Tutorial 1: Basic Functions & Derivatives

1. Find the limit of $\lim_{x \to 5} \frac{x^2 - 25}{x^2 + x - 30}$.

$$\begin{aligned}
\lim_{x \to 5} \frac{x^2 - 25}{x^2 + x - 30} &= \lim_{x \to 5} \frac{\cancel{(x-5)}(x+5)}{\cancel{(x-5)}(x+6)} \\
&= \lim_{x \to 5} \frac{x+5}{x+6} \\
&= \frac{5+5}{5+6} \\
&= \mathbf{\frac{10}{11}}
\end{aligned}$$

2. Find the limit of $\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9}$.

$$\begin{aligned}
\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9} &= \lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9} \times \frac{\sqrt{x} + 3}{\sqrt{x} + 3} \\
&= \lim_{x \to 9} \frac{\cancel{x-9}}{\cancel{(x-9)}(\sqrt{x}+3)} \\
&= \lim_{x \to 9} \frac{1}{(\sqrt{x}+3)} \\
&= \frac{1}{\sqrt{9}+3} \\
&= \mathbf{\frac{1}{6}}
\end{aligned}$$

3. Find the limit of $\lim_{x \to 0} \frac{\sqrt{2+x} - \sqrt{2}}{x}$.

$$\begin{aligned}
\lim_{x \to 0} \frac{\sqrt{2+x} - \sqrt{2}}{x} &= \lim_{x \to 0} \frac{\sqrt{2+x} - \sqrt{2}}{x} \times \frac{\sqrt{2+x} + \sqrt{2}}{\sqrt{2+x} + \sqrt{2}} \\
&= \lim_{x \to 0} \frac{(\sqrt{2+x})^2 - (\sqrt{2})^2}{x(\sqrt{2+x} + \sqrt{2})} \\
&= \lim_{x \to 0} \frac{2+x-2}{x(\sqrt{2+x} + \sqrt{2})} \\
&= \lim_{x \to 0} \frac{\cancel{x}}{\cancel{x}(\sqrt{2+x} + \sqrt{2})} \\
&= \frac{1}{\sqrt{2} + \sqrt{2}} \\
&= \frac{1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\
&= \mathbf{\frac{\sqrt{2}}{4}}
\end{aligned}$$

4. Find the limit of $\lim_{\theta \to \frac{\pi}{2}} \frac{\tan\theta}{\sec\theta}$.

$$\begin{aligned}\lim_{\theta \to \frac{\pi}{2}} \frac{\tan\theta}{\sec\theta} &= \lim_{\theta \to \frac{\pi}{2}} \frac{\left(\frac{\sin\theta}{\cancel{\cos\theta}}\right)}{\left(\frac{1}{\cancel{\cos\theta}}\right)} \\ &= \lim_{\theta \to \frac{\pi}{2}} \sin\theta \\ &= \sin\frac{\pi}{2} \\ &= \mathbf{1}\end{aligned}$$

5. Find the limit of $\lim_{\theta \to 0} \frac{\cos\theta - 1}{\sin\theta}$.

$$\begin{aligned}\lim_{\theta \to 0} \frac{\cos\theta - 1}{\sin\theta} &= \lim_{\theta \to 0} \frac{\cos\theta - 1}{\sin\theta} \times \frac{\theta}{\theta} \\ &= \lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} \times \frac{\theta}{\sin\theta} \\ &= \lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} \times \lim_{\theta \to 0} \frac{\theta}{\sin\theta} \\ &= \lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} \times \frac{1}{\lim_{\theta \to 0} \frac{\sin\theta}{\theta}} \\ &= 0 \times 1 \\ &= \mathbf{0}\end{aligned}$$

6. If $2x \le g(x) \le x^2 - x + 2$ for all x, evaluate $\lim_{x \to 1} g(x)$.

$$\lim_{x \to 1} 2x \le \lim_{x \to 1} g(x) \le \lim_{x \to 1} (x^2 - x + 2)$$

Therefore,

$$2 \le \lim_{x \to 1} g(x) \le 2$$

Hence,

$$\lim_{x \to 1} g(x) = \mathbf{2}$$

7. Solve y' if $y = \sqrt{3x^2 - 2x + 3}$.

Let $u = 3x^2 - 2x + 3$, therefore $\frac{du}{dx} = 6x - 2$.

Substitute $y = \sqrt{u} = u^{\frac{1}{2}}$, therefore $\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$.

Putting back together,

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\
&= \frac{1}{2}u^{-\frac{1}{2}} \cdot 6x - 2 \\
&= \frac{1}{2}(3x^2 - 2x + 3)^{-\frac{1}{2}} \cdot 6x - 2 \\
&= \frac{6x - 2}{2\sqrt{3x^2 - 2x + 3}} \\
&= \boldsymbol{\frac{3x - 1}{\sqrt{3x^2 - 2x + 3}}}
\end{aligned}
$$

8. Solve y' if $y = 5\sqrt[3]{x^2 + \sqrt{x^3}}$.

$y = 5\sqrt[3]{x^2 + \sqrt{x^3}}$ can be rewrite as $y = 5(x^2 + x^{\frac{3}{2}})^{\frac{1}{3}}$.

Let $u = x^2 + x^{\frac{3}{2}}$, therefore $\frac{du}{dx} = 2x + \frac{3}{2}x^{\frac{1}{2}}$.

Substitute $y = 5u^{\frac{1}{3}}$, therefore $\frac{dy}{du} = \frac{5}{3}u^{-\frac{2}{3}}$.

Putting back together,

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\
&= \frac{5}{3}u^{-\frac{2}{3}} \cdot \left(2x + \frac{3}{2}x^{\frac{1}{2}}\right) \\
&= \boldsymbol{\frac{5}{3}\left(x^2 + x^{\frac{3}{2}}\right)^{-\frac{2}{3}} \cdot \left(2x + \frac{3}{2}x^{\frac{1}{2}}\right)}
\end{aligned}
$$

Alternatively, the equation can be written as

$$
\frac{5\left(2x + \frac{3}{2}\sqrt{x}\right)}{3\left(x^2 + \sqrt{x^3}\right)^{\frac{2}{3}}} = \frac{5\left(\frac{2x}{3} + \frac{1}{2}\sqrt{x}\right)}{\left(x^2 + \sqrt{x^3}\right)^{\frac{2}{3}}} = \frac{5\left(\frac{2x}{3} + \frac{\sqrt{x^3}}{2x}\right)}{\left(x^2 + \sqrt{x^3}\right)^{\frac{2}{3}}}
$$

9. Solve y if $y = \ln \cos x^2$.

Let $u = \cos x^2$, therefore $\frac{du}{dx} = (-\sin x^2)(2x)$.

Substitute $y = \ln u$, therefore $\frac{dy}{du} = \frac{1}{u}$.

Putting back together,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \frac{1}{u} \cdot (-\sin x^2)(2x) \\ &= -\frac{2x \sin x^2}{\cos x^2} \\ &= \mathbf{-2x \tan x^2}\end{aligned}$$

10. Differentiate $y = \log(4 + \cos x)$.

Note that $\frac{d}{dx}(\log_b(x)) = \frac{1}{\ln(b) \cdot x} = \frac{1}{x} \cdot \log e$

$$\begin{aligned}\frac{d}{dx}(\log 4 + \cos x) &= \frac{1}{4 + \cos x} \cdot \log e \cdot \frac{d}{dx}(4 + \cos x) \\ &= \frac{1}{4 + \cos x} \cdot \log e \cdot \sin x \\ &= \boldsymbol{\frac{-(\log e)(\sin x)}{4 + \cos x}}\end{aligned}$$

11. Find y' for $10e^{2xy} = e^{15y} + e^{13x}$.

$$\begin{aligned}10e^{2xy} &= e^{15y} + e^{13x} \\ 10e^{2xy}(2x \cdot y' + 2y) &= e^{15y}(15y') + e^{13x}(13) \\ 10e^{2xy}(2x \cdot y' + 2y) &= 15y'e^{15y} + 13e^{13x} \\ (20e^{2xy} - 15e^{15y})y' &= 13e^{13x} - 20ye^{2xy} \\ y' &= \boldsymbol{\frac{13e^{13x} - 20ye^{2xy}}{20e^{2xy} - 15e^{15y}}}\end{aligned}$$

12. Find $f'(x)$ if $f(x) = 2x(\arctan 5x)^2 + 6\tan(\cos 6x)$.

- $\frac{d}{dx}(\tan^{-1} 5x)^2 = 2\tan^{-1} 5x \cdot \frac{1}{1+(5x)^2} \cdot 5 = \frac{10 \tan^{-1} 5x}{1+25x^2}$
- $\frac{d}{dx} 2x(\tan^{-1} 5x)^2 = 2x(\frac{10 \tan^{-1} 5x}{1+25x^2}) + (\tan^{-1} 5x)^2 \cdot 2 = \frac{20x \tan^{-1} 5x}{1+25x^2} + 2(\tan^{-1} 5x)^2$
- $\frac{d}{dx} 6\tan(\cos 6x) = 6\sec^2(\cos 6x) \frac{d}{dx} \cos 6x = 6\sec^2(\cos 6x) \cdot (-\sin 6x) \cdot 6 = -36(\sec^2(\cos 6x)) \sin 6x$

$$\therefore f'(x) = \boldsymbol{\frac{20x \tan^{-1} 5x}{1 + 25x^2} + 2(\tan^{-1} 5x)^2 - 36(\sec^2(\cos 6x)) \sin 6x}$$

13. Solve y' if $y = 4x\sinh^{-1}\left(\frac{x}{6}\right) + \tanh^{-1}\left(\frac{x}{6}\right) + \tanh^{-1}(\cos 10x)$.

- $\frac{d}{dx}4x\sinh^{-1}(\frac{x}{6}) = 4x\left[\frac{1}{\sqrt{1+(\frac{x}{6})^2}}\cdot\frac{1}{6}\right] + \sinh^{-1}(\frac{x}{6}) = \frac{\frac{2x}{3}}{\sqrt{1+\frac{x^2}{36}}} + 4\sinh^{-1}(\frac{x}{6})$
- $\frac{d}{dx}\tanh^{-1}(\cos 10x) = \frac{1}{1-(\cos 10x)^2}\cdot\frac{d}{dx}\cos 10x = \frac{1}{1-\cos^2 10x}\cdot(-10\sin 10x) = \frac{-10\sin 10x}{\sin^2 10x} = -10\csc 10x$

$$\therefore f'(x) = \frac{\frac{2x}{3}}{\sqrt{1+\frac{x^2}{36}}} + 4\sinh^{-1}(\frac{x}{6}) - 10\csc 10x$$

14. Differentiate $y = \dfrac{1}{\sin^{-1}x}$.

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$$

$$\begin{aligned} y &= (\sin^{-1}x)^{-1} \\ y' &= -(\sin^{-1}x)^{-2}\frac{d}{dx}(\sin^{-1}x) \\ &= -\frac{1}{(\sin^{-1}x^2)\sqrt{1-x^2}} \end{aligned}$$

15. Differentiate $y = (x^3-1)^{100}$.

$$\begin{aligned} y' &= 100(x^3-1)^{99}\frac{d}{dx}(x^3-1) \\ &= 100(x^3-1)^{99}\cdot 3x^2 \\ &= 300x^2(x^3-1)^{99} \end{aligned}$$