\infty} n c_n x^{n-1}}_{k=n-1} \\ &= \sum_{k=1}^{\infty} (k+1)k c_{k+1} x^k - \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2} x^k + \sum_{k=0}^{\infty} (k+1)c_{k+1} x^k \\ &= -2c_2 + c_1 + \sum_{k=1}^{\infty} [(k+1)k c_{k+1} - (k+2)(k+1)c_{k+2} + (k+1)c_{k+1}] x^k \end{aligned}$$

Thus,

$$\begin{aligned} -2c_2 + c_1 &= 0 \\ (k+1)^2 c_{k+1} - (k+2)(k+1)c_{k+2} &= 0 \end{aligned}$$

and

$$\begin{aligned} c_2 &= \frac{1}{2} c_1 \\ c_{k+2} &= \frac{k+1}{k+2} c_{k+1}, \quad k = 1, 2, 3, \dots \end{aligned}$$

Choosing  $c_0 = 1$  and  $c_1 = 0$ , we find  $c_2 = c_3 = c_4 = \dots = 0$ . For  $c_0 = 0$  and  $c_1 = 1$ , we obtain

$$c_2 = \frac{1}{2}, \quad c_3 = \frac{1}{3}, \quad c_4 = \frac{1}{4}$$

Thus, the two solutions are

$$y_1 = 1 \quad y_2 = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \dots$$

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(c)

Substituting  $y = \sum_{n=0}^{\infty} c_n x^n$  into the differential equation we have,

$$\begin{aligned} y'' - (x+1)y' - y &= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - \underbrace{\sum_{n=1}^{\infty} n c_n x^n}_{k=n} - \underbrace{\sum_{n=1}^{\infty} n c_n x^{n-1}}_{k=n-1} - \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n} \\ &= \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2} x^k - \sum_{k=1}^{\infty} k c_k x^k - \sum_{k=0}^{\infty} (k+1)c_{k+1} x^k - \sum_{k=0}^{\infty} c_k x^k \\ &= 2c_2 - c_1 - c_0 + \sum_{k=1}^{\infty} [(k+2)(k+1)c_{k+2} - (k+1)c_{k+1} - (k+1)c_k] x^k \end{aligned}$$

Thus,

$$\begin{aligned} 2c_2 - c_1 - c_0 &= 0 \\ (k+2)(k+1)c_{k+2} - (k+1)(c_{k+1} + c_k) &= 0 \end{aligned}$$

and,

$$\begin{aligned} c_2 &= \frac{c_1 + c_0}{2} \\ c_{k+2} &= \frac{c_{k+1} + c_k}{k+2} c_k, \quad k = 2, 3, 4, \dots \end{aligned}$$

Choosing  $c_0 = 1$  and  $c_1 = 0$ , we find,

$$c_2 = \frac{1}{2}, \quad c_3 = \frac{1}{6}, \quad c_4 = \frac{1}{6}$$

and so on. For  $c_0 = 0$  and  $c_1 = 1$ , we obtain,

$$c_2 = \frac{1}{2}, \quad c_3 = \frac{1}{2}, \quad c_4 = \frac{1}{4}$$

and so on. Thus, the two solutions are,

$$y_1 = 1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{6}x^4 + \dots \quad \text{and} \quad y_2 = x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{4}x^4 + \dots$$

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(d)

Let  $y(x) = \sum_{n=0}^{\infty} c_n x^n \implies y''(x) = \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2}$  and  $xy'' = \sum_{n=0}^{\infty} nc_n x^n$ . Thus,

$$(x^2 + 1)y'' = \sum_{n=0}^{\infty} n(n-1)c_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n$$

The differential equation becomes,

$$\sum_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} + [n(n-1) + n-1]c_n]x^n = 0$$

The recursion relation is

$$c_{n+2} = -\frac{(n-1)c_n}{n+2}, \quad n = 0, 1, 2, \dots$$

Given  $c_0$  and  $c_1$ ,  $c_2 = \frac{c_0}{2}$ ,  $c_4 = -\frac{c_0}{2^2 \cdot 2!}$ ,  $c_6 = -\frac{3c_4}{6} = (-1)^2 \frac{3c_0}{2^3 \cdot 3!}, \dots$

$$c_{2n} = (-1)^{n-1} \frac{1 \cdot 3 \cdots (2n-3)c_0}{2^n n!} = (-1)^{n-1} \frac{(2n-3)!c_0}{2^n 2^{n-2} n! (n-2)!} = (-1)^{n-1} \frac{(2n-3)!c_0}{2^{2n-2} n! (n-2)!} \text{ for } n \geq 2$$

$c_3 = \frac{0 \cdot c_1}{3} = 0 \implies c_{2n+1} = 0$  for  $n = 1, 2, \dots$ . Thus the solution is,

$$y(x) = c_0 + c_1 x + c_0 \frac{x^2}{2} + c_0 \sum_{n=2}^{\infty} \frac{(-1)^{n-1} (2n-3)!}{2^{2n-2} n! (n-2)!} x^{2n}$$

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**5. Use the power series method to solve the given initial-value problem.**

**a.**  $y'' - xy' - y = 0, \quad y(0) = 1, \quad y'(0) = 0$

**b.**  $y'' + x^2y' + xy = 0, \quad y(0) = 0, \quad y'(0) = 1$

**c.**  $(x+1)y'' - (2-x)y' + y = 0, \quad y(0) = 2, \quad y'(0) = -1$

(a)

Let  $y(x) = \sum_{n=0}^{\infty} c_n x^n$ . Then  $-xy'(x) = -x \sum_{n=1}^{\infty} n c_n x^{n-1} = - \sum_{n=1}^{\infty} n c_n x^n = - \sum_{n=0}^{\infty} n c_n x^n$ .

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$$y''(x) = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n$$

(b)

Assuming that  $y(x) = \sum_{n=0}^{\infty} c_n x^n$ , we have  $xy = x \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+1}$ ,

$$x^2 y' = x^2 \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} n c_n x^{n+1}$$

$$\begin{aligned} y''(x) &= \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} = \sum_{n=-1}^{\infty} (n+3)(n+2) c_{n+3} x^{n+1} \\ &= 2c_2 + \sum_{n=0}^{\infty} (n+3)(n+2) c_{n+3} x^{n+1} \end{aligned}$$

Thus, the equation  $y'' + x^2 y' + xy = 0$  becomes

$$2c_2 + \sum_{n=0}^{\infty} [(n+3)(n+2)c_{n+3} + n c_n + c_n] x^{n+1} = 0$$

So,  $c_2 = 0$  and the recursion relation is,

$$c_{n+3} = \frac{-n c_n - c_n}{(n+3)(n+2)} = -\frac{(n+1)c_n}{(n+3)(n+2)}, \quad n = 0, 1, 2, \dots$$

Also,  $c_1 = y'(0) = 1$ , so,

$$c_4 = -\frac{2c_1}{4 \cdot 3} = -\frac{2}{4 \cdot 3}, \quad c_7 = -\frac{5c_4}{7 \cdot 6} = (-1)^2 \frac{2 \cdot 5}{7 \cdot 6 \cdot 4 \cdot 3} = (-1)^2 \frac{2^2 5^2}{7!}, \dots$$

$$c_{3n+1} = (-1)^n \frac{2^2 5^2 \dots (3n-1)^2}{(3n+1)!}$$

Thus, the solution is,

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = x + \sum_{n=1}^{\infty} \left[ (-1)^n \frac{2^2 5^2 \dots (3n-1)^2 x^{3n+1}}{(3n+1)!} \right]$$

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(c)

Substituting  $y = \sum_{n=0}^{\infty} c_n x^n$  into the differential equation, we have,

$$\begin{aligned} & (x+1)y'' - (2-x)y' + y \\ &= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1}}_{k=n-1} + \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - 2 \underbrace{\sum_{n=2}^{\infty} n c_n x^{n-1}}_{k=n-1} + \underbrace{\sum_{n=1}^{\infty} n c_n x^n}_{k=n} + \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n} \\ &= \sum_{k=1}^{\infty} (k+1)k c_{k+1} x^k + \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2} x^k - 2 \sum_{k=0}^{\infty} (k+1)c_{k+1} x^k + \sum_{k=1}^{\infty} k c_k x^k + \sum_{k=0}^{\infty} c_k x^k \\ &= 2c_2 - 2c_1 + c_0 + \sum_{k=1}^{\infty} [(k+2)(k+1)c_{k+2} - (k+1)c_{k+1} + (k+1)c_k] x^k = 0 \end{aligned}$$

Thus,

$$\begin{aligned} 2c_2 - 2c_1 + c_0 &= 0 \\ (k+2)(k+1)c_{k+2} - (k+1)c_{k+1} + (k+1)c_k &= 0 \end{aligned}$$

and,

$$\begin{aligned} c_2 &= c_1 - \frac{1}{2}c_0 \\ c_{k+2} &= \frac{1}{k+2}c_{k+1} - \frac{1}{k+2}c_k, \quad k = 1, 2, 3, \dots \end{aligned}$$

Choosing  $c_0 = 1$  and  $c_1 = 0$ , we find

$$c_2 = -\frac{1}{2}, \quad c_3 = -\frac{1}{6}, \quad c_4 = \frac{1}{12}$$

and so on. For  $c_0 = 0$  and  $c_1 = 1$ , we obtain,

$$c_2 = 1, \quad c_3 = 0, \quad c_4 = -\frac{1}{4}$$

and do on. Thus,

$$y = C_1 \left( 1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \dots \right) + C_2 \left( x + x^2 - \frac{1}{4}x^4 + \dots \right)$$

and

$$y' = C_1 \left( -x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots \right) + C_2 (1 + 2x - x^3 + \dots)$$

The initial condition imply  $C_1 = 2$  and  $C_2 = -1$ , so

$$y = 2 \left( 1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \dots \right) - \left( x + x^2 - \frac{1}{4}x^4 + \dots \right)$$

$$y = 2 \left( 1 - \frac{x}{2} - \frac{x^2}{6} + \frac{x^3}{12} + \dots \right) - \left( x + x^2 - \frac{x^3}{4} + \dots \right)$$

$$= 2 - x - 2x^2 - \frac{1}{3}x^3 + \frac{5}{12}x^4 + \dots$$

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**6. Determine the singular points of the given differential equation. Classify each singular point as regular or irregular.**

**a.**  $x^3y'' - 4x^2y' + 3y = 0$

**b.**  $(x^2 - 9)^2y'' + (x + 3)y' + 2y = 0$

**c.**  $(2x^2 - 5x - 3)y'' + (2x + 1)y' + \frac{6}{(x - 3)}y = 0$

**d.**  $(x^3 - 2x^2 - 3x)^2y'' + x(x - 3)^2y' - (x + 1)y = 0$

(a)

Dividing the equation with  $x^3$

$$y'' + \frac{4x^2}{x^3}y' + \frac{3}{x^3}y = 0$$

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$$P(x) = \frac{4}{x} \quad Q(x) = \frac{3}{x^3}$$

(c)

Factor the equation  $(2x^2 - 5x - 3) = (x - 3)(2x + 1)$ ,

$$y'' + \frac{(2x + 1)}{(x - 3)(2x + 1)}y' + \frac{6}{(x - 3)^2(2x + 1)}y = 0$$

$$P(x) = \frac{1}{(x - 3)} \quad Q(x) = \frac{6}{(x - 3)^2(2x + 1)}$$

- The factor  $(x - 3)$  appears at most to the first power in  $P(x)$  and second power in the denominator of  $Q(x)$
- $x = 3$  is regular singular point
- The factor  $(2x + 1)$  appears at to the first power in the denominator of  $Q(x)$
- $x = -\frac{1}{2}$  is regular singular point

(d)

Divide the equation with  $(x^3 - 2x^2 - 3x)^2 = (x(x^2 - 2x - 3))^2 = x^2(x - 3)^2(x + 1)^2$ ,

$$y'' + \frac{x(x - 3)^2}{x^2(x - 3)^2(x + 1)^2}y' - \frac{(x + 1)}{x^2(x - 3)^2(x + 1)^2}y = 0$$

$$P(x) = \frac{1}{x(x + 1)^2} \quad Q(x) = \frac{1}{x^2(x - 3)^2(x + 1)}$$

- The factor  $x$  appears at most to the first power in the denominator of  $P(x)$  and at second power in the denominator of  $Q(x)$
- $x = 0$  is regular singular point
- The factor  $x - 3$  appears to the second power in the denominator of  $Q(x)$
- $x = 3$  is regular singular point
- The factor  $x + 1$  appears to the second power in the denominator of  $P(x)$  and to the first power in the denominator of  $Q(x)$
- $x = -1$  is irregular singular point

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7. Find the indicial roots for the given differential equations where regular singular point.

a.  $2xy'' - y' + 2y = 0$

b.  $3xy'' + (2 - x)y' - y = 0$

c.  $9x^2y'' + 9x^2y' + 2y = 0$

d.  $x^2y'' + xy' + \left(x^2 - \frac{4}{9}\right)y = 0$

e.  $xy'' + (1 - x)y' - y = 0$

(a)

Divide the equation with  $2x$ ,

$$y'' - \frac{1}{2x}y' + \frac{2}{2x}y = y'' - \frac{\frac{1}{2}}{x}y' + \frac{1}{x}y = y'' - \frac{\frac{1}{2}}{x}y' + \frac{x}{x^2}y = 0$$

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Hence,  $b_0 = -\frac{1}{2}$  and  $r_2 = 0$ ,

(c)

Divide the equation with  $9x^2$ ,

$$y'' + \frac{9x^2}{9x^2}y' + \frac{2}{9x^2}y = y'' + y' + \frac{\frac{2}{9}}{x^2}y = y'' + \frac{x}{x}y' + \frac{\frac{2}{9}}{x^2}y = 0$$

Hence,  $b_0 = 0$  and  $c_0 = \frac{2}{9}$ ,

$$r(r-1) + b_0r + c_0 = r^2 - r + \frac{2}{9} = \left(r - \frac{1}{3}\right)\left(r - \frac{2}{3}\right) = 0$$

The indicial roots are  $r_1 = \frac{2}{3}$  and  $r_2 = \frac{1}{3}$ .

(d)

Divide the equation with  $x^2$ ,

$$y'' + \frac{x}{x^2}y' + \frac{\left(x^2 - \frac{4}{9}\right)}{x^2}y = y'' + \frac{1}{x}y' + \frac{-\frac{4}{9} + x^2}{x^2}y = y'' + \frac{x}{x}y' + \frac{\frac{2}{9}}{x^2}y = 0$$

Hence,  $b_0 = 1$  and  $c_0 = -\frac{4}{9}$ ,

$$r(r-1) + b_0r + c_0 = r^2 - r + r - \frac{4}{9} = r^2 - \frac{4}{9} = \left(r + \frac{2}{3}\right)\left(r - \frac{2}{3}\right) = 0$$

The indicial roots are  $r_1 = \frac{2}{3}$  and  $r_2 = -\frac{2}{3}$ .

(e)

Divide the equation with  $x$ ,

$$y'' + \frac{(1-x)}{x}y' - \frac{1}{x}y = y'' + \frac{(1-x)}{x}y' + \frac{x}{x^2}y = 0$$

Hence,  $b_0 = 1$  and  $c_0 = 0$ ,

$$r(r-1) + b_0r + c_0 = r^2 - r + r = r^2 = 0$$

The indicial roots are  $r_1 = 0$  and  $r_2 = 0$ .

