Tutorial 1: Basic Functions & Derivatives

1. **Find the limit of** $\displaystyle\lim_{x \to 5} \frac{x^2 - 25}{x^2 + x - 30}$.

$$\lim_{x \to 5} \frac{x^2 - 25}{x^2 + x - 30} = \lim_{x \to 5} \frac{\cancel{(x-5)}(x+5)}{\cancel{(x-5)}(x+6)}$$

$$= \lim_{x \to 5} \frac{x+5}{x+6}$$

$$= \frac{5+5}{5+6}$$

$$= \frac{10}{11}$$

2. **Find the limit of** $\displaystyle\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9}$.

$$\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9} = \lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9} \times \frac{\sqrt{x} + 3}{\sqrt{x} + 3}$$

$$= \lim_{x \to 9} \frac{\cancel{x - 9}}{\cancel{(x - 9)}(\sqrt{x} + 3)}$$

$$= \lim_{x \to 9} \frac{1}{(\sqrt{x} + 3)}$$

$$= \frac{1}{\sqrt{9} + 3}$$

$$= \frac{1}{6}$$

3. **Find the limit of** $\displaystyle\lim_{x \to 0} \frac{\sqrt{2 + x} - \sqrt{2}}{x}$.

$$\lim_{x \to 0} \frac{\sqrt{2 + x} - \sqrt{2}}{x} = \lim_{x \to 0} \frac{\sqrt{2 + x} - \sqrt{2}}{x} \times \frac{\sqrt{2 + x} + \sqrt{2}}{\sqrt{2 + x} + \sqrt{2}}$$

$$= \lim_{x \to 0} \frac{(\sqrt{2 + x})^2 - (\sqrt{2})^2}{x(\sqrt{2 + x} + \sqrt{2})}$$

$$= \lim_{x \to 0} \frac{2 + x - 2}{x(\sqrt{2 + x} + \sqrt{2})}$$

$$= \lim_{x \to 0} \frac{\cancel{x}}{\cancel{x}(\sqrt{2 + x} + \sqrt{2})}$$

$$= \frac{1}{\sqrt{2} + \sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{2}}{4}$$

4. Find the limit of $\lim\limits_{\theta \to \frac{\pi}{2}} \dfrac{\tan \theta}{\sec \theta}$.

$$
\begin{aligned}
\lim_{\theta \to \frac{\pi}{2}} \frac{\tan \theta}{\sec \theta} &= \lim_{\theta \to \frac{\pi}{2}} \frac{\left(\frac{\sin \theta}{\cos \theta}\right)}{\left(\frac{1}{\cos \theta}\right)} \\
&= \lim_{\theta \to \frac{\pi}{2}} \sin \theta \\
&= \sin \frac{\pi}{2} \\
&= 1
\end{aligned}
$$

5. Find the limit of $\lim\limits_{\theta \to 0} \dfrac{\cos \theta - 1}{\sin \theta}$.

$$
\begin{aligned}
\lim_{\theta \to 0} \frac{\cos \theta - 1}{\sin \theta} &= \lim_{\theta \to 0} \frac{\cos \theta - 1}{\sin \theta} \times \frac{\theta}{\theta} \\
&= \lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} \times \frac{\theta}{\sin \theta} \\
&= \lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} \times \lim_{\theta \to 0} \frac{\theta}{\sin \theta} \\
&= \lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} \times \frac{1}{\lim_{\theta \to 0} \frac{\sin \theta}{\theta}} \\
&= 0 \times 1 \\
&= 0
\end{aligned}
$$

6. If $2x \le g(x) \le x^2 - x + 2$ for all x, evaluate $\lim\limits_{x \to 1} g(x)$.

$$
\lim_{x \to 1} 2x \le \lim_{x \to 1} g(x) \le \lim_{x \to 1} (x^2 - x + 2)
$$

Therefore,

$$
2 \le \lim_{x \to 1} g(x) \le 2
$$

Hence,

$$
\lim_{x \to 1} g(x) = 2
$$

7. Solve y' if $y = \sqrt{3x^2 - 2x + 3}$.

Let $u = 3x^2 - 2x + 3$, therefore $\frac{du}{dx} = 6x - 2$.

Substitute $y = \sqrt{u} = u^{\frac{1}{2}}$, therefore $\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$.

Putting back together,

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\
&= \frac{1}{2}u^{-\frac{1}{2}} \cdot 6x - 2 \\
&= \frac{1}{2}(3x^2 - 2x + 3)^{-\frac{1}{2}} \cdot 6x - 2 \\
&= \frac{6x - 2}{2\sqrt{3x^2 - 2x + 3}} \\
&= \frac{3x - 1}{\sqrt{3x^2 - 2x + 3}}
\end{aligned}
$$

8. Solve y' if $y = 5\sqrt[3]{x^2 + \sqrt{x^3}}$.

$y = 5\sqrt[3]{x^2 + \sqrt{x^3}}$ can be rewrite as $y = 5(x^2 + x^{\frac{3}{2}})^{\frac{1}{3}}$.

Let $u = x^2 + x^{\frac{3}{2}}$, therefore $\frac{du}{dx} = 2x + \frac{3}{2}x^{\frac{1}{2}}$.

Substitute $y = 5u^{\frac{1}{3}}$, therefore $\frac{dy}{du} = \frac{5}{3}u^{-\frac{2}{3}}$.

Putting back together,

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\
&= \frac{5}{3}u^{-\frac{2}{3}} \cdot \left(2x + \frac{3}{2}x^{\frac{1}{2}}\right) \\
&= \frac{5}{3}\left(x^2 + x^{\frac{3}{2}}\right)^{-\frac{2}{3}} \cdot \left(2x + \frac{3}{2}x^{\frac{1}{2}}\right)
\end{aligned}
$$

Alternatively, the equation can be written as

$$
\frac{5\left(2x + \frac{3}{2}\sqrt{x}\right)}{3\left(x^2 + \sqrt{x^3}\right)^{\frac{2}{3}}} = \frac{5\left(\frac{2x}{3} + \frac{1}{2}\sqrt{x}\right)}{\left(x^2 + \sqrt{x^3}\right)^{\frac{2}{3}}} = \frac{5\left(\frac{2x}{3} + \frac{\sqrt{x^3}}{2x}\right)}{\left(x^2 + \sqrt{x^3}\right)^{\frac{2}{3}}}
$$

9. Solve y if $y = \ln \cos x^2$.

Let $u = \cos x^2$,   therefore $\frac{du}{dx} = (-\sin x^2)(2x)$.

Substitute $y = \ln u$,   therefore $\frac{dy}{du} = \frac{1}{u}$.

Putting back together,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\
&= \frac{1}{u} \cdot (-\sin x^2)(2x) \\
&= -\frac{2x \sin x^2}{\cos x^2} \\
&= -2x \tan x^2
\end{aligned}$$

10. Differentiate $y = \log(4 + \cos x)$.

Note that $\frac{d}{dx}(\log_b(x)) = \frac{1}{\ln(b) \cdot x} = \frac{1}{x} \cdot \log e$

$$\begin{aligned}
\frac{d}{dx}(\log 4 + \cos x) &= \frac{1}{4 + \cos x} \cdot \log e \cdot \frac{d}{dx}(4 + \cos x) \\
&= \frac{1}{4 + \cos x} \cdot \log e \cdot (-\sin x) \\
&= \frac{-(\log e)(\sin x)}{4 + \cos x}
\end{aligned}$$

11. Find y' for $10e^{2xy} = e^{15y} + e^{13x}$.

$$\begin{aligned}
10e^{2xy} &= e^{15y} + e^{13x} \\
10e^{2xy}(2x \cdot y' + 2y) &= e^{15y}(15y') + e^{13x}(13) \\
10e^{2xy}(2x \cdot y' + 2y) &= 15y'e^{15y} + 13e^{13x} \\
(20xe^{2xy} - 15e^{15y})y' &= 13e^{13x} - 20ye^{2xy} \\
y' &= \frac{13e^{13x} - 20ye^{2xy}}{20xe^{2xy} - 15e^{15y}}
\end{aligned}$$

12. Find $f'(x)$ if $f(x) = 2x(\arctan 5x)^2 + 6\tan(\cos 6x)$.

- $\frac{d}{dx}(\tan^{-1} 5x)^2 = 2\tan^{-1} 5x \cdot \frac{1}{1+(5x)^2} \cdot 5 = \frac{10\tan^{-1} 5x}{1+25x^2}$

- $\frac{d}{dx} 2x(\tan^{-1} 5x)^2 = 2x(\frac{10\tan^{-1} 5x}{1+25x^2}) + (\tan^{-1} 5x)^2 \cdot 2 = \frac{20x\tan^{-1} 5x}{1+25x^2} + 2(\tan^{-1} 5x)^2$

- $\frac{d}{dx} 6\tan(\cos 6x) = 6\sec^2(\cos 6x)\frac{d}{dx}\cos 6x = 6\sec^2(\cos 6x) \cdot (-\sin 6x) \cdot 6 = -36(\sec^2(\cos 6x))\sin 6x$

$$\therefore f'(x) = \frac{20x\tan^{-1} 5x}{1+25x^2} + 2(\tan^{-1} 5x)^2 - 36(\sec^2(\cos 6x))\sin 6x$$

13. Solve y' **if** $y = 4x \sinh^{-1}\left(\dfrac{x}{6}\right) + \tanh^{-1}\left(\dfrac{x}{6}\right) + \tanh^{-1}(\cos 10x)$.

- $\dfrac{d}{dx} 4x \sinh^{-1}(\frac{x}{6}) = 4x\left[\dfrac{1}{\sqrt{1+(\frac{x}{6})^2}} \cdot \dfrac{1}{6}\right] + \sinh^{-1}(\frac{x}{6}) \cdot 4 = \dfrac{\frac{2x}{3}}{\sqrt{1+\frac{x^2}{36}}} + 4\sinh^{-1}(\frac{x}{6})$

- $\dfrac{d}{dx} \tanh^{-1}(\cos 10x) = \dfrac{1}{1-(\cos 10x)^2} \cdot \dfrac{d}{dx}\cos 10x = \dfrac{1}{1-\cos^2 10x} \cdot (-10\sin 10x) = \dfrac{-10\sin 10x}{\sin^2 10x} = -10\csc 10x$

$$\therefore f'(x) = \dfrac{\frac{2x}{3}}{\sqrt{1+\frac{x^2}{36}}} + 4\sinh^{-1}(\frac{x}{6}) - 10\csc 10x$$

14. Differentiate $y = \dfrac{1}{\sin^{-1}x}$.

$$\dfrac{d}{dx}(\sin^{-1}x) = \dfrac{1}{\sqrt{1-x^2}}$$

$$y = (\sin^{-1}x)^{-1}$$
$$y' = -(\sin^{-1}x)^{-2}\dfrac{d}{dx}(\sin^{-1}x)$$
$$= -\dfrac{1}{(\sin^{-1}x^2)\sqrt{1-x^2}}$$

15. Differentiate $y = \left(x^3 - 1\right)^{100}$.

$$y' = 100(x^3-1)^{99}\dfrac{d}{dx}(x^3-1)$$
$$= 100(x^3-1)^{99} \cdot 3x^2$$
$$= 300x^2(x^3-1)^{99}$$