

Tutorial 2: Functions & Derivatives

1. Find y' for $(x-y)^2 = x+y-1$ by using implicit differentiation.

$$\begin{aligned}\mathcal{D}(x-y)^2 &= \mathcal{D}(x+y-1) \\ \mathcal{D}(x-y)^2 &= \mathcal{D}(x) + \mathcal{D}(y) - \mathcal{D}(1) \\ 2(x-y)\mathcal{D}(x-y) &= 1 + y' \\ 2(x-y)(1-y') &= 1 + y' \\ 2(x-y) - 2(x-y)y' &= 1 + y' \\ -2(x-y)y' - y' &= 1 - 2(x-y) \\ y'[-2(x-y) - 1] &= 1 - 2(x-y) \\ y' &= \frac{1 - 2(x-y)}{-2(x-y) - 1} \\ y' &= \frac{2y - 2x + 1}{2y - 2x - 1}\end{aligned}$$

2. Implicit differentiation to find an equation of the tangent line to the curve $5x^2 + 10xy^2 + 5y = 20$ at point $(2, 2)$.

$$\begin{aligned}\mathcal{D}(5x^2 + 10xy^2 + 5y) &= \mathcal{D}(20) \\ 10x + 10x(2y \cdot y') + y^2 \cdot 10 + 5y' &= 0 \\ 10x + 20xy \cdot y' + 10y^2 + 5y' &= 0 \\ (20xy + 5)y' &= -10y^2 - 10x \\ y' &= \frac{-10y^2 - 10x}{20xy + 5}\end{aligned}$$

$$\text{At point } (2, 2), y' = \frac{-10(2)^2 - 10(2)}{20(2)(2) + 5} = -\frac{12}{17}$$

Knowing $y = mx + c$,

$$2 = -\frac{12}{17}(2) + c \rightarrow c = \frac{58}{17}$$

$$\therefore \text{Tangent Line: } y = -\frac{12}{17}x + \frac{58}{17}$$

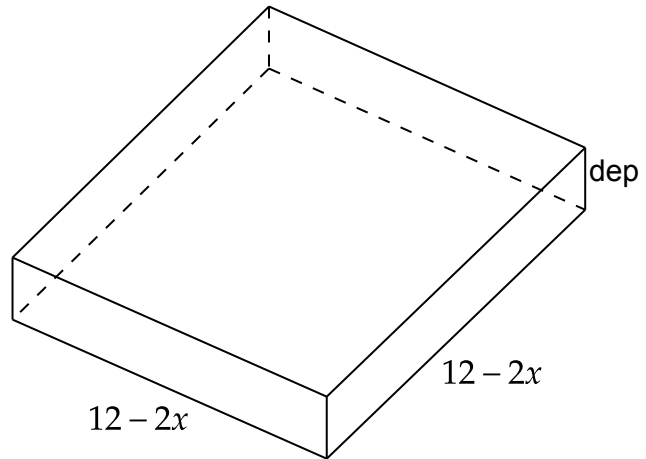
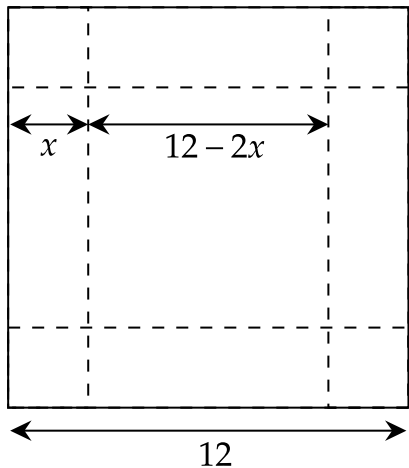
3. $y \cos x = 1 + \sin(xy)$. Find $\frac{dy}{dx}$ by implicit differentiation.

$$\begin{aligned}\mathcal{D}(y \cos x) &= \mathcal{D}(1 + \sin(xy)) \\ y(-\sin x) + (\cos x)y' &= \cos(xy) \cdot (xy' + y) \\ -y \sin x + y' \cos x &= (xy' + y)\cos(xy) \\ y' &= \frac{y \cos(xy) + y \sin x}{\cos x - x \cos xy}\end{aligned}$$

4. Find $\frac{dy}{dx}$ by implicit differentiation $4 \cos x \sin y = 1$.

$$\begin{aligned}4 \cos x \sin y &= 1 \\ (4 \cos x) \cdot (\cos y) \cdot y' + (\sin y) \cdot (-4 \sin x) &= 0 \\ y' &= \frac{4 \sin x \sin y}{4 \cos x \cos y} \\ &= \tan x \tan y\end{aligned}$$

5. An open box is to be made from cutting a square from each corner of a 12 in by 12 in piece of metal and then folding up the sides. What size square should be cut from each corner to produce a box of maximum volume?



The volume is to be maximized, $V = lwh = x(12 - 2x)(12 - 2x) = 4x^3 - 48x^2 + 144x$.

$$V(x) = 4x^3 - 48x^2 + 144x$$

$$V'(x) = 12x^2 - 96x + 144$$

The critical point can be calculate such that, $V'(x) = 0$.

$$0 = 12x^2 - 96x + 144$$

$$0 = 12(x^2 - 8x + 12)$$

$$0 = 12(x - 2)(x - 6)$$

$$\therefore x = 2, 6$$

When $x = 2$, $V(2) = 128$.

When $x = 6$, $V(6) = 0$.

Thus, when $x = 2$, the maximum volume is 128 in^3 .

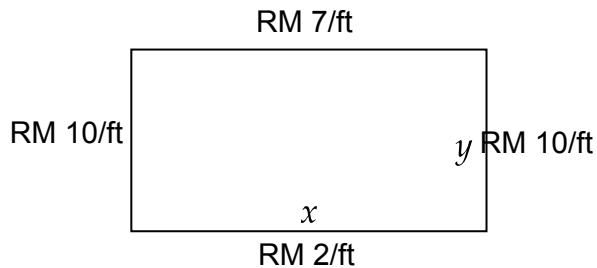
To prove that, using second derivation $V''(x) = 24x - 96$.

When $x = 2$, $V''(2) < 0$ (Maximum Volume)

When $x = 6$, $V''(6) > 0$ (Minimum Volume)

Thus, $x = 2$.

6. We are going to fence in a rectangular field. If we look at the field from above the cost of the vertical sides are RM 10/ft, the cost of the bottom is RM2/ft and the cost of the top is RM 7 /ft. If we have RM 700 determind the dimensions of the field that will maximize the enclosed area.



We have RM 700, therefore

$$700 = 10y + 2x + 10y + 7x = 20y + 9x$$

In terms of y , $y = 35 - \frac{9}{20}x$.

Since $A = xy$, plugging in yields

$$A(x) = x \left(35 - \frac{9}{20}x \right)$$

The critical point can be calculate such that, $A'(x) = 0$.

$$A'(x) = 35 - \frac{9}{10}(x)$$

$$0 = 35 - \frac{9}{10}(x)$$

$$\therefore x = \frac{350}{9}$$

The second derivation,

$$A''(x) = -\frac{9}{10}$$

This means that the second derivative always negative and so $A(x)$ will always be concave down and so the single critical point. Therefore the value obtained in $A'(x)$ must be the value that yield maximum area.

Substituting $x = \frac{350}{9}$ into the equation of y ,

$$y = 35 - \frac{9}{20} \left(\frac{350}{9} \right) = \frac{35}{2}$$

Thus, the final dimension is, $x = \frac{350}{9}$, $y = \frac{35}{2}$.