Tutorial 10: Multiple Integrals

1. Evaluate each of the following integrals over the given region D.

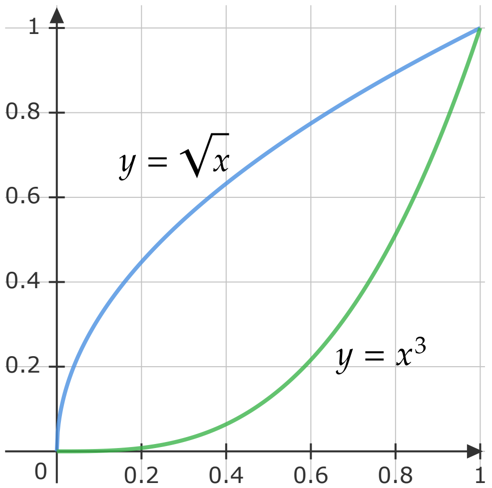
(i) $\iint_D e^{\frac{x}{y}}\, dA$ $\qquad D = \left\{(x,y) \mid 1 \leqslant y \leqslant 2,\ y \leqslant x \leqslant y^3\right\}$

(ii) $\iint_D 4xy - y^3\, dA$, D **is the region bounded by** $y = \sqrt{x}$ **and** $y = x^3$

(i)

$$\begin{aligned}
\iint_D e^{\frac{x}{y}}\, dA &= \int_1^2 \int_y^{y^3} e^{\frac{x}{y}}\, dxdy \\
&= \int_1^2 ye^{xy}\Big|_y^{y^3}\, dy \\
&= \int_1^2 \left(ye^{y^4} - ye^{y^2}\right) dy \\
&= \left(\frac{1}{2}e^{y^2} - \frac{1}{2}y^2e^1\right)\Bigg|_1^2 \\
&= \frac{1}{2}e^4 - 2e^1
\end{aligned}$$

(ii)

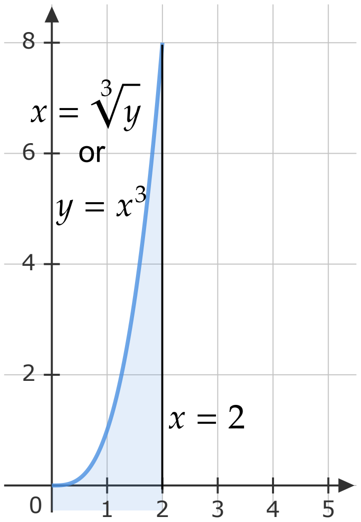


Two inequalities:

$0 \leqslant x \leqslant 1 \qquad x^3 \leqslant y \leqslant \sqrt{x}$

$$\begin{aligned}
\iint_D 4xy - y^3\, dA &= \int_0^1 \int_{x^3}^{\sqrt{x}} 4xy - y^3\, dydx \\
&= \int_0^1 \left(2xy^2 - \frac{1}{4}y^4\right)\Bigg|_{x^3}^{\sqrt{x}} dx \\
&= \int_0^1 2x^2 - \frac{1}{4}x^2 - 2x^7 + \frac{1}{4}x^{12} dx \\
&= \int_0^1 \frac{1}{4}x^{12} - 2x^7 + \frac{7}{4}x^2 dx \\
&= \left(\frac{1}{52}x^{13} - \frac{2}{8}x^8 + \frac{7}{12}x^3\right)\Bigg|_0^1 \\
&= \frac{1}{52} - \frac{2}{8} + \frac{7}{24} \\
&= \frac{55}{156}
\end{aligned}$$

2. Evaluate $\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4+1}\, dxdy$.



Two inequalities:

$\sqrt[3]{y} \leqslant x \leqslant 2 \qquad 0 \leqslant y \leqslant 8$

Reversing the inequalities:

$0 \leqslant x \leqslant 2 \qquad 0 \leqslant y \leqslant x^3$

$$\begin{aligned}\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4+1}\, dxdy &= \int_0^2 \int_0^{x^3} \sqrt{x^4+1}\, dydx \\ &= \int_0^2 y\sqrt{x^4+1}\Big|_0^{x^3} dx \\ &= \int_0^2 x^3\sqrt{x^4+1}\, dx\end{aligned}$$

Let $u = x^4+1$, $du = 4x^3dx$. Therefore, $x^3dx = \frac{1}{4}du$.

Thus, the lower bound is now $u = 0^4+1 = 1$ and upper bound is now $u = 2^4+1 = 17$

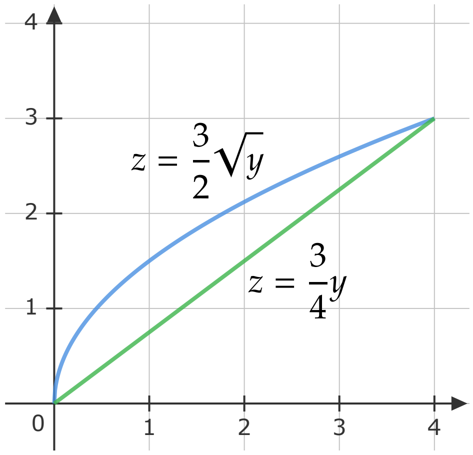
Hence,

$$\begin{aligned}\int_0^2 x^3\sqrt{x^4+1}\, dx &= \int_1^{17} \frac{1}{4}\sqrt{u}\, du \\ &= \frac{1}{4}\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_1^{17} \\ &= \frac{1}{6}\left(17^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) \\ &= \mathbf{\frac{1}{6}\left(17\sqrt{17}-1\right)}\end{aligned}$$

3. Evaluate the following integral $\iiint_B 8xyz\, dV$, $B = [2,3]\times[1,2]\times[0,1]$.

$$\begin{aligned}
\iiint_B 8xyz \;\; dV &= \int_1^2 \int_2^3 \int_0^1 8xyz\, dz\, dx\, dy \\
&= \int_1^2 \int_2^3 4xyz^2 \Big|_0^1 \, dx\, dy \\
&= \int_1^2 \int_2^3 4xy\, dx\, dy \\
&= \int_1^2 2x^2y\Big|_2^3 \;\, dy \\
&= \int_1^2 \left(2\left(3^2\right)y - 2(2)^2y\right)\, dy \\
&= \int_1^2 10y\, dy \\
&= \; 5y^2\Big|_1^2 \\
&= 5(2)^2 - 5(1)^1 \\
&= \mathbf{15}
\end{aligned}$$

4. Determine the volume of the region that lies behind the plane $x+y+z=8$ and in front of the region in the yz-plane that is bounded by $z=\frac{3}{2}\sqrt{y}$ and $z=\frac{3}{4}y$.



Limit of the variable:

$0 \leqslant y \leqslant 4$

$\frac{3}{4}y \leqslant z \leqslant \frac{3}{2}\sqrt{y}$

$0 \leqslant x \leqslant 8-y-z$

Therefore the volume:

$$\iiint\limits_B dV = \iint\limits_D \left[\int_0^{8-y-z} dx\right] dA$$

$$= \int_0^4 \int_{\frac{3}{4}y}^{\frac{3}{2}\sqrt{y}} (8-y-z)\,dz\,dy$$

$$= \int_0^4 8z - yz - \frac{z^2}{2}\Bigg|_{\frac{3}{4}y}^{\frac{3}{2}\sqrt{y}} dy$$

$$= \int_0^4 \left[8\left(\frac{3}{2}\sqrt{y}\right) - y\left(\frac{3}{2}\sqrt{y}\right) - \frac{\left(\frac{3}{2}\sqrt{y}\right)^2}{2}\right] - \left[8\left(\frac{3}{4}y\right) - y\left(\frac{3}{4}y\right) - \frac{\left(\frac{3}{4}y\right)^2}{2}\right] dy$$

$$= \int_0^4 \left(12y^{\frac{1}{2}} - \frac{3}{2}y^{\frac{3}{2}} - \frac{9}{8}y - 6y + \frac{3}{4}y^2 + \frac{9}{32}y^2\right) dy$$

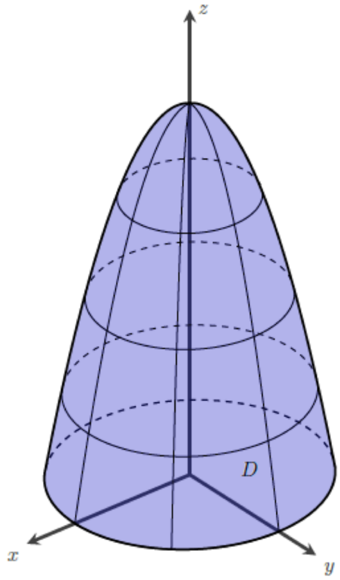
$$= \int_0^4 \left(\frac{33}{32}y^2 - \frac{3}{2}y^{\frac{3}{2}} - \frac{57}{8}y + 12y^{\frac{1}{2}}\right) dy$$

$$= \frac{33}{96}y^3 - \frac{6}{10}y^{\frac{5}{2}} - \frac{57}{16}y^2 + 8y^{\frac{3}{2}}\Bigg|_0^4$$

$$= \frac{33}{96}(4)^3 - \frac{6}{10}(4)^{\frac{5}{2}} - \frac{57}{16}(4)^2 + 8(4)^{\frac{3}{2}}$$

$$= \mathbf{\frac{49}{5}}$$

5. Find the volume of the region bounded by the paraboloid $z = 4 - 4x^2 - 4y^2$ and the plane $z = 0$.



The region D is on xy-plane, so $z = 0$.

Given $z = 0$,

$$\begin{aligned} z &= 4 - 4x^2 - 4y^2 \\ 0 &= 4 - 4x^2 - 4y^2 \\ 4x^2 + 4y^2 &= 4 \\ x^2 + y^2 &= 1 \end{aligned}$$

$x^2 + y^2 = 1$ is a circle equation with $r = 1$

Since D is a circular disk, the integration will be in polar coordinate.

The limit: $0 \leqslant \theta \leqslant 2\pi$, $\quad 0 \leqslant r \leqslant 1$

$$\begin{aligned} V &= \iint_D \left(4 - 4x^2 - 4y^2\right) dA \\ &= \iint_D 4\left(1 - \left(x^2 - y^2\right)\right) dA \\ &= \int_0^{2\pi} \int_0^1 4\left(1 - r^2\right) r \, dr d\theta \\ &= 4 \int_0^{2\pi} d\theta \int_0^1 \left(r - r^3\right) \, dr \\ &= 4 \cdot \theta\Big|_0^{2\pi} \cdot \left.\frac{r^2}{2} - \frac{r^4}{4}\right|_0^1 \\ &= 4 \cdot (2\pi) \cdot \left(\frac{1}{2} - \frac{1}{4}\right) \\ &= 8\pi\left(\frac{1}{4}\right) \\ &= \mathbf{2\pi} \end{aligned}$$

6. Evaluate the following integral

$$\iiint xy^2 \cos(xyz)\, dx\, dy\, dz;\; B = [0,\pi] \times \left[0, \frac{1}{2}\right] \times [0,1]$$

$$\int_0^{\pi} \int_0^{\frac{1}{2}} \int_0^{1} xy^2 \cos(xyz)\, dz\, dy\, dx$$

Let $u = xyz, du = xy\, dz \implies dz = \dfrac{du}{xy}$,

$$\begin{aligned}\int xy^2 \cos(xyz)\, dz &= \int xy^2 \cos(u)\, \frac{du}{xy} \\ &= \int y \cos(u)\, du \\ &= y \sin(u) \\ &= y \sin(xyz)\end{aligned}$$

Subtituting back to the original integral,

$$\begin{aligned}\int_0^{\pi} \int_0^{\frac{1}{2}} \int_0^{1} xy^2 \cos(xyz)\, dz\, dy\, dx &= \int_0^{\pi} \int_0^{\frac{1}{2}} y \sin(xyz)|_{z=0}^{z=1}\, dy\, dx \\ &= \int_0^{\pi} \int_0^{\frac{1}{2}} [y \sin(xy) - y \sin(0)]\, dy\, dx \\ &= \int_0^{\pi} \int_0^{\frac{1}{2}} y \sin(xy)\, dy\, dx \\ &= \int_0^{\frac{1}{2}} -\frac{y}{y} \cos(xy) \Big|_{x=0}^{x=\pi} dy \\ &= \int_0^{\frac{1}{2}} [-\cos(\pi y) + \cos 0]\; dy \\ &= \int_0^{\frac{1}{2}} [-\cos(\pi y) + 1]\; dy \\ &= -\frac{1}{\pi} \sin(\pi y) + y \Big|_0^{\frac{1}{2}} \\ &= -\frac{1}{\pi} \sin\left(\frac{\pi}{2}\right) + \frac{1}{2} + \frac{1}{\pi} \sin 0 \\ &= \boldsymbol{-\frac{1}{\pi} + \frac{1}{2}}\end{aligned}$$