Tutorial 4: Vector Algebra I

1. Graphical interpretation and the effect of scalar multiplication

(i) For the position vector $\underset{\sim}{a} = \langle 2, 4 \rangle$**, compute** $3\underset{\sim}{a}$**,** $\frac{1}{2}\underset{\sim}{a}$**, and** $-2\underset{\sim}{a}$**. Sketch all four vectors on the same axis system. Discuss the effect of scalar multiplication on the magnitude and direction of the original vector.**

(ii) Determine if the sets of vectors are parallel or not.

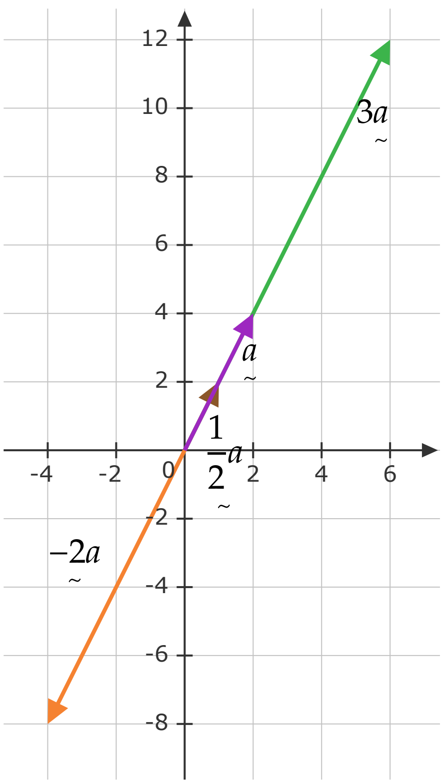
a. $\underset{\sim}{a} = \langle 2, 4, -1 \rangle$, $\underset{\sim}{b} = \langle -6, -12, 3 \rangle$

b. $\underset{\sim}{a} = \langle 4, 10 \rangle$, $\underset{\sim}{b} = \langle 2, -9 \rangle$

(iii) Find unit vector that has the same direction $\underset{\sim}{u} = \langle -5, 2, 1 \rangle$

(i)

$3\underset{\sim}{a} = \langle 6, 12 \rangle$, $\frac{1}{2}\underset{\sim}{a} = \langle 1, 2 \rangle$, $-2\underset{\sim}{a} = \langle -4, -8 \rangle$



- The scalar multiplication affects the magnitude of vectors if the scalar is a negative value.

  For example, $-2\underset{\sim}{a}$ switch the direction in exactly the opposite direction.

- The scalar multiplication affects the magnitude of vectors if the scalar is not equal to 1.

  When scalar > 1, the length of the vector is stretched. (magnitude increased)

  When scalar < 1, the length of the vector is shrinked. (magnitude decreased)

(ii) (a) These two vectors are **parallel** since $\underset{\sim}{b} = -3\underset{\sim}{a}$.

(b) These two vectors are **not parallel** since there is no scalar that can fulfil the scalar multiplication.

(iii) We use $\hat{\underset{\sim}{u}} = \dfrac{\underset{\sim}{u}}{|\underset{\sim}{u}|} = \dfrac{1}{|\underset{\sim}{u}|}\underset{\sim}{u}$.

The chosen scalar is $\dfrac{1}{\underset{\sim}{u}}$ so when multiply to vector $\underset{\sim}{u}$, which has magnitude of $|\underset{\sim}{u}|$, a vector of magnitude 1 is obtained.

Besides, $\hat{\underset{\sim}{u}} || \underset{\sim}{u}$ as scalar multiplication works within them.

2. Let $A(1,3,5)$ and $B(4,6,2)$. Find the point C so that it is located on the line segment AB which divides AB into two segments which are in the ratio 1:3.

$$
\begin{aligned}
\underset{\sim}{c} &= \frac{1}{m+n}\left(n\underset{\sim}{a} + m\underset{\sim}{b}\right) \\
&= \frac{1}{1+3}\left(3\underset{\sim}{a} + \underset{\sim}{b}\right) \\
&= \frac{1}{4}\left(3\underset{\sim}{a} + \underset{\sim}{b}\right) \\
&= \frac{1}{4}(3\langle 1,3,5\rangle + \langle 4,6,2\rangle) \\
&= \frac{1}{4}\langle 7,15,17\rangle \\
&= \langle \mathbf{\tfrac{7}{4}}, \mathbf{\tfrac{15}{4}}, \mathbf{\tfrac{17}{4}} \rangle
\end{aligned}
$$

3. Find the following equation of line for the line L passing through the point $P(3,1,-2)$ and $Q(-2,7,-4)$.

(i) vector equation,

(ii) parametric equation, and

(iii) Cartesian equation

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \langle -2,7,-4\rangle + \langle 3,1,-2\rangle = \langle -5,6,-2\rangle$ is a vector parallel to ine L

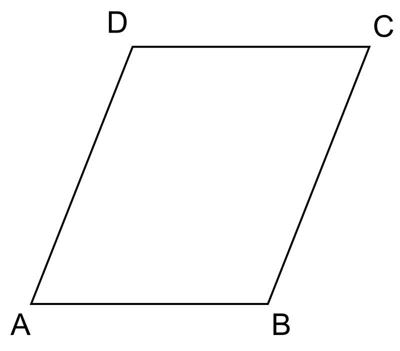
(i) $\underset{\sim}{r} = \underset{\sim}{a} + t\underset{\sim}{v} = \langle 3,1,-2\rangle + t\langle -5,6,-2\rangle$ or $r = \langle -2,7,-4\rangle + t\langle -5,6,-2\rangle$

(ii) Using $\underset{\sim}{r} = \langle 3,1,-2\rangle + t\langle -5,6,-2\rangle$ and comparing each component, we obtain:

$$\begin{aligned} x &= 3-5t \\ y &= 1+6t \\ z &= -2-t \end{aligned}$$

(iii) $\dfrac{x-3}{-5} = \dfrac{y-1}{6} = \dfrac{z+2}{-2}$

4. Let $ABCD$ be a parallelogram. If $A(3,2,-5)$, $B(4,1,0)$ and $C(1,1,4)$ are three vertices of parallelogram. Find point D.



$\overrightarrow{AD}||\overrightarrow{BC}$ or $\overrightarrow{AB}||\overrightarrow{CD}$

$\overrightarrow{AD} = \alpha\overrightarrow{BC}$ where $\alpha = 1$ (Same magnitude and direction)

$$\begin{aligned} \overrightarrow{AD} &= \overrightarrow{BC} \\ \overrightarrow{OD} - \overrightarrow{OA} &= \overrightarrow{OC} - \overrightarrow{OB} \\ \overrightarrow{OD} - \langle 3,2,-5\rangle &= \langle 1,1,4\rangle - \langle 4,1,0\rangle \\ \overrightarrow{OD} &= \langle 0,2,-1\rangle \end{aligned}$$

$\therefore D = (0,2,-1)$

5. Let $\overrightarrow{OP} = \underset{\sim}{i} + 3\underset{\sim}{j} - 7\underset{\sim}{k}$ and $\overrightarrow{OQ} = 5\underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}$

(i) Find the unit vector in the direction of $\overrightarrow{PQ}$

(ii) Find the direction cosines of $\overrightarrow{PQ}$

(iii) Find the vector of magnitude 5 in the direction of $\overrightarrow{QP}$ in polar form

(i)

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{OQ} - \overrightarrow{OP}\\ &= \langle 5, -2, 4\rangle - \langle 1, 3, -7\rangle\\ &= \langle 4, -5, 11\rangle\end{aligned}$$

$$\begin{aligned}\hat{\overrightarrow{PQ}} &= \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|}\\ &= \frac{\langle 4, -5, 11\rangle}{\sqrt{(4)^2 + (-5)^2 + (11)^2}}\\ &= \mathbf{\frac{1}{9\sqrt{2}}\langle 4, -5, 11\rangle}\end{aligned}$$

(ii) Comparing each component, we obtain:

- For component $\underset{\sim}{i}$, $\cos\alpha = \dfrac{4}{9\sqrt{2}}$
- For component $\underset{\sim}{j}$, $\cos\beta = \dfrac{-5}{9\sqrt{2}}$
- For component $\underset{\sim}{k}$, $\cos\gamma = \dfrac{11}{9\sqrt{2}}$

(iii)

$$\begin{aligned}\cos\alpha &= \frac{4}{9\sqrt{2}} \implies \alpha = 71.68°\\ \cos\beta &= \frac{-5}{9\sqrt{2}} \implies \beta = 113.13°\\ \cos\gamma &= \frac{11}{9\sqrt{2}} \implies \gamma = 30.20°\end{aligned}$$

Direction $\overrightarrow{PQ}$ is in the direction of its unit vector, $\langle\cos(71.68°), \cos(113.13°), \cos(30.20°)\rangle$.

Direction $\overrightarrow{QP}$ is in the direction of its unit vector,
$\langle\cos(180 - 71.68°), \cos(180 - 113.13°), \cos(180 - 30.20°)\rangle$
$= \langle\cos(108.32°), \cos(66.87°), \cos(149.80°)\rangle$

Vector of magintude 5 in the direction $\overrightarrow{QP}$ is $5\langle\cos(108.32°), \cos(66.87°), \cos(149.80°)\rangle$.

6. Let $A(1,2)$ and $B(3,4)$

(i) Find the vector equation of the line L passing points A and B

(ii) Sketch the line for $t = 0:1:5$ and indicate its direction and initial point.

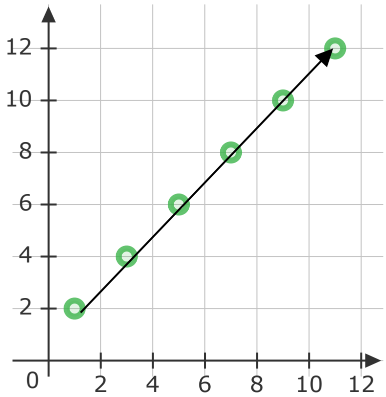
(i)

Direction of vector is parallel with $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \langle 3,4 \rangle - \langle 1,2 \rangle = \boldsymbol{\langle 2,2 \rangle}$

Vector equation of line L is $\underset{\sim}{r} = \underset{\sim}{a} + t\underset{\sim}{v} = \boldsymbol{\langle 1,2 \rangle + \mathrm{t}\langle 2,2 \rangle}$

(ii)

t	$\underset{\sim}{r}$
0	$\langle 1,2 \rangle$
1	$\langle 3,4 \rangle$
2	$\langle 5,6 \rangle$
3	$\langle 7,8 \rangle$
4	$\langle 9,10 \rangle$
5	$\langle 11,12 \rangle$



7. If a unit vector $\vec{a}$ makes angles $\dfrac{\pi}{3}$ with i, $\dfrac{\pi}{4}$ with j and acute angle θ with k then find θ and hence components of $\vec{a}$.

Let α, β, γ to represent angles of $\vec{a}$, and $\vec{a} = a_1 i + a_2 j + a_3 k$.

Given $|\vec{a}| = 1$,

$$\cos\alpha = \frac{a_1}{|a_1|} \qquad \cos\left(\frac{\pi}{3}\right) = \frac{a_1}{1} \qquad a_1 = \frac{1}{2}$$

$$\cos\beta = \frac{a_2}{|a_2|} \qquad \cos\left(\frac{\pi}{4}\right) = \frac{a_1}{1} \qquad a_1 = \frac{1}{\sqrt{2}}$$

$$\cos\gamma = \frac{a_3}{|a_3|} \qquad \cos\theta = \frac{a_3}{1} \qquad a_3 = \cos\theta$$

$$\therefore\ \vec{a} = \frac{1}{2}i + \frac{1}{\sqrt{2}}j + \cos\theta k$$

$$\begin{aligned}
|\vec{a}| &= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + (\cos\theta)^2} \\
1 &= \sqrt{\frac{1}{4} + \frac{1}{2} + \cos^2\theta} \\
1 &= \frac{3}{4} + \cos^2\theta \\
\cos^2\theta &= \frac{1}{4} \\
\cos\theta &= \pm\frac{1}{2} \\
\theta &= 60^\circ\ or\ 120^\circ
\end{aligned}$$

$$\therefore\ \vec{a} = \frac{1}{2}i + \frac{1}{\sqrt{2}}j + \frac{1}{2}k \quad or \quad \vec{a} = \frac{1}{2}i + \frac{1}{\sqrt{2}}j - \frac{1}{2}k$$

8. **If $\vec{a}$ is a unit vector and $(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a}) = 8$, then find $|\vec{x}|$.**

Since $\vec{a}$ is a unit vector, $|\vec{a}| = 1$.

$$\begin{aligned}(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a}) &= 8\\ \vec{x}\cdot\vec{x}+\vec{x}\cdot\vec{a}-\vec{a}\cdot\vec{x}-\vec{a}\cdot\vec{a} &= 8\\ |\vec{x}|^2-|\vec{a}|^2 &= 8\\ |\vec{x}|^2-1 &= 8\\ |\vec{x}|^2 &= 9\\ |\vec{x}| &= 3\end{aligned}$$

Note that magnitude of vector is non-negative.