

## Tutorial 1: Basic Functions & Derivatives

1. Find the limit of  $\lim_{x \rightarrow 5} \frac{x^2 - 25}{x^2 + x - 30}$ .

$$\begin{aligned}\lim_{x \rightarrow 5} \frac{x^2 - 25}{x^2 + x - 30} &= \lim_{x \rightarrow 5} \frac{\cancel{(x-5)}(x+5)}{\cancel{(x-5)}(x+6)} \\ &= \lim_{x \rightarrow 5} \frac{x+5}{x+6} \\ &= \frac{5+5}{5+6} \\ &= \frac{10}{11}\end{aligned}$$

2. Find the limit of  $\lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9}$ .

$$\begin{aligned}\lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9} &= \lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9} \times \frac{\sqrt{x} + 3}{\sqrt{x} + 3} \\ &= \lim_{x \rightarrow 9} \frac{\cancel{x-9}}{\cancel{(x-9)}(\sqrt{x} + 3)} \\ &= \lim_{x \rightarrow 9} \frac{1}{(\sqrt{x} + 3)} \\ &= \frac{1}{\sqrt{9} + 3} \\ &= \frac{1}{6}\end{aligned}$$

3. Find the limit of  $\lim_{x \rightarrow 0} \frac{\sqrt{2+x} - \sqrt{2}}{x}$ .

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sqrt{2+x} - \sqrt{2}}{x} &= \lim_{x \rightarrow 0} \frac{\sqrt{2+x} - \sqrt{2}}{x} \times \frac{\sqrt{2+x} + \sqrt{2}}{\sqrt{2+x} + \sqrt{2}} \\ &= \lim_{x \rightarrow 0} \frac{(\sqrt{2+x})^2 - (\sqrt{2})^2}{x(\sqrt{2+x} + \sqrt{2})} \\ &= \lim_{x \rightarrow 0} \frac{2+x-2}{x(\sqrt{2+x} + \sqrt{2})} \\ &= \lim_{x \rightarrow 0} \frac{\cancel{x}}{\cancel{x}(\sqrt{2+x} + \sqrt{2})} \\ &= \frac{1}{\sqrt{2} + \sqrt{2}} \\ &= \frac{1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{4}\end{aligned}$$

4. Find the limit of  $\lim_{\theta \rightarrow \frac{\pi}{2}} \frac{\tan \theta}{\sec \theta}$ .

$$\begin{aligned}\lim_{\theta \rightarrow \frac{\pi}{2}} \frac{\tan \theta}{\sec \theta} &= \lim_{\theta \rightarrow \frac{\pi}{2}} \frac{\left(\frac{\sin \theta}{\cos \theta}\right)}{\left(\frac{1}{\cos \theta}\right)} \\ &= \lim_{\theta \rightarrow \frac{\pi}{2}} \sin \theta \\ &= \sin \frac{\pi}{2} \\ &= 1\end{aligned}$$

5. Find the limit of  $\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\sin \theta}$ .

$$\begin{aligned}\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\sin \theta} &= \lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\sin \theta} \times \frac{\theta}{\theta} \\ &= \lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} \times \frac{\theta}{\sin \theta} \\ &= \lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} \times \lim_{\theta \rightarrow 0} \frac{\theta}{\sin \theta} \\ &= \lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} \times \frac{1}{\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}} \\ &= 0 \times 1 \\ &= 0\end{aligned}$$

6. If  $2x \leq g(x) \leq x^2 - x + 2$  for all  $x$ , evaluate  $\lim_{x \rightarrow 1} g(x)$ .

$$\lim_{x \rightarrow 1} 2x \leq \lim_{x \rightarrow 1} g(x) \leq \lim_{x \rightarrow 1} (x^2 - x + 2)$$

Therefore,

$$2 \leq \lim_{x \rightarrow 1} g(x) \leq 2$$

Hence,

$$\lim_{x \rightarrow 1} g(x) = 2$$

## 7. Solve $y'$ if $y = \sqrt{3x^2 - 2x + 3}$ .

Let  $u = 3x^2 - 2x + 3$ , therefore  $\frac{du}{dx} = 6x - 2$ .

Substitute  $y = \sqrt{u} = u^{\frac{1}{2}}$ , therefore  $\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$ .

Putting back together,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \frac{1}{2}u^{-\frac{1}{2}} \cdot 6x - 2 \\ &= \frac{1}{2}(3x^2 - 2x + 3)^{-\frac{1}{2}} \cdot 6x - 2 \\ &= \frac{6x - 2}{2\sqrt{3x^2 - 2x + 3}} \\ &= \frac{3x - 1}{\sqrt{3x^2 - 2x + 3}}\end{aligned}$$

## 8. Solve $y'$ if $y = 5\sqrt[3]{x^2 + \sqrt{x^3}}$ .

$y = 5\sqrt[3]{x^2 + \sqrt{x^3}}$  can be rewrite as  $y = 5(x^2 + x^{\frac{3}{2}})^{\frac{1}{3}}$ .

Let  $u = x^2 + x^{\frac{3}{2}}$ , therefore  $\frac{du}{dx} = 2x + \frac{3}{2}x^{\frac{1}{2}}$ .

Substitute  $y = 5u^{\frac{1}{3}}$ , therefore  $\frac{dy}{du} = \frac{5}{3}u^{-\frac{2}{3}}$ .

Putting back together,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \frac{5}{3}u^{-\frac{2}{3}} \cdot \left(2x + \frac{3}{2}x^{\frac{1}{2}}\right) \\ &= \frac{5}{3}\left(x^2 + x^{\frac{3}{2}}\right)^{-\frac{2}{3}} \cdot \left(2x + \frac{3}{2}x^{\frac{1}{2}}\right)\end{aligned}$$

Alternatively, the equation can be written as

$$\frac{5\left(2x + \frac{3}{2}\sqrt{x}\right)}{3\left(x^2 + \sqrt{x^3}\right)^{\frac{2}{3}}} = \frac{5\left(\frac{2x}{3} + \frac{1}{2}\sqrt{x}\right)}{\left(x^2 + \sqrt{x^3}\right)^{\frac{2}{3}}} = \frac{5\left(\frac{2x}{3} + \frac{\sqrt{x^3}}{2x}\right)}{\left(x^2 + \sqrt{x^3}\right)^{\frac{2}{3}}}$$

### 9. Solve $y$ if $y = \ln \cos x^2$ .

Let  $u = \cos x^2$ , therefore  $\frac{du}{dx} = (-\sin x^2)(2x)$ .

Substitute  $y = \ln u$ , therefore  $\frac{dy}{du} = \frac{1}{u}$ .

Putting back together,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \frac{1}{u} \cdot (-\sin x^2)(2x) \\ &= -\frac{2x \sin x^2}{\cos x^2} \\ &= -2x \tan x^2\end{aligned}$$

### 10. Differentiate $y = \log(4 + \cos x)$ .

Note that  $\frac{d}{dx}(\log_b(x)) = \frac{1}{\ln(b) \cdot x} \implies \frac{d}{dx}(\log_{10}(x)) = \frac{1}{\ln(10) \cdot x} = \frac{1}{x} \cdot \log e \cdot \left(\frac{1}{\ln 10} = \log e\right)$

$$\begin{aligned}\frac{d}{dx}(\log 4 + \cos x) &= \frac{1}{4 + \cos x} \cdot \log e \cdot \frac{d}{dx}(4 + \cos x) \\ &= \frac{1}{4 + \cos x} \cdot \log e \cdot (-\sin x) \\ &= \frac{-(\log e)(\sin x)}{4 + \cos x}\end{aligned}$$

### 11. Find $y'$ for $10e^{2xy} = e^{15y} + e^{13x}$ .

$$\begin{aligned}10e^{2xy} &= e^{15y} + e^{13x} \\ 10e^{2xy}(2x \cdot y' + 2y) &= e^{15y}(15y') + e^{13x}(13) \\ 10e^{2xy}(2x \cdot y' + 2y) &= 15y'e^{15y} + 13e^{13x} \\ (20xe^{2xy} - 15e^{15y})y' &= 13e^{13x} - 20ye^{2xy} \\ y' &= \frac{13e^{13x} - 20ye^{2xy}}{20xe^{2xy} - 15e^{15y}}\end{aligned}$$

### 12. Find $f'(x)$ if $f(x) = 2x(\arctan 5x)^2 + 6 \tan(\cos 6x)$ .

$$\begin{aligned}\bullet \quad \frac{d}{dx}(\tan^{-1}5x)^2 &= 2 \tan^{-1}5x \cdot \frac{1}{1+(5x)^2} \cdot 5 = \frac{10 \tan^{-1}5x}{1+25x^2} \\ \bullet \quad \frac{d}{dx}2x(\tan^{-1}5x)^2 &= 2x\left(\frac{10 \tan^{-1}5x}{1+25x^2}\right) + (\tan^{-1}5x)^2 \cdot 2 = \frac{20x \tan^{-1}5x}{1+25x^2} + 2(\tan^{-1}5x)^2 \\ \bullet \quad \frac{d}{dx}6 \tan(\cos 6x) &= 6 \sec^2(\cos 6x) \frac{d}{dx} \cos 6x = 6 \sec^2(\cos 6x) \cdot (-\sin 6x) \cdot 6 = -36(\sec^2(\cos 6x))\sin 6x \\ \therefore f'(x) &= \frac{20x \tan^{-1}5x}{1+25x^2} + 2(\tan^{-1}5x)^2 - 36(\sec^2(\cos 6x))\sin 6x\end{aligned}$$

**13. Solve  $y'$  if  $y = 4x \sinh^{-1}\left(\frac{x}{6}\right) + \tanh^{-1}(\cos 10x)$ .**

$$\begin{aligned} \bullet \frac{d}{dx} 4x \sinh^{-1}\left(\frac{x}{6}\right) &= 4x \left[ \frac{1}{\sqrt{1 + \left(\frac{x}{6}\right)^2}} \cdot \frac{1}{6} \right] + \sinh^{-1}\left(\frac{x}{6}\right) \cdot 4 = \frac{\frac{2x}{3}}{\sqrt{1 + \frac{x^2}{36}}} + 4 \sinh^{-1}\left(\frac{x}{6}\right) \\ \bullet \frac{d}{dx} \tanh^{-1}(\cos 10x) &= \frac{1}{1 - (\cos 10x)^2} \cdot \frac{d}{dx} \cos 10x = \frac{1}{1 - \cos^2 10x} \cdot (-10 \sin 10x) = \frac{-10 \sin 10x}{\sin^2 10x} = -10 \csc 10x \\ \therefore f'(x) &= \frac{\frac{2x}{3}}{\sqrt{1 + \frac{x^2}{36}}} + 4 \sinh^{-1}\left(\frac{x}{6}\right) - 10 \csc 10x \end{aligned}$$

**14. Differentiate  $y = \frac{1}{\sin^{-1}x}$ .**

$$\begin{aligned} \frac{d}{dx}(\sin^{-1}x) &= \frac{1}{\sqrt{1-x^2}} \\ y &= (\sin^{-1}x)^{-1} \\ y' &= -(\sin^{-1}x)^{-2} \frac{d}{dx}(\sin^{-1}x) \\ &= -\frac{1}{(\sin^{-1}x)^2 \sqrt{1-x^2}} \end{aligned}$$

**15. Differentiate  $y = (x^3 - 1)^{100}$ .**

$$\begin{aligned} y' &= 100(x^3 - 1)^{99} \frac{d}{dx}(x^3 - 1) \\ &= 100(x^3 - 1)^{99} \cdot 3x^2 \\ &= 300x^2(x^3 - 1)^{99} \end{aligned}$$