LAPLACE TRANSFORM

WEEK 8: LAPLACE TRANSFORM

8.1 INTRODUCTION

Laplace transforms are invaluable for any engineer's mathematical toolbox as they make solving linear ODEs and related initial value problems, as well as systems of linear ODEs, much easier. The key motivation for learning about Laplace transforms is that the process of solving an ODE is simplified to an algebraic problem (and transformations). Applications abound: electrical networks, springs, mixing problems, signal processing, and other areas of engineering and physics.

8.2 DEFINITION

Laplace transform of a function *f(t)* is defined as

$$F(s) = \mathcal{L}\{f(t)\} = \int_{0}^{\infty} f(t)e^{-st}\,dt$$

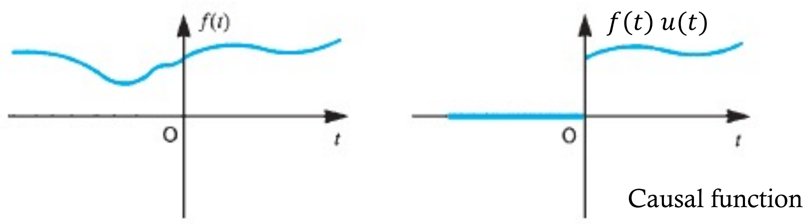
Laplace transform is called an **integral transform** because it transforms (changes) a function in one space to a function in another space by a *process of integration* that involves a kernel, $k(s,t) = e^{-st}$.

$$F(s) = \int_{0}^{\infty} k(s,t)f(t)\ dt$$

One can imagine $f(t)$ to be a function in the time domain (t). By performing Laplace transform, one can transform the time domain function to the frequency doman $F(s)$ to be in frequency domain (s).

The purpose of doing that is because it is easier to solve integrals and ordinary differential equations with constant coefficients in the frequency domain than it is in the time domain. Once solved (easily) in the frequency domain, one needs to revert it back to the time domain (using inverse Laplace transform) to obtain the solution. This will be topic to cover in Week 9, but before doing so, one needs to be familiar with the mechanics of forward and inverse Laplace transform.

Transforming f(t) to a causal function:



Note that since the limit of the above integral begins at t = 0, the behavior of the function $f(t)$ at t < 0 is ignored. This is so-called causal function (where $f(t) = 0$ at any value $t < 0$). In other words, we only deal with real time, $t \geq 0$.

Example 8.1:

1. Let $f(t) = 1$ when $t \geq 0$. Find $F(s)$.

Solution:

$$F(s) = \mathcal{L}(1)$$

$$= \int_0^\infty 1 \cdot e^{-st}\,dt$$

$$= -\frac{1}{s}e^{-st}\Big|_0^\infty$$

$$= \frac{1}{s}$$

2. Let $f(t) = e^{at}$ when $t \geq 0$, where a is a constant. Find $\mathcal{L}\{f(t)\}$.

Solution:

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{at} \cdot e^{-st}\,dt$$

$$= -\frac{1}{s-a}e^{-(s-a)t}\Big|_0^\infty$$

$$= \frac{1}{s-a}$$

8.3 LINEARITY

For any functions $f(t)$ and $g(t)$ whose transforms exist and any constants a and b, the transform of $af(t) + bg(t)$ is given by

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

Proof:

$$\mathcal{L}\{af(t) + bg(t)\} = \int_0^\infty e^{-st}\big(af(t) + bg(t)\big)\,dt$$

$$= a\int_0^\infty e^{-st}f(t)\,dt + b\int_0^\infty e^{-st}g(t)\,dt$$

$$= a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

Example 8.2:

1. Find the Lapalce transforms of $\cosh(at)$ and $\sinh(at)$.

Solution:

$$\mathcal{L}\{\cosh(at)\} = \mathcal{L}\left(\frac{e^{at}+e^{-at}}{2}\right)$$

$$= \frac{1}{2}\mathcal{L}(e^{at}) + \frac{1}{2}\mathcal{L}(e^{-at})$$

$$= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right)$$

$$= \frac{s}{s^2-a^2}$$

$$\mathcal{L}\{\sinh(at)\} = \mathcal{L}\left(\frac{e^{at}-e^{-at}}{2}\right)$$

$$= \frac{1}{2}\mathcal{L}(e^{at}) - \frac{1}{2}\mathcal{L}(e^{-at})$$

$$= \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right)$$

$$= \frac{a}{s^2-a^2}$$

8.3.1 LAPLACE TRANSFORM PAIRS

	$f(t)$	$\mathscr{L}(f)$		$f(t)$	$\mathscr{L}(f)$
1	1	$1/s$	7	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
2	t	$1/s^2$	8	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
3	t^2	$2!/s^3$	9	$\cosh at$	$\frac{s}{s^2-a^2}$
4	t^n $(n = 0, 1, \cdots)$	$\frac{n!}{s^{n+1}}$	10	$\sinh at$	$\frac{a}{s^2-a^2}$
5	t^a (a positive)	$\frac{\Gamma(a+1)}{s^{a+1}}$	11	$e^{at}\cos \omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
6	e^{at}	$\frac{1}{s-a}$	12	$e^{at}\sin \omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$

Exercise

Based on the Laplace transform pairs, compute the Laplace transform of the following functions, $f(t)$:

(i) 4 Ans: $\frac{4}{s}$

(ii) 10 t^6 Ans: $\frac{7200}{s^7}$

(iii) $20 \sin 5t$ Ans: $\frac{100}{s^2+25}$

(iv) $7 \cos(\frac{t}{2})$ Ans: $\frac{28s}{4s^2+1}$

(v) $2 \sin^2 6t$ Ans: $\frac{1}{s} - \frac{s}{s^2+144}$

8.4 INVERSE TRANSFORM

If $F(s)$ represents the Laplace transform of a function $f(t)$, that is, $\mathcal{L}\{f(t)\} = F(s)$, we say $f(t)$ is the inverse Laplace transform of $F(s)$.

The inverse Laplace transform is denoted as

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

Note: $\mathcal{L}^{-1}\{\mathcal{L}(f(t))\} = f(t)$ and $\mathcal{L}\{\mathcal{L}^{-1}(F(s))\} = F(s)$

In determining the inverse Laplace transform, some manipulations must be done to get $F(s)$ into a form suitable for the direct use of the Laplace transform table.

Example 8.3:

Evaluate (a) $\mathcal{L}^{-1}\left(\frac{1}{s^5}\right)$ (b) $\mathcal{L}^{-1}\left(\frac{1}{s^2+7}\right)$ (c) $\mathcal{L}^{-1}\left(\frac{-2s+6}{s^2+4}\right)$.

Solution:

(a) $\mathcal{L}^{-1}\left(\frac{1}{s^5}\right)=\frac{1}{4!}\mathcal{L}^{-1}\left(\frac{4!}{s^5}\right)=\frac{1}{24}t^4$

(b) $\mathcal{L}^{-1}\left(\frac{1}{s^2+7}\right)=\frac{1}{\sqrt{7}}\mathcal{L}^{-1}\left(\frac{\sqrt{7}}{s^2+7}\right)=\frac{1}{\sqrt{7}}\sin\sqrt{7}t$

(c) $\mathcal{L}^{-1}\left(\frac{-2s+6}{s^2+4}\right)=\mathcal{L}^{-1}\left(\frac{-2s}{s^2+4}+\frac{6}{s^2+4}\right)=-2\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right)+\frac{6}{2}\mathcal{L}^{-1}\left(\frac{2}{s^2+4}\right)=-2\cos 2t+3\sin 2t$

8.5 PARTIAL FRACTION

The solution $F(s)$ usually comes out as a general form of

$$F(s)=\frac{P(s)}{Q(s)}=S(s)+\frac{R(s)}{Q(s)}=\frac{a_n s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}{b_m s^m+b_{m-1}s^{m-1}+\cdots+b_1 s+b_0}$$

where $P(s)$ and $R(s)$ are the numerators and $Q(s)$ is the denominator.

A proper rational function of $F(s)$ should be expanded as a sum of partial fraction before its inverse Laplace transform can be found.

Depending on the roots of $Q(s)$ we have fraction expansion in the form:

Denominator $Q(s)$	Example of $F(s)$	Partial fraction expansion
1. Distinct & real roots $Q(s)=(a_1s+b_1)(a_2s+b_2)\ldots(a_ks+b_k)$	$\frac{96s}{s(s+8)(s+6)}$	$\frac{A}{s}+\frac{B}{s+8}+\frac{C}{s+6}$
2. Repeated & real roots $Q(s)=(a_1s+b_1)^r$	$\frac{s+30}{(s+3)^2}$	$\frac{A}{(s+3)^2}+\frac{B}{s+3}$
3. Distinct irreducible quadratic factors $Q(s)=as^2+bs+c$, where $b^2-4ac<0$	$\frac{s+3}{s^2+6s+25}$	$\frac{As+B}{s^2+6s+25}$
4. Repeated irreducible quadratic factors $Q(s)=(as^2+bs+c)^r$, where $b^2-4ac<0$	$\frac{s+6}{(s^2+6s+25)^2}$	$\frac{As+B}{s^2+6s+25}+\frac{Cs+D}{(s^2+6s+25)^2}$

where *A*, *B*, *C* and *D* are constants.

Example 8.4:

Find the inverse Laplace transform of $F(s) = \frac{10s^2+4}{s(s+1)(s+2)^2}$.

Solution:

$$F(s) = \frac{10s^2+4}{s(s+1)(s+2)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+2)^2} + \frac{D}{s+2}$$

$$10s^2 + 4 = A(s+1)(s+2)^2 + Bs(s+2)^2 + Cs(s+1) + Ds(s+1)(s+2)$$

If we set $s = 0$, $s = -1$, $s = -2$ and $s = 1$, we obtain

$$A = 1,\ B = -14,\ C = 22,\ D = 13 \text{ respectively.}$$

Therefore,

$$\mathcal{L}^{-1}\left\{\frac{10s^2+4}{s(s+1)(s+2)^2}\right\} = \mathcal{L}^{-1}\left(\frac{1}{s} - \frac{14}{s+1} + \frac{22}{(s+2)^2} + \frac{13}{s+2}\right) = 1 - 14e^{-t} + 22te^{-2t} + 13e^{-2t}$$

Exercise

Solve the inverse Laplace transform of the followings:

(i) $\mathcal{L}^{-1}\left\{\frac{4s^2-5s-41}{(s-1)(s+2)(s-3)}\right\}$ Ans: $7e^t - e^{-2t} - 2e^{3t}$

(ii) $\mathcal{L}^{-1}\left\{\frac{1}{s^3+s}\right\}$ Ans: $1 - \cos t$

(iii) $\mathcal{L}^{-1}\left\{\frac{s^2+s-52}{(s+1)(s^2+25)}\right\}$ Ans: $-2e^{-t} + 3\cos 5t - \frac{2}{5}\sin 5t$

8.6 LAPLACE TRANSFORM OF DERIVATIVES

8.6.1 LAPLACE TRANSFORM OF FIRST DERIVATIVE

If $f(t)$ is continuous for all $t \geq 0$, satisfies the growth condition and $f'(t)$ is piecewise continuous on every finite interval on the semi-axis $t \geq 0$, then

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

Proof:

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= e^{-st} f(t)\big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + s\mathcal{L}\{f(t)\}$$

8.6.2 LAPLACE TRANSFORM OF SECOND DERIVATIVE

If $f(t)$ and $f'(t)$ are continuous for all $t \geq 0$, satisfies the growth condition and $f''(t)$ is piecewise continuous on every finite interval on the semi-axis $t \geq 0$, then

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

Proof:

$$\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0)$$
$$= s\big(s\mathcal{L}\{f(t)\} - f(0)\big) - f'(0)$$
$$= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

8.6.3 LAPLACE TRANSFORM OF THE DERIVATIVE $f^{(n)}$ OF ANY ORDER

If $f, f', \ldots f^{(n-1)}$ are continuous for all $t \geq 0$, satisfies the growth condition and $f^{(n)}$ is piecewise continuous on every finite interval on the semi-axis $t \geq 0$, then

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{n-1}(0)$$

8.7 LAPLACE TRANSFORM OF INTEGRAL

If $f(t)$ is piecewise continuous for all $t \geq 0$,

$$\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{1}{s}F(s)$$

Proof:

$$\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\}' = s\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\} - \int_0^0 f(\tau)\, d\tau$$

$$\mathcal{L}\{f(\tau)\} = s\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\}$$

$$\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(\tau)\}$$

Example 8.5:

Find the inverse of $\frac{1}{s(s^2+\omega^2)}$ and $\frac{1}{s^2(s^2+\omega^2)}$.

Solution:

We know that $\mathcal{L}^{-1}\left(\frac{1}{s^2+\omega^2}\right) = \frac{\sin \omega t}{\omega} = f(\tau)$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+\omega^2)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\big(f(\tau)\big)\right\}$$

$$= \int_0^t \frac{\sin \omega\tau}{\omega} d\tau$$

$$= \frac{1}{\omega^2}(1 - \cos \omega t)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+\omega^2)}\right\} = \frac{1}{\omega^2}\int_0^t (1 - \cos \omega\tau)\, d\tau$$

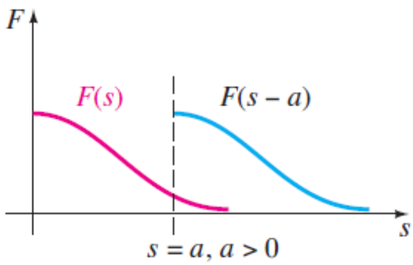
$$= \frac{t}{\omega^2} - \frac{\sin \omega t}{\omega^3}$$

8.8 FIRST SHIFT THEOREM: s-SHIFTING

If $f(t)$ has the transform $F(s)$ (where $s > k$ for some k), then $e^{at}f(t)$ has the transform

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

where $(s - a) > k$.



Proof:

$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t) dt$$

$$= \int_0^\infty \left(e^{at}f(t)\right)e^{-st}\, dt$$

$$= \mathcal{L}\{e^{at}f(t)\}$$

Example 8.6:

1. From previous example, we know that $\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2}$ and $\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}$ then

   $$\mathcal{L}\{e^{at}\cos(\omega t)\} = \frac{s-a}{(s-a)^2+\omega^2}$$

   $$\mathcal{L}\{e^{at}\sin(\omega t)\} = \frac{\omega}{(s-a)^2+\omega^2}$$

2. Using first shift theorem, find the inverse of the transform $\mathcal{L}\{f(t)\} = \frac{3s-137}{s^2+2s+401}$.

Solution:

$$\mathcal{L}^{-1}\left(\frac{3s-137}{s^2+2s+401}\right) = \mathcal{L}^{-1}\left(\frac{3(s+1)-140}{(s+1)^2+400}\right)$$

$$= 3\mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+400}\right) - 7\mathcal{L}^{-1}\left(\frac{20}{(s+1)^2+400}\right)$$

$$= 3e^{-t}\cos 20t - 7\,e^{-t}\sin 20t$$

Exercise

Solve the inverse Laplace transform of the followings:

(i) $\mathcal{L}^{-1}\left\{\frac{6}{s^2+8s+25}\right\}$ Ans: $2\,e^{-4t}\sin 3t$

(ii) $\mathcal{L}^{-1}\left\{\frac{3s-5}{s^2+2s+5}\right\}$ Ans: $3e^{t}\cos 2t - 4\,\mathrm{e}^{-\mathrm{t}}\sin 2t$

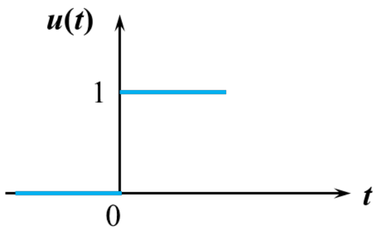
(iii) $\mathcal{L}^{-1}\left\{\frac{s}{(s+5)^4}\right\}$ Ans: $\frac{1}{2}e^{-5t}t^2 - \frac{5}{3!}e^{-5t}t^3$

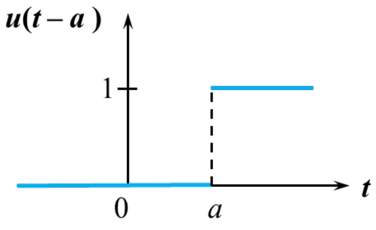
8.9 UNIT STEP FUNCTION (HEAVISIDE FUNCTION)

Mathematical definition of a unit step function is

$$u(t-a)=\begin{cases}0 & \text{if } t<a\\ 1 & \text{if } t>a\end{cases}$$

The unit step function has a discontinuity, or jump, at the origin for $u(t)$ or at the position a for $u(t-a)$ where a is an arbitrary positive.





The transform of $u(t-a)$ follows directly from the defining integral:

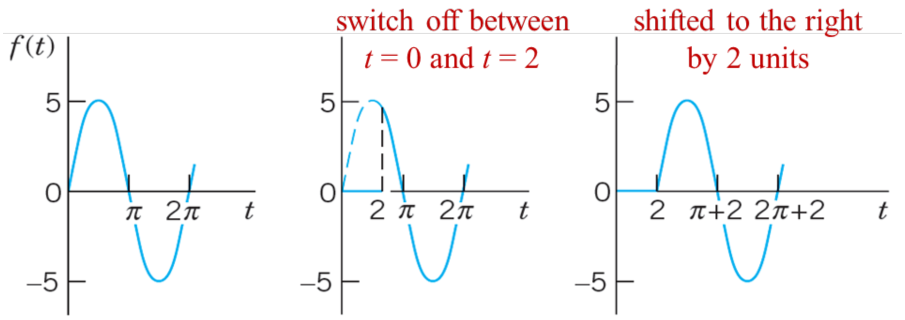
$$\mathcal{L}\{u(t-a)\} = \int_0^{\infty} e^{-st}\, u(t-a)dt$$

$$= \int_a^{\infty} e^{-st}\cdot 1\; dt$$

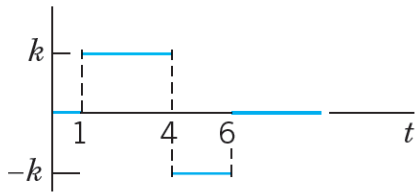
$$= -\frac{1}{s}e^{-st}\Big|_a^{\infty}$$

$$= \frac{e^{-as}}{s}$$

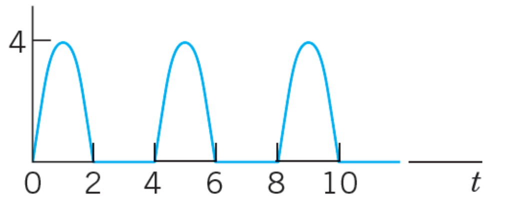
Example 8.7:



(A) $f(t) = 5 \sin t$ (B) $f(t)u(t-2)$ (C) $f(t-2)u(t-2)$



(A) $k[u(t-1) - 2u(t-4) + u(t-6)]$



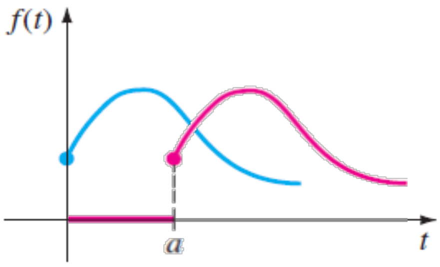
(B) $4 \sin (\frac{1}{2}\pi t)[u(t) - u(t-2) + u(t-4) - + \cdots]$

8.10 SECOND SHIFT THEOREM: TIME SHIFTING (t-SHIFTING)

If $f(t)$ has the transform $F(s)$, then the "shifted-function"

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$$

$$\mathcal{L}\{f(t)u(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}$$



Proof:

Let $\tau = t - a$,

$$\mathcal{L}\{f(t-a)u(t-a)\} = \int_0^\infty e^{-st} f(t-a)u(t-a)dt$$

$$= \int_{-a}^\infty e^{-s(\tau+a)} f(\tau)u(\tau)d\tau$$

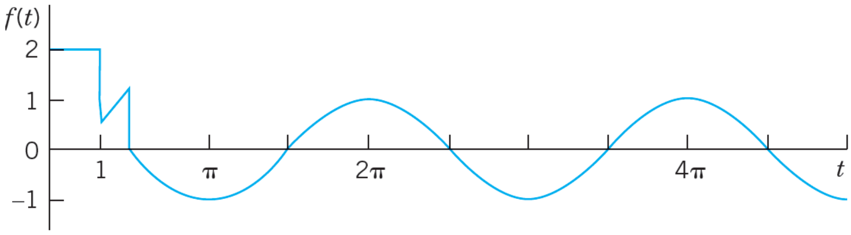
$$= e^{-as} \int_0^\infty e^{-s\tau} f(\tau) d\tau$$

$$= e^{-as} F(s)$$

Example 8.8:

1. Write the following function using unit step functions and find its transform.

$$f(t) = \begin{cases} 2 & \text{if } 0 < t < 1 \\ \frac{1}{2}t^2 & \text{if } 1 < t < \frac{1}{2}\pi \\ \cos t & \text{if } t > \frac{1}{2}\pi \end{cases}$$



Solution:

$$f(t) = \underbrace{2\big(u(t) - u(t-1)\big)}_{part\ (a)} + \frac{1}{2}t^2\left(\underbrace{u(t-1)}_{part\ (b)} - \underbrace{u\left(t - \frac{1}{2}\pi\right)}_{part\ (c)}\right) + \underbrace{(\cos t)\, u\left(t - \frac{1}{2}\pi\right)}_{part\ (d)}$$

Part (a): $\mathcal{L}\{2\big(u(t) - u(t-1)\big)\} = 2\left(\frac{1}{s} - \frac{e^{-s}}{s}\right)$

Part (b): $\mathcal{L}\left\{\frac{1}{2}t^2 u(t-1)\right\} = \frac{1}{2}\mathcal{L}\{(t-1)^2 u(t-1) + 2(t-1)\, u(t-1) + u(t-1)\}$

$$= \frac{1}{2}e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right)$$

$$= e^{-s}\left(\frac{1}{s^3} + \frac{1}{s^2} + \frac{1}{2s}\right)$$

Part (c):

$$\mathcal{L}\left\{\frac{1}{2}t^2 u\left(t - \frac{1}{2}\pi\right)\right\} = \frac{1}{2}\mathcal{L}\left\{\left(t - \frac{1}{2}\pi\right)^2 u\left(t - \frac{1}{2}\pi\right) + \pi\left(t - \frac{1}{2}\pi\right) u\left(t - \frac{1}{2}\pi\right) + \frac{1}{4}\pi^2\, u\left(t - \frac{1}{2}\pi\right)\right\}$$

$$= \frac{1}{2}e^{-\pi s/2}\left(\frac{2}{s^3} + \frac{\pi}{s^2} + \frac{\pi^2}{4s}\right)$$

$$= e^{-\pi s/2}\left(\frac{1}{s^3} + \frac{\pi}{2s^2} + \frac{\pi^2}{8s}\right)$$

Part (d): $\mathcal{L}\left\{(\cos t)\, u\left(t - \frac{1}{2}\pi\right)\right\} = \mathcal{L}\left\{-\sin\left(t - \frac{1}{2}\pi\right)\, u\left(t - \frac{1}{2}\pi\right)\right\}$

$$= -\frac{e^{-\pi s/2}}{s^2+1}$$

Combining all the terms:

$$\mathcal{L}\{f(t)\} = \frac{2}{s} - \frac{2}{s}e^{-s} + e^{-s}\left(\frac{1}{s^3} + \frac{1}{s^2} + \frac{1}{2s}\right) - e^{-\frac{\pi s}{2}}\left(\frac{1}{s^3} + \frac{\pi}{2s^2} + \frac{\pi^2}{8s}\right) - \frac{e^{-\pi s/2}}{s^2+1}$$

2. Find the inverse transform $f(t)$ of $F(s) = \frac{e^{-s}}{s^2+\pi^2} + \frac{e^{-2s}}{s^2+\pi^2} + \frac{e^{-3s}}{(s+2)^2}$.

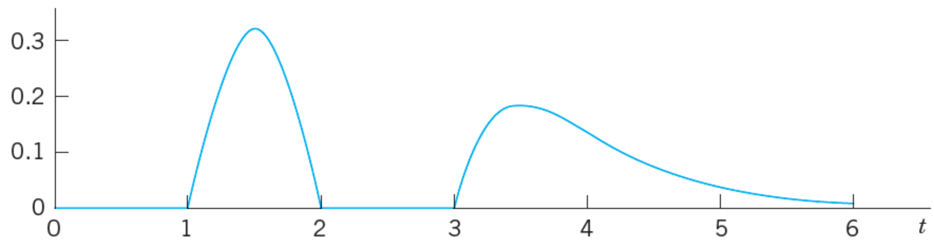
Solution:

First, we consider without the exponential functions in the numerator.

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+\pi^2}\right) = \frac{\sin \pi t}{\pi}, \qquad \mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right) = te^{-2t}$$

By second shift theorem,

$$f(t) = \frac{1}{\pi}\sin\big(\pi(t-1)\big)\,u(t-1) + \frac{1}{\pi}\sin\big(\pi(t-2)\big)\,u(t-2) + (t-3)e^{-2(t-3)}u(t-3)$$



LAPLACE TRANSFORM SOLUTIONS FOR DIFFERENTIAL EQUATIONS

WEEK 9: LAPLACE TRANSFORM SOLUTIONS FOR DIFFERENTIAL EQUATIONS

9.1 SOLVING LINEAR ODES

Having mastered the mechanics of forward and inverse Laplace transform, this Chapter applies such skills to solve linear differential equations.

The Laplace transform of a linear differential equation (in t-domain) with constant coefficient yields an algebraic equation, Y(s) in s-domain, which can be solved easily. Once solved, the solution is s-domain can be easily reconverted to t-domain to obtain the final solution.

Initial condition(s) is(are) required to solve differential equations using Laplace transform.

Example 9.1: Solving first-order initial value problem

Use the Laplace transform to solve the initial-value problem

$$\frac{dy}{dt} + 3y = 13\sin 2t\,,\ \ y(0) = 6$$

Solution:

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} + 3\mathcal{L}\{y\} = 13\mathcal{L}\{\sin 2t\} \tag{1}$$

From Table of Laplace Transform,

$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0) = sY(s) - 6$ and $\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$

Equation (1) becomes

$$sY(s) - 6 + 3Y(s) = 13\left(\frac{2}{s^2+4}\right)$$

$$(s+3)Y(s) = 6 + 13\left(\frac{2}{s^2+4}\right)$$

$$Y(s) = \frac{6}{s+3} + \frac{26}{(s+3)(s^2+4)} = \frac{6s^2+50}{(s+3)(s^2+4)}$$

Performing partial fraction:

$$\frac{6s^2+50}{(s+3)(s^2+4)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+4}$$

$$A = 8, B = -2, C = 6$$

$Y(s) = \frac{8}{s+3} + \frac{-2s+6}{s^2+4}$

Therefore, by inverse Laplace transform,

$y(t) = \mathcal{L}^{-1}\left\{\frac{8}{s+3}\right\} - 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 3\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$

$y(t) = 8e^{-3t} - 2\cos 2t + 3\sin 2t$

Example 9.2: Solving second-order initial value problem

Solve $y'' - 3y' + 2y = e^{-4t},\ y(0) = 1,\ y'(0) = 5.$

Solution:

$\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^{-4t}\}$

$s^2Y(s) - sy(0) - y'(0) - 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s+4}$

$s^2Y(s) - s - 5 - 3[sY(s) - 1] + 2Y(s) = \frac{1}{s+4}$

$(s^2 - 3s + 2)Y(s) = s + 2 + \frac{1}{s+4}$

$Y(s) = \frac{s+2}{s^2-3s+2} + \frac{1}{(s^2-3s+2)(s+4)} = \frac{s^2+6s+9}{(s-1)(s-2)(s+4)}$

Performing partial fraction:

$Y(s) = \frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = -\frac{16}{5}\frac{1}{s-1} + \frac{25}{6}\frac{1}{s-2} + \frac{1}{30}\frac{1}{s+4}$

Therefore, by inverse Laplace transform,

$y(t) = \mathcal{L}^{-1}\{Y(s)\} = -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}$

Exercise

Solve the following linear differential equations using Laplace transform method

(i) $y'' + 14\,y' + 49\,y = 40\,t^3e^{-7t}$ where, $y'(0) = -5$ and $y(0) = 2$

Ans: $y(t) = e^{-7t}(2t^5 + 9t + 2)$

(ii) $y'' + y' - 2y = 4t$ where, $y'(0) = 0$ and $y(0) = 1$

Ans: $y(t) = -1 - 2t + 2e^{t}$

9.2 DIFFERENTIATION OF TRANSFORMS

If $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$, then

$$F'(s) = -\int_0^\infty tf(t)e^{-st}\,dt = \mathcal{L}\{-tf(t)\}$$

Proof:

$F(s) = \int_0^\infty f(t)e^{-st}\,dt$

$F'(s) = \frac{d}{ds}\left(\int_0^\infty f(t)e^{-st}\,dt\right)$

$= \int_0^\infty \frac{d}{ds}(e^{-st})\,f(t)\,dt$

$= \int_0^\infty -te^{-st}\,f(t)\,dt$

$= \mathcal{L}\{-tf(t)\}$

In other words, if $F(s) = \mathcal{L}\{f(t)\}$ and n = 1, 2, 3, ..., then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$$

Example 9.3:

1. Given $\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2} = F(s)$,
   Then $\mathcal{L}\{-t\sin(\omega t)\} = F'(s) = -\frac{2s\omega}{(s^2+\omega^2)^2}$

2. Find the inverse transform of $\ln\left(1+\frac{\omega^2}{s^2}\right)$.

Solution:

Let $F(s) = \ln\left(1+\frac{\omega^2}{s^2}\right) = \ln\left(\frac{s^2+\omega^2}{s^2}\right) = \ln(s^2+\omega^2) - \ln(s^2)$

$F'(s) = \frac{d}{ds}(\ln(s^2+\omega^2) - \ln(s^2)) = \frac{2s}{s^2+\omega^2} - \frac{2s}{s^2} = \mathcal{L}\{-tf(t)\}$

Taking inverse transform,

$\mathcal{L}^{-1}\{F'(s)\} = -tf(t) = \mathcal{L}^{-1}\left(\frac{2s}{s^2+\omega^2} - \frac{2s}{s^2}\right)$

$-tf(t) = 2\cos(\omega t) - 2$

$\mathcal{L}^{-1}\left\{\ln\left(1+\frac{\omega^2}{s^2}\right)\right\} = f(t) = \frac{2}{t}(1-\cos(\omega t))$

9.3 INTEGRATION OF TRANSFORMS

If $f(t)$ satisfies the assumptions of the existence theorem and the limit of $f(t)/t$ exists when t approaches 0 from the right, then

$$\mathcal{L}^{-1}\left(\int_s^\infty F(\tilde{s})\,d\tilde{s}\right) = \frac{f(t)}{t}$$

Proof:

$$\int_s^\infty F(\tilde{s})\,d\tilde{s} = \int_s^\infty \left(\int_0^\infty e^{-\tilde{s}t} f(t)\,dt\right) d\tilde{s}$$

$$= \int_0^\infty \left(\int_s^\infty e^{-\tilde{s}t}\,d\tilde{s}\right) f(t)\,dt$$

$$= \int_0^\infty \left(-\frac{1}{t} e^{-\tilde{s}t}\Big|_s^\infty\right) f(t)\,dt$$

$$= \int_0^\infty \frac{1}{t} e^{-st} f(t)\,dt$$

$$= \mathcal{L}\left\{\frac{f(t)}{t}\right\}$$

9.4 DIRAC DELTA FUNCTION

Mechanical systems are often acted on by an external force (or electromotive force in an electrical circuit) of large magnitude that acts only for a very short period of time. We can model such phenomena and problems by "Dirac delta function," and solve them very effectively by the Laplace transform.

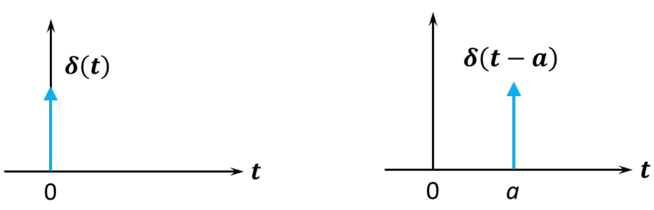
To model situations of that type, we consider the function

$$f_k(t-a) = \begin{cases} 1/k & \text{if } a \le t \le a+k \\ 0 & \text{otherwise} \end{cases}$$

As $k \to 0$, this limit is denoted by $\delta(t-a)$, that is

$$\delta(t-a) = \lim_{k\to 0} f_k(t-a)$$

$\delta(t-a)$ is called the Dirac delta function or the unit impulse function.



The Laplace transform of the Dirac delta function is given by

$$\mathcal{L}\{\delta(t-a)\} = e^{-as} \qquad \text{for } a > 0$$

Example 9.4:

Solve $y'' + y = 4\delta(t-2\pi)$ subject to $y(0) = 1, y'(0) = 0$.

Solution:

The Laplace transform of the differential equation is

$s^2Y(s) - sy(0) - y'(0) + Y(s) = 4e^{-2\pi s}$

$(s^2+1)Y(s) - s = 4e^{-2\pi s}$

$Y(s) = \frac{4e^{-2\pi s}+s}{s^2+1}$

$y(t) = 4\sin(t-2\pi)\,u(t-2\pi) + \cos t$

$$y(t) = \begin{cases} \cos t & 0 \le t \le 2\pi \\ 4\sin t + \cos t & t \ge 2\pi \end{cases}$$

Exercise

Solve the linear differential equation using Laplace transform method: $y' + y = \delta(t-1)$, where $y(0) = 1$

Ans: $y(t) = e^{-(t-1)}u(t-1) + e^{-t}$

9.5 CONVOLUTION

The convolution of two functions $f(t)$ and $g(t)$ is denoted by the standard notation $f * g$ and defined by the integral $(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$.

The Laplace transform is given by $\mathcal{L}\{(f*g)(t)\} = F(s)G(s)$.

Proof: $F(s)G(s) = \left(\int_0^\infty e^{-s\tau}f(\tau)\,d\tau\right)\left(\int_0^\infty e^{-s\sigma}g(\sigma)\,d\sigma\right)$

$$= \int_0^\infty\left(\int_0^\infty e^{-s(\tau+\sigma)}g(\sigma)\,d\sigma\right)f(\tau)\,d\tau$$

$$= \int_0^\infty\left(\int_\tau^\infty e^{-st}g(t-\tau)\,dt\right)f(\tau)\,d\tau \qquad [t = \sigma+\tau]$$

$$= \int_0^\infty e^{-st}\left(\int_0^t f(\tau)g(t-\tau)\,d\tau\right)dt$$

$$= \mathcal{L}\{(f*g)\}$$

9.5.1 PROPERTIES OF CONVOLUTION

i. Commutative law: $f * g = g * f$

ii. Distributive law: $f * (g_1 + g_2) = f * g_1 + f * g_2$

iii. Associative law: $(f * g) * v = f * (g * v)$

iv. $f * 0 = 0 * f = 0$

v. $f * 1 \neq f$

9.5.2 INTEGRAL EQUATIONS

Convolution also helps in solving certain integral equations, that is, equations in which the unknown function $y(t)$ appears in an integral.

Example 9.5: Volterra integral equation of the second kind

Solve $y(t) - \int_0^t y(\tau) \sin(t - \tau)\, d\tau = t$.

Solution:

$y - y * \sin t = t$

Applying Laplace transform and convolution theorem, we obtain

$Y(s) - Y(s)\frac{1}{s^2+1} = \frac{1}{s^2}$

$Y(s)\frac{s^2}{s^2+1} = \frac{1}{s^2}$

$Y(s) = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4}$

Therefore, $\quad y(t) = t + \frac{t^3}{6}$

9.6 SYSTEM OF ODEs

We consider a first-order linear system with constant coefficients:

$$y_1' = a_{11}y_1 + a_{12}y_2 + g_1$$

$$y_2' = a_{21}y_1 + a_{22}y_2 + g_2$$

If we transform it,

$$sY_1 - y_1(0) = a_{11}Y_1 + a_{12}Y_2 + G_1$$

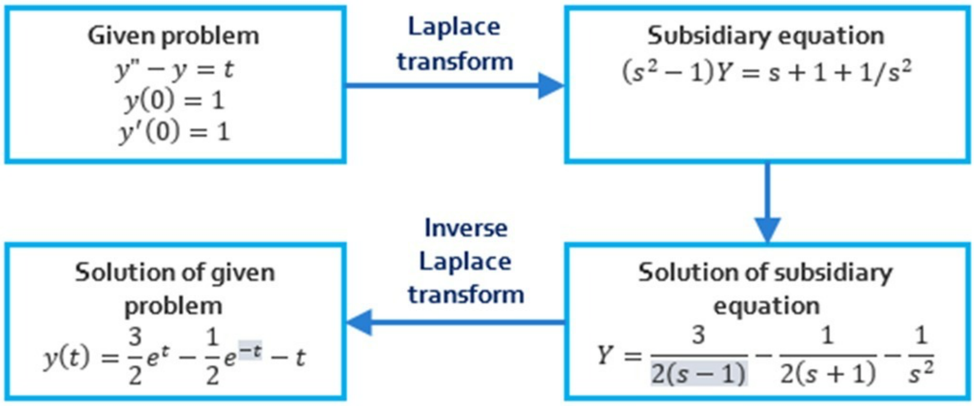
$$sY_2 - y_2(0) = a_{21}Y_1 + a_{22}Y_2 + G_2$$

By collecting the Y_1- and Y_2-terms we have

$$(a_{11} - s)Y_1 + a_{12}Y_2 = -y_1(0) - G_1(s)$$

$$a_{21}Y_1 + (a_{22} - s)Y_2 = -y_2(0) - G_2(s)$$

By solving this system algebraically for $Y_1(s)$, $Y_2(s)$ and taking the inverse transform we obtain the solution y_1 and y_2 of the system.

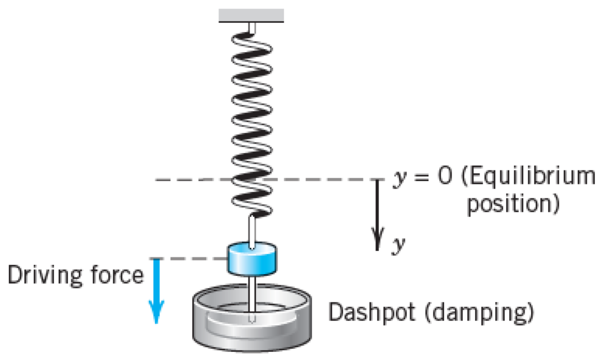


Steps of the Laplace transform method

Example 9.6:

1. Damped Forced Vibrations

Solve the initial value problem for a damped mass–spring system acted upon by a sinusoidal force for some time interval.



Mechanical system

$y'' + 2y' + 2y = r(t)$, $r(t) = 10 \sin 2t$ if $0 < t < \pi$ and 0 if $t > \pi$;

$y(0) = 1,\ y'(0) = -5$

Solution:

$y'' + 2y' + 2y = 10 \sin 2t\big(u(t) - u(t-\pi)\big)$

Using Laplace transform,

$(s^2Y - s + 5) + 2(sY - 1) + 2Y = 10\frac{2}{s^2+4}(1 - e^{-\pi s})$

$(s^2 + 2s + 2)Y = s - 3 + 10\frac{2}{s^2+4}(1 - e^{-\pi s})$

$$Y = \underbrace{\frac{s-3}{(s^2+2s+2)}}_{part\ (a)} + \underbrace{\frac{20}{(s^2+2s+2)(s^2+4)}}_{part\ (b)} - \underbrace{\frac{20e^{-\pi s}}{(s^2+2s+2)(s^2+4)}}_{part\ (b1)}$$

Applying inverse Laplace transform,

Part (a): $\mathcal{L}^{-1}\left(\frac{s-3}{s^2+2s+2}\right) = \mathcal{L}^{-1}\left(\frac{(s+1)-4}{(s+1)^2+1}\right)$

$$= e^{-t}(\cos t - 4\sin t)$$

Part (b): Partial fraction expansion: $\frac{20}{(s^2+2s+2)(s^2+4)} = \frac{As+B}{(s+1)^2+1} + \frac{Ms+N}{s^2+4}$

$20 = (As + B)(s^2 + 4) + (Ms + N)(s^2 + 2s + 2)$

$20 = (A + M)s^3 + (2M + B + N)s^2 + (4A + 2M + 2N)s + (4B + 2N)$

Equating the coefficients of each power of *s* on both sides gives the four equations:

$A + M = 0;$ $\qquad 2M + B + N = 0;$

$4A + 2M + 2N = 0;$ $\qquad 4B + 2N = 20;$

We determine *A* = 2, *B* = 6, *M* = -2, *N* = -2

$$\frac{20}{(s^2+2s+2)(s^2+4)} = \frac{2s+6}{(s+1)^2+1} - \frac{(2s+2)}{s^2+4} = \frac{2(s+1)+4}{(s+1)^2+1} - \frac{2s+2}{s^2+4}$$

$$\mathcal{L}^{-1}\left\{\frac{20}{(s^2+2s+2)(s^2+4)}\right\} = e^{-t}(2\cos t + 4\sin t) - 2\cos 2t - \sin 2t$$

Part (b1): From second shift theorem, we have

$$\mathcal{L}^{-1}\left\{\frac{20e^{-\pi s}}{(s^2+2s+2)(s^2+4)}\right\} = e^{-(t-\pi)}(2\cos(t-\pi) + 4\sin(t-\pi)) - 2\cos 2(t-\pi) - \sin 2(t-\pi)$$

$$= e^{-(t-\pi)}(-2\cos t - 4\sin t) - 2\cos 2t - \sin 2t$$

$$\boxed{\begin{array}{l}\cos(t-\pi) = -\cos t \\ \sin(t-\pi) = -\sin t\end{array}}$$

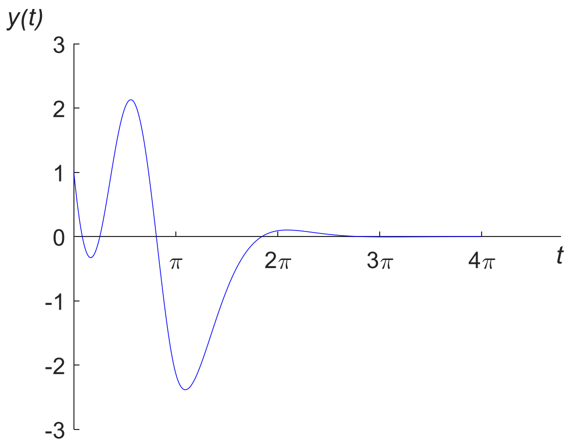
Therefore, the solution is

$$y(t) = e^{-t}(\cos t - 4\sin t) + e^{-t}(2\cos t + 4\sin t) - 2\cos 2t - \sin 2t \qquad \text{if } 0 < t < \pi$$

$$= 3e^{-t}\cos t - 2\cos 2t - \sin 2t \qquad \text{if } 0 < t < \pi$$

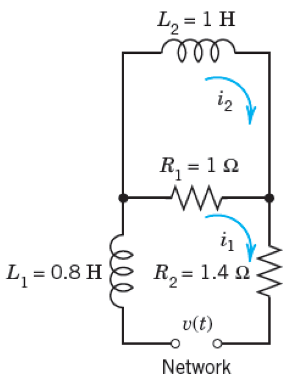
$$y(t) = 3e^{-t}\cos t - 2\cos 2t - \sin 2t - \left[e^{-(t-\pi)}(-2\cos t - 4\sin t) - 2\cos 2t - \sin 2t\right] \qquad \text{if } t > \pi$$

$$= e^{-t}\left((3 + 2e^{\pi})\cos t + 4e^{\pi}\sin t\right) \qquad \text{if } t > \pi$$



2. **Electrical Network**

Find the currents $i_1(t)$ and $i_2(t)$ in the network with L and R measured in terms of the usual units, $v(t) = 100$ volts if $0 \le t \le 0.5$ sec and 0 thereafter, and $i(0) = 0$, $i'(0) = 0$.



Solution:

The model of the network is obtained from Kirchhoff's Voltage Law:

For the lower circuit:

$$0.8i_1' + 1(i_1 - i_2) + 1.4i_1 - 100[1 - u(t - 0.5)] = 0$$

For the upper circuit:

$$1i_2' + 1(i_2 - i_1) = 0$$

Applying Laplace transform,

$$0.8sI_1 + (I_1 - I_2) + 1.4I_1 = 100\left[\frac{1}{s} - \frac{e^{-0.5s}}{s}\right]$$

$$sI_2 + (I_2 - I_1) = 0$$

Solving algebraically for I_1 and I_2:

$$I_1 = \left(\frac{500}{7s} - \frac{125}{3(s+0.5)} - \frac{625}{21(s+3.5)}\right)(1 - e^{-0.5s})$$

$$I_2 = \left(\frac{500}{7s} - \frac{250}{3(s+0.5)} + \frac{250}{21(s+3.5)}\right)(1 - e^{-0.5s})$$

The inverse transform for $0 \le t \le 0.5$

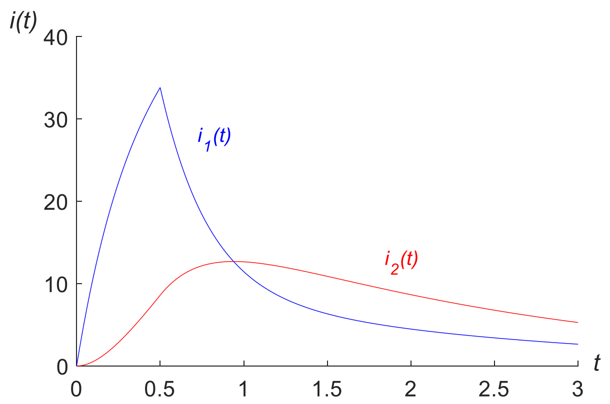
$$i_1(t) = \frac{500}{7} - \frac{125}{3}e^{-0.5t} - \frac{625}{21}e^{-3.5t}$$

$$i_2(t) = \frac{500}{7} - \frac{250}{3}e^{-0.5t} + \frac{250}{21}e^{-3.5t}$$

The inverse transform for $t > 0.5$

$$i_1(t) = i_1(t) - i_1(t - 0.5) = -\frac{125}{3}(1 - e^{0.25})e^{-0.5t} - \frac{625}{21}(1 - e^{1.75})e^{-3.5t}$$

$$i_2(t) = i_2(t) - i_2(t - 0.5) = -\frac{250}{3}(1 - e^{0.25})e^{-0.5t} + \frac{250}{21}(1 - e^{1.75})e^{-3.5t}$$



Exercise

Solve the following ODEs using Laplace transform method:

$y_1' = 5y_1 + y_2$ and $y_2' = y_1 + 5y_2 + 2\,\delta(t-1)$

Initial conditions: $y_1(0) = 0;\ y_2(0) = 10$

Ans:

$y_1 = 5e^{6t} - 5e^{4t} + \left\{e^{6(t-1)} - e^{4(t-1)}\right\}u(t-1)$ and $y_2 = 5e^{6t} + 5e^{4t} + \left\{e^{6(t-1)} - e^{4(t-1)}\right\}u(t-1)$

Table of Laplace Transform

$f(t)$	$F(s)$
1	$\frac{1}{s}$
$\delta(t)$	1
t	$\frac{1}{s^2}$
$t^n,\ \ n = 1,2,3,\dots$	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
te^{at}	$\frac{1}{(s-a)^2}$
$t^n e^{at},\ \ n = 1,2,3,\dots$	$\frac{n!}{(s-a)^{n+1}}$
$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
$e^{at}\sin \omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$
$e^{at}\cos \omega t$	$\dfrac{s-a}{(s-a)^2+\omega^2}$
$\sinh \omega t$	$\dfrac{\omega}{s^2-\omega^2}$
$\cosh \omega t$	$\dfrac{s}{s^2-\omega^2}$

$f(t)$	$F(s)$
$af(t)+bg(t)$	$aF(s)+bG(s)$
$u(t-a)$	$\frac{e^{-as}}{s}$
$\delta(t-a)$	e^{-as}
$f(t-a)u(t-a)$	$e^{-as}F(s)$
$e^{at}f(t)$	$F(s-a)$
$\dfrac{df}{dt}$	$sF(s)-f(0)$
$\dfrac{d^2f}{dt^2}$	$s^2F(s)-sf(0)-f'(0)$
$\dfrac{d^nf}{dt^n}$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$
$\int_0^t f(\tau)\,d\tau$	$\dfrac{1}{s}F(s)$
$tf(t)$	$-\dfrac{d}{ds}F(s)$
$\dfrac{f(t)}{t}$	$\int_s^\infty F(s)ds$
$f(t)*g(t)$	$F(s)G(s)$