Tutorial 9: Engineering Applications of Integral

1. Find the arc length L of the curve $y = x^{\frac{3}{2}}$ from $x = 0$ to $x = 5$.

$$y' = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$$

$$\begin{aligned}
L &= \int_0^5 \sqrt{1 + (y')^2}\, dx \\
&= \int_0^5 \sqrt{1 + \left(\frac{3}{2}\sqrt{x}\right)^2}\, dx \\
&= \int_0^5 \sqrt{1 + \frac{9}{4}x}\, dx \\
&= \int_0^5 \left(1 + \frac{9}{4}x\right)^{\frac{1}{2}} dx \\
&= \left. \frac{\left(1 + \frac{9}{4}x\right)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)\left(\frac{9}{4}\right)} \right|_0^5 \\
&= \frac{\left(1 + \frac{9}{4}(5)\right)^{\frac{3}{2}}}{\frac{27}{8}} - \frac{\left(1 + \frac{9}{4}(0)\right)^{\frac{3}{2}}}{\frac{27}{8}} \\
&= \frac{335}{27}
\end{aligned}$$

2. Find the area bounded by the parabola $x = 8+2y-y^2$, the y-axis, and the lines $y = -1$ and $y = 3$.

$$\begin{aligned} x &= 8+2y-y^2 \\ &= -\left(y^2-2y-8\right) \\ &= -\left((y-1)^2-9\right) \\ &= (4-y)(2+y) \end{aligned}$$

$\therefore$ The vortex of the parabola is $(9, 1)$ and cuts the y-axis at $y = 4$, $y = -2$.

Area bounded by the parabola:

$$\begin{aligned} \int_{-1}^{3} \left(8+2y-y^2\right)\, dy &= 8y+y^2-\frac{1}{3}y^3 \Big|_{-1}^{3} \\ &= \left(8(3)+3^2-\frac{1}{3}(3)^3\right) - \left(8(-1)+(-1)^2-\frac{1}{3}(-1)^3\right) \\ &= \mathbf{\frac{92}{3}} \end{aligned}$$

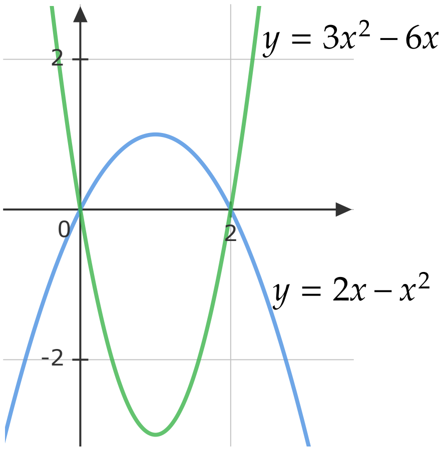
3. Find the arc length of the catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ from $x = 0$ to $x = a$.

$$
\begin{aligned}
y' &= \frac{a}{2}\left(\frac{1}{a}e^{\frac{x}{a}} - \frac{1}{a}e^{-\frac{x}{a}}\right) + 0\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right) \\
&= \frac{a}{2}\left(\frac{1}{a}\right)\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right) \\
&= \frac{1}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right)
\end{aligned}
$$

Arc length:

$$
\begin{aligned}
L &= \int_0^a \sqrt{1 + (y')^2}\, dx \\
&= \int_0^a \sqrt{1 + \left[\frac{1}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right)\right]^2}\, dx \\
&= \int_0^a \sqrt{1 + \frac{1}{4}\left(e^{\frac{2x}{a}} - 2e^0 + e^{-\frac{2x}{a}}\right)}\, dx \\
&= \int_0^a \sqrt{1 + \frac{1}{4}e^{\frac{2x}{a}} - \frac{1}{2} + \frac{1}{4}e^{-\frac{2x}{a}}}\, dx \\
&= \int_0^a \sqrt{\frac{1}{4}e^{\frac{2x}{a}} + \frac{1}{2} + \frac{1}{4}e^{-\frac{2x}{a}}}\, dx \\
&= \int_0^a \sqrt{\frac{1}{4}\left[e^{\frac{2x}{a}} + 2 + e^{-\frac{2x}{a}}\right]}\, dx \\
&= \int_0^a \sqrt{\frac{1}{4}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)^2}\, dx \\
&= \int_0^a \frac{1}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right) dx \\
&= \frac{1}{2}\int_0^a \left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right) dx \\
&= \frac{1}{2}\left[\frac{e^{\frac{x}{a}}}{\frac{1}{a}} - \frac{e^{-\frac{x}{a}}}{\frac{1}{a}}\Bigg|_0^a\right] \\
&= \frac{1}{2} \div a\left[e^1 - e^{-1} - 1 + 1\right] \\
&= \mathbf{\frac{a}{2}\left[e - e^{-1}\right]}
\end{aligned}
$$

4. Find the centroid of the plane area bounded by the parabolas $y = 2x - x^2$ and $y = 3x^2 - 6x$.



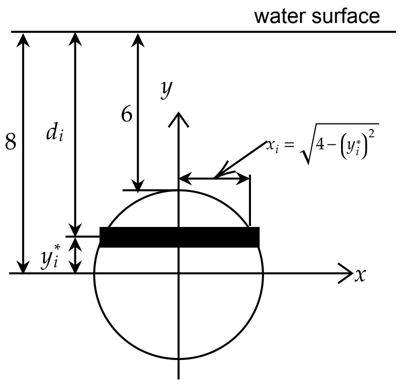
$$
\begin{aligned}
A &= \iint_R dA \\
&= \int_0^2 \int_{3x^2-6x}^{2x-x^2} dydx \\
&= \int_0^2 \left[2x - x^2 - 3x^2 + 6x\right] dx \\
&= \int_0^2 \left[-4x^2 + 8x\right] dx \\
&= \frac{-4x^3}{3} + 4x^2 \Big|_0^2 \\
&= \frac{-4(2)^3}{3} + 4(2)^2 \\
&= \mathbf{\frac{16}{3}}
\end{aligned}
$$

$$
\begin{aligned}
M_y &= \iint_R x\, dA \\
&= \int_0^2 \int_{3x^2-6x}^{2x-x^2} x\, dydx \\
&= \int_0^2 x\left(2x - x^2 - 3x^2 + 6x\right)\, dx \\
&= \int_0^2 \left[-4x^3 + 8x^2\right] dx \\
&= -x^4 + \frac{8}{3}x^3 \Big|_0^2 \\
&= -2^4 + \frac{8}{3}(2)^3 \\
&= \mathbf{\frac{16}{3}}
\end{aligned}
$$

$$
\begin{aligned}
M_x &= \iint_R y\, dA \\
&= \int_0^2 \int_{3x^2-6x}^{2x-x^2} y\, dydx \\
&= \frac{1}{2}\int_0^2 \left[\left(2x - x^2\right)^2 - \left(3x^2 - 6x\right)^2\right] dx \\
&= \frac{1}{2}\int_0^2 \left[4x^2 - 4x^3 + x^4 - 9x^4 + 36x^3 - 36x^2\right] dx \\
&= \frac{1}{2}\int_0^2 \left[-8x^4 + 32x^3 - 32x^2\right] dx \\
&= \frac{1}{2}\left[-\frac{8}{5}x^5 + 8x^4 - \frac{32}{3}x^3 \Big|_0^2\right] \\
&= \frac{1}{2}\left[-\frac{8}{5}(2)^5 + 8(2)^4 - \frac{32}{3}(2)^3\right] \\
&= -\frac{64}{15}
\end{aligned}
$$

Hence, $x = \frac{M_y}{A} = \frac{\frac{16}{3}}{\frac{16}{3}} = 1$, $y = \frac{M_x}{A} = \frac{-\frac{64}{15}}{\frac{16}{3}} = -\frac{4}{5}$. Thus, **the centroid is** $\left(1, -\frac{4}{5}\right)$.

5. Find the hydrostatic force on a circular plate of radius 2 that is submerged 6 meters in the water.



Assume that the top of the circular plate is 6 meters under the water.

Setting up the axis system such that the origin of the axis is at the center of the plate.

Finally, split up the plate into n horizontal strips each of width Δy. Choosing a point y_i^* from each strip. Assume that the strips are rectangular.

The depth below the water surface of each strip is,

$$d_i = 8 - y_i^*$$

and that in turn gives us the pressure on the strip,

$$P_i = \rho g d_i = 9810\left(8 - y_i^*\right)$$

The area of each strip is,

$$A_i = 2\sqrt{4-\left(y_i^*\right)^2}\ \Delta y$$

The hydrostatic force on each strip is,

$$F_i = P_i A_i = 9810\left(8 - y_i^*\right)(2)\sqrt{4-\left(y_i^*\right)^2}\ \Delta y$$

The total force on the plate is,

$$\begin{aligned} F &= \lim_{n\to\infty}\sum_{i=1}^{n} 19620\left(8-y_i^*\right)\sqrt{4-\left(y_i^*\right)^2}\ \Delta y \\ &= 19620\int_{-2}^{2}(8-y)\sqrt{4-y^2}\,dy \\ &= 19620\int_{-2}^{2}8\sqrt{4-y^2}\,dy - 19620\int_{-2}^{2}y\sqrt{4-y^2}\,dy \end{aligned}$$

The first integral requires the trig substitution $y = 2\sin\theta$ and the second integral needs the substitution $v = 4 - y^2$.

$$\begin{aligned} F &= 627840\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta + 9810\int_{0}^{0}\sqrt{v}\,dv \\ &= 313920\int_{-\pi/2}^{\pi/2}1+\cos(2\theta)\,d\theta + 0 \\ &= 313920\left(\theta + \frac{1}{2}\sin(2\theta)\right)\Bigg|_{-\pi/2}^{\pi/2} \\ &= \mathbf{313920\pi} \end{aligned}$$