

Tutorial 9: Engineering Applications of Integral

1. Find the arc length L of the curve $y = x^{\frac{3}{2}}$ from $x = 0$ to $x = 5$.

$$\begin{aligned}y' &= \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x} \\L &= \int_0^5 \sqrt{1 + (y')^2} \, dx \\&= \int_0^5 \sqrt{1 + \left(\frac{3}{2}\sqrt{x}\right)^2} \, dx \\&= \int_0^5 \sqrt{1 + \frac{9}{4}x} \, dx \\&= \int_0^5 \left(1 + \frac{9}{4}x\right)^{\frac{1}{2}} \, dx \\&= \frac{\left(1 + \frac{9}{4}x\right)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)\left(\frac{9}{4}\right)} \bigg|_0^5 \\&= \frac{\left(1 + \frac{9}{4}(5)\right)^{\frac{3}{2}}}{\frac{27}{8}} - \frac{\left(1 + \frac{9}{4}(0)\right)^{\frac{3}{2}}}{\frac{27}{8}} \\&= \frac{335}{27}\end{aligned}$$

2. Find the area bounded by the parabola $x = 8 + 2y - y^2$, the y-axis, and the lines $y = -1$ and $y = 3$.

$$\begin{aligned}x &= 8 + 2y - y^2 \\&= -(y^2 - 2y - 8) \\&= -((y - 1)^2 - 9) \\&= (4 - y)(2 + y)\end{aligned}$$

\therefore The vertex of the parabola is $(9, 1)$ and cuts the y-axis at $y = 4, y = -2$.

Area bounded by the parabola:

$$\begin{aligned}\int_{-1}^3 (8 + 2y - y^2) dy &= 8y + y^2 - \frac{1}{3}y^3 \Big|_{-1}^3 \\&= \left(8(3) + 3^2 - \frac{1}{3}(3)^3 \right) - \left(8(-1) + (-1)^2 - \frac{1}{3}(-1)^3 \right) \\&= \frac{92}{3}\end{aligned}$$

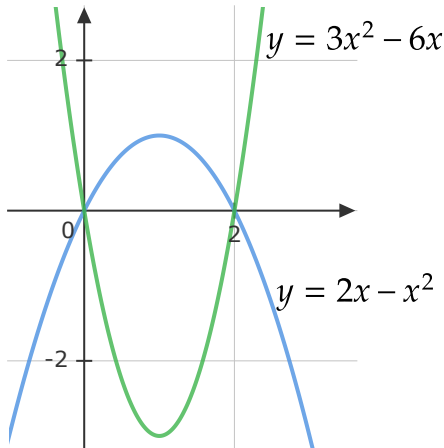
3. Find the arc length of the catenary $y = \frac{a}{2} \left(e^{\frac{x}{a}} + e^{-\frac{x}{a}} \right)$ from $x = 0$ to $x = a$.

$$\begin{aligned} y' &= \frac{a}{2} \left(\frac{1}{a} e^{\frac{x}{a}} - \frac{1}{a} e^{-\frac{x}{a}} \right) + 0 \left(e^{\frac{x}{a}} + e^{-\frac{x}{a}} \right) \\ &= \frac{a}{2} \left(\frac{1}{a} \right) \left(e^{\frac{x}{a}} - e^{-\frac{x}{a}} \right) \\ &= \frac{1}{2} \left(e^{\frac{x}{a}} - e^{-\frac{x}{a}} \right) \end{aligned}$$

Arc length:

$$\begin{aligned} L &= \int_0^a \sqrt{1 + (y')^2} \, dx \\ &= \int_0^a \sqrt{1 + \left[\frac{1}{2} \left(e^{\frac{x}{a}} - e^{-\frac{x}{a}} \right) \right]^2} \, dx \\ &= \int_0^a \sqrt{1 + \frac{1}{4} \left(e^{\frac{2x}{a}} - 2e^0 + e^{-\frac{2x}{a}} \right)} \, dx \\ &= \int_0^a \sqrt{1 + \frac{1}{4} e^{\frac{2x}{a}} - \frac{1}{2} + \frac{1}{4} e^{-\frac{2x}{a}}} \, dx \\ &= \int_0^a \sqrt{\frac{1}{4} e^{\frac{2x}{a}} + \frac{1}{2} + \frac{1}{4} e^{-\frac{2x}{a}}} \, dx \\ &= \int_0^a \sqrt{\frac{1}{4} \left[e^{\frac{2x}{a}} + 2 + e^{-\frac{2x}{a}} \right]} \, dx \\ &= \int_0^a \sqrt{\frac{1}{4} \left(e^{\frac{x}{a}} - e^{-\frac{x}{a}} \right)^2} \, dx \\ &= \int_0^a \frac{1}{2} \left(e^{\frac{x}{a}} - e^{-\frac{x}{a}} \right) \, dx \\ &= \frac{1}{2} \int_0^a \left(e^{\frac{x}{a}} - e^{-\frac{x}{a}} \right) \, dx \\ &= \frac{1}{2} \left[\frac{e^{\frac{x}{a}}}{\frac{1}{a}} - \frac{e^{-\frac{x}{a}}}{\frac{1}{a}} \right]_0^a \\ &= \frac{1}{2} \div a [e^1 + e^{-1} - 1 + 1] \\ &= \frac{a}{2} [e - e^{-1}] \end{aligned}$$

4. Find the centroid of the plane area bounded by the parabolas $y = 2x - x^2$ and $y = 3x^2 - 6x$.



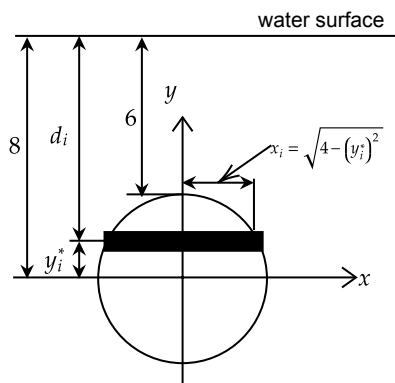
$$\begin{aligned}
 A &= \iint_R dA \\
 &= \int_0^2 \int_{3x^2-6x}^{2x-x^2} dy dx \\
 &= \int_0^2 [2x - x^2 - 3x^2 + 6x] dx \\
 &= \int_0^2 [-4x^2 + 8x] dx \\
 &= \left. \frac{-4x^3}{3} + 4x^2 \right|_0^2 \\
 &= \frac{-4(2)^3}{3} + 4(2)^2 \\
 &= \frac{16}{3}
 \end{aligned}$$

$$\begin{aligned}
 M_y &= \iint_R x dA \\
 &= \int_0^2 \int_{3x^2-6x}^{2x-x^2} x dy dx \\
 &= \int_0^2 x(2x - x^2 - 3x^2 + 6x) dx \\
 &= \int_0^2 [-4x^3 + 8x^2] dx \\
 &= \left. -x^4 + \frac{8}{3}x^3 \right|_0^2 \\
 &= -2^4 + \frac{8}{3}(2)^3 \\
 &= \frac{16}{3}
 \end{aligned}$$

$$\begin{aligned}
 M_x &= \iint_R y dA \\
 &= \int_0^2 \int_{3x^2-6x}^{2x-x^2} y dy dx \\
 &= \frac{1}{2} \int_0^2 [(2x - x^2)^2 - (3x^2 - 6x)^2] dx \\
 &= \frac{1}{2} \int_0^2 [4x^2 - 4x^3 + x^4 - 9x^4 + 36x^3 - 36x^2] dx \\
 &= \frac{1}{2} \int_0^2 [-8x^4 + 32x^3 - 32x^2] dx \\
 &= \frac{1}{2} \left[-\frac{8}{5}x^5 + 8x^4 - \frac{32}{3}x^3 \right]_0^2 \\
 &= \frac{1}{2} \left[-\frac{8}{5}(2)^5 + 8(2)^4 - \frac{32}{3}(2)^3 \right] \\
 &= -\frac{64}{15}
 \end{aligned}$$

Hence, $x = \frac{M_y}{A} = \frac{\frac{16}{3}}{\frac{16}{3}} = 1$, $y = \frac{M_x}{A} = \frac{-\frac{64}{15}}{\frac{16}{3}} = -\frac{4}{5}$. Thus, the centroid is $\left(1, -\frac{4}{5}\right)$.

5. Find the hydrostatic force on a circular plate of radius 2 that is submerged 6 meters in the water.



Assume that the top of the circular plate is 6 meters under the water.

Setting up the axis system such that the origin of the axis is at the center of the plate.

Finally, split up the plate into n horizontal strips each of width Δy . Choosing a point y_i^* from each strip. Assume that the strips are rectangular.

The depth below the water surface of each strip is,

$$d_i = 8 - y_i^*$$

and that in turn gives us the pressure on the strip,

$$P_i = \rho g d_i = 9810(8 - y_i^*)$$

The area of each strip is,

$$A_i = 2\sqrt{4 - (y_i^*)^2} \Delta y$$

The hydrostatic force on each strip is,

$$F_i = P_i A_i = 9810(8 - y_i^*)(2)\sqrt{4 - (y_i^*)^2} \Delta y$$

The total force on the plate is,

$$\begin{aligned} F &= \lim_{n \rightarrow \infty} \sum_{i=1}^n 19620(8 - y_i^*)\sqrt{4 - (y_i^*)^2} \Delta y \\ &= 19620 \int_{-2}^2 (8 - y)\sqrt{4 - y^2} dy \\ &= 19620 \int_{-2}^2 8\sqrt{4 - y^2} dy - 19620 \int_{-2}^2 y\sqrt{4 - y^2} dy \end{aligned}$$

The first integral requires the trig substitution $y = 2 \sin \theta$ and the second integral needs the substitution $v = 4 - y^2$.

$$\begin{aligned} F &= 627840 \int_{-\pi/2}^{\pi/2} \cos^2 \theta d\theta + 9810 \int_0^0 \sqrt{v} dv \\ &= 313920 \int_{-\pi/2}^{\pi/2} 1 + \cos(2\theta) d\theta + 0 \\ &= 313920 \left(\theta + \frac{1}{2} \sin(2\theta) \right) \Big|_{-\pi/2}^{\pi/2} \\ &= 313920\pi \end{aligned}$$