Tutorial 14: Frobenius Method

Part 1: Using method of reduction of order, find y_2 such that y_1, y_2 basis.

1. $2t^2y'' + ty' - 3y = 0;$ $\quad\quad y_1(t) = t^{-1}, t \neq 0$

$$\begin{aligned} y_2(t) &= v(t)y_1(t) \\ y_2(t) &= t^{-1}v \\ y_2{}'(t) &= -t^{-2}v + t^{-1}v' \\ y_2{}''(t) &= 2t^{-3}v - 2t^{-2}v' + t^{-1}v'' \end{aligned}$$

Substituting back yields,

$$\begin{aligned} 2t^2(2t^{-3}v - 2t^{-2}v' + t^{-1}v'') + t(-t^{-2}v + t^{-1}v') - 3(t^{-1}v) &= 0 \\ 2tv'' + (-4+1)v' + (4t^{-1} - t^{-1} - 3t^{-1})v &= 0 \end{aligned}$$

[△ Ignoring the non-differentiated terms]

$$2tv'' - 3v' = 0$$

Changing variables, $w = v' \rightarrow w' = v''$, Thus,

$$\begin{aligned} 2tw' - 3w &= 0 \\ 2t\frac{dw}{dt} &= 3w \\ \int \frac{1}{3w}dw &= \int \frac{1}{2t}dt \\ \frac{1}{3}\ln|w| &= \frac{1}{2}\ln|t| + C \\ \ln|w| &= \ln t^{\frac{3}{2}} + C \\ w &= Ct^{\frac{3}{2}} \end{aligned}$$

And,

$$v' = w = Ct^{\frac{3}{2}} \rightarrow v = \int Ct^{\frac{3}{2}}dt = \frac{2}{5}Ct^{\frac{2}{5}} + K$$

Letting $C = \dfrac{5}{2}$ and $K = 0$,

$$v(t) = t^{\frac{5}{2}} \rightarrow y_2(t) = t^{-1}t^{\frac{5}{2}} = t^{\frac{3}{2}}$$

$$\therefore y_2(t) = t^{\frac{3}{2}}$$

The $y(t)$ is $y(t) = C_1 t^{-1} + C_2 t^{\frac{3}{2}}$

2. $t^2y'' - (t^2 + 2t)y' + (t + 2)y = 0$ **;** $y_1(t) = t$

$$\begin{aligned} y_2(t) &= v(t)y_1(t) \\ y_2(t) &= tv \\ y_2'(t) &= v + tv' \\ y_2''(t) &= v' + v' + tv'' = 2v' + tv'' \end{aligned}$$

Substituting back yields,

$$\begin{aligned} t^2(2v' + tv'') - (t^2 + 2t)(v + tv') + (t + 2)(tv) &= 0 \\ 2t^2v' + t^3v'' - t^2v - t^3v' - 2tv - 2t^2v' + t^2v + 2tv &= 0 \\ v'(2t^2 - t^3 - 2t^2) + v''(t^3) - t^2v + t^2v - 2tv + 2tv &= 0 \\ t^3v'' - t^3v' &= 0 \\ v'' - v' &= 0 \end{aligned}$$

A solution to the equation is $v' = e^t$. Integrate this yield,

$$v(t) = e^t$$

$$\therefore \boldsymbol{y_2(t) = te^t}$$

The $y(t)$ is: $y(t) = C_1t + C_2te^t$

3. $y'' + 6y' + 9y = 0;$ $\quad\quad$ $y_1(t) = e^{-3t}$

$$
\begin{aligned}
y_2(t) &= v(t)y_1(t) \\
y_2(t) &= ve^{-3t} \\
y_2{}'(t) &= v'e^{-3t} - 3e^{-3t}v \\
y_2{}''(t) &= v''e^{-3t} - 6v'e^{-3t} + 9ve^{-3t}
\end{aligned}
$$

Substituting back yields,

$$
(v''e^{-3t} - 6v'e^{-3t} + 9ve^{-3t}) + 6(v'e^{-3t} - 3e^{-3t}v) + 9(ve^{-3t}) = 0
$$

[△ Ignoring the non-differentiated terms]

$$
v''e^{-3t} = 0
$$

This gives two possibilities, $v'' = 0$ or $e^{-3t} = 0$. However, $e^{-3t} = 0$ only true if $t = \infty$.

$$
\begin{aligned}
v'' &= 0 \\
\frac{d}{dt}(\frac{dv}{dt}) &= 0 \\
\iint d(dv) &= \iint dt \\
\int v &= \int Adt \\
v &= At + B
\end{aligned}
$$

Pick A and B so that second linearly independent solution is obtained.

- If $A = 0$, $y_2 = Be^{-3t}$. $y_1$and y_2 are linearly dependent.
- If $B = 0$, $y_2 = Ate^{-3t}$. $y_1$and y_2 are linearly independent. This is chosen and let A

$$
\therefore y_2(t) = te^{-3t}
$$

The $y(t)$ is $y(t) = (C_1t + C_2)e^{-3t}$

4. $(x-1)y'' - xy' + y = 0\,;\qquad y_1(x) = e^x, x > 1$

$$
\begin{aligned}
y_2(x) &= v(x)y_1(x) \\
y_2(x) &= ve^x \\
y_2{}'(x) &= v'e^x + ve^x \\
y_2{}''(x) &= v''e^x + 2v'e^x + ve^x
\end{aligned}
$$

Substituting back yields,

Let $C_1 = 1$ and $C_2 = 0$,

$$v(x) = xe^{-x}$$

Thus,

$$y_2(t) = v(x)y_1(x) = xe^{-x}e^{x}$$

$$\therefore \boldsymbol{y_2(x) = x}$$

The $y(x)$ is $y(x) = C_1 e^{x} + C_2 x$

5. $xy'' - y' + 4x^3y = 0$; $\quad y_1(x) = \sin x^2, x > 0$

$$\begin{aligned} y_2(t) &= v(t)y_1(t) \\ y_2(t) &= v\sin x^2 \\ y_2{}'(t) &= v'\sin x^2 + 2xv\cos x^2 \\ y_2{}''(t) &= v''\sin x^2 + 2xv'\cos x^2 + (2v + 2xv')\cos x^2 - 4x^2v\sin x^2 \\ &= v''\sin x^2 + 4xv'\cos x^2 + 2v\cos x^2 - 4x^2v\sin x^2 \end{aligned}$$

Substituting back yields,

$$\begin{aligned} x(v''\sin x^2 + 4xv'\cos x^2 + 2v\cos x^2 - 4x^2v\sin x^2)- \\ v'\sin x^2 - 2xv\cos x^2 + 4x^3(v\sin x^2) &= 0 \\ xv''\sin x^2 + (4x^2\cos x^2 - \sin x^2)v' &= 0 \end{aligned}$$

Let $w = v'$ and so $w' = v''$.

$$\begin{aligned} xw'sinx^2 + (4x^2\cos x^2 - \sin x^2)w &= 0 \\ w' + \frac{(4x^2\cos x^2 - \sin x^2)w}{x\sin x^2} &= 0 \\ \frac{dw}{dx} &= \frac{(4x^2\cos x^2 - \sin x^2)w}{x\sin x^2} \\ \frac{dw}{w} &= \int \frac{1}{x}dx - \int 4x\frac{\cos x^2}{\sin x^2}dx \\ \ln|w| &= \ln|x| - \left|\left(\sin x^2\right)^2\right| + C \\ \ln|w| &= \ln\left|\frac{x}{(\sin x^2)^2}\right| + C \\ w &= C\frac{x}{(\sin x^2)^2} \end{aligned}$$

Therefore,

$$\begin{aligned} v'(x) &= w = C\frac{x}{(\sin x^2)^2} \\ v(x) &= \int C\frac{x}{(\sin x^2)^2}dx \\ &= -\frac{1}{2}C\cot x^2 + A \end{aligned}$$

Let $C = -2$ and $K = 0$,

$$v(x) = \cot x^2$$

$$\therefore y_2(t) = \cot x^2 \sin x^2 = \cos x^2$$

The $y(t)$ is $y(t) = C_1 \sin x^2 + C_2 \cos x^2$

**Part 2: Discuss whether two Frobenius series solutions exist or do
for the following equations.**

1. $2x^2y'' + x(x+1)y' - (\cos x)y = 0$

$$\begin{aligned} 2x^2y'' + x(x+1)y' - (\cos x)y &= 0 \\ x^2y'' + \frac{(x+1)xy'}{2} - \frac{\cos x}{2}y &= 0 \end{aligned}$$

$$p(x) = \frac{x+1}{2} \Rightarrow p(0) = \frac{1}{2}$$

$$q(x) = \frac{-\cos x}{2} \Rightarrow q(0) = -\frac{1}{2}$$

Substituting r to find r_1 and r_2,

$$r^2 + (p(0)-1)r + q(0) = 2r^2 - r - 1 = 0$$

$$\Rightarrow r_1 = 1, r_2 = -\frac{1}{2}$$

$$\therefore r_1 - r_2 = 1 - (-\frac{1}{2}) = \frac{3}{2} \quad \triangleright \text{ not a zero or positive integer}$$

Two Frobenius series solutions exist.

2. $x^4y'' - (x^2 \sin x)y' + 2(1 - \cos x)y = 0$

$$x^4y'' - (x^2 \sin x)y' + 2(1 - \cos x)y = 0$$

$$x^2y'' - \frac{\sin x}{x}xy' + 2\frac{1 - \cos x}{x^2}y = 0$$

$$p(x) = \frac{-\sin x}{x} \Rightarrow p(0) = -1$$

$$q(x) = \frac{2(1 - \cos x)}{x^2} \Rightarrow q(0) = 1$$

Substituting r to find r_1 and r_2,

$$r^2 + (p(0) - 1)r + q(0) = r^2 - 2r - 1 = 0$$

$$\Rightarrow r_1 = 1, r_2 = 1$$

One Frobenius series solution exists since $r_1 = r_2$.

Part 3: Apply Frobenius Method to find the basis of solutions of the fol differential equations.

1. $2xy'' + y' + y = 0$

$x = 0$ is a regular singular solution.

Assuming the solution is $y = \sum_{k=0}^{\infty} a_k x^{k+r}$,

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

∞

For a_0,

$$[2(r^2-r)+r]a_0 = 0$$

Since $a_0 \neq 0 \Rightarrow r(2r-1) = 0 \Rightarrow r_2 = 0, r_1 = \frac{1}{2}$,

The recurrence relation is then,

$$\begin{aligned}[2(n+r+1)(n+r)+(n+r+1)]a_{n+1}+a_n &= 0\\ (n+r+1)(2n+2r+1)a_{n+1} &= -a_n\\ a_{n+1} &= \frac{-a_n}{(n+r+1)(2n+2r+1)}, n = 0,1,2\end{aligned}$$

For $r = \frac{1}{2}$,

$$a_{n+1} = -\frac{a_n}{(n+\frac{3}{2})(2n+2)}$$

$$n = 0, a_1 = -\frac{a_0}{(\frac{3}{2})(2)} = -\frac{a_0}{3}$$

$$n = 1, a_2 = \frac{a_1}{(\frac{5}{2})(4)} = \frac{a_0}{10\cdot 3}$$

$$n = 2, a_3 = -\frac{a_2}{(\frac{7}{2})(6)} = -\frac{a_0}{21\cdot 10\cdot 3}$$

$$y_1(x) = \sum_{k=0}^{\infty} a_k x^{k+r} = a_0 x^{\frac{1}{2}} + a_1 x^{\frac{3}{2}} + \ldots$$

$$y_1(x) = a_0 x^{\frac{1}{2}} - \frac{a_0}{3}x^{\frac{3}{2}} + \frac{a_0}{10\cdot 3}x^{\frac{5}{2}} - \frac{a_0}{21\cdot 10\cdot 3}x^{\frac{7}{2}} + \ldots$$

$$\boldsymbol{y_1(x) = a_0 x^{\frac{1}{2}}\left[1 - \frac{x}{3} + \frac{x^2}{10\cdot 3} - \frac{x^3}{21\cdot 10\cdot 3} + \ldots\right]}$$

For $r = 0$,

$$a_{n+1} = -\frac{a_n}{(n+1)(2n+1)}$$

$$n = 0, a_1 = -\frac{a_0}{} = -a_0$$

$$(1)(1)$$

$$n = 1, a_2 = \frac{a_1}{(2)(3)} = \frac{a_0}{6}$$

$$n = 2, a_3 = -\frac{a_2}{(3)(5)} = -\frac{a_0}{15 \cdot 6}$$

$$y_2(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 x^0 + a_1 x^1 + \ldots$$

$$y_2(x) = a_0 - a_0 x + \frac{a_0}{6} x^2 - \frac{a_0}{15 \cdot 6} x^3 + \ldots$$

$$\boldsymbol{y_2(x) = a_0 \left[1 - x + \frac{x^2}{6} - \frac{x^3}{15 \cdot 6} + \ldots \right]}$$

2. $xy'' + 2y' + xy = 0$

$x = 0$ is a regular singular solution.

Assuming the solution is $y = \sum_{k=0}^{\infty} a_k x^{k+r}$,

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

∞

For a_0, $(r^2 - r + 2r)a_0 = 0$.

Since

$$a_0 \neq 0 \Rightarrow r^2 + r = 0 \Rightarrow r_1 = 0, r_2 = -1$$

The recurrence relation is then,

$$\begin{aligned}
[(n+r+1)(n+r) + 2(n+r+1)]a_{n+1} + a_{n-1} &= 0 \\
(n+r+1)(n+r+2)a_{n+1} &= -a_{n-1} \\
a_{n+1} &= \frac{-a_{n-1}}{(n+r+1)(n+r+2)}, n = 0, 1, 2
\end{aligned}$$

For $r = 0$,

$$a_{n+1} = -\frac{a_{n-1}}{(n+1)(n+2)}$$

$$\begin{aligned}
n &= 0, [(r^2 + r) + 2(r+1)]a_1 = 0 \Rightarrow a_1 = 0 \\
n &= 1, a_2 = \frac{-a_0}{3 \cdot 2} = \frac{-a_0}{3!} \\
n &= 2, a_3 = -\frac{-a_1}{12} = 0 \\
n &= 3, a_4 = \frac{-a_2}{20 \cdot 3!} = \frac{-a_0}{5!} \\
n &= 4, a_5 = -\frac{-a_3}{30} = 0
\end{aligned}$$

$$\begin{aligned}
y_1(x) &= \sum_{k=0}^{\infty} a_k x^{k+r} = a_0 x^0 + a_1 x^1 + \dots \\
y_1(x) &= a_0 + 0 - \frac{a_0}{3!}x^2 + 0 + \frac{a_0}{5!} + \dots \\
\boldsymbol{y_1(x)} &= \boldsymbol{a_0\left(1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 + \dots\right)}
\end{aligned}$$

For $r = -1$,

$$a_{n+1} = \frac{-a_{n-1}}{n(n+1)}, n = 1, 2, 3, \ldots$$

$$n = 0, [(r^2 + r) + 2(r+1)]a_1 = 0 \Rightarrow 0 \cdot a_1 = 0 \;\; \triangleright \; a_1 \neq 0 \text{(unknown)}$$

$$n = 1, a_2 = \frac{-a_0}{1 \cdot 2} = \frac{-a_0}{2!}$$

$$n = 2, a_3 = -\frac{-a_1}{2 \cdot 3} = 0$$

$$n = 3, a_4 = \frac{-a_2}{3 \cdot 4} = \frac{a_0}{4!}$$

$$n = 4, a_5 = -\frac{-a_3}{4 \cdot 5} = \frac{a_1}{5!}$$

$$y_2(x) = \sum_{k=0}^{\infty} a_k x^{k+r} = a_0 x^{-1} + a_1 x^0 + a_2 x^1 + \ldots$$

$$y_2(x) = \frac{1}{x} a_0 + a_1 - \frac{a_0}{2!} x - \frac{a_1}{3!} x^2 + \ldots$$

$$y_2(x) = a_0 \left(\frac{1}{x} - \frac{1}{2!} x + \frac{1}{4!} x^3 + \ldots\right) + a_1 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \ldots\right)$$

$$\boldsymbol{y_2(x) = a_0 \frac{1}{x} \cos x + a_1 \frac{1}{x} \sin x}$$

3. $xy'' + (1 - 2x)y' + (x - 1)y = 0$

$x = 0$ is a regular singular solution.

Assuming the solution is $y = \sum_{k=0}^{\infty} a_k x^{k+r}$,

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

∞

Collecting the non-summation terms,

$$([(r^2+r)+(r+1)]a_1+(-2r-1)a_0) = 0$$
$$(r^2+2r+1)a_1-(2r+1)a_0 = 0 \rhd \text{ will be used to find } a_0, a_1$$

Since $a_0 \neq 0 \Rightarrow r^2 = 0 \Rightarrow r_1 = 0, r_2 = 0$.

The recurrence relation is then,

$$([(n+r)(n+r+1)+(n+r+1)]a_{n+1}+a_{n-1}-[2(n+r)+1]a_n) = 0$$
$$([(n+r+1)(n+r+1)]a_{n+1}+a_{n-1}-[2(n+r+1)a_n]) = 0$$
$$a_{n+1} = \frac{[2(n+r)+1]a_n-a_{n-1}}{(n+r+1)^2}, n = 1,2,3...$$

For $r = 0$,

$$a_{n+1} = \frac{(2n+1)a_n-a_{n-1}}{(n+1)^2}, n = 1,2,3,...$$

From Eq(1), $(0+2(0)+1)a_1-(2(0)+1)a_0 = 0 \Rightarrow a_1 = a_0$

$$n = 1, a_2 = \frac{3a_1-a_0}{2^2} = \frac{3a_0-a_0}{2^2} = \frac{a_0}{2}$$

$$n = 2, a_3 = -\frac{5a_2-a_1}{3^2} = \frac{5(\frac{a_0}{2})-a_1}{3^2} = \frac{a_0}{3!}$$

$$n = 3, a_4 = \frac{7a_3-a_2}{4^2} = \frac{7(\frac{a_0}{3!})-\frac{a_0}{2}}{4^2} = \frac{a_0}{4!}$$

$$y_1(x) = \sum_{k=0}^{\infty} a_k x^{k+r} = a_0x^0+a_1x^1+...$$

$$y_1(x) = a_0+a_0x+\frac{a_0}{2!}x^2+\frac{a_0}{3!}x^3 <u> ...$$

$$y_1(x) = a_0(1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^4+...)$$

$$\boldsymbol{y_1(x) = a_0e^x}$$

For the second $r = 0$, using reduction of order.

Knowing $y_1 = e^x$,

$$
\begin{aligned}
y_2 &= ve^x \\
y_2' &= ve^x + v'e^x \\
y_2'' &= ve^x + 2v'e^x + v''e^x
\end{aligned}
$$

Substituting into main equation,

$$\begin{aligned}
x(v+2v'+v'')+(1-2x)(ve^x+v'e^x)+(x-1)ve^x &= 0 \\
x(v+2v'+v'')+(1-2x)(v+v')+(x-1)v &= 0 \\
\text{[} \triangle \text{ Diving out } x^r \text{]} & \\
xv+2xv'+xv''+v-2xv'+v'-2xv+xv-v &= 0 \\
v'+xv'' &= 0 \\
xv'' &= -v' \\
\frac{v''}{v'} &= -\frac{1}{x} \\
\int \frac{v''}{v'}dv &= -\int \frac{1}{x}dx \\
\ln v' &= -\ln x = \ln x^{-1} \\
v' &= x^{-1} \\
\int v'dv &= \int \frac{1}{x}dx \\
v &= \ln x
\end{aligned}$$

$$\therefore \boldsymbol{y_2(x) = (\ln x)e^x}$$

4. $2ty'' + (1 + t)y' + y = 0$

$x = 0$ is a regular singular solution.

Assuming the solution is $y = \sum_{k=0}^{\infty} a_k x^{k+r}$.

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

∞

The recurrence relation is then,

$$\begin{aligned}
[2(n+r)(n+r+1)+(n+r+1)]a_{n+1}+[(n+r)+1]a_n &= 0\\
(n+r+1)(2n+2r+1)a_{n+1} &= -[(n+r)+1]a_n\\
a_{n+1} &= \frac{-[(n+r)+1]a_n}{(n+r+1)(2n+2r+1)}, n =
\end{aligned}$$

For $r=-\frac{1}{2}$,

$$a_{n+1} = \frac{a_n}{2n+2}, n=0,1,2,3,...$$

$$\begin{aligned}
n &= 0, a_1 = -\frac{a_0}{2}\\
n &= 1, a_2 = -\frac{a_1}{2\cdot 2} = \frac{a_0}{2^2\cdot 2!}\\
n &= 2, a_3 = -\frac{a_2}{2\cdot 3} = -\frac{a_0}{2^3\cdot 3!}\\
n &= 3, a_4 = -\frac{a_3}{2\cdot 4} = \frac{a_0}{2^4\cdot 4!}
\end{aligned}$$

$$\begin{aligned}
y_1(t) &= \sum_{k=0}^{\infty} a_k t^{k+r} = a_0 t^{\frac{1}{2}} + a_1 t^{\frac{3}{2}} + ...\\
y_1(t) &= a_0 t^{\frac{1}{2}} - \frac{a_0}{2!}t^{\frac{3}{2}} + \frac{a_0}{2^2\cdot 2!}t^{\frac{5}{2}} - \frac{a_0}{2^3\cdot 3!}t^{\frac{7}{2}} + ...\\
\boldsymbol{y_1(t)} &= \boldsymbol{a_0 t^{\frac{1}{2}}\left(1-\frac{1}{2!}t+\frac{1}{2^2\cdot 2!}t^2-\frac{1}{2^3\cdot 3!}t^3+...\right)}
\end{aligned}$$

For $r=0$,

$$a_{n+1} = -\frac{a_n}{2n+1}, n=0,1,2,3,...$$

$$\begin{aligned}
n &= 0, a_1 = -a_0\\
n &= 1, a_2 = -\frac{a_1}{3} = \frac{a_0}{3}
\end{aligned}$$

$$n = 2, a_3 = -\frac{a_2}{5} = -\frac{a_0}{5 \cdot 3}$$
$$n = 3, a_4 = -\frac{a_3}{7} = \frac{a_0}{7 \cdot 5 \cdot 3}$$

$$y_2(t) = \sum_{k=0}^{\infty} a_k t^{k+r} = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots$$

$$y_2(t) = a_0 - a_0 t + \frac{a_0}{3} t^2 - \frac{a_0}{5 \cdot 3} t^3 + \ldots$$

$$\boldsymbol{y_2(t) = a_0 \left(1 - t + \frac{1}{3} t^2 - \frac{1}{5 \cdot 3} t^3 + \frac{1}{7 \cdot 5 \cdot 3} t^4 - \ldots \right)}$$

5. $x(1-x)y'' - 3xy' - y = 0$

The motivation behind Frobenius method is to seek a power series solution to ordinary diff
equations.

Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

Then we get that

First note that $a_0 = 0$. Choose a_1 arbitrarily. Then we get that $a_2 = 2a_1$, $a_3 = 3a_1$, $a_4 = 4a$
general, $a_n = na_1$.

Hence, the solution is given by

$$y_1(x) = a_1\left(x + 2x^2 + 3x^3 + \cdots\right)$$

This power series is valid only within $|x| < 1$. In this region, we can simplify the power ser

$$\begin{aligned} y_1(x) &= a_1 x\left(1 + 2x + 3x^2 + \cdots\right) \\ &= a_1 x \frac{d}{dx}\left(x + x^2 + x^3 + \cdots\right) \\ &= a_1 x \frac{d}{dx}\left(\frac{x}{1-x}\right) \\ &= a_1 \frac{x}{(1-x)^2} \end{aligned}$$

Taking $a_1 = 1$,

$$\boldsymbol{y_1(x) = \frac{x}{(1-x)^2}}$$

The order reduction method seeks a second basis solution in the form $y = y_1 u$, where

$$y_1(x) = \frac{x}{(1-x)^2}$$

is the already found basis solution.

$$x(1-x)[y_1 u'' + 2y_1' u'] - 3x[y_1 u'] = 0 \Longrightarrow \frac{u''}{u'} = \frac{3y_1 - 2(1-x)y_1'}{(1-x)y_1}$$

Insert $y_1(x) = \dfrac{1}{(1-x)^2} - \dfrac{1}{1-x}$, $y_1' = \dfrac{2}{(1-x)^3} - \dfrac{1}{(1-x)^2}$, $y_1'' = \dfrac{6}{(1-x)^4} - \dfrac{2}{(1-x)^3}$ int
formula to find

$$\frac{u''}{u'} = \frac{-\frac{1}{(1-x)^2} - \frac{1}{1-x}}{\frac{x}{1-x}} = -\frac{2-x}{x(1-x)} = -\frac{2}{x} + \frac{1}{1-x}$$

$1 \qquad 1 + x \qquad 1$

$$\implies u' = \frac{1}{x^2(1-x)} = \frac{1+x}{x^2} - \frac{1}{1-x}$$

$$\implies u = -\frac{1}{x} + \ln|x(1-x)|$$

So that the second basis solution is

$$y_2 = \frac{x \ln|x(1-x)| - 1}{(1-x)^2}$$