Tutorial 6: Matrix Algebra

1. Determine a scalar r such that $Ax = rx$ where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix} \text{ and } x = \begin{bmatrix} -1/2 \\ 1/4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}\begin{bmatrix} -1/2 \\ 1/4 \\ 1 \end{bmatrix} = r\begin{bmatrix} -1/2 \\ 1/4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 1/2 \\ 2 \end{bmatrix} = r\begin{bmatrix} -1/2 \\ 1/4 \\ 1 \end{bmatrix}$$

$$\mathbf{r = 2}$$

2. Let

$$A = \begin{bmatrix} 6 & -1 & 1 \\ 0 & 13 & -16 \\ 0 & 8 & -11 \end{bmatrix} \text{ and } x = \begin{bmatrix} 10.5 \\ 21.0 \\ 10.5 \end{bmatrix}$$

(a) Determine a scalar r such that $Ax = rx$

(b) Is it true $A^T x = rx$ for the value of r determined in part (a)

$$\begin{bmatrix} 6 & -1 & 1 \\ 0 & 13 & -16 \\ 0 & 8 & -11 \end{bmatrix}\begin{bmatrix} 10.5 \\ 21.0 \\ 10.5 \end{bmatrix} = r\begin{bmatrix} 10.5 \\ 21.0 \\ 10.5 \end{bmatrix}$$

$$\begin{bmatrix} 525 \\ 1050 \\ 525 \end{bmatrix} = r\begin{bmatrix} 10.5 \\ 21.0 \\ 10.5 \end{bmatrix}$$

$$\mathbf{r} = 5$$

(a) $r = 5$

(b) No.

3. Solve each of the following systems of linear equations using Gaussian Elimination technique.

(a)
$$\begin{aligned} x + 2y + 3z &= 9 \\ 2x - y + z &= 8 \\ 3x - z &= 3 \end{aligned}$$

(b)
$$\begin{aligned} -3x + 2y - 6z &= 6 \\ 5x + 7y - 5z &= 6 \\ x + 4y - 2z &= 8 \end{aligned}$$

(c)
$$\begin{aligned} 2x + y + 3z &= 1 \\ 2x + 6y + 8z &= 3 \\ 6x + 8y + 18z &= 5 \end{aligned}$$

(d)
$$\begin{aligned} 2x - 3y + z &= 5 \\ 3x + 2y - z &= 7 \\ x + 4y - 5z &= 3 \end{aligned}$$

(e)
$$\begin{aligned} x + y + z &= 6 \\ 2x - y + z &= 3 \\ 3x - z &= 0 \end{aligned}$$

(a)

$$\begin{bmatrix} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 1 & 0 & -1 & 3 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ 0 & -6 & -10 & -24 \end{bmatrix} \xrightarrow{R_2 \to -1/5 R_2} \begin{bmatrix} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & -6 & -10 & -24 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + 6R_2} \begin{bmatrix} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -4 & -12 \end{bmatrix}$$

$$\begin{aligned} -4z &= -12 \\ \mathbf{z} &\mathbf{= 3} \end{aligned} \qquad \begin{aligned} y + z &= 2 \\ y + 3 &= 2 \\ \mathbf{y} &\mathbf{= -1} \end{aligned} \qquad \begin{aligned} x + 2y + 3z &= 9 \\ x + 2(-1) + 3(3) &= 9 \\ \mathbf{x} &\mathbf{= 2} \end{aligned}$$

(b)

$$\begin{bmatrix} -3 & 2 & -6 & 6 \\ 5 & 7 & -5 & 6 \\ 1 & 4 & -2 & 8 \end{bmatrix} \xrightarrow{R_1 \to 3R_3 + 1} \begin{bmatrix} 0 & 14 & -12 & 30 \\ 5 & 7 & -5 & 6 \\ 1 & 4 & -2 & 8 \end{bmatrix} \xrightarrow[R_2 \to -5R_3 + R_2]{R_1 \to 1/2 R_1} \begin{bmatrix} 0 & 7 & -6 & 15 \\ 0 & -13 & -5 & -34 \\ 1 & 4 & -2 & 8 \end{bmatrix}$$

$$\xrightarrow[R_2 \to 2R_1 + R_2]{R_1 \to 7R_2 + R_1} \begin{bmatrix} 0 & 0 & 43 & 43 \\ 0 & 1 & -7 & -4 \\ 1 & 4 & -2 & 8 \end{bmatrix}$$

$$\begin{aligned} 43z &= 43 \\ \mathbf{z} &\mathbf{= 1} \end{aligned} \qquad \begin{aligned} y - 7z &= -4 \\ y - 7 &= -4 \\ \mathbf{y} &\mathbf{= 3} \end{aligned} \qquad \begin{aligned} x + 4y - 2z &= 8 \\ x + 4(3) - 2(1) &= 8 \\ \mathbf{x} &\mathbf{= -2} \end{aligned}$$

(c)

$$\left[\begin{array}{cccc}2&1&3&1\\2&6&8&3\\6&8&18&5\end{array}\right]\xrightarrow[R_3\to -3R_1+R_3]{R_2\to -R_1+R_2}\left[\begin{array}{cccc}2&1&3&1\\0&5&5&2\\0&5&9&2\end{array}\right]\xrightarrow{R_3\to -R_2+R_3}\left[\begin{array}{cccc}2&1&3&1\\0&5&5&2\\0&0&4&0\end{array}\right]$$

$$\begin{array}{ccc}4z=0 & 5y+5z=2 & 2x+y+3z=1\\ \mathbf{z=0} & 5y=2 & 2x+(2/5)+3(0)=1\\ & \mathbf{y=2/5} & \mathbf{x=3/5}\end{array}$$

(d)

$$\left[\begin{array}{cccc}2&-3&1&-5\\3&2&-1&7\\1&4&-5&3\end{array}\right]\xrightarrow[R_3\to 2R_3-R_1]{R_2\to R_2-3R_3}\left[\begin{array}{cccc}2&-3&1&-5\\0&-10&14&-2\\0&11&-11&11\end{array}\right]\xrightarrow{R_3\to \frac{1}{11}R_3}\left[\begin{array}{cccc}2&-3&1&-5\\0&-10&14&-2\\0&1&-1&1\end{array}\right]$$

$$\xrightarrow{R_3\to 10R_3+R_2}\left[\begin{array}{cccc}2&-3&1&-5\\0&-10&14&-2\\0&0&4&8\end{array}\right]$$

$$\begin{array}{ccc}4z=8 & -10y+14z=-2 & 2x-3y+z=-5\\ \mathbf{z=2} & -10y+14(2)=-2 & 2x-3(3)+2=-5\\ & \mathbf{y=3} & \mathbf{x=1}\end{array}$$

(e)

$$\left[\begin{array}{cccc}1&1&1&6\\2&-1&1&3\\3&0&-1&0\end{array}\right]\xrightarrow[R_3\to R_3-3R_1]{R_2\to R_2-2R_1}\left[\begin{array}{cccc}1&1&1&6\\0&-3&-1&-9\\0&-3&-4&-18\end{array}\right]\xrightarrow{R_3\to R_3-R_2}\left[\begin{array}{cccc}1&1&1&6\\0&-3&-1&-9\\0&0&-3&-9\end{array}\right]$$

$$\begin{array}{ccc}-3z=-9 & -3y-z=-9 & x+y+z=6\\ \mathbf{z=3} & -3y-3=-9 & x+2+3=6\\ & \mathbf{y=2} & \mathbf{x=1}\end{array}$$

4. Find the eigenvalues and their associated eigenvectors

(a) $A = \begin{bmatrix} 7 & 0 & -3 \\ -9 & -2 & 3 \\ 18 & 0 & -8 \end{bmatrix}$

(b) $A = \begin{bmatrix} -5 & 0 & 0 \\ 3 & 7 & 0 \\ 4 & -2 & 3 \end{bmatrix}$

(a) Compute $\det(\mathbf{A} - \lambda\mathbf{I})$ via a cofactor expansion along the second column:

$$\begin{aligned}
\det \begin{vmatrix} 7-\lambda & 0 & -3 \\ -9 & -2-\lambda & 3 \\ 18 & 0 & -8-\lambda \end{vmatrix} &= (-2-\lambda) \begin{vmatrix} 7-\lambda & -3 \\ 18 & -8-\lambda \end{vmatrix} \\
&= (-2-\lambda)[(7-\lambda)(-8-\lambda) - 18(-3)] \\
&= (-2-\lambda)\left[-56 - 7\lambda + 8\lambda + \lambda^2 + 54\right] \\
&= (-2-\lambda)\left[\lambda^2 + \lambda - 2\right] \\
&= -(\lambda+2)(\lambda+2)(\lambda-1) \\
&= -(\lambda+2)^2(\lambda-1)
\end{aligned}$$

Thus A has two distinct eigenvalues, $\lambda_1 = -2$ and $\lambda_2 = 1$.

When $\lambda_1 = 1$,

$$\begin{bmatrix} 6 & 0 & -3 \\ -9 & -3 & 3 \\ 18 & 0 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6x_1 - 3x_2 \\ -9x_1 - 3x_2 + 3x_3 \\ 18x_1 - 9x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies x_3 = 2x_1$ and $x_2 = x_3 - 3x_1$

$\implies x_3 = 2x_1$ and $x_2 = -x_1$

When $\lambda_2 = -2$,

$$\begin{bmatrix} 9 & 0 & -3 \\ -9 & 0 & 3 \\ 18 & 0 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 9x_1 - 3x_3 \\ -9x_1 + 3x_3 \\ 18x_1 - 6x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies x_3 = 3x_1$

(b) Compute $\det(\mathbf{A} - \lambda\mathbf{I})$ via a cofactor expansion along the first row

$$\begin{aligned}
\det \begin{vmatrix} -5-\lambda & 0 & 0 \\ 3 & 7-\lambda & 0 \\ 4 & -2 & 3-\lambda \end{vmatrix} &= (-5-\lambda) \begin{vmatrix} 7-\lambda & 0 \\ -2 & 3-\lambda \end{vmatrix} \\
&= (-5-\lambda)[(7-\lambda)(3-\lambda) - 2(0)] \\
&= (-5-\lambda)\left[21 - 7\lambda - 3\lambda + \lambda^2\right] \\
&= (-5-\lambda)\left[\lambda^2 - 10\lambda - 21\right] \\
&= (-\lambda-5)(\lambda-7)(\lambda-3)
\end{aligned}$$

Thus A has three eigenvalues, $\lambda_1 = -5$, $\lambda_2 = 7$, $\lambda_3 = 3$.

When $\lambda_1 = -5$,

$$\begin{bmatrix} 0 & 0 & 0 \\ 3 & 12 & 0 \\ 4 & -2 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 3x_1 + 12x_2 \\ 4x_1 - 2x_2 + 8x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies x_1 = -4x_2$ and $x_2 = x_3 - 3x_1$

$\implies x_3 = 2x_1$ and $x_2 = -x_1$

When $\lambda_1 = -2$,

$$\begin{bmatrix} -3 & 0 & 0 \\ 3 & 5 & 0 \\ 4 & -2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3x_1 \\ 3x_1 + 5x_2 \\ 4x_1 - 2x_2 + x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies x_3 = 3x_1$

5. Using Cayley-Hamilthon approach, find A^{-1} for the following matrix:

(a) $\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

(a)

$$\begin{aligned}
|A - \lambda I| &= 0 \\
\left| \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| &= 0 \\
(7-\lambda)[(-1-\lambda)(-1-\lambda) - 4] - 2[-6(-1-A) - 12] - 2[-12 - 6(-1-\lambda)] &= 0 \\
(7-\lambda)\left[1 + \lambda + \lambda + \lambda^2 - 4\right] - 2[6 + 6\lambda - 12] - 2[-12 + 6 + 6\lambda] &= 0 \\
(7-\lambda)\left(\lambda^2 + 2\lambda - 3\right) - \lambda\left(\lambda^2 + 2\lambda - 3\right) - 12\lambda + 12 - 12\lambda + 12 &= 0 \\
7\lambda^2 + 14\lambda - 21 - \lambda^3 - 2\lambda^2 + 3\lambda - 12\lambda + 12 - 12\lambda + 12 &= 0 \\
-\lambda^3 + 5\lambda^2 - 7\lambda + 3 &= 0 \\
\lambda^3 - 5\lambda^2 + 7\lambda - 3 &= 0
\end{aligned}$$

Replacing $\lambda = A$,

$$\begin{aligned}
A^3 - 5A^2 + 7A - 3I &= 0 \\
A^2 - 5A + 7I - 3A^{-1} &= 0 \\
A^{-1} &= \frac{1}{3}\left[A^2 - 5A + 7I\right]
\end{aligned}$$

$$A^2 = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix}$$

$$\begin{aligned} A^{-1} &= \frac{1}{3}\begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} - 5\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \boldsymbol{\frac{1}{3}\begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}} \end{aligned}$$

(b)

$$\begin{aligned} |A - \lambda I| &= 0 \\ \left| \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| &= 0 \\ (2-\lambda)[(1-\lambda)(2-\lambda) - 0] - 0 + 1[-(1-\lambda)] &= 0 \\ 2\left(\lambda^2 - 3\lambda + 2\right) - \lambda\left(\lambda^2 + 3\lambda + 2\right) - 1 + \lambda &= 0 \\ 2\lambda^2 - 6\lambda + 4 - \lambda^3 - 3\lambda^2 - 2\lambda - 1 + \lambda &= 0 \\ -\lambda^3 - 5\lambda^2 - 7\lambda + 3 &= 0 \\ \lambda^3 - 5\lambda^2 + 7\lambda - 3 &= 0 \end{aligned}$$

Replacing $\lambda = A$,

$$\begin{aligned} A^3 + 5A^2 + 7A - 3I &= 0 \\ A^2 - 5A + 7I - 3A^{-1} &= 0 \\ A^{-1} &= \frac{1}{3}\left[A^2 - 5A + 7I\right] \end{aligned}$$

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$\begin{aligned} A^{-1} &= \frac{1}{3}\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - 5\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \boldsymbol{\frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}} \end{aligned}$$

6. Diagonalize the following matrix, if possible

(a) $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 4 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix}$

(a)

Step 1: Find the eigenvalue,

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ -1 & 0 & 1-\lambda \end{bmatrix} = (2-\lambda)^2(1-\lambda) = 0$$

Eigenvalue: $\lambda = 1$ and $\lambda = 2$.

Step 2: Find three linearly independent eigenvector.

Basis of $\lambda = 1$, $v_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Basis of $\lambda = 2$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Step 3: Construct P from Step 2:

$$P = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Step 4: Construct D from corresponding eigenvalue:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Step 5: Verify $AP = PD$:

$$AP = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$$PD = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

(b)

Step 1: Find the eigenvalue,

Since the matrix is triangle,

Eigenvalue: $\lambda = 2$ and $\lambda = 4$.

Step 2: Find three linearly independent eigenvector.

Basis of $\lambda = 2$, $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Basis of $\lambda = 4$, $v_2 = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$

Every eigenvector of A is a multiple of v_1 or v_2. There are not three linearly independent eigenvector of A. A is not diagonalizable.