# POWER SERIES SOLUTIONS FOR DIFFERENTIAL EQUATIONS

## WEEK 6: POWER SERIES SOLUTIONS FOR DIFFERENTIAL EQUATIONS

### 6.1 Power series method

Power Series Method

The power series method is the standard basic method for solving linear differential equations with **variable** coefficients. It gives solutions in the form of power series.

Power Series

A power series is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots \qquad (1)$$

where $a_0$, $a_1$, $a_2$, ... are real constants, called the coefficients of the series, $x_0$ is a constant, called the center of the series, and $x$ is a variable.

In particular, if $x_0 = 0$, a power series in powers of $x$ is obtained

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Familiar examples of power series:

(i) $$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \cdots$$

(ii) $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

(iii) $$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \pm \cdots$$

(iv) $$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} \pm \cdots$$

(v) $$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

## 6.1.1 Basic concepts of power series

The *n*th partial sum of (1) is

$$s_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n \tag{2}$$

where $n$ = 0, 1, … . If the terms of $s_n$ are from (1), the remaining expression is

$$R_n(x) = a_{n+1}(x - x_0)^{n+1} + a_{n+2}(x - x_0)^{n+2} + \cdots$$

and is called the remainder of (1) after the term $a_n(x - x_0)^n$.

Example:

For the geometric series

$$1 + x + x^2 + \cdots + x^n + \cdots$$

Then:

$s_1 = 1 + x$ $\quad$ $R_1 = x^2 + x^3 + x^4 + \cdots$

$s_2 = 1 + x + x^2$ $\quad$ $R_2 = x^2 + x^3 + x^4 + \cdots$

etc.

If for some $x = x_1$, $s_n(x)$ converges, that is, $\lim_{n\to\infty} s_n(x_1) = s(x_1)$ then the series (1) **converges**, or is called **convergent** at $x = x_1$; and the number $s(x_1)$ is called the value or sum of (1) at $x_1$, and can be written as

$$s(x_1) = \sum_{n=0}^{\infty} a_n\,(x_1 - x_0)^n$$

If the sequence is divergent at $x = x_1$, then the series (1) is said to **diverge**, or to be **divergent** at $x = x_1$.

Note:

1. The series (1) converges at $x = x_0$ when all its terms except for the first ao are zero. In unusual cases this may be the only $x$ for which (1) converges.

2. If there are further values of $x$ for which the series (1) converges, these value form an interval, called the **convergence interval**. If this interval is finite, it has the midpoint $x_0$ so that it is of the form

$$|x - x_0| < R$$

and the series (1) converges for all $x$ such that $|x - x_0| < R$ and diverges for all $x$ such that $|x - x_0| > R$. The number $R$ is called the **radius of convergence** of (1). It can be obtained from either of the following formulas:

$$(a)\quad R = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{|a_m|}} \qquad\qquad (b)\quad R = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \qquad (3)$$

provided these limits exist and are not zero. [If they are infinite, then (1) converges only at the center $x_0$.]

3. The convergence interval may sometimes be infinite, that is, (1) converges for all x. For example, if the limit in (3a) and (3b) is zero. Then $R = \infty$, for convenience.

4. Since power series are functions of *x* and we know that not every series will in fact exist, it then makes sense to ask if a power series will exist for all *x*. This question is answered by looking at the convergence of the power series. We say that a power series **converges** for *x* = *c* if the series,

$$\sum_{n=0}^{\infty} a_n\,(c - x_0)^n$$

converges. Recall that this series will converge if the limit of partial sums,

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n\,(c - x_0)^n$$

exists and is finite. In other words, a power series will converge for *x* = *c* if

$$\sum_{n=0}^{\infty} a_n\,(c - x_0)^n$$

is a finite number.

5. A power series will always converge if $x = x_o$. In this case the power series will become

$$\sum_{n=0}^{\infty} a_n\,(c - x_0)^n = a_0$$

With this it is known now that power series are guaranteed to exist for at least one value of *x*. The following fact about the convergence of a power series is derived.

**Fact**

Given a power series, (1), there will exist a number $0 \leq \rho \leq \infty$ so that the power series will converge for $|x - x_0| < \rho$ and diverge for $|x - x_0| > \rho$. This number is called the **radius of convergence**.

### 6.1.2 Test for convergence

1. If $\lim_{n\to\infty} u_n = 0$ the series may be convergent; and

   if $\lim_{n\to\infty} u_n \neq 0$ the series is certainly divergent.

2. Comparison test – useful standard series

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \frac{1}{5^p} + \cdots + \frac{1}{n^p} + \cdots$$

   For $p > 1$, the series converges; for $p < 1$, the series diverges.

3. D'Alembert's Ratio Test for positive terms

   Let $u_1 + u_2 + u_3 + u_4 + \ldots + u_n + \ldots$ be a series of positive terms. Find expressions for $u_n$ and $u_{n+1}$, that is, the $n^{th}$ term and the $(n + 1)^{th}$ term, respectively, and form the ratio

$$\frac{u_{n+1}}{u_n}$$

   Then, find the limiting value for this ratio,

$$\rho = \lim_{n\to\infty} \frac{u_{n+1}}{u_n}$$

   If $\rho < 1$, the series converges;

   $\rho > 1$, the series diverges;

   $\rho = 1$, the series may converge or diverge and the test gives no definite information.

4. For general series:

   (i) if $\sum|u_n|$ converges, $\sum u_n$ is absolutely convergent

   (ii) if $\sum|u_n|$ diverges, but $\sum u_n$ converges, then $\sum u_n$ is conditionally convergent

Example 6.1:

Find the radius of convergence of the following series.

1. $$\sum_{k=1}^{\infty} \frac{k}{2^k}$$

*Solution*:

$$\rho = \lim_{k\to\infty}\left|\frac{k+1}{2^{k+1}}\cdot\frac{2^k}{k}\right| = \lim_{k\to\infty}\left|\frac{k+1}{2k}\right| = \frac{1}{2}\lim_{k\to\infty}\left|\frac{k+1}{k}\right| = \frac{1}{2}\lim_{k\to\infty}\left|1+\frac{1}{k}\right| = \frac{1}{2}$$

The series converges

2. $$\sum_{k=1}^{\infty} \frac{(-1)^k (x-3)^k}{3^k (k+1)}$$

*Solution*:

$$\rho = \lim_{k\to\infty}\left|\frac{(-1)^{k+1}(x-3)^{k+1}}{3^{k+1}(k+1+1)}\cdot\frac{3^k(k+1)}{(-1)^k(x-3)^k}\right| = \lim_{k\to\infty}\left|\frac{(-1)(x-3)(k+1)}{3(k+2)}\right|$$

$$= \frac{|(x-3)|}{3}\lim_{k\to\infty}\left|\frac{(-1)(k+1)}{(k+2)}\right| = \frac{|(x-3)|}{3}\cdot 1 = \frac{|(x-3)|}{3}$$

Series converges when $\rho < 1$

$$\frac{|(x-3)|}{3} < 1$$

$$|(x-3)| < 3$$

$$-(x-3) < 3 \quad \Rightarrow \quad x > 0$$

or

$$(x-3) < 3 \quad \Rightarrow \quad x < 6$$

Convergence interval (0, 6)

Radius of convergence 3

### 6.1.3 Operations of power series

Three permissible operations on power series: differentiation, addition, and multiplication.

(1) Termwise differentiation

A power series may be differentiated term by term. If

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

converges for $|x - x_0| < R$ where $R > 0$, then the series obtained by differentiating term by term also converges for those $x$ and represents the derivative $y'$ of $y$ for those $x$, that is,

$$y'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}$$

Similarly,

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2}$$

and so on.

(2) Termwise addition

Two power series may be added term by term. If the series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n \qquad \text{and} \qquad \sum_{n=0}^{\infty} b_n (x - x_0)^n$$

have positive radii of convergence and their sums are $f(x)$ and $g(x)$, respectively, then the series

$$\sum_{n=0}^{\infty} (a_n + b_n)(x - x_0)^n$$

converges and represent $f(x) + g(x)$ for each $x$ that lies in the interior of the convergence interval of each of the given series.

(3) Termwise multiplication

Two power series may be multiplied term by term. Suppose that

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n \qquad \text{and} \qquad \sum_{n=0}^{\infty} b_n (x - x_0)^n$$

have positive radii of convergence and let $f(x)$ and $g(x)$ be their sums, respectively. Then the series obtained by multiplying each term of the first series by each term of the second series and collecting like powers of $x - x_0$, that is,

$$\sum_{n=0}^{\infty}(a_0b_n + a_1b_{n-1} + \cdots + a_nb_0)(x-x_0)^n$$
$$= a_0b_0 + (a_0b_1 + a_1b_0)(x-x_0) + (a_0b_2 + a_1b_1 + a_2b_0)(x-x_0)^2 + \cdots$$

converges and represents $f(x)g(x)$ for each $x$ in the interior of convergence interval of each of the given series.

### 6.1.4 Vanishing all coefficients – a condition that is a basic tool of the power series method

*If a power series has a positive radius of convergence and a sum that is identically zero throughout its interval of convergence, then each coefficient of the series is zero.*

<u>Sifting summation indices</u>

(1) An index of summation is a dummy and can be changed.

Example:

$$\sum_{n=1}^{\infty}\frac{3^n n^2}{n!} = \sum_{k=1}^{\infty}\frac{3^k k^2}{k!} = 1 + 18 + \frac{81}{2} + \cdots.$$

(2) An index of summation can be "shifted".

If set $n = s + 2$, then $s = n - 2$, and

$$\sum_{n=2}^{\infty} n(n-1)a_n\,(x-x_0)^{n-2} = \sum_{s=0}^{\infty}(s+2)(s+1)a_{s+2}x^s = 2a_2 + 6a_3x + \ 12a_4x^2 + \cdots$$

When writing the sum of two series,

$$x^2\sum_{n=2}^{\infty} n(n-1)a_n\,x^{n-2} + 2\sum_{n=1}^{\infty} na_n\,x^{n-1}$$
$$= x^2(2a_2 + 6a_3x + 12a_4x^2 + \cdots) + 2(a_1 + 2a_2x + 3a_3x^2 + \cdots$$

as a single series; firstly, take $x^2$ and 2, respectively, inside the summation, obtaining

$$\sum_{n=2}^{\infty} n(n-1)x^n + \sum_{n=1}^{\infty} 2na_nx^{n-1}$$

and then set $n = s$ and $n - 1 = s$, respectively, obtaining

$$\sum_{s=2}^{\infty} s(s-1)a_s\, x^s + \sum_{s=0}^{\infty} 2(s+1)\, a_{s+1}x^s$$

where $s$ = 2 can be replaced by $s$ = 0, so that

$$\sum_{s=0}^{\infty}[s(s-1)a_s + 2(s+1)a_{s+1}]\, x^s = 2a_1 + 4a_2x + (2a_2 + 6a_3)x^2 + (6a_3 + 8a_4)x^3 + \cdots$$

**Theorem** (*Existence of power series solution*)

If the functions $p$, $q$, and $r$ in the differential equation

$$(4) \qquad y'' + p(x)y' + q(x)y = r(x)$$

are analytic at $x = x_0$, then every solution $y(x)$ of (4) is analytic at $x = x_0$ and can thus be represented by a power series in powers $x - x_0$ with radius of convergence $R > 0$.

Example 6.2:

Find the series of the following functions.

1. $e^{x^2}$

*Solution:*

$$e^{x^2} = \sum_{m=0}^{\infty} \frac{(x^2)^m}{m!} = 1 + x^2 + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \cdots$$

$$= 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \cdots$$

2. $e^x + \sin x$

*Solution:*

$$e^x + \sin x = \sum_{m=0}^{\infty} \frac{x^m}{m!} + \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + x - \frac{x^3}{3!} + \frac{x^5}{5!} \pm \cdots$$

$$= 1 + 2x + \frac{x^2}{2} + \frac{x^4}{4!} + \frac{2x^5}{5!} + \cdots$$

3. $e^x(\cos x)$

*Solution:*

$$e^x(\cos x) = \left(\sum_{m=0}^{\infty} \frac{x^m}{m!}\right)\left(\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!}\right)$$

$$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \pm \cdots\right)$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + x - \frac{x^3}{2!} + \frac{x^5}{4!} + \cdots$$

### 6.1.5 Idea of the power series method

Before finding series solutions to differential equations; we need to determine when we can find series solutions to differential equations with nonconstant coefficients. So, let's start with the differential equation,

$$p(x)y'' + q(x)y' + r(x)y = 0 \tag{5}$$

To this point we've only dealt with constant coefficients. However, with series solutions we can now have nonconstant coefficient differential equations. Also, here we will be dealing only with polynomial coefficients.

Now, we say that $x = x_o$ is an **ordinary point** if provided both

$$\frac{q(x)}{p(x)} \qquad \text{and} \qquad \frac{r(x)}{p(x)}$$

are analytic at $x = x_o$. That is to say that these two quantities have Taylor series around $x = x_o$. Since, we are only dealing with coefficients that are polynomials so this will be equivalent to saying that

$$p(x_0) \neq 0$$

for most of the problems.

If a point is not an ordinary point we call it a **singular point**.

The basic idea to finding a series solution to a differential equation is to assume that we can write the solution as a power series in the form,

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{6}$$

and then try to determine what the $a_n$'s need to be. We will only be able to do this if the point $x = x_o$, is an ordinary point. We will usually say that (6) is a series solution around $x = x_o$.

Example 6.3:

1. Find a series solution around $x_o$ = 0 for the following differential equation.

$$y'' - xy = 0$$

*Solution:*

In this case, $p(x) = 1$; hence for this differential equation every point is an ordinary point.

Assume solution:

$$y = \sum_{n=0}^{\infty} a_n (x-0)^n = \sum_{n=0}^{\infty} a_n x^n$$

Then,

$$y' = \sum_{n=1}^{\infty} n\, a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n\,(n-1) a_n x^{n-2}$$

Step1: Plugging into the differential equation

$$\sum_{n=2}^{\infty} n\,(n-1) a_n x^{n-2} - x \sum_{n=0}^{\infty} a_n\, x^n = 0$$

Step 2: Get all the coefficients moved into the series.

$$\sum_{n=2}^{\infty} n\,(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n\, x^{n+1} = 0$$

Step 3: Shift the first series down by 2 and the second series up by 1 to get both of the series in terms of $x^n$

$$\sum_{n=0}^{\infty} (n+2)\,(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1}\, x^n = 0$$

Step 4: Get the two series starting at the same value of $n$. The only way to do that for this problem is to strip out the $n$ = 0 term

$$(2)(1) a_2 x^0 + \sum_{n=1}^{\infty} (n+2)\,(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1}\, x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} [(n+2)(n+1) a_{n+2} - a_{n-1}] x^n = 0$$

Step 5: Set all the coefficients equal to zero. The $n$ = 0 coefficient is in front of the series and the $n$ = 1,2,3… are all in the series. So, setting coefficient equal to zero gives,

$$n = 0 \qquad 2a_2 = 0$$

$$n = 1, 2, 3, \ldots \qquad (n+2)(n+1)a_{n+2} - a_{n-1} = 0$$

Step 6: Solving the first as well as **the recurrence relation** gives

$$n = 0 \qquad a_2 = 0$$

$$n = 1, 2, 3, \ldots \qquad a_{n+2} -= \frac{a_{n-1}}{(n+2)(n+1)}$$

Step 7: Start plugging in values of *n*

$$n = 1 \qquad a_3 = \frac{a_0}{(3)(2)}$$

$$n = 2 \qquad a_4 = \frac{a_1}{(4)(3)}$$

$$n = 3 \qquad a_5 = \frac{a_2}{(5)(4)} = 0$$

$$n = 4 \qquad a_6 = \frac{a_3}{(6)(5)} = \frac{a_0}{(6)(5)(3)(2)}$$

$$n = 5 \qquad a_7 = \frac{a_4}{(7)(6)} = \frac{a_1}{(7)(6)(4)(3)}$$

$$n = 6 \qquad a_8 = \frac{a_5}{(8)(7)} = 0$$

$$\vdots$$

$$a_{3k} = \frac{a_0}{(2)(3)(5)(6)\cdots(3k-1)(3k)} \qquad k = 1, 2, 3, \cdots$$

$$a_{3k+1} = \frac{a_1}{(3)(4)(6)(7)\cdots(3k)(3k+1)} \qquad k = 1, 2, 3, \cdots$$

$$a_{3k+2} = 0 \qquad k = 0, 1, 2, \cdots$$

Note: Every third coefficient is zero. The formulas here are somewhat unpleasant and not all that easy to see the first time around. These formulas will not work for *k = o*.

Step 8: Get the solution

$$\begin{aligned} y(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots + a_{3k} x^{3k} + a_{3k+1} x^{3k+1} + \cdots \\ &= a_0 + a_1 x + \frac{a_0}{6} x^2 + \frac{a_1}{12} x^4 + \cdots + \frac{a_0}{(2)(3)(5)(6)\cdots(3k-1)(3k)} x^{3k} \\ &\qquad + \frac{a_1}{(3)(4)(6)(7)\cdots(3k)(3k+1)} x^{3k+1} + \cdots \end{aligned}$$

Step 9: Collect up the terms that contain the same coefficient, factor the coefficient out and write the results as a new series

$$y(x) = a_0\left[1+\sum_{k=1}^{\infty}\frac{x^{3k}}{(2)(3)(5)(6)\cdots(3k-1)(3k)}\right]+a_1\left[x+\sum_{k=1}^{\infty}\frac{x^{3k+1}}{(3)(4)(6)(7)\cdots(3k)(3k+1)}\right]$$

Note: The series could not start at *k* = 0 since the general term doesn't hold for *k* = 0

2. Find the first four terms in each portion of the series solution around $x_o$ = -2 for the following differential equation

$$y'' - xy = 0$$

*Solution:*

In this case, $p(x) = 1$; hence for this differential equation every point is an ordinary point.

Assume solution:

$$y = \sum_{n=0}^{\infty} a_n\,(x-(-2))^n = \sum_{n=0}^{\infty} a_n\,(x+2)^n$$

Then,

$$y' = \sum_{n=1}^{\infty} n\,a_n(x+2)^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n\,(n-1)a_n(x+2)^{n-2}$$

Step 1: Plugging into the differential equation

$$\sum_{n=2}^{\infty} n\,(n-1)a_n(x+2)^{n-2} - x\sum_{n=0}^{\infty} a_n\,(x+2)^n = 0$$

Step 2: Get all the coefficients moved into the series. There is a difference between this example and the previous example. In this case we can't just multiply the *x* into the second series since in order to combine with the series it must be *x* + 2. Therefore we will first need to modify the coefficient of the second series before multiplying it into the series.

$$\sum_{n=2}^{\infty} n\,(n-1)a_n(x+2)^{n-2} - (x+2-2)\sum_{n=0}^{\infty} a_n\,(x+2)^n = 0$$

$$\sum_{n=2}^{\infty} n\,(n-1)a_n(x+2)^{n-2} - (x+2)\sum_{n=0}^{\infty} a_n\,(x+2)^n + 2\sum_{n=0}^{\infty} a_n\,(x+2)^n = 0$$

$$\sum_{n=2}^{\infty} n\,(n-1)a_n(x+2)^{n-2} - \sum_{n=0}^{\infty} a_n\,(x+2)^{n+1} + \sum_{n=0}^{\infty} 2a_n\,(x+2)^n = 0$$

Note: Now have three series to work with.

Step 3: Need to shift the first series down by 2 and the second series up by 1 to get common exponents in all the series

$$\sum_{n=0}^{\infty}(n+2)\,(n+1)a_{n+2}(x+2)^n - \sum_{n=1}^{\infty} a_{n-1}\,(x+2)^n + \sum_{n=0}^{\infty} 2a_n\,(x+2)^n = 0$$

Step 4: Combine the series by stripping out the $n$ = 0 terms from both the first and third series

$$2a_2 + \sum_{n=1}^{\infty}(n+2)\,(n+1)a_{n+2}(x+2)^n - \sum_{n=1}^{\infty} a_{n-1}\,(x+2)^n + 2a_0 + \sum_{n=1}^{\infty} 2a_n\,(x+2)^n = 0$$

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} - a_{n-1} + 2a_n](x+2)^n = 0$$

Step 5: Set all the coefficients equal to zero.

$$n = 0 \qquad\qquad 2a_2 + 2a_0 = 0$$

$$n = 1, 2, 3, \dots \qquad\qquad (n+2)(n+1)a_{n+2} - a_{n-1} + 2a_n = 0$$

Step 6: Solve the first as well as **the recurrence relation**. In the first case there are two options, we can solve for $a_2$ or we can solve for $a_0$. Out of habit I'll solve for $a_0$. In the recurrence relation we'll solve for the term with the largest subscript

$$n = 0 \qquad\qquad a_2 = -a_0$$

$$n = 1, 2, 3, \dots \qquad\qquad a_{n+2} = \frac{a_{n-1} - 2a_n}{(n+2)(n+1)}$$

Note 1: This example we won't be having every third term drop out as we did in the previous example.

Note 2: At this point we'll also acknowledge that the instructions for this problem are different as well. We aren't going to get a general formula for the $a_n$'s this time so we'll have to be satisfied with just getting the first couple of terms for each portion of the solution. This is often the case for series solutions. Getting general formulas for the $a_n$'s is the exception rather than the rule in these kinds of problems.

Step 7: Start plugging in values of $n$. To get the first four terms we'll just start plugging in terms until we've got the required number of terms. Note that we will already be starting with an $a_0$ and an $a_1$ from the first two terms of the solution so all we will need are three more terms with an $a_0$ in them and three more terms with an $a_1$ in them

$$n = 0 \qquad\qquad a_2 = -a_0$$

$$n = 1 \qquad\qquad a_3 = \frac{a_0 - 2a_1}{(3)(2)} = \frac{a_0}{6} - \frac{a_1}{3}$$

$n = 2$ $$a_4 = \frac{a_1 - 2a_2}{(4)(3)} = \frac{a_1 - 2(-a_0)}{(4)(3)} = \frac{a_0}{6} + \frac{a_1}{12}$$

$n = 3$ $$a_5 = \frac{a_2 - 2a_3}{(5)(4)} = \frac{a_0}{20} - \frac{1}{10}\left(\frac{a_0}{6} - \frac{a_1}{3}\right) = -\frac{a_0}{15} + \frac{a_1}{30}$$

Step 8: Get the solution

$$\begin{aligned} y(x) &= a_0 + a_1(x+2) + a_2(x+2)^2 + a_3(x+2)^3 + a_4(x+2)^4 + a_5(x+2)^5 + \cdots \\ &= a_0 + a_1(x+2) - a_0(x+2)^2 + \left(\frac{a_0}{6} - \frac{a_1}{3}\right)(x+2)^3 + \left(\frac{a_0}{6} + \frac{a_1}{12}\right)(x+2)^4 \\ &\quad + \left(-\frac{a_0}{15} + \frac{a_1}{30}\right)(x+2)^5 + \cdots \end{aligned}$$

Step 9: Collect up the terms that contain the same coefficient, factor the coefficient out and write the results as a new series

$$\begin{aligned} y(x) &= a_0\left\{1 - (x+2)^2 + \frac{1}{6}(x+2)^3 + \frac{1}{6}(x+2)^4 - \frac{1}{15}(x+2)^5 + \cdots\right\} \\ &\quad + a_1\left\{(x+2) - \frac{1}{3}(x+2)^3 + \frac{1}{12}(x+2)^4 + \frac{1}{30}(x+2)^5 + \cdots\right\} \end{aligned}$$

Note: That's the solution for this problem as far as we're concerned. Notice that this solution looks nothing like the solution to the previous example. It's the same differential equation, but changing $x_o$ completely changed the solution.

3. Determine a series solution about $x_o = 0$ for the following initial value problem.

$$y'' - 2xy' + y = 0, \qquad y(0) = 1, \qquad y'(0) = 1$$

*Solution:*

Assume solution:

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then,

$$y' = \sum_{n=1}^{\infty} n\, a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Step 1: Plugging into the differential equation

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

Step 2: Get all the coefficients moved into the series.

$$\sum_{n=2}^{\infty} n\,(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} 2na_n\, x^n \; + \sum_{n=0}^{\infty} a_n\, x^n = 0$$

Step 3: Need to shift the first series down by 2 to get common exponents in all the series

$$\sum_{n=0}^{\infty} (n+2)\,(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} 2na_n\, x^n + \sum_{n=0}^{\infty} a_n\, x^n = 0$$

Step 4: Combine the series by stripping out the $n$ = 0 terms from both the first and third series

$$2a_2x^0 + \sum_{n=1}^{\infty} (n+2)\,(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} 2na_n\, x^n + a_0x^0 + \sum_{n=1}^{\infty} a_n\, x^n = 0$$

$$(a_0 + 2a_2)x^0 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} - 2na_n + a_n]\, x^n = 0 = \; 0x^0 + \sum_{n=0}^{\infty} [0]\, x^n$$

Step 5: Set all the coefficients equal to zero

| | |
|---|---|
| $n = 0$ | $a_0 + 2a_2 = 0$ |
| $n = 1, 2, 3, \ldots$ | $(n+2)(n+1)a_{n+2} - 2na_n + a_n = 0$ |

Step 6: Solve the first as well as **the recurrence relation**.

| | |
|---|---|
| $n = 0$ | $a_2 = -\dfrac{a_0}{2}$ |
| $n = 1, 2, 3, \ldots$ | $a_{n+2} = \dfrac{(2n-1)a_n}{(n+2)(n+1)}$ |

Step 7: Start plugging in values of $n$.

| | |
|---|---|
| $n = 0$ | $a_2 = -\dfrac{a_0}{2}$ |
| $n = 1$ | $a_3 = \dfrac{a_1}{6}$ |
| $n = 2$ | $a_4 = \dfrac{3a_2}{12} = \dfrac{a_2}{4} = -\dfrac{a_0}{8}$ |
| $n = 3$ | $a_5 = \dfrac{5a_3}{20} = \dfrac{a_3}{4} = \dfrac{a_1}{24}$ |

Note: Can choose any arbitrary constants for $a_0$ and $a_1$

Step 8: Get the solution

$$y(x) = \sum_{n=0}^{\infty} a_n\, x^n = \; a_0 + \; a_1x + \; a_2x^2 + \; a_3x^3 + \cdots$$

$$= a_0 + a_1 x + \left(-\frac{a_0}{2}\right) x^2 + \frac{a_1}{6} x^3 + \left(-\frac{a_0}{8}\right) x^4 + \frac{a_1}{24} x^5 + \cdots$$

Step 9: Collect up the terms that contain the same coefficient, factor the coefficient out and write the results as a new series

$$y(x) = a_0 \left[1 - \frac{x^2}{2} - \frac{x^4}{8} + \cdots\right] + a_1 \left[x + \frac{x^3}{6} + \frac{x^5}{24} + \cdots\right]$$

Step 10: Applying the initial conditions gives values for $a_0$ and $a_1$

$$y(0) = 1 \quad \Rightarrow \quad a_0 = 1$$
$$y'(0) = 1 \quad \Rightarrow \quad a_1 = 1$$

Step 11: Write out the particular solution

$$y(x) = \left[1 - \frac{x^2}{2} - \frac{x^4}{8} + \cdots\right] + \left[x + \frac{x^3}{6} + \frac{x^5}{24} + \cdots\right]$$

Solutions About Singular Points

The power series method for solving linear differential equations with variable coefficients no longer works when solving the differential equation about a singular point. It appears that some features of the solutions of such equations of the most importance for applications are largely determined by their behavior near their singular points. Frobenius method is usually used to solve the differential equation about a regular singular point. This method does not always yield two infinite series solutions. When only one solution is found, a certain formula can be used to get the second solution.

The two differential equations

$$(a) \qquad y'' + xy = 0 \qquad\qquad (b) \qquad xy'' + y = 0 \tag{7}$$

are similar only in that they are both examples of simple linear second-order differential equations with variable coefficients. For (7a), $x = 0$ is an ordinary point; hence, there is no problem in finding two distinct power series solution centered at that point. In contrast, $x = 0$ is a singular point for (7b), finding two infinite series solutions about that point becomes more difficult task.

For the homogeneous second-order linear differential equation

$$A(x)y'' + B(x)y' + C(x)y = 0 \tag{8}$$

The singular points are simply points where $A(x) = 0$ if the functions $A$, $B$, and $C$ are polynomials having no common factors.

For example, $x$ = 0 is the only singular point of the Bessel equation of order $n$,

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$

whereas the Legendre equation of order $n$,

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$$

has two singular points $x$ = -1 and $x$ = 1.

Note: Usually, only the case in which $x$ = 0 is a singular point of Equation (7) is considered. A differential equation having $x = a$ as a singular point is easily transformed by the substitution $t = x - a$ into one having a corresponding singular point at 0.

<u>Types of Singular Points</u>

A differential equation having a singular point at 0 ordinarily will not have power series solutions of the form

$$y(x) = \sum c_n x^n$$

So the straightforward method of power series fails in this case.

A *singular point* $x_0$ of a linear differential equation

$$A(x)y'' + B(x)y' + C(x)y = 0$$

is further classified as either *regular* or *irregular*. The classification depends on the functions $P$ and $Q$ in the standard form

$$y'' + P(x)y' + Q(x)y = 0$$

**Definition** (Regular or Irregular Singular Points)

A singular point $x_0$ is said to be a regular singular point of the differential equation (8) if the functions

$$p(x) = (x - x_0)P(x) \qquad \text{and} \qquad q(x) = (x - x_0)^2Q(x)$$

are both analytic at $x_0$. A singular point that is not regular is said to be irregular singular point of the equation.

**Quick Visual Check** (Regular or Irregular Singular Points)

If $x - x_0$ appears at most to the first power in the denominator of P(x) and at most to the second power in the denominator of Q(x), then $x - x_0$ is a regular singular point.

Example 6.4:

Find the singular point(s) for the differential equation

$$(x^2-4)^2y''+3(x-2)y'+5y=0$$

*Answer*

Divide the equation with

$$(x^2-4)^2=(x-2)^2(x+2)^2$$

and reduce the coefficients to the lowest terms, produce

$$P(x)=\frac{3}{(x-2)(x+2)^2} \qquad \text{and} \qquad Q(x)=\frac{5}{(x-2)^2(x+2)^2}$$

Test *P*(*x*) and *Q*(*x*)

(i) For $x = 2$ to be a regular point, the factor $x - 2$ can appear at most to the first power in the denominator of *P*(*x*) and at most to the second power in the denominator of *Q*(*x*). A check of the denominators of *P*(*x*) and *Q*(*x*) shows that both these conditions are satisfied, so *x* _ 2 is a regular singular point. Alternatively, the same conclusion is madeby noting that both rational functions

$$p(x)=(x-2)P(x)=\frac{3}{(x+2)^2} \qquad \text{and} \qquad q(x)=(x-2)^2Q(x)=\frac{5}{(x+2)^2}$$

are analytic at *x* -2.

(ii) Now since the factor $x - (-2) = x + 2$ appears to the second power in the denominator of *P*(*x*), we can conclude immediately that $x = -2$ is an irregular singular point of the equation. This also follows from the fact that

$$p(x)=(x+2)P(x)=\frac{3}{(x-2)(x+2)}$$

is not analytic at $x = -2$.

# FROBENIUS METHOD

## WEEK 7: FROBENIUS METHOD

### 7.1 Solutions about singular points

The power series method for solving linear differential equations with variable coefficients no longer works when solving the differential equation about a singular point. It appears that some features of the solutions of such equations of the most importance for applications are largely determined by their behavior near their singular points. Frobenius method is usually used to solve the differential equation about a regular singular point. *This method does not always yield two infinite series solutions. When only one solution is found, a certain formula can be used to get the second solution*.

**Reduction of Order**

The "reduction of order method" is a method for converting any linear differential equation to another linear differential equation of lower order, and then constructing the general solution to the original differential equation using the general solution to the lower-order equation.

**Reduction of Order for Homogeneous Linear Second-Order Equations**

This method is for finding a general solution to some homogeneous linear second-order differential equation

$$ay'' + by' + cy = 0$$

where $a$, $b$, and $c$ are known functions with $a(x)$ never being zero on the interval of interest. Then assume that there is already one nontrivial particular solution $y_1(x)$ to this generic differential equation.

Now, the details in using the reduction of order method to solve the above:

Step 1: Let

$$y = y_1 u$$

Then, using the product rule, derive $y'$ and $y''$:

$$y' = (y_1 u)' = y_1' u + y_1 u'$$

and

$$\begin{aligned} y'' = (y')' &= (y_1' u + y_1 u')' \\ &= (y_1' u)' + (y_1 u')' \\ &= (y_1'' u + y_1' u') + (y_1' u' + y_1 u'') \\ &= y_1'' u + 2y_1' u' + y_1 u'' \end{aligned}$$

Step 2: Plug the formulas just computed for $y$ , $y'$ and $y''$ into the differential equation, group together the coefficients for u and each of its derivatives, and simplify as far as possible.

$$\begin{aligned}0 &= ay'' + by' + cy \\ &= a[y_1''u + 2y_1'u' + y_1u''] + b[y_1'u + y_1u'] + c[y_1u] \\ &= ay_1''u + 2ay_1'u' + ay_1u'' + by_1'u + by_1u' + cy_1u \\ &= ay_1u'' + [2ay_1' + by_1']u' + [ay_1'' + by_1' + cy_1]u\end{aligned}$$

The differential equation becomes

$$Au'' + Bu' + Cu = 0$$

where

$$A = ay_1$$

$$B = 2ay_1' + by_1'$$

$$C = ay_1'' + by_1' + cy_1$$

But remember $y_1$ is a solution to the homogeneous equation

$$ay'' + by' + cy = 0$$

Consequently,

$$C = ay_1'' + by_1' + cy_1 = 0$$

and the differential equation for *u* automatically reduces to

$$Au'' + Bu' = 0$$

The *u* term always drops out.

Step 3: Now find the general solution to the second-order differential equation just obtained for *u*

$$Au'' + Bu' = 0$$

via the substitution method:

(a) Let $u' = v$.

Thus,

$$u'' = v' = \frac{dv}{dx}$$

To convert the second-order differential equation for *u* to the first-order differential equation for *v*:

$$A\frac{dv}{dx} + Bv = 0$$

Note: This first-order differential equation will be both linear and separable.

(b) Find the general solution *v*(*x*) to this first-order equation.

(c) Using the formula just found for v , integrate the substitution formula $u' = v$ to obtain the formula for $u$

$$u(x) = \int v(x)dx$$

Don't forget all the arbitrary constants.

Step 4: Finally, plug the formula just obtained for $u(x)$ into the first substitution $y = y_1 u$ used to convert the original differential equation for y to a differential equation for $u$. The resulting formula for $y(x)$ will be a general solution for that original differential equation.

To illustrate the method, use the differential equation

$$x^2y'' - 3xy' + 4y = 0$$

Note that the first coefficient, $x^2$ , vanishes when $x = 0$. So $x = 0$ ought not be in any interval of interest for this equation and solution should be found over the intervals (0, ∞) and (−∞, 0). Before starting the reduction of order method, one nontrivial solution $y_1$ is needed to the differential equation. Ways for finding that first solution will be discussed in later chapters. For now let us just observe that if

$$y_1(x) = x^2$$

then

$$\begin{aligned} x^2y_1'' - 3xy_1' + 4y_1 &= x^2\frac{d^2}{dx^2}[x^2] - 3x\frac{d}{dx}[x^2] + 4[x^2] \\ &= x^2[2 \cdot 1] - 3x[2x] + 4x^2 \\ &= x^2[2 - (3 \cdot 2) + 4] = 0 \end{aligned}$$

Thus, one solution to the above differential equation is $y_1(x) = x^2$

Step 1:

$$y = y_1u = x^2u$$

The derivatives of $y$ are:

$$y' = (x^2u)' = 2xu + x^2u'$$

and

$$\begin{aligned} y'' &= (y')' = (2xu + x^2u')' \\ &= (2xu)' + (x^2u')' \\ &= (2u + 2xu') + (2xu' + x^2u'') \\ &= 2u + 4xu' + x^2u'' \end{aligned}$$

Step 2:

$$0 = x^2y'' - 3xy' + 4y$$

$$= x^2[2u + 4xu' + x^2u''] - 3x[2xu + x^2u'] + 4[x^2u]$$
$$= 2x^2u + 4x^3u' + x^4u'' - 6x^2u - 3x^3u' + 4x^2u$$
$$= x^4u'' + [4x^3 - 3x^3]u' + [2x^2 - 6x^2 + 4x^2]u$$
$$= x^4u'' + x^3u' + 0 \cdot u$$

So, the resulting differential equation for *u* is

$$x^4u'' + x^3u' = 0$$

Further simplify by dividing $x^4$

$$u'' + \frac{1}{x}u' = 0$$

Step 3: Let $v = u'$ and $v' = u''$. Hence, the above differential equation becomes

$$\frac{dv}{dx} + \frac{1}{x}v = 0$$

Equivalently,

$$\frac{dv}{dx} = -\frac{1}{x}v$$

This is separable first-order differential equation.

$$\frac{1}{v}\frac{dv}{dx} = -\frac{1}{x}$$
$$\int \frac{1}{v}dv = \int -\frac{1}{x}dx$$
$$ln|v| = -ln|x| + C_0$$
$$v = \pm e^{-ln|x| + C_0}$$
$$v = \pm x^{-1}e^{C_0} = \frac{C_1}{x}$$

Since $u' = v$, then

$$u(x) = \int v(x)dx = \int \frac{C_1}{x}dx = C_1 ln|x| + C_2$$

Step 4: Here, $y_1(x) = x^2$

$$y = y_1u = x^2[C_1 ln|x| + C_2]$$
$$= C_1x^2 ln|x| + C_2x^2$$

This is the general solution to the differential equation $x^2y'' - 3xy' + 4y = 0$. The general solution obtained can be viewed as a linear combination of the two functions

$$y_1(x) = x^2 \qquad \text{and} \qquad y_2 = x^2 ln|x|$$

Since the $C_1$ and $C_2$ in the above formula for $y(x)$ are arbitrary constants, and $y_2$ is given by that formula for y with $C_1 = 1$ and $C_2 = 0$, it must be that this $y_2$ is another particular solution to our original homogeneous linear differential equation. What's more, it is clearly not a constant multiple of $y_1$.

The two differential equations

$$(a) \qquad y'' + xy = 0 \qquad\qquad (b) \qquad xy'' + y = 0 \qquad\qquad (7)$$

are similar only in that they are both examples of simple linear second-order differential equations with variable coefficients. For (7a), $x = 0$ is an ordinary point; hence, there is no problem in finding two distinct power series solution centered at that point. In contrast, $x = 0$ is a singular point for (7b), finding two infinite series solutions about that point becomes more difficult task.

## 7.2 Frobenius method

If $x = x_0$ is a singular point of the differential equation (8), then there exists <u>at least one solution</u> of the form

$$y(x) = (x - x_0)^r \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r}$$

where the number $r$ and the $c_k$s are constants to be determined. The series will converge at least on some interval $0 < x - x_0 < R$.

<u>An Introduction to the Method of Frobenius</u>

Before actually starting the method, there are two "pre-steps":

Pre-step 1: Choose a value for $x_0$. If conditions are given for $y(x)$ at some point, then use that point for $x_0$. Otherwise, choose $x_0$ as convenient — which usually choose $x_0 = 0$.

Pre-step 2: Get the differential equation into the form

$$A(x)y'' + B(x)y' + C(x)y = 0$$

where $A$, $B$, and $C$ are polynomials.

Now for the basic method of Frobenius:

Step 1: (a) Start by assuming a solution of the form

$$y = y(x) = (x - x_0)^r \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

where $a_0$ is an arbitrary constant. Since it is arbitrary, we can and will assume $a_0 \neq 0$ in the following computations.

(b) Then simplify the formula for the following computations by bringing the $(x - x_0)^r$ factor into the summation,

$$y = y(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^{k+r}$$

(c) And then compute the corresponding modified power series for $y'$ and $y''$ from the assumed series for y by differentiating "term-by-term".

Step 2: Plug these series for $y$ , $y'$ , and $y''$ back into the differential equation, "multiply things out", and divide out the $(x - x_0)^r$ to get the left side of your equation in the form of the sum of a few power series.

Some Notes:

ii. Absorb any *x*'s in *A*, *B* and *C* (of the differential equation) into the series.

iii. Dividing out the $(x - x_0)^r$ isn't necessary, but it simplifies the expressions slightly and reduces the chances of silly errors later.

iv. You may want to turn your paper sideways for more room!

Step 3: For each series in your last equation, do a change of index so that each series looks like

$$\sum_{n=\text{something}}^{\infty} [\text{something not involving } x](x - x_0)^n$$

Be sure to appropriately adjust the lower limit in each series.

Step 4: Convert the sum of series in your last equation into one big series. The first few terms will probably have to be written separately. Simplify what can be simplified.

Observe that the end result of this step will be an equation of the form

some big power series = 0

This, in turn, tells that each term that big power series must be 0.

Step 5: The first term in the last equation just derived will be of the form

$$a_0[\text{formula of } r](x - x_0)^{\text{something}}$$

But, remember, each term in that series must be 0. So we must have

$$a_0[\text{formula of } r] = 0$$

Moreover, since $a_0 \neq 0$ (by assumption), the above must reduce to

$$\text{formula of } r = 0$$

This is the *indicial equation* for *r*. It will always be a quadratic equation for *r* (i.e., of the form $\alpha r^2 + \beta r + \delta = 0$). Solve this equation for *r*. You will get two solutions (sometimes called either the *exponents* of the solution or the *exponents* of the singularity). Denote them by $r_2$ and $r_1$ with $r_2 \leq r_1$

Step 6: Using $r_1$, the larger *r* just found:

(a) Plug $r_1$ into the last series equation (and simplify, if possible). This will give you an equation of the form

$$\sum_{n=n_0}^{\infty} [n^{th} \text{ formula of } a_k{}'s\,](x - x_0)^n = 0$$

Since each term must vanish, we have

$$n^{th} \text{ formula of } a_k's = 0 \qquad\qquad \text{for } \; n_0 \leq n$$

(b) Solve this for

$$a_{\text{highest index}} = \text{formula of } n \text{ and lower indexed } a_k{}'s$$

A few of these equations may need to be treated separately, but you will also obtain a relatively simple formula that holds for all indices above some fixed value. This formula is the recursion formula for computing each coefficient $a_n$ from the previously computed coefficients.

(c) To simplify things just a little, do another change of indices so that the recursion formula just derived is rewritten as

$$a_k = \text{formula of } k \text{ and lower indexed coefficients}$$

Step 7: Use the recursion formula (and any corresponding formulas for the lower-order terms) to find all the $a_k$'s in terms of $a_0$ and, possibly, one other $a_m$. Look for patterns!

Step 8: Using $r = r_1$ along with the formulas just derived for the coefficients, write out the resulting series for *y*. Try to simplify it and factor out the arbitrary constant(s).

Step 9: If the indicial equation had two distinct solutions, now repeat steps 6 through 8 with the smaller *r*, $r_2$. Sometimes (but not always) this will give you a second independent solution to the differential equation. Sometimes, also, the series formula derived in this mega-step will include the series formula already derived.

Step 10: If the last step yielded $y$ as an arbitrary linear combination of two different series, then that is the general solution to the original differential equation. If the last step yielded $y$ as just one arbitrary constant times a series, then the general solution to the original differential equation is the linear combination of the two series obtained at the end of steps 8 and 9. Either way, write down the general solution (using different symbols for the two different arbitrary constants!). If step 9 did not yield a new series solution, then at least write down the one solution previously derived, noting that a second solution is still needed for the general solution to the differential equation.

Last Step: See if you recognize the series as the series for some well-known function (you probably won't!).

The following Bessel's equation of order ½ will be solved to illustrate the method.

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left[1 - \frac{1}{4x^2}\right]y = 0$$

Pre-step 1: There is no initial values at any point, so we will choose $x_0$ as simply as possibly; namely, $x_0 = 0$, which is a regular singular point.

Pre-step 2: To get the given differential equation into the form desired, we multiply the equation by $4x^2$. That gives us the differential equation

$$4x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + [4x^2 - 1]y = 0$$

Step 1: Since we've already decided $x_0 = 0$, we assume

$$y(x) = x^r\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k x^{k+r}$$

with $a_0 \neq 0$. Differentiating this term-by-term, we see that

$$y' = \frac{d}{dx}\sum_{k=0}^{\infty} a_k x^{k+r} = \sum_{k=0}^{\infty}\frac{d}{dx}[a_k x^{k+r}] = \sum_{k=0}^{\infty}(k+r)a_k x^{k+r-1}$$

$$y'' = \frac{d}{dx}\sum_{k=0}^{\infty}(k+r)a_k x^{k+r-1} = \sum_{k=0}^{\infty}\frac{d}{dx}[(k+r)a_k x^{k+r-1}] = \sum_{k=0}^{\infty}(k+r)(k+r-1)a_k x^{k+r-2}$$

Step 2: Combining the above series formulas for $y$, $y'$ and $y''$ with our differential equation, we get

$$0 = 4x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + [4x^2 - 1]y$$

$$= 4x^2\sum_{k=0}^{\infty}(k+r)(k+r-1)a_kx^{k+r-2} + 4x\sum_{k=0}^{\infty}(k+r)a_kx^{k+r-1} + [4x^2-1]\sum_{k=0}^{\infty}a_kx^{k+r}$$

$$= 4x^2\sum_{k=0}^{\infty}(k+r)(k+r-1)a_kx^{k+r-2} + 4x\sum_{k=0}^{\infty}(k+r)a_kx^{k+r-1} + 4x^2\sum_{k=0}^{\infty}a_kx^{k+r}$$
$$-1\sum_{k=0}^{\infty}a_kx^{k+r}$$

$$= \sum_{k=0}^{\infty}(k+r)(k+r-1)4a_kx^{k+r} + \sum_{k=0}^{\infty}(k+r)4a_kx^{k+r} + \sum_{k=0}^{\infty}4a_kx^{k+2+r} + \sum_{k=0}^{\infty}(-1)a_kx^{k+r}$$

Dividing out the $x^r$ from each term then yields

$$= \sum_{k=0}^{\infty}(k+r)(k+r-1)4a_kx^{k} + \sum_{k=0}^{\infty}(k+r)4a_kx^{k} + \sum_{k=0}^{\infty}4a_kx^{k+2} + \sum_{k=0}^{\infty}(-1)a_kx^{k}$$

Step 3: In all but the third series, the "change of index" is trivial, $n = k$ . In the third series, we will set $n = k + 2$ (equivalently, $n - 2 = k$). This means, in the third series, replacing $k$ with $n - 2$ , and replacing $k = 0$ with $n = 0 + 2 = 2$:

$$0 = \sum_{k=0}^{\infty}(k+r)(k+r-1)4a_kx^{k} + \sum_{k=0}^{\infty}(k+r)4a_kx^{k} + \sum_{k=0}^{\infty}4a_kx^{k+2} + \sum_{k=0}^{\infty}(-1)a_kx^{k}$$

$\boxed{n = k}$ $\boxed{n = k}$ $\boxed{n = k+2}$ $\boxed{n = k}$

$$= \sum_{n=0}^{\infty}(n+r)(n+r-1)4a_nx^{n} + \sum_{n=0}^{\infty}(n+r)4a_nx^{n} + \sum_{n=2}^{\infty}4a_{n-2}x^{n} + \sum_{n=0}^{\infty}(-1)a_nx^{n}$$

Step 4: Since one of the series in the last equation begins with $n = 2$, we need to separate out the terms corresponding to $n = 0$ and $n = 1$ in the other series before combining series:

$$0 = 4a_0(0+r)(0+r-1)x^0 + 4a_1(1+r)(1+r-1)x^1 + \sum_{n=2}^{\infty}(n+r)(n+r-1)4a_nx^{n}$$
$$+ 4a_0(0+r)x^0 + 4a_1(1+r)x^1 + \sum_{n=2}^{\infty}(n+r)4a_nx^{n} + \sum_{n=2}^{\infty}4a_{n-2}x^{n} - a_0x^0$$
$$- a_1x^1 + \sum_{n=2}^{\infty}(-1)a_nx^{n}$$

So our differential equation reduces to

$$a_0[4r^2 - 1]x^0 + a_1[4r^2 + 8r + 3]x^1 + \sum_{n=2}^{\infty} a_n[4(n+r)^2 - 1) + 4a_{n-2}]x^n = 0 \qquad (*)$$

Observe that the end result of this step will be an equation of the form

some big power series = 0

This, in turn, tells us that each term that big power series must be 0.

Step 5: The first term in the "big series" is the first term in the equation

$$a_0[4r^2 - 1]x^0$$

Since this must be zero (and $a_0 \neq 0$ by assumption) the indicial equation is

$$4r^2 - 1 = 0$$

Thus,

$$r = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}$$

Following the convention given,

$$r_2 = -\frac{1}{2} \qquad \text{and} \qquad r_1 = \frac{1}{2}$$

Step 6: Letting $r = r_1 = \frac{1}{2}$, equation (*) yields

$$a_0\left[4\left(\frac{1}{2}\right)^2 - 1\right]x^0 + a_1\left[4\left(\frac{1}{2}\right)^2 + 8\left(\frac{1}{2}\right) + 3\right]x^1 + \sum_{n=2}^{\infty} a_n\left[4\left(n+\frac{1}{2}\right)^2 - 1) + 4a_{n-2}\right]x^n = 0$$

$$a_0 0x^0 + a_1 8x^1 + \sum_{n=2}^{\infty}[a_n(4n^2 + 4n + 1 - 1) + 4a_{n-2}]x^n = 0$$

The first term vanishes (as it should since $r = 1/2$ satisfies the indicial equation, which came from making the first term vanish). Doing a little more simple algebra, we see that, with r = 1/2 , equation (*) reduces to

$$0a_0x^0 + 8a_1x^1 + \sum_{n=2}^{\infty} 4[n(n+1)a_n + a_{n-2}]x^n = 0 \qquad (**)$$

From the above series, we must have

$$n(n+1)a_n + a_{n-2} = 0 \qquad \text{for} \qquad n = 2, 3, 4, \ldots$$

Solving for $a_n$ leads to the recursion formula

$$a_n = \frac{-1}{n(n+1)}a_{n-2} \qquad \text{for} \qquad n = 2, 3, 4, \ldots$$

Using the trivial change of index, $k = n$, this is

$$a_k = \frac{-1}{k(k+1)} a_{k-2} \qquad \text{for} \qquad k = 2, 3, 4, \ldots$$

Step 7: From the first two terms in equation (**),

$$0a_0 = 0 \qquad \Longrightarrow \qquad a_0 \text{ is arbitrary}$$

$$8a_1 = 0 \qquad \Longrightarrow \qquad a_1 = 0$$

Using these values and the recursion formula with $k = 2, 3, 4, \ldots$ (and looking for patterns):

$$a_2 = \frac{-1}{2(2+1)} a_{2-2} = \frac{-1}{2 \cdot 3} a_0$$

$$a_3 = \frac{-1}{3(3+1)} a_{3-2} = \frac{-1}{3 \cdot 4} a_1 = \frac{-1}{3 \cdot 4} \cdot 0 = 0$$

$$a_4 = \frac{-1}{4(4+1)} a_{4-2} = \frac{-1}{4 \cdot 5} a_2 = \frac{-1}{4 \cdot 5} \cdot \frac{-1}{2 \cdot 3} a_0 = \frac{(-1)^2}{5 \cdot 4 \cdot 3 \cdot 2} a_0 = \frac{(-1)^2}{5!} a_0$$

$$a_5 = \frac{-1}{5(5+1)} a_{5-2} = \frac{-1}{5 \cdot 6} a_3 = \frac{-1}{5 \cdot 6} \cdot 0 = 0$$

$$a_6 = \frac{-1}{6(6+1)} a_{6-2} = \frac{-1}{6 \cdot 7} a_4 = \frac{-1}{7 \cdot 6} \cdot \frac{(-1)^2}{5!} a_0 = \frac{(-1)^3}{7!} a_0$$

$$\vdots$$

The patterns should be obvious here:

$$a_k = 0 \qquad \text{for} \quad k = 1, 3, 5, 7, \ldots$$

and

$$a_k = \frac{(-1)^{k/2}}{(k+1)!} a_0 \qquad \text{for} \quad k = 2, 4, 6, 8, \ldots$$

Using $k = 2m$, this be written more conveniently as

$$a_{2m} = (-1)^m \frac{a_0}{(2m+1)!} \qquad \text{for} \quad m = 1, 2, 3, 4, 5, \ldots$$

Step 8: Plugging $r = 1/2$ and the formulas just derived for the $a_n$'s into the formula originally assumed for $y$, we get

$$y = x^r \sum_{k=0}^{\infty} a_k x^k$$

$$= x^r \left[ \sum_{\substack{k=0 \\ k\ odd}}^{\infty} a_k x^k + \sum_{\substack{k=0 \\ k\ even}}^{\infty} a_k x^k \right]$$

$$= x^{1/2} \left[ \sum_{\substack{k=0 \\ k\ odd}}^{\infty} 0 \cdot x^k + \sum_{m=0}^{\infty} (-1)^m \frac{a_0}{(2m+1)!} x^{2m} \right]$$

$$= x^{1/2} \left[ 0 + a_0 \sum_{m=0}^{\infty} (-1)^m \frac{1}{(2m+1)!} x^{2m} \right]$$

So one solution to Bessel's equation of order 1/2 is given by

$$y = a_0 x^{1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m}$$

Step 9: Letting $r = r_2 = -\frac{1}{2}$ in equation (*) yields

$$a_0 \left[ 4\left(-\frac{1}{2}\right)^2 - 1 \right] x^0 + a_1 \left[ 4\left(-\frac{1}{2}\right)^2 + 8\left(-\frac{1}{2}\right) + 3 \right] x^1 + \sum_{n=2}^{\infty} a_n \left[ 4\left(n - \frac{1}{2}\right)^2 - 1) + 4a_{n-2} \right] x^n = 0$$

$$a_0 0 x^0 + a_1 0 x^1 + \sum_{n=2}^{\infty} [a_n (4n^2 - 4n + 1 - 1) + 4a_{n-2}] x^n = 0$$

Then …

⋮ {"Fill in the dots" in the last statement. That is, do all the computations that were omitted.}

yielding

$$y = a_0 x^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m} + a_1 x^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1}$$

Note that the second series term is the same series (slightly rewritten) since $x^{-1/2}x^{2m+1} = x^{1/2}x^{2m}$

Step 10: We are in luck. In the last step we obtained *y* as the linear combination of two different series. So

$$y = a_0 x^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m} + a_1 x^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1}$$

is the general solution to the original differential equation — Bessel's equation of order 1/2.

Last Step: Our luck continues! The two series are easily recognized as the series for the sine and the cosine functions:

$$y = a_0 x^{-1/2} \cos x + a_1 x^{-1/2} \sin x$$

So the general solution to Bessel's equation of order 1/2 is

$$= a_0 \frac{\cos x}{\sqrt{x}} + a_1 \frac{\sin x}{\sqrt{x}}$$

<u>Advice and Comments</u>

1. If you get something like

$$2a_1 = 0$$

then you know $a_1 = 0$. On the other hand, if you get something like

$$0a_1 = 0$$

then you have an equation that tells you nothing about $a_1$ . This means that $a_1$ is an arbitrary constant (unless something else tells you otherwise).

2. If the recursion formula blows up at some point, then some of the coefficients must be zero. For example, if

$$a_n = \frac{3}{(n+2)(n-5)} a_{n-2}$$

then, for $n = 5$ ,

$$a_n = \frac{3}{(7)(0)} a_3 = \infty \cdot a_3$$

which can only make sense if $a_3 = 0$. Note also, that, unless otherwise indicated, $a_5$ here would be arbitrary. (Remember, the last equation is equivalent to $(7)(0)a_5 = 3a_3$ .)

3 If you get a coefficient being zero, it is a good idea to check back using the recursion formula to see if any of the previous coefficients must also be zero, or if many of the following coefficients are zero. In some cases, you may find that an "infinite" series solution only contains a finite number of nonzero terms, in which case we have a "terminating series"; i.e., a solution which is simply a polynomial.

On the other hand, obtaining $a_0 = 0$, contrary to our basic assumption that $a_0 \neq 0$, tells you that there is no series solution of the form assumed for the basic Frobenius method using that value of *r*.

4. It is possible to end up with a three term recursion formula, say,

$$a_n = \frac{1}{n^2+1} a_{n-1} + \frac{2}{3n(n+3)} a_{n-2}$$

This, naturally, makes "finding patterns" rather difficult.

5. Keep in mind that, even if you find that "finding patterns" and describing them by "nice" formulas is beyond you, you can always use the recursion formulas to compute (or have a computer compute) as many terms as you wish of the series solutions.

6. The computations can become especially messy and confusing when $a_0 \neq 0$. In this case, simplify matters by using the substitutions

$$Y(X) = y(x) \qquad \text{with} \qquad X = x - x_0$$

You can then easily verify that, under these substitutions,

$$Y'^{(X)} = \frac{dY}{dX} = \frac{dy}{dx} = y(x)$$

and the differential equation

$$A(x)\frac{d^2y}{dx^2} + B(x)\frac{dy}{dx} + C(x)y = 0$$

becomes

$$A_1(X)\frac{d^2Y}{dX^2} + B_1(X)\frac{dY}{dX} + C_1(X)Y = 0$$

with

$$A_1(X) = A(X + x_0)\,, \quad B_1(X) = B(X + x_0) \quad \text{and} \quad C_1(X) = C(X + x_0)$$

Use the method of Frobenius to find the modified power series solutions

$$Y(X) = (X)^r \sum_{k=0}^{\infty} a_k X^k$$

for equation $A_1(X)\frac{d^2Y}{dX^2} + B_1(X)\frac{dY}{dX} + C_1(X)Y = 0$. The corresponding solutions to the original differential equation, equation $A(x)\frac{d^2y}{dx^2} + B(x)\frac{dy}{dx} + C(x)y = 0$, are then given from this via the above substitution,

$$y(x) = Y(X) = (X)^r \sum_{k=0}^{\infty} a_k X^k = (x - x_0)^r \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

The Big Theorem on the Frobenius Method

Let $x_0$ be a regular singular point (on the real line) for

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = 0$$

Then the indicial equation arising in the basic method of Frobenius exists and is a quadratic equation with two solutions $r_1$ and $r_2$ (which may be one solution, repeated). If $r_2$ and $r_1$ are real, assume $r_2 \leq r_1$. Then:

1. The basic method of Frobenius will yield at least one solution of the form

$$y_1(x) = (x - x_0)^{r_1} \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

where $a_0$ is the one and only arbitrary constant.

2. If $r_1 - r_2$ is not an integer, then the basic method of Frobenius **will** yield a second independent solution of the form

$$y_2(x) = (x - x_0)^{r_2} \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

where $a_0$ is an arbitrary constant.

3. If $r_1 - r_2 = N$ is positive integer, then the method of Frobenius **might** yield a second independent solution of the form

$$y_2(x) = (x - x_0)^{r_2} \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

where $a_0$ is an arbitrary constant. If it doesn't, then a second independent solution exists of the form

$$y_2(x) = y_1(x) \ln|x - x_0| + (x - x_0)^{r_2} \sum_{k=0}^{\infty} b_k (x - x_0)^k$$

or, equivalently

$$y_2(x) = y_1(x) \left[ \ln|x - x_0| + (x - x_0)^{-N} \sum_{k=0}^{\infty} c_k (x - x_0)^k \right]$$

where $b_0$ and $c_0$ are nonzero constants.

4. If $r_1 = r_2$, then there is a second solution of the form

$$y_2(x) = y_1(x) \ln|x - x_0| + (x - x_0)^{1+r_1} \sum_{k=0}^{\infty} b_k (x - x_0)^k$$

or, equivalently

$$y_2(x) = y_1(x) \left[ \ln|x - x_0| + (x - x_0)^{l} \sum_{k=0}^{\infty} c_k (x - x_0)^k \right]$$

In this case, $b_0$ and $c_0$ might be zero.

Moreover, if we let $R$ be the distance between xo and the nearest singular point (other than $x_o$ ) in the complex plane (with $R = \infty$ if $x_o$ is the only singular point), then the series solutions described above converge at least on the intervals ($x_o - R$, $x_o$) and ($x_o$, $x_o + R$).

Example 7.1: Apply the power series method to the following differential equations:

$$y'' + \frac{1}{2x}y' + \frac{1}{4x}y = 0$$

*Answer*:

Regular singular point at $x = 0$.

Multiply the equation with $4x$:

$$4xy'' + 2y' + y = 0$$

Assume solution:

$$y = \sum_{m=0}^{\infty} a_m\, x^{m+r}$$

Then,

$$y' = \sum_{m=0}^{\infty} (m+r)\, a_m x^{m+r-1}$$

$$y'' = \sum_{m=0}^{\infty} (m+r)\,(m+r-1) a_m x^{m+r-2}$$

Hence,

$$4x \sum_{m=0}^{\infty} (m+r)\,(m+r-1) a_m x^{m+r-2} + 2 \sum_{m=0}^{\infty} (m+r)\, a_m x^{m+r-1} + \sum_{m=0}^{\infty} a_m\, x^{m+r} = 0$$

$$4 \sum_{m=0}^{\infty} (m+r)\,(m+r-1) a_m x^{m+r-1} + 2 \sum_{m=0}^{\infty} (m+r)\, a_m x^{m+r-1} + \sum_{m=0}^{\infty} a_m\, x^{m+r} = 0$$

For the first and second summation, let $k + r = m + r - 1$ which implies $m = k + 1$

For the third summation, let $k + r = m + r$, which implies $m = k$

$$4 \sum_{k=-1}^{\infty} (k+1+r)\,(k+r) a_{k+1} x^{k+r} + 2 \sum_{k=-1}^{\infty} (k+1+r)\, a_{k+1} x^{k+r} + \sum_{k=0}^{\infty} a_k\, x^{k+r} = 0$$

$$4(r)(r-1)a_0x^{r-1} + 4\sum_{k=0}^{\infty}(k+1+r)\,(k+r)a_{k+1}x^{k+r} + 2(r)a_0\,x^{r-1} + 2\sum_{k=0}^{\infty}(k+1+r)\,a_{k+1}x^{k+r}$$
$$+\sum_{k=0}^{\infty} a_k\, x^{k+r} = 0$$

$$[4r(r-1)+2r]a_0x^{r-1} + \left(\left[\sum_{k=0}^{\infty} 4(k+1+r)\,(k+r)a_{k+1} + 2(k+1+r)a_{k+1} + a_k\right]x^{k+r}\right) = 0$$

Indicial Equation:

$$4r(r-1)+2r = 0$$

$$\Rightarrow \quad 4r^2 - 4r + 2r = 0$$

$$\Rightarrow \quad r^2 - \frac{1}{2}r = 0$$

$$\Rightarrow \quad r\left(r-\frac{1}{2}\right) = 0$$

$$\therefore \quad r_2 = 0\,;\ r_1 = \tfrac{1}{2}$$

**Distinct roots not differing by integer (including complex conjugates)**

$$4(k+1+r)(k+r)a_{k+1} + 2(k+1+r)a_{k+1} + a_k = 0$$

$$a_{k+1} = \frac{-a_k}{4(k+1+r)(k+r)+2(k+1+r)} \qquad k = 0, 1, 2, \ldots$$

First solution: $r = \frac{1}{2}$

$$a_{k+1} = \frac{-a_k}{4\left(k+1+\frac{1}{2}\right)\left(k+\frac{1}{2}\right)+2\left(k+1+\frac{1}{2}\right)} \qquad k = 0, 1, 2, \ldots$$

$$a_1 = \frac{-a_0}{3\cdot 2} = -\frac{a_0}{3!}$$

$$a_2 = \frac{-a_1}{5\cdot 4} = -\frac{-a_0}{5\cdot 4\cdot 3!} = \frac{a}{5!}$$

$$a_3 = \frac{-a_2}{7\cdot 6} = -\frac{a_0}{7\cdot 6\cdot 5!} = -\frac{a_0}{7!}$$

$\vdots$

In general, let $a_0 = 1$

$$y_1(x) = x^{1/2}\left(1 - \frac{1}{6}x + \frac{1}{120}x^2 - \frac{1}{5040}x^3 \pm \cdots\right) = x^{1/2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}x^m$$

Second solution: $r = 0$

$$A_{k+1} = \frac{-A_k}{4(k+1)(k) + 2(k+1)} \qquad k = 0, 1, 2, \ldots$$

$$A_1 = \frac{-A_0}{2 \cdot 1} = -\frac{A_0}{2!}$$

$$A_2 = \frac{-A_1}{4 \cdot 3} = -\frac{-A_0}{4 \cdot 3 \cdot 2!} = \frac{A_0}{4!}$$

$$A_3 = \frac{-A_2}{6 \cdot 5} = -\frac{A_0}{6 \cdot 5 \cdot 4!} = -\frac{A_0}{6!}$$

$\vdots$

In general, let $A_0 = 1$

$$y_2(x) = x^0\left(1 - \frac{1}{2}x + \frac{1}{24}x^2 - \frac{1}{720}x^3 \pm \cdots\right) = \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}x^m$$

Example 7.2: Solve the following differential equations using power series method:

$$x(x-1)y'' + (3x-1)y' + y = 0$$

*Answer*:

Assume solution:

$$y = \sum_{m=0}^{\infty} a_m\, x^{m+r}$$

Hence,

$$x(x-1)\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-2} + (3x-1)\sum_{m=0}^{\infty}(m+r)\,a_m x^{m+r-1} + \sum_{m=0}^{\infty}a_m\,x^{m+r} = 0$$

$$\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r} - \sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-1} + 3\sum_{m=0}^{\infty}(m+r)a_m\,x^{m+r}$$
$$- \sum_{m=0}^{\infty}(m+r)\,a_m x^{m+r-1} + \sum_{m=0}^{\infty}a_m\,x^{m+r} = 0$$

For the second and fourth summation, let *k* + *r* = *m* + *r* – 1 which implies *m* = *k* + 1

For the first, third, and fifth summation, let *k* + *r* = *m* + *r*, which implies *m* = *k*

$$\sum_{k=0}^{\infty}(k+r)\,(k+r-1)a_k x^{k+r} - \sum_{k=-1}^{\infty}(k+1+r)(k+r)\,a_{k+1}x^{k+r} + 3\sum_{k=0}^{\infty}(k+r)\,a_k x^{k+r} - \sum_{k=-1}^{\infty}(k+1+r)\,a_{k+1}x^{k+r} + \sum_{k=0}^{\infty}a_k\,x^{k+r} = 0$$

$$\sum_{k=0}^{\infty}(k+r)\,(k+r-1)a_k x^{k+r} - (r)(r-1)a_0x^{r-1} - \sum_{k=0}^{\infty}(k+1+r)(k+r)\,a_{k+1}x^{k+r} + 3\sum_{k=0}^{\infty}(k+r)\,a_k x^{k+r} - (r)a_0x^{r-1} - \sum_{k=0}^{\infty}(k+1+r)\,a_{k+1}x^{k+r} + \sum_{k=0}^{\infty}a_k\,x^{k+r} = 0$$

$$(-(r)(r-1)-r)a_0x^{r-1} + \left[\sum_{k=0}^{\infty}(k+r)\,(k+r-1)a_k - (k+1+r)(k+r)a_{k+1} + 3(k+r)a_k - (k+1+r)a_{k+1} + a_k\right]x^{k+r} = 0$$

Indicial Equation:

$-(r)(r-1) - r = 0$

$\Rightarrow \qquad -r^2 + r - r = 0$

$\Rightarrow \qquad r^2 = 0$

$\therefore \qquad r_1 = 0\,;\; r_2 = 0$ $\boxed{\textbf{Double root}}$

$$(k+r)(k+r-1)a_k - (k+1+r)(k+r)a_{k+1} + 3(k+r)a_k - (k+1+r)a_{k+1} + a_k = 0$$

$$a_{k+1} = a_k \qquad k = 0, 1, 2, \ldots$$

First solution: *r* = 0

$a_{k+1} = a_k \qquad k = 0, 1, 2, \ldots$

$a_1 = a_0$

$a_2 = a_1 = a_0$

$a_3 = a_2 = a_0$

$\vdots$

In general, let $a_0 = 1$

$$y_1(x) = x^0(1 + x + x^2 + x^3 + \cdots) = \sum_{m=0}^{\infty} x^m = \frac{1}{1-x}$$

Second solution: $r = 0$

$A_{k+1} = A_k \qquad k = 0, 1, 2, \dots$

$A_1 = A_0$

$A_2 = A_1 = A_0$

$A_3 = A_2 = A_0$

$\vdots$

In general, let $A_0 = 1$

$$y_2(x) = y_1(x)\ln x + x^1\left(\sum_{m=0}^{\infty} x^m\right) = \frac{1}{1-x}\ln x + x^1\left(\frac{1}{1-x}\right)$$

$$= \frac{1}{1-x}(\ln x - x)$$

<u>Example 7.3:</u> Solve the following differential equations using power series method:

$$(x^2 - 1)x^2 y'' - (x^2 + 1)xy' + (x^2 + 1)y = 0$$

<u>*Answer*</u>:

Assume solution:

$$y = \sum_{m=0}^{\infty} a_m\, x^{m+r}$$

Hence,

$$(x^2 - 1)x^2 \sum_{m=0}^{\infty} (m + r)\,(m + r - 1)a_m x^{m+r-2} - (x^2 + 1)x \sum_{m=0}^{\infty} (m + r)\, a_m x^{m+r-1} + (x^2 + 1) + \sum_{m=0}^{\infty} a_m\, x^{m+r} = 0$$

$$\sum_{m=0}^{\infty}(m+r)\,(m+r-1)a_m x^{m+r+2} - \sum_{m=0}^{\infty}(m+r)\,(m+r-1)a_m x^{m+r} - \sum_{m=0}^{\infty}(m+r)\,a_m x^{m+r+2}$$

$$+\sum_{m=0}^{\infty}(m+r)\,a_m x^{m+r} + \sum_{m=0}^{\infty}a_m\,x^{m+r+2} + \sum_{m=0}^{\infty}a_m\,x^{m+r} = 0$$

$$\sum_{m=0}^{\infty}(m+r-1)\,a_m x^{m+r+2} - \sum_{m=0}^{\infty}(m+r-1)(m+r+1)\,a_m x^{m+r} = 0$$

For the first summation, let $k + r = m + r + 2$ which implies $m = k - 2$

For the second summation, let $k + r = m + r$, which implies $m = k$

$$\sum_{k=2}^{\infty}(k+r-3)^2\,a_{k-2}x^{k+r} - \sum_{k=0}^{\infty}(k+r-1)(k+r+1)\,a_k x^{k+r} = 0$$

$$\sum_{k=2}^{\infty}[(k+r-3)^2 a_{k-2} - (k+r-1)(k+r+1)a_k]\,x^{k+r} - (r-1)(r+1)a_0x^r - (r)(r+2)a_1x^{r+1}$$
$$= 0$$

Indicial Equation:

$-(r-1)(r+1) = 0$

$\therefore \qquad r_1 = 1\,;\; r_2 = -1$ 

**Roots differing by an integer**

$(k+r-3)^2 a_{k-2} - (k+r-1)(k+r+1)a_k = 0$

$(k+r-1)(k+r+1)a_k = 0$

$$a_k = \frac{(k+r-3)^2 a_{k-2}}{(k+r-1)(k+r+1)} \qquad k = 2, 3, 4, \ldots$$

$-(r)(r+2)a_1 = 0$

$(-r^2 - 2r)a_1 = 0$

$a_1 = 0 \qquad \text{since} \;\; (-r^2 - 2r) \neq 0$

First solution: $r = 1$

$$a_k = \frac{(k-2)^2 a_{k-2}}{(k)(k+2)} \qquad k = 2, 3, 4, \ldots$$

$a_1 = 0$

$a_2 = 0$

$a_3 = 0$

$\vdots$

In general,

$y_1(x) = x^1(a_0 + a_1 x + a_2 x^2 + \cdots) = a_0 x$

Second solution: $r$ = -1

$$A_k = \frac{(k-4)^2 A_{k-2}}{(k-2)(k)} \qquad k = 2, 3, 4, \ldots$$

When $k$ = 2, there is no solution. Hence, there will be no solution in the form of power series and reduction of order should be used for getting the second solution.

Let $y_2(x) = xu(x)$ be a solution to the differential equation

Then $y_2'(x) = xu'(x) + u(x)$

And $y_2''(x) = xu''(x) + u'(x) + u'(x) = xu''(x) + 2u'(x)$

Substitute into the equation,

$$(x^2-1)x^2[xu''(x) + 2u'(x)] - (x^2+1)x[xu'(x) + u(x)] + (x^2+1)[xu(x)] = 0$$

$$x^2(x^3 - x)u'' + x^2(x^2 - 3)u' = 0$$

$$(x^3 - x)u'' + (x^2 - 3)u' = 0$$

$$\frac{u''}{u'} = \frac{(x^2-3)}{(x^3-x)} = -\frac{3}{x} + \frac{1}{x+1} + \frac{1}{x-1}$$

$$\int \frac{u''}{u'}\, du = \int -\frac{3}{x}\, dx + \int \frac{1}{x+1}\, dx + \int \frac{1}{x-1} dx$$

$$\ln u' = -3\ln x + \ln(x+1) + \ln(x-1) = \ln\frac{(x+1)(x-1)}{x^3} = \ln\frac{x^2-1}{x^3}$$

$$u = \ln x + \frac{1}{2x^2}$$

Therefore, the second solution is

$$y_2(x) = xu(x) = x\left(\ln x + \frac{1}{2x^2}\right) = x\ln x + \frac{1}{2x}$$