APPENDIX 10.1 SINUSOIDAL FUNCTION

A sinusoidal function is given by the following equation.

$$f(x) = A_n \, sin(n\omega x + \varphi_n)$$

Where A_n = amplitude of the function

φ_n = phase shift of the function for nth harmonic

$n = 1,2,3, \dots, \infty$

ω = angular frequency $= 2\pi f = \frac{2\pi}{p} = \frac{\pi}{L}$

[**Note 1**: f = frequency [Hz]; p =period; L =half of the period]

[**Note 2**: Fundamental angular frequency, ω_1 is the lowest angular frequency when $= 1$]

[**Note 3**: 2nd harmonic angular frequency, ω_2 is the 2 times lowest angular frequency when $= 2$]

An illustration of zero phase, positive phase and negative phase are given in the Figure A10.1.

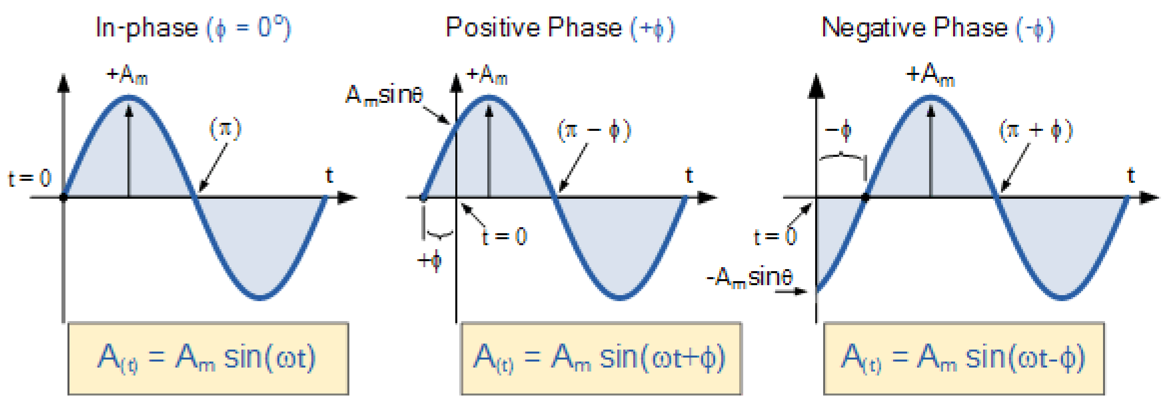


Figure 10.1. Sinusoidal waves with zero phase, positive phase and negative phase.

A sinusoidal wave with the magnitude of $f(x) = 5 \, sin(7x - \pi) + 4$ is illustrated in the Figure 10.2

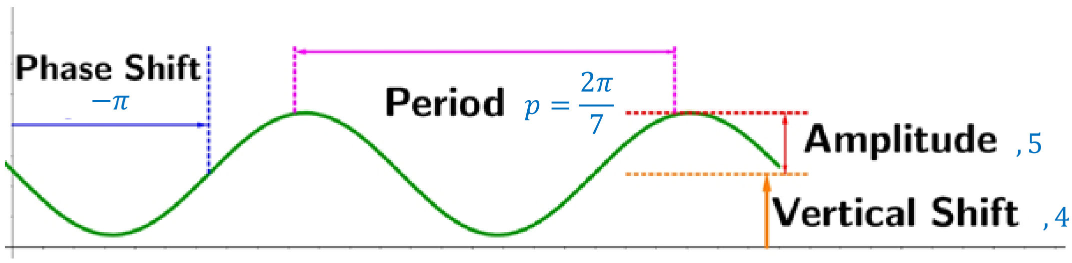


Figure 10.2. Sinusoidal waves of $f(x) = 5 \, sin(7x - \pi) + 4$

A sinusoidal wave can be also represented in other forms such as

$$f(x) = A_n \sin(n\omega x + \varphi_n) = a_n \cos(n\omega x) + b_n \sin(n\omega x)$$

Proof: $f(x) = A_n \sin(n\omega x + \varphi_n) = a_n \cos(n\omega x) + b_n \sin(n\omega x)$

From Angle Summation Identity, LHS:

$$A_n \sin(n\omega x + \varphi_n) = A_n[\sin(n\omega x) cos\varphi_n + \cos(n\omega x) sin\varphi_n]$$

Rearrange we obtain

$$A_n \sin(n\omega x + \varphi_n) = A_n cos\varphi_n(\sin(n\omega x)) + A_n sin\varphi_n(\cos(n\omega x))$$

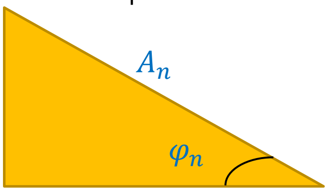
Since $A_n cos\varphi_n$ & $A_n sin\varphi_n$ are constants, they can be represented by

$$A_n \sin(n\omega x + \varphi_n) = a_n(\cos(n\omega x)) + b_n(\sin(n\omega x)) \qquad \text{(Proven)}$$

where $a_n = A_n cos\varphi_n$ & $b_n = A_n cos\varphi_n$

A trigonometric relationship is drawn based on the information

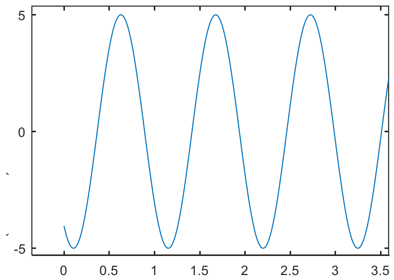
$a_n = A_n cos\varphi_n$



$b_n = A_n cos\varphi_n$

where the magnitude, $A_n = \sqrt{(A_n cos\varphi_n)^2 + (A_n sin\varphi_n)^2}$

Phase, $\varphi_n = tan^{-1}\frac{a_n}{b_n}$

Convert $A_n \sin(n\omega x + \varphi_n)$ to $A_n[\sin(n\omega x) cos\varphi_n + \cos(n\omega x) sin\varphi_n]$	Convert $A_n[\sin(n\omega x) cos\varphi_n + \cos(n\omega x) sin\varphi_n]$ to $A_n \sin(n\omega x + \varphi_n)$
Example: $5\sin(6x + 1.3\pi)$ $= 5(\sin(6x)\cos(1.3\pi) + \cos(6x)\sin(1.3\pi))$ $\therefore -2.939\sin(6x) - 4.045\cos(6x)$ 	***Example***: $-2.939\sin(6x) - 4.045\cos(6x)$ $Magnitude, A_n = \sqrt{(2.939)^2 + (4.045)^2} = 5$ Phase, $\emptyset = tan^{-1}\frac{a_n}{b_n} = tan^{-1}\frac{0.706}{-4.95}$ $= \frac{-4.045}{-2.939} = 0.94 rad\ (3rd\ quadrant)$ Actual Phase, $\emptyset$ =4.08rad=1.3π $\therefore$ 5sin($6x$+ 1.3π) [**Comment:** Same answer if you convert from one form to another]

More info: The sinusoidal wave can be represented in the following form as well.

$$f(x) = A_n \sin(n\omega x + \varphi_n) = B_n \cos(n\omega x + \varphi_n) = \left(c_n e^{(i(n\omega))x} + d_n e^{(-i(n\omega))x}\right)$$

APPENDIX 10.2 ORTHOGONALITY CHARACTERISTIC OF FOURIER SERIES

Previously, it was given the Euler Formulae to find the Fourier coefficients, i.e. a_0, a_n & b_n are:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos n\omega x \ dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin n\omega x \ dx$$

where ω =angular frequency, L =half of period, $n = 1,2,3, \ldots$

In this section, we will learn how to use the orthogonality characteristic and the odd/even function characteristic to derive the Euler Formulae. This derivation serves as the additional material to further understand the Fourier series, no memorization of formula is needed in this section.

(i) ***<u>Important Properties from the odd/even function</u>***

From the characteristic of odd and even functions, 4 useful properties of the even and odd functions that are useful for the derivation of Euler Formulae are given as follow.

Let $f(t)$ to be any function, i.e. it can be an odd function or even function;
$cos(nt)$ to be an even function;
$sin(nt)$ to be an odd function.

Property	Function 1	Function 2	Operation	Result
1	$f(t) =$ Odd function	$sin(nt) =$ Odd function	Summation $f(t) + sin(nt) =$	Odd function
2	$f(t) =$ Odd function	$sin(nt) =$ Odd function	Product or Multiplication $f(t). sin(nt) =$	Even function
3	$f(t) =$ Even function	$cos(nt) =$ Even function	Product $f(t). cos(nt) =$	Even function
4	$sin(nt) =$ Odd function	$cos(nt) =$ Even function	Product $sin(nt). cos(nt) =$	Odd function

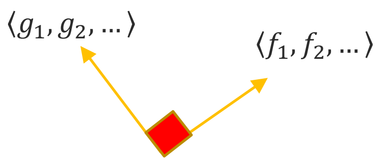
(ii) ***Orthogonality Characteristic of Fourier Series***

Definition of orthogonal function #1:

Two non-zero functions are said to be orthogonal on $a \le x \le b$ if

$$\int_a^b f(x)g(x)dx = 0$$

- ***Analogy:*** *Think orthogonal as 90 deg different between 2 functions*

> *If $f(x)$ & $g(x)$ are orthogonal (90^0 apart) to each other*
>
> 
>
>
> *Dot product formula,*
>
> $$\cos(90^0) = \frac{\langle f_1, f_2, \dots \rangle . \langle g_1, g_2, \dots \rangle}{|\langle f_1, f_2, \dots \rangle . ||\langle g_1, g_2, \dots \rangle|} = 0$$

Definition of orthogonal function #2:

Two sets of non-zero functions are said to be mutually orthogonal / orthogonal set on $a \le x \le b$ if f_m & g_n are orthogonal for $m \ne n$

$$\int_a^b f_m(x)g_n(x)dx = \begin{cases} 0 & \text{if } m \ne n \\ c > 0 & \text{if } \ m = n \end{cases}$$

With the definition of orthogonal function, we can show the trigonometric system of Fourier series has the orthogonality characteristic.

- For example, let a ***Fourier series with period of 2π:***

$$f(x) = \underset{h}{1} + \underset{f_1}{cosx} + \underset{g_1}{sinx} + \underset{f_2}{cos2x} + \underset{g_2}{sin2x} + \cdots$$

Where $h = constant\ function$

$f\ = cosine\ function\ set$

$g\ = sine\ function\ set$

- **Remark 1:** Constant function, cosine functions set and sine function set are orthogonal to each other as illustrated in the Eqs. [i]-[v]. The proof is given below.
- **Remark 2:** $\{1, cosx, cos2x, \dots, cosnx, sinx, \ sin2x, \dots, sinnx\}$ is an orthogonal set of functions on the interval $-\pi \le x \le \pi$ (hence also on $0 \le x \le 2\pi$) or any other interval of length *2π*

- **Proof:** Eqs [i]-[v] indicates the orthogonality characteristic of Fourier series:

  [i] $\int_{-\pi}^{\pi} 1.cosnxdx = 0 \quad \text{if } n \neq 0$

  [ii] $\int_{-\pi}^{\pi} 1.sinnxdx = 0 \quad \text{if } n \neq 0$

  [iii] $\int_{-\pi}^{\pi} sinmx.sinnxdx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$

  [iv] $\int_{-\pi}^{\pi} cosmx.cosnxdx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$

  [v] $\int_{-\pi}^{\pi} cosmx.sinnxdx = 0 \quad for\ all\ m,n$

Proof of [i] :

$\int_{-\pi}^{\pi} 1.cosnxdx = \left[\frac{\sin(nx)}{n}\right]_{-\pi}^{\pi} = \frac{\sin(n\pi)}{n} - \frac{\sin(n(-\pi))}{n} = 0$,where $n = 1,2,3, \ldots$ ([i] Proven)

Proof of [ii] :

$\int_{-\pi}^{\pi} 1.sinnxdx = \left[\frac{-\cos(nx)}{n}\right]_{-\pi}^{\pi} = \frac{-\cos(n(\pi))}{n} - \frac{-\cos(n(\pi))}{n} = 0$,where $n = 1,2, \ldots$ ([ii] Proven)

Proof of [iii]:

Hint: $sinAsinB = \frac{\cos(A-B)-\cos(A+B)}{2}$

<u>*Assume* $m \neq n$</u>

$$\int_{-\pi}^{\pi} sinmx.sinnxdx = \int_{-\pi}^{\pi} \frac{\cos(n-m)x-\cos(n+m)x}{2} dx$$
$$= \frac{1}{2}\left[\frac{\sin(n-m)x)}{n-m} - \frac{\sin(n+m)x)}{n+m}\right]_{-\pi}^{\pi}$$
$$= \frac{1}{2}\left[\left(\frac{\sin((n-m)\pi)}{n-m} - \frac{\sin((n+m)\pi)}{n+m}\right) - \left(\frac{\sin((n-m)(-\pi))}{n-m} + \frac{\sin((n+m)(-\pi))}{n+m}\right)\right]$$

$= 0$ if $m \neq n$ ([iii] Proven)

<u>*Assume* $m = n$</u>

$$\int_{-\pi}^{\pi} sinmx.sinnxdx = \int_{-\pi}^{\pi} \frac{\cos(n-m)x-\cos(n+m)x}{2} dx$$
$$= \int_{-\pi}^{\pi} \frac{\cos(0)x-\cos(n+m)x}{2} dx$$
$$= \frac{1}{2}\left[x - \frac{\sin(n+m)x)}{n+m}\right]_{-\pi}^{\pi}$$

$= \pi$ if $m = n$ ([iii] Proven)

Proof of [iv]:

Hint 1: $cosAcosB = \frac{\cos(A+B)+\cos(A-B)}{2}$

Hint 2: $sinAcosB = \frac{sin(A-B)+sin(A+B)}{2}$

<u>*Assume* $m \neq n$</u>

$$\int_{-\pi}^{\pi} cosmx.cosnxdx = \int_{-\pi}^{\pi} \frac{\cos(n+m)x+\cos(n-m)x}{2}dx$$

$$= \frac{1}{2}\left[\frac{\sin((n+m)x)}{n+m} + \frac{\sin((n-m)x)}{n-m}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left[\left(\frac{\sin((n+m)\pi)}{n+m} + \frac{\sin((n-m)\pi)}{n-m}\right) - \left(\frac{\sin((n+m)(-\pi))}{n+m} + \frac{\sin((n-m)(-\pi))}{n-m}\right)\right]$$

If $m \neq n$*, then* $sin\,(c\,\pi) = sin\,(-c\pi) = 0$ *for any arbitrary constant c*

$$= \frac{1}{2}\left[\left(\frac{\cancel{\sin((n+m)\pi)}}{n+m} + \frac{\cancel{\sin((n-m)\pi)}}{n-m}\right) - \left(\frac{\cancel{\sin((n+m)(-\pi))}}{n+m} + \frac{\cancel{\sin((n-m)(-\pi))}}{n-m}\right)\right]$$

$$= 0$$

([iv] Proven)

<u>*Assume* $m = n$,</u>

$$\int_{-\pi}^{\pi} cosmx.cosnxdx = \int_{-\pi}^{\pi} \frac{\cos(n+m)x+\cos(n-m)x}{2}dx$$

$$= \int_{-\pi}^{\pi} \frac{\cos(0)x+\cos(n-m)x}{2}dx$$

$$= \frac{1}{2}\left[x + \frac{\sin((n-m)x)}{n-m}\right]_{-\pi}^{\pi}$$

$$= \pi$$

([iv] Proven)

Proof of [v]:

Hint: $sinAcosB = \frac{sin(A-B)+sin(A+B)}{2}$

<u>*Assume* $m \neq n$</u>

$$\int_{-\pi}^{\pi} cosmx.sinnxdx = \int_{-\pi}^{\pi} \frac{\sin(n-m)x+\sin(n+m)x}{2}dx$$

$$= \frac{1}{2}\left[\frac{-\cos((n-m)x)}{n-m} + \frac{-\cos((n+m)x)}{n+m}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left[\left(\frac{-\cos((n-m)\pi)}{n-m} + \frac{-\cos((n+m)\pi)}{n+m}\right) - \left(\frac{-\cos((n-m)(-\pi))}{n-m} + \frac{-\cos((n+m)(-\pi))}{n+m}\right)\right]$$

If $m \neq n$*, then* $cos\,(c\,\pi) = cos\,(-c\pi)$ *for any arbitrary constant c*

$$= \frac{1}{2}\left[\left(\frac{-\cos((n-m)\pi)}{n-m} + \frac{\cos((n-m)(\pi))}{n-m}\right) + \left(\frac{-\cos((n+m)\pi)}{n+m} + \frac{\cos((n+m)(\pi))}{n+m}\right)\right]$$

$$= 0$$

([v] Proven)

Assume $m = n$

$$\int_{-\pi}^{\pi} cosmx.sinnxdx = \int_{-\pi}^{\pi} \frac{\sin(0)x+\sin(n+m)x}{2} dx$$

$$= \frac{1}{2}\left[\frac{-\cos((n+m)x)}{n+m}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left[\left(\frac{-\cos((n+m)\pi)}{n+m}\right) - \left(\frac{-\cos((n+m)(-\pi))}{n+m}\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{-\cos((n+m)\pi)}{n+m}\right) + \left(\frac{\cos((n+m)(\pi))}{n+m}\right)\right]$$

$$= 0$$ ([v] Proven)

By using the ***Important Properties from the odd/even function*** and ***Orthogonality Characteristic of Fourier Series***, we will demonstrate how to derive the Euler's Formulae:

Problem:

Fourier Series expression of a periodic signal with arbitrary period:

$$f(x) = \underbrace{a_0}_{arbitrary\ constant} + \underbrace{\sum_{n=1}^{\infty}(a_n \cos n\omega x + b_n \sin n\omega x)}_{sinusoidal\ function}$$

where a_0, a_n & b_n *are unknown to be solved.*

$$\omega = \frac{2\pi}{p}, L = \frac{p}{2}$$

Solution:

Step 1: ***Integrate both sides of the Fourier series within a period interval,*** $-L \leq x \leq L$

$$\int_{-L}^{L} f(x)dx = \int_{-L}^{L} a_0 dx + \sum_{n=1}^{\infty}\left(\int_{-L}^{L} a_n \cos n\omega x\, dx + \int_{-L}^{L} b_n sinn\omega xdx\right)$$

Using orthogonality characteristic [ii] & [i]

$\int_{-L}^{L} a_n \cos n\omega x\, dx = 0, n = 1,2,3, \ldots$ using orthogonality [ii] $= 0$ if $n \neq 0$

$\int_{-L}^{L} b_n sinn\omega xdx = 0, n = 1,2,3, \ldots$ using orthogonality [i] $= 0$ if $n \neq 0$

$$\int_{-L}^{L} f(x)dx = \int_{-L}^{L} a_0 dx = a_0(2L)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$ (Euler's Formulae [1])

Step 2: ***Integrate both sides of the (Fourier series multiply cosmx)***

$$\int_{-L}^{L} f(x) cosm\omega x dx$$

$$= \int_{-L}^{L} a_0 cosm\omega x dx + \sum_{n=1}^{\infty} \left(\int_{-L}^{L} a_n \cos n\omega x \, cosm\omega x dx + \int_{-L}^{L} b_n sinn\omega x cosm\omega x dx \right)$$

Using orthogonality characteristic [ii] , [iv] & [v]

$\int_{-L}^{L} a_0 cosm\omega x dx = 0, n = 1,2,3, \ldots$ using orthogonality [ii] $= 0$ if $n \neq 0$

$\int_{-L}^{L} a_n \cos n\omega x \, cosm\omega x dx = \begin{cases} 0 & if\ m \neq n \\ L & if\ m = n \end{cases}, n \,\&\, m = 1,2,3, \ldots$ using orthogonality [iv]

$\int_{-L}^{L} b_n sinn\omega x cosm\omega x dx = 0, n \,\&\, m = 1,2,3, \ldots$ using orthogonality [v] $= 0\ for\ all\ m, n$

Since the RHS is non-zero only when $m = n$

$$\int_{-L}^{L} f(x) cosm\omega x dx = a_n L = a_m L$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) cosn\omega x dx$$ (Euler's Formulae [2])

Step 3: ***Integrate both sides of the (Fourier series multiply sinmx)***

$$\int_{-L}^{L} f(x) sinm\omega x dx$$

$$= \int_{-L}^{L} a_0 sinm\omega x dx + \sum_{n=1}^{\infty} \left(\int_{-L}^{L} a_n \cos n\omega x \, sinm\omega x dx + \int_{-L}^{L} b_n sinn\omega x sinm\omega x dx \right)$$

Using orthogonality characteristic [i] , [v] & [iii]

$\int_{-L}^{L} a_0 sinm\omega x dx = 0, n = 1,2,3, \ldots$ using orthogonality [i] $= 0$ if $n \neq 0$

$\int_{-L}^{L} a_n \cos n\omega x \, sinm\omega x dx = 0, n \,\&\, m = 1,2,3, \ldots$ using orthogonality [v] $= 0\ for\ all\ m, n$

$\int_{-L}^{L} b_n sinn\omega x sinm\omega x dx = \begin{cases} 0 & if\ m \neq n \\ L & if\ m = n \end{cases}, n \,\&\, m = 1,2,3, \ldots$ using orthogonality [iii]

Since the RHS is non-zero only when $m = n$

$$\int_{-L}^{L} f(x) sinm\omega x dx = b_n L = b_m L$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) sinn\omega x dx$$ (Euler's Formulae [3])

APPENDIX 10.3 CONVERGENCE OF FOURIER SERIES

A full discussion of the convergence of a Fourier series is beyond of scope of this study. Not all the function $f(x)$ can have the Fourier series that is valid and converged.

To check whether or not the Fourier series thus obtained is a valid representation of the periodic function $f(x)$, we use Dirichlet's conditions to ensure that $f(x)$ has a convergent Fourier series expression.

Dirichlet's conditions:

- Bounded periodic function of $f(x)$ that has the following characteristics is said to have a valid Fourier series (i.e. converge)

(i) a finite number of points of finite discontinuity

Example:

Function $f(x) = \frac{1}{3-x}$ in the interval $0 < x < 2\pi$ *does not satisfy Direchlet's condition* and thus it has invalid Fourier series (i.e. divergence)

Explanation:

This is because it has infinite number of points of infinite discontinuity. For example, at $x = 3,\ f(x) = \infty$ (infinite discontinuity)

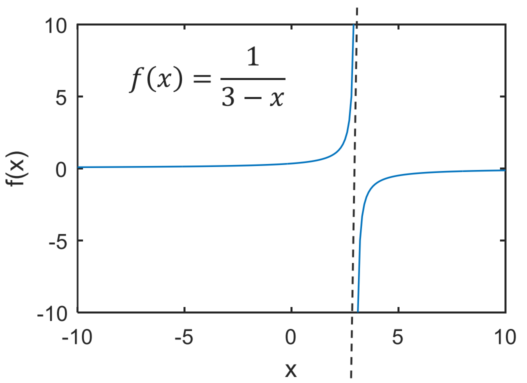
(ii) a finite number of isolated maxima and minima

Example:

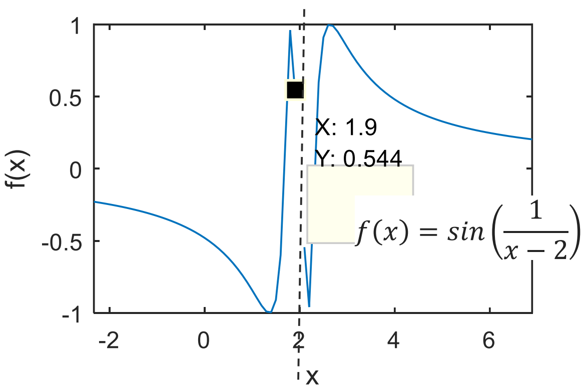
Function $f(x) = sin\left(\frac{1}{x-2}\right)$ in the interval $0 < x < 2\pi$ *does not satisfy Direchlet's condition* and thus it has invalid Fourier series (i.e. divergence)

Explanation:

This is because it has infinite number of isolated maxima and minima. For example, at $x = 2,\ f(x) = \sin(\infty)$ (infinite maxima and minima)



$f(3) = \infty\ (undefined)$

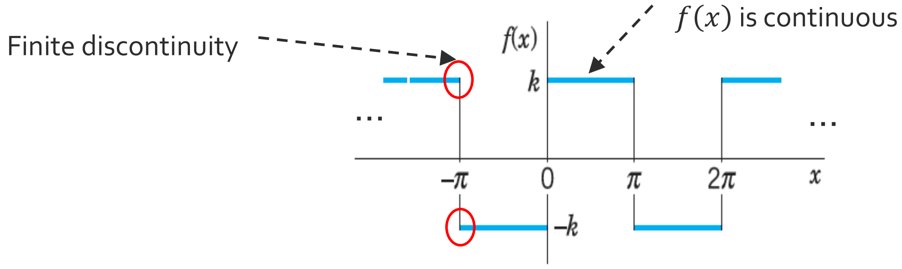


$f(2) = \infty\ (undefined)$

In other words, we will have the Fourier series expression of $f(x)$ that

(i) converges to its function, $f(x)$ at all points where $f(x)$ is a continuous; and
(ii) converges to the average of the right- and left-hand limits of $f(x)$ at points where $f(x)$ is discontinuous (i.e. to the mean of discontinuity).

For example, the Fourier series of the periodical rectangular wave is given.



$$\therefore f(x) = \frac{4k}{\pi}\sin x + \frac{4k}{3\pi}\sin 3x + \frac{4k}{5\pi}\sin 5x + \frac{4k}{7\pi}\sin 7x + \ldots$$

Note 1: The Fourier series expression will converge to its function at all points where the function is continuous, for example:

$$\ldots, f(x) = \frac{4k}{\pi}\sin x + \frac{4k}{3\pi}\sin 3x + \frac{4k}{5\pi}\sin 5x + \frac{4k}{7\pi}\sin 7x + \ldots = -k \textit{ is valid} \text{ at } -\pi < x < 0,$$

$$f(x) = \frac{4k}{\pi}\sin x + \frac{4k}{3\pi}\sin 3x + \frac{4k}{5\pi}\sin 5x + \frac{4k}{7\pi}\sin 7x + \ldots = k \textit{ is valid} \text{ at } 0 < x < \pi,$$

$$f(x) = \frac{4k}{\pi}\sin x + \frac{4k}{3\pi}\sin 3x + \frac{4k}{5\pi}\sin 5x + \frac{4k}{7\pi}\sin 7x + \ldots = -k \textit{ is valid} \text{ at } \pi < x < 2\pi,$$

$$f(x) = \frac{4k}{\pi}\sin x + \frac{4k}{3\pi}\sin 3x + \frac{4k}{5\pi}\sin 5x + \frac{4k}{7\pi}\sin 7x + \ldots = k \textit{ is valid} \text{ at } 2\pi < x < 3\pi, \ldots$$

Observation: At $\ldots, -\pi < x < 0, 0 < x < \pi, \pi < x < 2\pi, 2\pi < x < 3\pi, \ldots,$

$f(x) = \frac{4k}{\pi}\sin x + \frac{4k}{3\pi}\sin 3x + \frac{4k}{5\pi}\sin 5x + \frac{4k}{7\pi}\sin 7x + \ldots$ is converged to $f(x)$ when $f(x)$ is continuous.

Note 2: The Fourier series expression converges to the average of the right- and left-hand limits of $f(x)$ at points, for example:

$$\ldots, \text{At } x = -\pi, f(x) = \frac{1}{2}[f(-\pi^-) + f(-\pi^+)] = \frac{1}{2}[k + (-k)] = 0$$

$$\text{At } x = 0, f(x) = \frac{1}{2}[f(0^-) + f(0^+)] = \frac{1}{2}[(-k) + k] = 0$$

$$\text{At } x = \pi, f(x) = \frac{1}{2}[f(\pi^-) + f(\pi^+)] = \frac{1}{2}[k + (-k)] = 0, \ldots$$

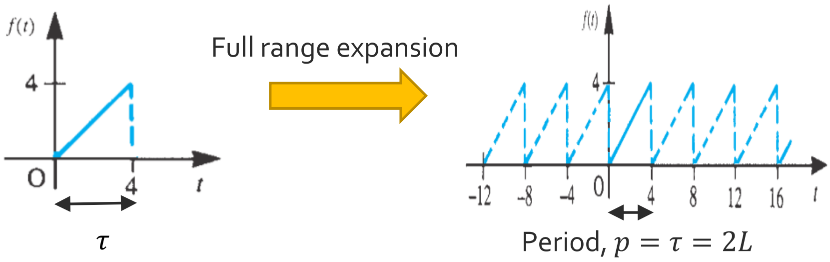
Observation: At $\ldots, x = -\pi, x = 0, x = \pi, \ldots$

$f(x) = \frac{4k}{\pi}\sin x + \frac{4k}{3\pi}\sin 3x + \frac{4k}{5\pi}\sin 5x + \frac{4k}{7\pi}\sin 7x + \ldots$ is converged to when $\frac{1}{2}[f(x^-) + f(x^+)$ when $f(x)$ have a jump discontinuous (finite discontinuity).

APPENDIX 11.1 SIMPLIFICATION OF THE CONVENTIONAL APPROACH OF FOURIER SERIES EXPANSION

In this session, the simplification of the conventional approach of the (i) Full-range Fourier Series Expansion, (ii) Half-range Fourier Cosine Series Expansion, (iii) Half-range Fourier Sine Series Expansion will be given.

(i) The Full-range Fourier Series Expansion of a finite interval signal is illustrated below.



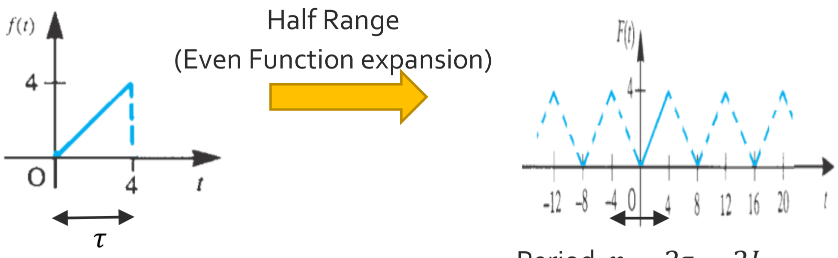
With that, the non-periodic signal with finite interval is converted to a periodic signal with infinite interval. Hence, the Full-range Fourier Series Expansion can be represented as

$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)$$

where the coefficient a_0, a_n & b_n can be simplified to alternative approach using several important properties of the Full range expansion.

	Conventional approach	***Important properties***	***Alternative approach***
1	$a_0 = \frac{1}{2L}\int_{-L}^{L} f(t)\, dt$	• $f(t)$ is not solely an odd or even function, but the mixture of them.	$a_0 = \frac{1}{2L}\int_{-L}^{L} f(t)\, dt$ $= \frac{1}{2L}\int_{0}^{2L} f(t)\, dt$ $= \frac{1}{2L}\int_{0}^{\tau} f(t)\, dt$ where $\tau = 2L$
2	$a_n = \frac{1}{L}\int_{-L}^{L} f(t) \cos n\omega t\ dt$	• The integration $\int_{-L}^{L} dt$ must be done within a period interval (i.e. $-L + c \leq t \leq L + c$) where c is arbitrary constant	$a_n = \frac{1}{L}\int_{-L}^{L} f(t) \cos n\omega t\ dt$ $= \frac{1}{L}\int_{0}^{2L} f(t) \cos n\omega t\ dt$ $= \frac{1}{L}\int_{0}^{\tau} f(t) \cos n\omega t\ dt$
3	$b_n = \frac{1}{L}\int_{-L}^{L} f(t) \sin n\omega t\, dt$	• Half period, L • Finite interval, τ • Period, $p = \tau = 2L$	$b_n = \frac{1}{L}\int_{-L}^{L} f(t) \sin n\omega t\, dt$ $= \frac{1}{L}\int_{0}^{2L} f(t) \sin n\omega t\, dt$ $= \frac{1}{L}\int_{0}^{\tau} f(t) \sin n\omega t\, dt$

(ii) The Half-range Fourier Cosine Series Expansion of a finite interval signal is illustrated below.



Period, $p = 2\tau = 2L$

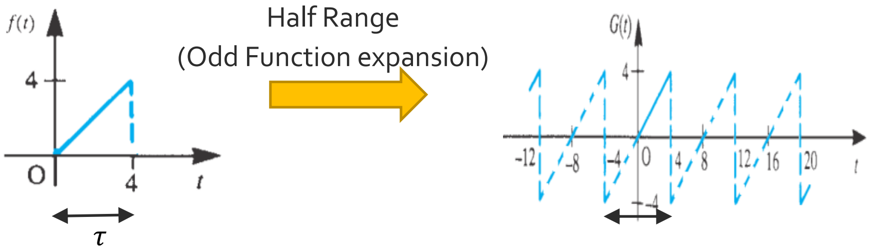
With that, the non-periodic signal with finite interval is converted to a periodic signal with infinite interval. Hence, the Half-range Fourier Cosine Series Expansion can be represented as

$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega t)$$

where the coefficient a_0, a_n & b_n can be simplified to alternative approach using several important properties of the Half range expansion & Even/Odd function.

	Conventional approach	***Important properties***	***Alternative approach***
1	$a_0 = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt$	• $f(t)$ is solely an even function, where half range of the signal, i.e. $0 \le t \le 4$ and half range of its mirror are assumed to be repeating itself within infinite interval.	$a_0 = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt$ $= \frac{1}{2L}\int_{-\tau}^{\tau} f(t)\,dt$ $= \frac{1}{L}\int_{0}^{\tau} f(t)\,dt$ where $\tau = L$
2	$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos n\omega t\,\,dt$	• For even function, $f(t)$ $$\int_{-\tau}^{\tau} f(t)\,dt = 2\int_{0}^{\tau} f(t)\,dt$$	$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos n\omega t\,\,dt$ $= \frac{1}{L}\int_{-\tau}^{\tau} f(t)\cos n\omega t\,\,dt$ $= \frac{2}{L}\int_{0}^{\tau} f(t)\cos n\omega t\,\,dt$
3	$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin n\omega t\,dt$	• Appendix 10.2: $f(t)\cos n\omega t$ is an even function. $\int_{-\tau}^{\tau} f(t)\,cos t dt = 2\int_{0}^{\tau} f(t)\,cos t\,dt$ • Appendix 10.2: $f(t)\sin n\omega t$ is an odd function. $\int_{-\tau}^{\tau} f(t)\,sin t dt = 0$ • The integration $\int_{-L}^{L} dt$ must be done within a period interval (i.e. $-L + c \le t \le L + c$) where c is arbitrary constant • Half period, L • Finite interval, τ • Period, $p = 2\tau = 2L$	$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin n\omega t\,dt$ $= \frac{1}{L}\int_{-\tau}^{\tau} f(t)\sin n\omega t\,dt$ $= 0$

(iii) The Half-range Fourier Sine Series Expansion of a finite interval signal is illustrated below.



Period, $p = 2\tau = 2L$

With that, the non-periodic signal with finite interval is converted to a periodic signal with infinite interval. Hence, the Half-range Fourier Sine Series Expansion can be represented as

$$f(t) = \sum_{n=1}^{\infty}(b_n \sin n\omega t)$$

where the coefficient a_0, a_n & b_n can be simplified to alternative approach using several important properties of the Half range expansion & Even/Odd function.

	Conventional approach	***Important properties***	***Alternative approach***
1	$a_0 = \frac{1}{2L}\int_{-L}^{L} f(t)\, dt$	• $f(t)$ is solely an odd function, where half range of the signal, i.e. $0 \le t \le 4$ and half range of its upside down mirror are assumed to be repeating itself within infinite interval.	$a_0 = \frac{1}{2L}\int_{-L}^{L} f(t)\, dt$ $= \frac{1}{2L}\int_{-\tau}^{\tau} f(t)\, dt$ $= 0$ where $\tau = L$
2	$a_n = \frac{1}{L}\int_{-L}^{L} f(t) \cos n\omega t\ \ dt$	• For odd function, $f(t)$ $$\int_{-\tau}^{\tau} f(t)\, dt = 0$$	$a_n = \frac{1}{L}\int_{-L}^{L} f(t) \cos n\omega t\ \ dt$ $= \frac{1}{L}\int_{-\tau}^{\tau} f(t) \cos n\omega t\ \ dt$ $= 0$
3	$b_n = \frac{1}{L}\int_{-L}^{L} f(t) \sin n\omega t\, dt$	• Appendix 10.2: $f(t) \cos n\omega t$ is an odd function. $\int_{-\tau}^{\tau} f(t)\ costdt = 0$ • Appendix 10.2: $f(t) \sin n\omega t$ is an even function. $\int_{-\tau}^{\tau} f(t)\ sintdt = 2\int_{0}^{\tau} f(t)\ sintdt$ • The integration $\int_{-L}^{L} dt$ must be done within a period interval (i.e. $-L + c \le t \le L + c$) where c is arbitrary constant • Half period, L • Finite interval, τ • Period, $p = 2\tau = 2L$	$b_n = \frac{1}{L}\int_{-L}^{L} f(t) \sin n\omega t\, dt$ $= \frac{1}{L}\int_{-\tau}^{\tau} f(t) \sin n\omega t\, dt$ $= \frac{2}{L}\int_{0}^{\tau} f(t) \sin n\omega t\, dt$