Tutorial 10: Multiple Integrals

1. Evaluate each of the following integrals over the given region D.

(i) $\iint_D e^{\frac{x}{y}}\, dA \qquad D = \left\{(x,y) \mid 1 \leqslant y \leqslant 2,\ y \leqslant x \leqslant y^3\right\}$

(ii) $\iint_D 4xy - y^3\, dA$, D **is the region bounded by** $y = \sqrt{x}$ **and** $y = x^3$

(i)

$$\begin{aligned}
\iint_D e^{\frac{x}{y}}\, dA &= \int_1^2 \int_y^{y^3} e^{\frac{x}{y}}\, dxdy \\
&= \int_1^2 ye^{\frac{x}{y}}\Big|_y^{y^3}\, dy \\
&= \int_1^2 \left(ye^{y^2} - ye^1\right) dy \ ** \\
&= \left(\frac{1}{2}e^{y^2} - \frac{1}{2}y^2e^1\right)\Bigg|_1^2 \\
&= \mathbf{\frac{1}{2}e^4 - 2e^1}
\end{aligned}$$

$$** \int ye^{y^2} - ye^1 dy = \int ye^{y^2} dy - \int ye^1 dy$$

For $\int ye^{y^2} dy$,

Let $u = y^2 \Longrightarrow y = \sqrt{u}$

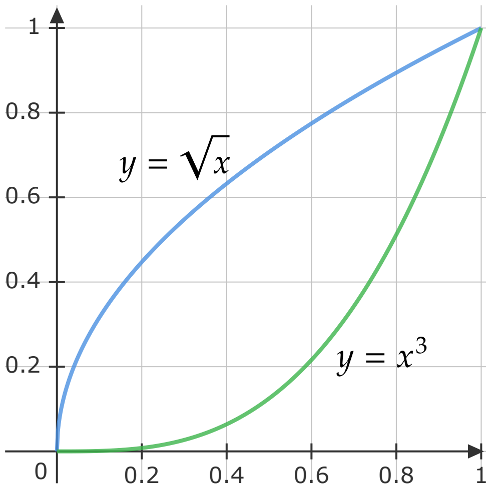
$$du = 2ydy \Longrightarrow dy = \frac{1}{2y}du = \frac{1}{2\sqrt{u}}du$$

$$\int ye^{y^2} dy = \int \sqrt{u}e^u\left(\frac{1}{2\sqrt{u}}du\right) = \frac{1}{2}\int e^u du$$

$$= \frac{1}{2}e^u = \frac{1}{2}e^{y^2}$$

For $\int ye^1 dy$, $\int ye^1 dy = \frac{1}{2}y^2e^1$

(ii)

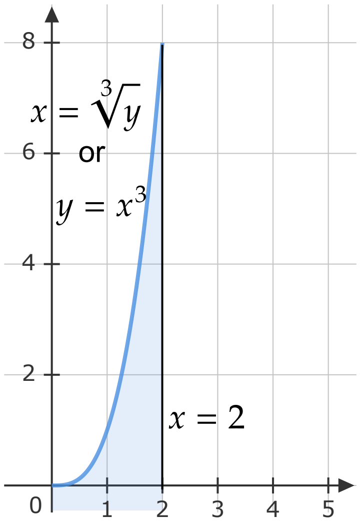


Two inequalities:

$0 \leqslant x \leqslant 1 \qquad x^3 \leqslant y \leqslant \sqrt{x}$

$$\begin{aligned}
\iint_D 4xy - y^3\, dA &= \int_0^1 \int_{x^3}^{\sqrt{x}} 4xy - y^3\, dydx \\
&= \int_0^1 \left(2xy^2 - \frac{1}{4}y^4\right)\Bigg|_{x^3}^{\sqrt{x}} dx \\
&= \int_0^1 2x^2 - \frac{1}{4}x^2 - 2x^7 + \frac{1}{4}x^{12} dx \\
&= \int_0^1 \frac{1}{4}x^{12} - 2x^7 + \frac{7}{4}x^2 dx \\
&= \left(\frac{1}{52}x^{13} - \frac{2}{8}x^8 + \frac{7}{12}x^3\right)\Bigg|_0^1 \\
&= \frac{1}{52} - \frac{2}{8} + \frac{7}{24} \\
&= \mathbf{\frac{55}{156}}
\end{aligned}$$

2. Evaluate $\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4+1}\, dxdy$.



Two inequalities:

$\sqrt[3]{y} \leqslant x \leqslant 2 \qquad 0 \leqslant y \leqslant 8$

Reversing the inequalities:

$0 \leqslant x \leqslant 2 \qquad 0 \leqslant y \leqslant x^3$

$$\begin{aligned}\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4+1}\, dxdy &= \int_0^2 \int_0^{x^3} \sqrt{x^4+1}\, dydx \\ &= \int_0^2 y\sqrt{x^4+1}\Big|_0^{x^3} dx \\ &= \int_0^2 x^3\sqrt{x^4+1}\, dx\end{aligned}$$

Let $u = x^4+1$, $du = 4x^3dx$. Therefore, $x^3dx = \frac{1}{4}du$.

Thus, the lower bound is now $u = 0^4+1 = 1$ and upper bound is now $u = 2^4+1 = 17$

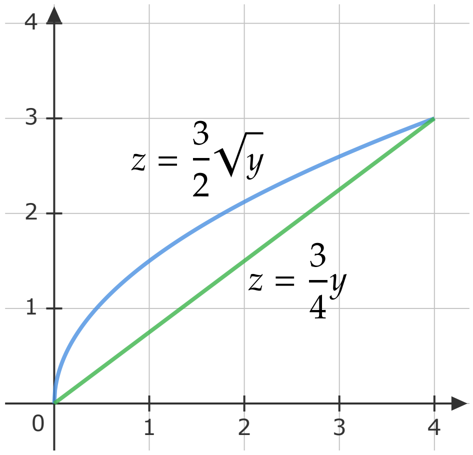
Hence,

$$\begin{aligned}\int_0^2 x^3\sqrt{x^4+1}\, dx &= \int_1^{17} \frac{1}{4}\sqrt{u}\, du \\ &= \frac{1}{4}\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_1^{17} \\ &= \frac{1}{6}\left(17^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) \\ &= \mathbf{\frac{1}{6}\left(17\sqrt{17}-1\right)}\end{aligned}$$

3. Evaluate the following integral $\iiint\limits_B 8xyz\,dV$, $B=[2,3]\times[1,2]\times[0,1]$.

$$\begin{aligned}
\iiint\limits_B 8xyz\ \ dV &= \int_1^2\int_2^3\int_0^1 8xyz\,dz\,dx\,dy \\
&= \int_1^2\int_2^3 4xyz^2\big|_0^1\,dx\,dy \\
&= \int_1^2\int_2^3 4xy\,dx\,dy \\
&= \int_1^2 2x^2y\big|_2^3\ \ dy \\
&= \int_1^2 \left(2\left(3^2\right)y-2(2)^2y\right)\,dy \\
&= \int_1^2 10y\,dy \\
&= 5y^2\big|_1^2 \\
&= 5(2)^2-5(1)^1 \\
&= \mathbf{15}
\end{aligned}$$

4. Determine the volume of the region that lies behind the plane $x+y+z=8$ and in front of the region in the yz-plane that is bounded by $z=\frac{3}{2}\sqrt{y}$ and $z=\frac{3}{4}y$.



Limit of the variable:

$0 \leqslant y \leqslant 4$

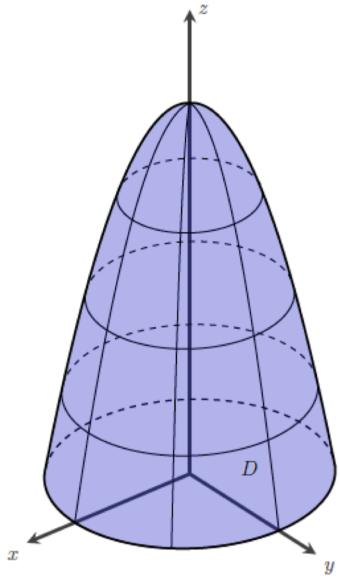
$\frac{3}{4}y \leqslant z \leqslant \frac{3}{2}\sqrt{y}$

$0 \leqslant x \leqslant 8-y-z$

Therefore the volume:

$$\begin{aligned}
\iiint\limits_B dV &= \iint\limits_D \left[\int_0^{8-y-z} dx\right] dA \\
&= \int_0^4 \int_{\frac{3}{4}y}^{\frac{3}{2}\sqrt{y}} (8-y-z)\, dz\, dy \\
&= \int_0^4 8z - yz - \frac{z^2}{2}\Bigg|_{\frac{3}{4}y}^{\frac{3}{2}\sqrt{y}} dy \\
&= \int_0^4 \left[8\left(\frac{3}{2}\sqrt{y}\right) - y\left(\frac{3}{2}\sqrt{y}\right) - \frac{\left(\frac{3}{2}\sqrt{y}\right)^2}{2}\right] - \left[8\left(\frac{3}{4}y\right) - y\left(\frac{3}{4}y\right) - \frac{\left(\frac{3}{4}y\right)^2}{2}\right] dy \\
&= \int_0^4 \left(12y^{\frac{1}{2}} - \frac{3}{2}y^{\frac{3}{2}} - \frac{9}{8}y - 6y + \frac{3}{4}y^2 + \frac{9}{32}y^2\right) dy \\
&= \int_0^4 \left(\frac{33}{32}y^2 - \frac{3}{2}y^{\frac{3}{2}} - \frac{57}{8}y + 12y^{\frac{1}{2}}\right) dy \\
&= \frac{33}{96}y^3 - \frac{6}{10}y^{\frac{5}{2}} - \frac{57}{16}y^2 + 8y^{\frac{3}{2}}\Bigg|_0^4 \\
&= \frac{33}{96}(4)^3 - \frac{6}{10}(4)^{\frac{5}{2}} - \frac{57}{16}(4)^2 + 8(4)^{\frac{3}{2}} \\
&= \mathbf{\frac{49}{5}}
\end{aligned}$$

5. Find the volume of the region bounded by the paraboloid $z = 4 - 4x^2 - 4y^2$ and the plane $z = 0$.



The region D is on xy-plane, so $z = 0$.

Given $z = 0$,

$$\begin{aligned} z &= 4 - 4x^2 - 4y^2 \\ 0 &= 4 - 4x^2 - 4y^2 \\ 4x^2 + 4y^2 &= 4 \\ x^2 + y^2 &= 1 \end{aligned}$$

$x^2 + y^2 = 1$ is a circle equation with $r = 1$

Since D is a circular disk, the integration will be in polar coordinate.

The limit: $0 \leqslant \theta \leqslant 2\pi, \qquad 0 \leqslant r \leqslant 1$

$$\begin{aligned} V &= \iint_D \left(4 - 4x^2 - 4y^2\right) dA \\ &= \iint_D 4\left(1 - \left(x^2 + y^2\right)\right) dA \\ &= \int_0^{2\pi} \int_0^1 4\left(1 - r^2\right) r \, dr d\theta \\ &= 4 \int_0^{2\pi} d\theta \int_0^1 \left(r - r^3\right) \, dr \\ &= 4 \cdot \theta \Big|_0^{2\pi} \cdot \frac{r^2}{2} - \frac{r^4}{4} \Bigg|_0^1 \\ &= 4 \cdot (2\pi) \cdot \left(\frac{1}{2} - \frac{1}{4}\right) \\ &= 8\pi \left(\frac{1}{4}\right) \\ &= \mathbf{2\pi} \end{aligned}$$

6. Evaluate the following integral

$$\iiint xy^2 \cos(xyz)\,dx\,dy\,dz;\ B = [0,\pi]\times\left[0,\frac{1}{2}\right]\times[0,1]$$

$$\int_0^{\pi}\int_0^{\frac{1}{2}}\int_0^{1} xy^2\cos(xyz)\,dz\,dy\,dx$$

Let $u \;=\; xyz, du = xy\,dz \implies dz = \dfrac{du}{xy}$,

$$\begin{aligned}\int xy^2\cos(xyz)\,dz &= \int xy^2\cos(u)\,\frac{du}{xy}\\ &= \int y\cos(u)\,du\\ &= y\sin(u)\\ &= y\sin(xyz)\end{aligned}$$

Subtituting back to the original integral,

$$\begin{aligned}\int_0^{\pi}\int_0^{\frac{1}{2}}\int_0^{1} xy^2\cos(xyz)\,dz\,dy\,dx &= \int_0^{\pi}\int_0^{\frac{1}{2}} y\sin(xyz)|_{z=0}^{z=1}\,dy\,dx\\ &= \int_0^{\pi}\int_0^{\frac{1}{2}}[y\sin(xy)-y\sin(0)]\,dy\,dx\\ &= \int_0^{\pi}\int_0^{\frac{1}{2}} y\sin(xy)\,dy\,dx\\ &= \int_0^{\frac{1}{2}} -\frac{y}{y}\cos(xy)\Big|_{x=0}^{x=\pi}\,dy\\ &= \int_0^{\frac{1}{2}}[-\cos(\pi y)+\cos 0]\;dy\\ &= \int_0^{\frac{1}{2}}[-\cos(\pi y)+1]\;dy\\ &= -\frac{1}{\pi}\sin(\pi y)+y\Big|_0^{\frac{1}{2}}\\ &= -\frac{1}{\pi}\sin\left(\frac{\pi}{2}\right)+\frac{1}{2}+\frac{1}{\pi}\sin 0\\ &= \boldsymbol{-\frac{1}{\pi}+\frac{1}{2}}\end{aligned}$$