# SOLVING GENERAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATION (PDE)

## WEEK 12: SOLVING GENERAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATION (PDE)

### 12.1 GENERAL SOLUTION & PARTICULAR SOLUTION OF PDE

In the differential equation chapter, you have learned how to differentiate between ODE and PDE and how to classify them in terms of the order, linearity, and homogeneity. In simple, PDE is an equation that involves partial derivatives (i.e. $\partial$ symbol). Recall that a linear PDE is *homogeneous* if each of its terms contains either $u$ *or* one of its partial derivatives on LHS while RHS=0. Otherwise, it is a *non-homogeneous* PDE. In this study, we will only focus on solving the 2nd order linear homogeneous PDE problem with constant coefficients.

The general equation is given below:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial t} + C\frac{\partial^2 u}{\partial t^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial t} + Fu = 0 \quad \text{, where } A - F \text{ are constants.}$$

(i) The general PDE solution, i.e. $u(x,t)$ in terms of unknown coefficients can be obtained by using separable of variable method.
For example: $u_{total}(x,t) = \sum_{n=1}^{\infty} A_{3,\mathrm{n}} \cos(n\pi t)\big(\sin(n\pi x)\big)$

(ii) The particular PDE solution, i.e. $u(x,t)$ in terms of known coefficients can be obtained by applying all the initial & boundary conditions, as well as the Fourier series expansion method.
For example: $u_{total}(x,t) = \sum_{n=1}^{\infty} -\frac{2n\pi \sin n\pi + 4\cos n\pi - 4}{n^3\pi^3} \cos(n\pi t)\big(\sin(n\pi x)\big)$

Notation of PDE:

Note that $\frac{\partial^2 u}{\partial x^2} \neq u''$ for $\frac{\partial u}{\partial x} \neq u'$ for PDE as it has more than 1 possibility. For example, $u'$ can be $\frac{\partial u}{\partial x}$ or $\frac{\partial u}{\partial t}$ while $u''$ can be $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial t}, or\ \frac{\partial^2 u}{\partial t^2}$.

Thus, instead of writing $u'$ or $u''$ for PDE, there is another alternative.

(i) Derivative and second derivative of $u(x,t)$ with *respect to t*

$$u_t = \frac{\partial}{\partial t}\{u(x,t)\} \ , \ u_{tt} = \frac{\partial^2}{\partial t^2}\{u(x,t)\}$$

(ii) Derivative and second derivative of $u(x,t)$ with *respect to x*

$$u_x = \frac{\partial}{\partial x}\{u(x,t)\} \ , \ u_{xx} = \frac{\partial^2}{\partial x^2}\{u(x,t)\}$$

(iii) Derivative of $u(x,t)$ with *respect to t and x*

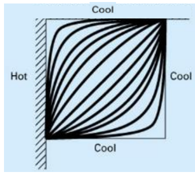
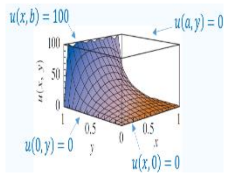
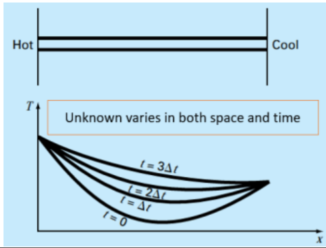
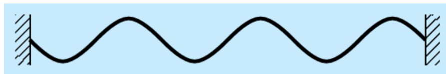
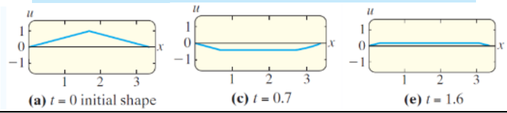
$$u_{tx}\ or\ u_{xt} = \frac{\partial^2}{\partial x \partial t}\{u(x,t)\}$$

Thus, we can rewrite and simplify the previous PDE using this notation:

$$Au_{xx} + Bu_{xt} + Cu_{tt} + Du_x + Eu_t + Fu = 0 \quad \text{, where } A - F \text{ are constants.}$$

## 12.2 CATEGORIES OF 2ND ORDER LINEAR HOMOGENEOUS PDE

Based on the $B^2 - 4AC$, the PDE can be categorized into 3 types:

| Category | Example | Application |
|---|---|---|
| Elliptic PDE<br><br>$B^2 - 4AC < 0$ | Laplace equation<br>$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \rightarrow u(x,y) = ?$<br><br>Characteristic:<br>Steady state/ Time invariant | To find the stable temperature distribution of a heated/cooled 2D plate<br> |
| Parabolic PDE<br><br>$B^2 - 4AC = 0$ | Heat conduction equation / Heat equation<br><br>$3\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad \rightarrow u(x,t) = ?$<br><br>Characteristic:<br>Time variant, non-oscillating | To find the temperature of a heated/cooled 1D rod that changes over time without oscillation<br> |
| Hyperbolic PDE<br><br>$B^2 - 4AC > 0$ | Wave equation<br>$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \quad \rightarrow u(x,t) = ?$<br><br>Characteristic:<br>Time variant, oscillating | To find the vibration of a string that changes over time with oscillation<br> |

Note: $\frac{\partial}{\partial t}$ = Time variant (change with time) or transient behavior

Time invariant means that the physical quantity will not change with time or steady state behavior

| **Description** | **Elliptic PDE Equation** | **Strategy to solve** |
|---|---|---|
| One-dimensional Laplace equation | $\frac{\partial^2 u}{\partial x^2} = 0$<br><br>$u(x) = ?$ | Integration, Solve PDE like ODE, Reduction of Order (Out of the scope – Appendix 12 for extra info) |
| Two-dimensional Laplace equation | $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$<br><br>$u(x,y) = ?$ | Separation of variables method (**Focus**) |
| Three-dimensional Laplace equation | $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$<br><br>$u(x,y,z) = ?$ | Separation of variables method can be applied to 3D cases, however these 2 cases (Out of the scope) |

| Description | Parabolic PDE Equation | Strategy to Solve |
| --- | --- | --- |
| One-dimensional heat equation | $\frac{1}{c^2}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$<br>$u(x,t) = ?$ | Separation of variables method (**Focus**) |
| Two-dimensional heat equation | $\frac{1}{c^2}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$<br>$u(x,y,t) = ?$ | Separation of variables method can be applied to 2D and 3D cases, however these 2 cases (out of the scope) |
| Three-dimensional heat equation | $\frac{1}{c^2}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$<br>$u(x,y,z,t) = ?$ | |

| Description | Hyperbolic PDE Equation | Strategy to solve |
| --- | --- | --- |
| One-dimensional wave equation | $\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$<br>$u(x,t) = ?$ | Separation of variables method (**Focus**) |
| Two-dimensional wave equation | $\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$<br>$u(x,y,t) = ?$ | Separation of variables method can be applied to 2D and 3D cases, however these 2 cases (out of the scope) |
| Three-dimensional wave equation | $\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$<br>$u(x,y,z,t) = ?$ | |

Note that solving non-homogeneous PDE problem is out of scope in this study.

For example: The non-zero RHS function, $f(x,y)$ $or$ $f(x,t) \neq 0$

| Description | Non-Homogeneous Elliptic PDE Equation |
| --- | --- |
| Two-dimensional Poisson equation with heat source/sink | $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$ |

| Description | Non-Homogeneous Parabolic PDE Equation |
| --- | --- |
| One-dimensional heat equation with heating element | $\frac{\partial u}{\partial t} - c^2\frac{\partial^2 u}{\partial x^2} = f(x,t)$ |

| Description | Non-Homogeneous Hyperbolic PDE Equation |
| --- | --- |
| One-dimensional wave equation with forcing function | $\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = f(x,t)$ |

## 12.3 SEPARATION OF VARIABLE METHOD

For the 2nd order linear homogeneous PDE problem with constant coefficients:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0$$

We assume that our solution to be:

$$u(x,y) = \underbrace{X(x)Y(y)}_{can\ be\ separated\ into\ x\ function\ and\ y\ function\ respectively}$$

Differentiate it,

| | | |
|---|---|---|
| $\frac{\partial u}{\partial x} = \frac{\partial X(x)}{\partial x} Y(y) = X'Y$ | ; $\frac{\partial u}{\partial y} = X(x)\frac{\partial Y(y)}{\partial y} = XY'$ | |
| $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 X(x)}{\partial x^2} Y(y) = X''Y$ | ; $\frac{\partial^2 u}{\partial y^2} = X(x)\frac{\partial^2 Y(y)}{\partial y^2} = XY''$ | ; $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial X(x)}{\partial x}\frac{\partial Y(y)}{\partial y} = X'Y'$ |

$$AX''Y + BX'Y' + CXY'' + DX'Y + EXY' + FXY = 0$$

$$Y(AX'' + DX' + FX) + Y'(BX' + EX) + Y''(CX) = 0$$

If the PDE can be simplified to $C = 0$ ; $B \,\&\, E = 0$; or $A, D \,\&\, F = 0$. The problem can be simplified to a separable differential equation. For example,

- If $C = 0$ >> $Y(AX'' + DX' + FX) + Y'(BX' + EX) = 0$

Rearrange the equation, we get the separation of variable result:

$$\frac{Y'}{Y} = \frac{-(AX''+DX'+FX)}{(BX'+EX)} = -\lambda$$

By assuming it is equal to a separation constant of $-\lambda$, we success to convert it into 2 ODE equations.

| **ODE #1**: $\frac{Y'}{Y} = -\lambda$ | **ODE #2**: $\frac{-(AX''+DX'+FX)}{(BX'+EX)} = -\lambda$ |
|---|---|
| $Y' + \lambda Y = 0$ | $(AX'' + DX' + FX) = (\lambda BX' + \lambda EX)$<br>$(A)X'' + (D - \lambda B)X' + (F - \lambda E)X = 0$ |

**Hint:** Based on experience, separation constant of $-\lambda$ can solve the problem easier. In fact, let <u>*separation constant of* $\lambda$</u> can also solve the problem with same answer but longer procedure.

The separation constant, $\lambda$ may be (i) zero, (ii) negative or (iii) positive. We can get three PDE solutions from these 3 cases.

| **Case #1** ($\lambda$=0) | **Case #2** ($\lambda = -\alpha^2$), $\alpha > 0$ | **Case #3** ($\lambda = +\alpha^2$), $\alpha > 0$ |
|---|---|---|
| $Y' = 0$<br>$Y_1(y) = ?$ | $Y' - \alpha^2 Y = 0$<br>$Y_2(y) = ?$ | $Y' + \alpha^2 Y = 0$<br>$Y_3(y) = ?$ |
| $AX'' + DX' + FX = 0$<br>$X_1(x) = ?$ | $AX'' + (D + \alpha^2 B)X' + (F + \alpha^2 E)X = 0$<br>$X_2(x) = ?$ | $AX'' + (D - \alpha^2 B)X' + (F - \alpha^2 E)X = 0$<br>$X_3(x) = ?$ |
| $u_1 = X_1(x)Y_1(y)$ | $u_2 = X_2(x)Y_2(y)$ | $u_3 = X_3(x)Y_3(y)$ |
| **Total PDE solution** can be obtained by superposition principle:<br>$u(x,y) = c_1u_1 + c_2u_2 + c_3u_3$ | | |

Recall for the 2nd order linear homogeneous ODE:

$$aX'' + bX' + cX = 0$$

Assume solution, $X = e^{rx}$ , Let $r$ =root

Characteristic equation: $ar^2 + br + c = 0$

Root of the characteristic equation, $r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (Note: you get 2 roots for 2nd order ODE)

<table>
<tr>
<td rowspan="2">For 2nd order ODE</td>
<td>If $r_1 \neq r_2$ for case of <u>distinct roots</u> or <u>complex conjugate roots</u><br>$X_1 = e^{r_1 x}$ ; $X_2 = e^{r_2 x}$<br>where $X_1$ & $X_2$ are linearly independent</td>
<td>Total solution can be obtained by superposition principle without treatment:<br>$X_{total} = c_1 e^{r_1 x} + c_2 e^{r_2 x}$</td>
</tr>
<tr>
<td>f $r_1 = r_2$ for case of <u>repeated roots</u><br>$X_1 = e^{r_1 x}$ ; $X_2 = e^{r_2 x} = e^{r_1 x}$<br>where $X_1$ & $X_2$ are linearly dependent<br><br>Treatment must be done by multiplying with the independent variable<br>$X_1 = e^{r_1 x}$ ; $X_{2,treat} = xe^{r_2 x} = xe^{r_1 x}$<br>where $X_1$ & $X_{2,treat}$ are linearly independent</td>
<td>Total solution can be obtained by superposition principle with treatment:<br>$X_{total} = c_1 e^{r_1 x} + c_2 x e^{r_2 x}$</td>
</tr>
</table>

Note that the same method can be used to solve 1st order linear homogeneous ODE:

$$bX' + cX = 0$$

Assume solution, $X = e^{rx}$ , Let $r$ =root

Characteristic equation: $br + c = 0$

Root of the characteristic equation, $r = \frac{-c}{b}$ (Note: you get 1 root for 1st order ODE)

| | | |
|---|---|---|
| For 1st order ODE | $X_1 = e^{r_1 x}$ | Total solution can be obtained by superposition principle:<br>$X_{total} = c_1 e^{r_1 x}$ |

Example: Solve the general solution of PDE below by using the separation of variable method

$$\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial y}$$

- ***Step 1:*** *Using separation of variable method:* Let $u(x,y) = X(x)Y(y)$

$$X''Y = 4XY'$$

- ***Step 2:*** *Obtain 2 ODE equations*

$$\frac{X''}{4X} = \frac{Y'}{Y} = -\lambda$$

($Hint$: Calculation is easier with the coefficient = 1 for the numerator components)

$$Y' + \lambda Y = 0 \quad \text{--- (ODE \#1)}$$

$$X'' + 4\lambda X = 0 \text{ --- (ODE \#2)}$$

- ***Step 3:*** *3 cases of* λ

3.1 Case #1 (λ=0)

| $Y' = 0$ | $X'' = 0$ |
|---|---|
| Let $r$ =root<br>Characteristic equation: $r = 0$<br>$\therefore Y(y) = c_1 e^{ry} = c_1$ | Let $r$ =root<br>Characteristic equation: $r^2 = 0$,<br>Repeated root case, $r_1 = r_2 = 0$<br>$\therefore X(x) = c_2 e^{r_1 x} + c_3 \underbrace{x}_{treament} e^{r_2 x}$<br>$X(x) = c_2 + c_3 x$ |
| PDE solution in Case 1:<br>$\therefore u_1 = X_1(x)Y_1(y) = (c_2 + c_3 x)(c_1) = A_1 x + B_1$ *where* $A_1, B_1$ =*constant* | |

3.2 Case #2 (λ= $-\alpha^2$), $\alpha > 0$

| $Y' - \alpha^2 Y = 0$ | $X'' - 4\alpha^2 X = 0$ |
|---|---|
| Let $r$ =root<br>*Characteristic equation:* $r - \alpha^2 = 0$<br>$r = \alpha^2$<br>$\therefore Y(y) = c_4 e^{\alpha^2 y}$ | Let $r$ =root<br>*Characteristic equation:* $r^2 - 4\alpha^2 = 0$<br>$r = \pm\sqrt{4\alpha^2}$<br>Distinct root case: $r_1 = +2\alpha$, $r_2 = -2\alpha$<br>$\therefore X(x) = c_5 e^{2\alpha x} + c_6 e^{-2\alpha x}$ |
| PDE solution in Case 2:<br>$\therefore u_2 = X_2(x)Y_2(y) = (c_5 e^{2\alpha x} + c_6 e^{-2\alpha x})(c_4 e^{\alpha^2 y})$<br>$u_2 = e^{\alpha^2 y}(A_2 e^{2\alpha x} + B_2 e^{-2\alpha x})$ *where* $A_2, B_2$ =*constant* | |

3.3 Case #3 ($\lambda = +\alpha^2$), $\alpha > 0$

| $Y' + \alpha^2 Y = 0$ | $X'' + 4\alpha^2 X = 0$ |
|---|---|
| Let $r$ =root<br>*Characteristic equation:* $r + \alpha^2 = 0$<br>$r = -\alpha^2$<br>$\therefore Y(y) = c_7 e^{-\alpha^2 y}$ | Let $r$ =root<br>*Characteristic equation:* $r^2 + 4\alpha^2 = 0$<br>$r = \pm\sqrt{-4\alpha^2}$<br>Complex conjugate root case:<br>$r_1 = +2\alpha i,\ r_2 = -2\alpha i$<br>$\therefore X(x) = c_8 e^{2\alpha x i} + c_9 e^{-2\alpha x i}$ |
| PDE solution in Case #3:<br>$\therefore u_3 = X_3(x)Y_3(y) = \left(c_8 e^{2\alpha x i} + c_9 e^{-2\alpha x i}\right)\left(c_7 e^{-\alpha^2 y}\right)$<br>$u_3 = e^{-\alpha^2 y}\left(A_3 e^{2\alpha x i} + B_3 e^{-2\alpha x i}\right)$ *where* $A_3, B_3$ =*constant* | |

- ***Step 4:*** *Using superposition principle to find the general PDE solution*

$$u(x,y) = \underbrace{A_1 x + B_1}_{Case\ 1\ solution} + \underbrace{e^{\alpha^2 y}\left(A_2 e^{2\alpha x} + B_2 e^{-2\alpha x}\right)}_{Case\ 2\ solution} + \underbrace{e^{-\alpha^2 y}\left(A_3 e^{2\alpha x i} + B_3 e^{-2\alpha x i}\right)}_{Case\ 3\ solution}$$

## 12.4 EXPRESSION OF PDE SOLUTION IN TERMS OF COS/SINE OR COSH/SINH

Previously in ODE chapter, we have learned that the exponential of complex conjugate roots can be expressed in terms of $cos$ and $sin$ via Euler formula.

$$\left(A_3 e^{2\alpha x i} + B_3 e^{-2\alpha x i}\right) = \left(C_3 cos(2\alpha x) + D_3 sin(2\alpha x)\right)$$

Similarly, exponential of distinct real roots can be expressed in terms of $cosh$ and $sinh$ via Euler formula. These two expressions are useful to find the particular solution for the PDE later.

$$\left(A_2 e^{2\alpha x} + B_2 e^{-2\alpha x}\right) = \left(C_2 cosh(2\alpha x) + D_2 sinh(2\alpha x)\right)$$

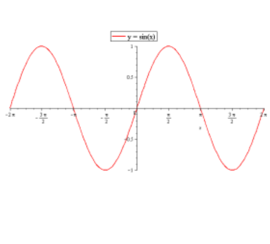
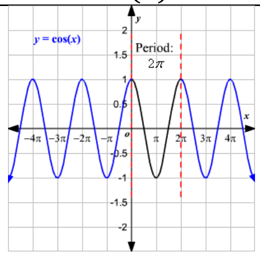
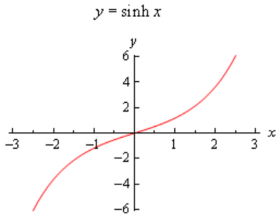
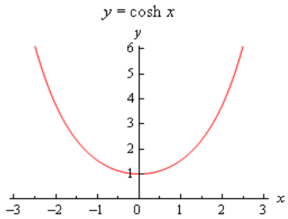
Derivation by using Euler Formula is given below:

| $A_3 e^{2\alpha x i} + B_3 e^{-2\alpha x i}$ | $A_2 e^{2\alpha x} + B_2 e^{-2\alpha x}$ |
|---|---|
| Since $e^{i\theta} = cos\theta + isin\theta$, we get<br>$\therefore A_3\left(cos(2\alpha x) + isin(2\alpha x)\right)$<br>$+B_3\left(cos(-2\alpha x) + isin(-2\alpha x)\right)$<br><br>Since $cos(-\theta) = cos\theta, sin(-\theta) = -sin\theta$<br>$\therefore A_3\left(cos(2\alpha x) + isin(2\alpha x)\right)$<br>$+B_3\left(cos(2\alpha x) - isin(2\alpha x)\right)$<br><br>Rearrange,<br>$\therefore cos(2\alpha x)(A_3 + B_3)$<br>$+sin(2\alpha x)(iA_3 - iB_3)$<br><br>$\therefore C_3 cos(2\alpha x) + D_3 sin(2\alpha x)$<br><br>where $C_3 = A_3 + B_3; D_3 = iA_3 - iB_3$ | Since $e^{\theta} = cosh\theta + sinh\theta$ , we get<br>$\therefore A_2\left(cosh(2\alpha x) + sinh(2\alpha x)\right)$<br>$+B_2\left(cosh(-2\alpha x) + sinh(-2\alpha x)\right)$<br><br>Since $cos\,h(-\theta) = cosh\theta, sinh(-\theta) = -sinh\theta$<br>$\therefore A_2\left(cosh(2\alpha x) + sinh(2\alpha x)\right)$<br>$+B_2\left(cosh(2\alpha x) - sinh(2\alpha x)\right)$<br><br>Rearrange,<br>$\therefore cosh(2\alpha x)(A_2 + B_2)$<br>$+sinh(2\alpha x)(A_2 - B_2)$<br><br>$\therefore C_2 cosh(2\alpha x) + D_2 sinh(2\alpha x)$<br><br>where $C_2 = A_2 + B_2; D_2 = A_2 - B_2$ |

Thus, the previous PDE solution can be expressed in the cos/sine & cosh/sinh formats:

$$u(x,y) = \underbrace{A_1 x + B_1}_{Case\ 1\ solution} + \underbrace{e^{\alpha^2 y}\left(C_2 cosh(2\alpha x) + D_2 sinh(2\alpha x)\right)}_{Case\ 2\ solution} + \underbrace{e^{-\alpha^2 y}\left(C_3 cos(2\alpha x) + D_3 sin(2\alpha x)\right)}_{Case\ 3\ solution}$$

**Important Characteristics** of Sine, Cosine, Hyperbolic Sine & Hyperbolic Cosine:

| Sine, $sin(x)$ | Cosine, $cos(x)$ | Hyperbolic Sine, $sinh(x)$ | Hyperbolic Cosine, $cosh(x)$ |
|---|---|---|---|
|  |  |  |  |
| *sin(0)=0* | *cos(0)=1* | *sinh(0)=0* | *cosh(0)=1* |
| *Odd function*<br>*sin(-x)= -sin(x)* | *Even function*<br>*cos(-x)=cos(x)* | *Odd function*<br>*sinh(-x)= -sinh(x)* | *Even function*<br>*cosh(-x)=cosh(x)* |
| $\frac{d}{dx}sin(x) = cos(x)$ | $\frac{d}{dx}cos(x) = -sin(x)$ | $\frac{d}{dx}sinh(x) = cosh(x)$ | $\frac{d}{dx}cosh(x) = sinh(x)$ |
| $sin(n\pi) = 0$<br>where $n = integer$ | $cos\left((2n-1)\frac{\pi}{2}\right) = 0$<br>where $n = integer$ | $sinh(0) = 0$ *only when* $x = 0$<br>$sinh(x) > 0$ *for* $x > 0$ | $cosh(x) \neq 0$ *for any* $x$<br>$cosh(x) > 0$ *for any* $x$ |

## 12.5 INITIAL/ BOUNDARY CONDITION OF PDE PROBLEM

Previously, we solve the following PDE: $\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial y}$ and obtain the general solution with a lot of unknowns $(A_1, B_1, A_2, B_2, A_3, B_3)$. By using initial or/and boundary conditions of the problem, we can continue to solve those unknowns and obtain the particular solution of the PDE.

Thus, it is important to formulate the initial/ boundary condition from a given problem. The three conditions that are found to occur most regularly are

| Cauchy conditions | Dirichlet conditions | Neumann conditions |
|---|---|---|
| $u$ and $\frac{\partial u}{\partial n}$ given on $C$ | $u$ given on $C$ | $\frac{\partial u}{\partial n}$ given on $C$ |

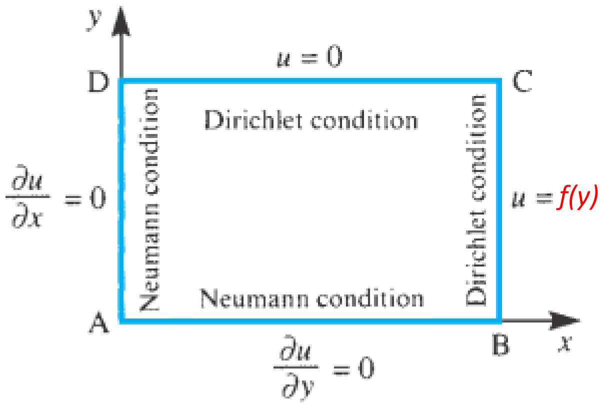
**Note:** A boundary C is said to be closed if conditions are specified on the whole of it, or open if conditions are only specified on part of it. Naming of the type of conditions is out of scope, it is sufficient as long as student is able to formulate the equations for the initial/ boundary conditions.

Example of **formulating the initial/ boundary condition** from Elliptic PDE, Parabolic PDE, and Hyperbolic PDE are given below:

i. **Elliptic PDE**: Set up the boundary value problem for the steady-state temperature $u(x, y)$ for a thin rectangular plate coincides with the region defined by $0 \le x \le 4$, $0 \le y \le 2$. The left end and the bottom of the plate are insulated. The top of the plate is held at temperature zero, and the right end of the plate is held at temperature $f(y)$. The PDE that governs the problem is given:
*2D Laplace Equation* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

Solution: Stable temperature distribution of 2D plate, $u(x, y)$

*The corresponding region and boundary conditions in the (x,y) plane for a steady state heated rectangular plate.*




***<u>Dirchlet condition for the top and right end:</u>***

$u(x, 2) = 0\,, \qquad 0 < x < 4$

$u(4,\ y) = f(y)\,,\ \ 0 < y < 2$

***<u>Neumann condition for the bottom and left end:</u>***

$\frac{\partial u(x,0)}{\partial y} = \frac{\partial u}{\partial y}\Big|_{y=0} = 0\,,\ 0 < x < 4$

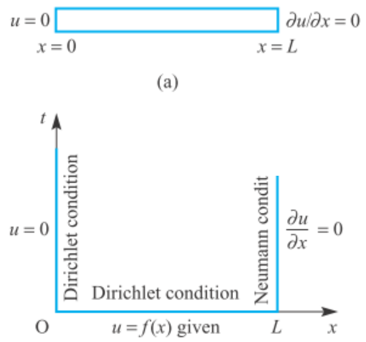
$\frac{\partial u(0,y)}{\partial x} = \frac{\partial u}{\partial x}\Big|_{x=0} = 0,\ \ 0 < y < 2$

Note: The heat flux can't flow in the x-direction, $q_x = 0$ if there is insulation on the left end. Since $q_x = -k \underbrace{\frac{\partial u}{\partial x}}_{Temperature\ gradient} = 0$, thus $\frac{\partial u}{\partial x}\Big|_{x=0} = 0$ indicates insulation on left end which blocks the heat flux to flow in the x-direction.

ii. **Parabolic PDE:** A rod of length *L* coincides with the interval [0, *L*] on the *x*-axis. <u>Set up the boundary value problem</u> for the temperature *u*(*x*, *t*) when the left end is held at temperature zero, and the right end is insulated. The initial temperature is *f*(*x*) throughout. The PDE that governs the problem is given: *1D Heat Equation* $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ where *c* = constant. Formulate the initial/ boundary condition.

Solution: Temperature of the 1D bar that changes over time, $u(x, t)$

*The corresponding region and boundary conditions in the (x,t) plane for a heated/cooled bar*




***<u>Dirchlet condition for the left end:</u>***

$$u(0,\ t) = 0\,, t > 0$$

***<u>Dirchlet condition for the initial temperature:</u>***

$$\left.\begin{matrix} u(x,\ 0) = f(x) \\ \underbrace{\frac{\partial u(x,t)}{\partial x} = \frac{\partial u}{\partial x}\Big|_x \neq 0}_{bar\ is\ not\ insulated} \end{matrix}\right\}\ 0 < x < L$$

***<u>Neumann condition for the right end:</u>***

$$u_x(L, t) = \frac{\partial u(L,t)}{\partial x} = \frac{\partial u}{\partial x}\Big|_{x=L} = 0\,, t > 0$$

iii. **Hyperbolic PDE:** *A string of length L coincides with interval [0, L] on the x-axis. Set up the boundary value problem for the displacement u(x, t) when the ends are secured to the x-axis. The string is released from rest from the initial displacement x(L – x).* The PDE that governs the problem is given: *1D Wave Equation* $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ where $c$ = constant. Formulate the initial/ boundary condition.

Solution: Vibration of the 1D string over time, $u(x,t)$

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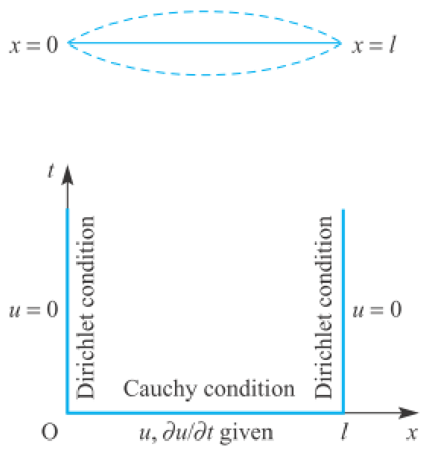
*The corresponding region and boundary conditions in the (x,t) plane for a vibrating string*

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***<u>Dirchlet condition for the fixed end:</u>***

$$\left.\begin{aligned} u(0,\ t) &= 0 \\ u(L,\ t) &= 0 \end{aligned}\right\} t > 0$$

***<u>Cauchy condition for initial displacement/velocity:</u>***

$$\left.\begin{aligned} u(x,\ 0) &= x(L-x) \\ \frac{\partial u(x,0)}{\partial t} = \left.\frac{\partial u}{\partial t}\right|_{t=0} &= 0 \end{aligned}\right\} \quad 0 < x < L$$

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# SOLVING PARTICULAR SOLUTION OF LAPLACE EQUATION

## WEEK 13: SOLVING PARTICULAR SOLUTION OF LAPLACE EQUATION

### 13.1 EIGENVALUE AND EIGENFUNCTION OF ODE (BACKGROUND - EXTRA INFO)

Find all the eigenvalues and eigenfunction of the following ODE, where $Y'' = \frac{d^2Y}{dy^2}$

$Y'' + \lambda Y = 0$ where $Y(0) = 0$ and $Y(2\pi) = 0$

The ODE above can be transformed to an eigenvalue problem:
Let $Y = A_1 sin(\omega t + \theta_1)$ and

$$Y'' = -\omega^2 A_1 sin(\omega t + \theta_1) = -\omega^2 Y$$

$-\omega^2 Y + \lambda Y = 0$

$(\lambda - \omega^2)Y = 0$

The solution $Y$ can't be zero and hence $|\lambda - \omega^2| = 0$,
where the **eigenvalue, $\boldsymbol{\lambda}$** $= \omega^2$ &
the **corresponding solution $\boldsymbol{Y}$ is the eigenfunction of the ODE**.

Recall that eigenfunction represents each of a set of independent functions, which are the solutions to a given differential equation.

| Case | General solution of the ODE | Particular solution of the ODE |
|---|---|---|
| Case #1: ($\lambda$=0) | $Y'' = 0$<br>Let $r$ =root<br>Characteristic equation: $r^2 = 0$<br>Repeated roots: $r_1 = 0,\ r_2 = 0$<br><br>$Y(y) = c_1 e^{0y} + c_2 y e^{0y}$<br>$\therefore Y(y) = c_1 + c_2 y$ | Using boundary condition,<br>$Y(0) = c_1 + c_2(0) = 0$<br>$c_1 = 0$<br>$\rightarrow Y = c_2 y$<br>$Y(2\pi) = 0 = c_2$<br><br>$\therefore Y = 0$ (No solution if $\lambda$=0) |
| Case #2: ($\lambda = -\alpha^2$) $\alpha > 0$ | $Y'' + (-\alpha^2)Y = 0$<br>Let $r$ =root<br>Characteristic equation:<br>$r^2 - \alpha^2 = 0$<br>$r = \pm\sqrt{\alpha^2} = \pm\alpha$<br><br>Distinct roots: $r_1 = \alpha,\ r_2 = -\alpha$<br><br>$\therefore Y = c_3 cosh(\alpha y) + c_4 sinh(\alpha y)$<br><br>*Hint: Refer section 12.4* | Using boundary condition,<br>$Y(0) = c_3 \cosh(0) + c_4 \sinh(0) = 0$<br>$c_3(1) + c_4(0) = 0$<br>$c_3 = 0$<br><br>$\rightarrow Y = c_4 sinh\alpha y$<br><br>$Y(2\pi) = c_4 sinh\alpha(2\pi) = 0$<br><br>Since $\alpha(2\pi) > 0$ & $sinh(+ve)$ is never equal to zero for all $\alpha(2\pi)$, thus $c_4 = 0$ |

| | | |
|---|---|---|
| | | $\therefore Y = 0$ (No solution if $\lambda = -\alpha^2$) |
| Case #3:<br>($\lambda = +\alpha^2$)<br>$\alpha > 0$ | $Y'' + (\alpha^2)Y = 0$<br>Let $r$ =root<br>Characteristic equation:<br>$r^2 + \alpha^2 = 0$<br>$r = \pm\sqrt{-\alpha^2} = \pm\alpha i$<br><br>Complex conjugate roots:<br>$r_1 = \alpha i, \quad r_2 = -\alpha i$<br><br>$\therefore Y(y) = c_5\cos(\alpha y) + c_6\sin(\alpha y)$<br><br>*Hint: Refer section 12.4* | Using boundary condition,<br>$Y(0) = c_5\cos(0) + c_6\sin(0) = 0$<br>$c_5(1) + c_6(0) = 0$<br>$c_5 = 0$<br>$\rightarrow Y = c_6\sin(\alpha y)$<br><br>$Y(2\pi) = c_6 sin(2\pi\alpha) = 0$<br><br>**Possibility 1:** If $c_6 = 0$, we will get no solution, $Y = 0$ .<br>**Possibility 2:** So, we check if $sin(2\pi\alpha)$ can be zero.<br><br>Since $2\pi\alpha > 0$ & $sin(2\pi\alpha) = 0$ when $2\pi\alpha = n\pi$, where integer $n = 1,2,3,\ldots$<br><br>Then, $c_6 \neq 0$ in this condition.<br><br>$\therefore Y_n = c_{6,n}\sin(\alpha y)$ where the $\alpha = \frac{n}{2}$ for $n = 1,2,3,\ldots$<br>(We have solution if $\lambda = +\alpha^2$)<br><br>Think: Can the solution valid for $n = \cdots, -2, -1, 0$ ? Hint: $2\pi\alpha > 0$ |

Eigenvalue for the ODE, $\lambda = +\alpha^2 = \frac{n^2}{4}$ for $n = 1,2,3,\ldots$

Eigenfunction for the ODE, $Y_n = c_{6,n}\sin(\frac{n}{2}y)$ for $n = 1,2,3,\ldots$

$n = 1 \rightarrow Y_1 = c_{6,1}\sin(\frac{1}{2}y)$

$n = 2 \rightarrow Y_2 = c_{6,2}\sin(\frac{2}{2}y)$

Thus, we have infinite solutions in the 3rd case, by using the superposition principle:

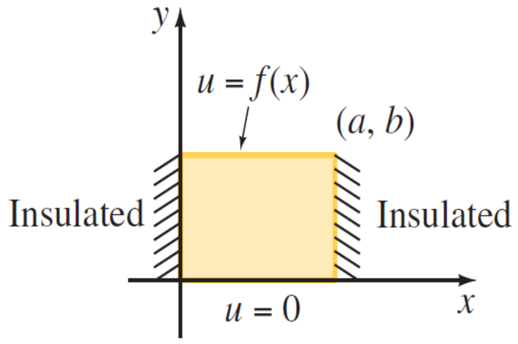
$Y_{total} = Y_1 + Y_2 + \cdots = \sum_{n=1}^{\infty} c_{6,n}\sin(\frac{n}{2}y)$

Note that $c_{6,n}$ can be solved further with Fourier series expansion & additional initial/boundary condition. Then, the complete particular solution can be obtained.

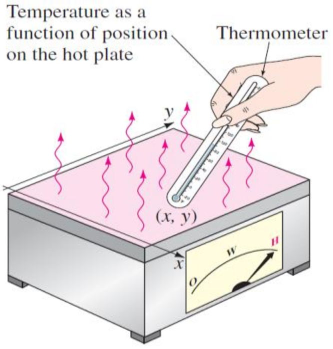
The similar concepts discussed in section 13.1 can be used to solve the PDE problem.

## 13.2 SOLVING PARTICULAR SOLUTION OF ELLIPTIC PDE (LAPLACE EQUATION)

*Consider a hot place of area (xy), find the* *<u>steady state temperature distribution over the x and y location</u>, i.e.* $u(x, y)$.




Top view of the 2D heated plate




3D view of the heated plate

- **Governing equation for the 2D Laplace equation**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

- **Boundary condition 1 & 2:** $\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0\,, \left.\frac{\partial u}{\partial x}\right|_{x=a} = 0$ for $0 < y < b$

- **Boundary condition 3 & 4:** $u(x,\ 0) = 0\,, u(x,\ b) = f(x)$ for $0 < x < a$

Solution:

***Step 1:*** *Using separation of variable method:* Let $u(x, y) = X(x)Y(y)$

$$X''Y + XY'' = 0$$

***Step 2:*** *Obtain 2 ODE equations*

$$\frac{Y''}{-Y} = \frac{X''}{X} = -\lambda$$

$Y'' - \lambda Y = 0$ --- (ODE #1)

$X'' + \lambda X = 0$ --- (ODE #2)

| Case | ODE #1 | ODE #2 | $u(x, y) = X(x)Y(y)$ |
|---|---|---|---|
| Case #1: (λ=0) | $Y'' = 0$<br>Let $r$ =root<br>Characteristic equation:<br>$r^2 = 0$<br>Repeated roots:<br>$r_1 = 0, \quad r_2 = 0$<br><br>$Y(y) = c_1 e^{0y} + c_2 y e^{0y}$ | $X'' = 0$<br>Let $r$ =root<br>Characteristic equation:<br>$r^2 = 0$<br>Repeated root: $r_1 = 0$, $r_2 = 0$<br><br>$X(x) = c_3 e^{0x} + c_4 x e^{0x}$ | $\therefore u_1 = X_1(x)Y_1(y)$<br>$= (c_1 + c_2 y)(c_3 + c_4 x)$ |

| | | | |
|---|---|---|---|
| | $\therefore Y(y) = c_1 + c_2 y$ | $\therefore X(x) = c_3 + c_4 x$ | |
| Case #2:<br>(λ= $-\alpha^2$)<br>$\alpha > 0$ | $Y'' + (\alpha^2)Y = 0$<br>Let $r$ =root<br>Characteristic equation:<br>$r^2 + \alpha^2 = 0$<br>$r = \pm\sqrt{-\alpha^2} = \pm\alpha i$<br><br>Complex conjugate roots: $r_1 = \alpha i,\ r_2 = -\alpha i$<br><br>$\therefore Y(y) = c_5 cos(\alpha y) + c_6 sin(\alpha y)$ | $X'' - \alpha^2 X = 0$<br>Let $r$ =root<br>Characteristic equation:<br>$r^2 - \alpha^2 = 0$<br>$r = \pm\sqrt{\alpha^2} = \pm\alpha$<br><br>Distinct roots:<br>$r_1 = \alpha, \quad r_2 = -\alpha$<br><br>$\therefore X(x) = c_7 cosh(\alpha x) + c_8 sinh(\alpha x)$ | $\therefore u_2 = X_2(x)Y_2(y)$<br>$= \left(c_5 cos(\alpha y) + c_6 sin(\alpha y)\right)(c_7 cosh(\alpha x) + c_8 sinh(\alpha x))$ |
| Case #3:<br>(λ= $+\alpha^2$)<br>$\alpha > 0$ | $Y'' + (-\alpha^2)Y = 0$<br>Let $r$ =root<br>Characteristic equation:<br>$r^2 - \alpha^2 = 0$<br>$r = \pm\sqrt{\alpha^2} = \pm\alpha$<br><br>Distinct roots:<br>$r_1 = \alpha, \quad r_2 = -\alpha$<br><br>$\therefore Y(y) = c_9 cosh(\alpha y) + c_{10} sinh(\alpha y)$ | $X'' + (\alpha^2)X = 0$<br>Let $r$ =root<br>Characteristic equation:<br>$r^2 + \alpha^2 = 0$<br>$r = \pm\sqrt{-\alpha^2} = \pm\alpha i$<br><br>Complex conjugate roots: $r_1 = \alpha i,\ r_2 = -\alpha i$<br><br>$\therefore X(x) = c_{11} cos(\alpha x) + c_{12} sin(\alpha x)$ | $\therefore u_3 = X_3(x)Y_3(y)$<br><br>$= \left(c_9 cosh(\alpha y) + c_{10} sinh(\alpha y)\right)\left(c_{11} cos(\alpha x) + c_{12} sin(\alpha x)\right)$ |

In fact, we can find the general PDE solution to the problem by using superposition principle:

$$u(x,y) = \underbrace{\left(c_1 + c_2 y\right)\left(c_3 + c_4 x\right)}_{Solution\ of\ Case\ 1} + \underbrace{\left(c_5 cos(\alpha y) + c_6 sin(\alpha y)\right)\left(c_7 cosh(\alpha x) + c_8 sinh(\alpha x)\right)}_{Solution\ of\ Case\ 2}$$

$$+ \underbrace{\left(c_9 cosh(\alpha y) + c_{10} sinh(\alpha y)\right)\left(c_{11} cos(\alpha x) + c_{12} sin(\alpha x)\right)}_{Solution\ of\ Case\ 3}$$

where there are 12 unknown coefficients ($c_1 - c_{12}$). Next we will continue to solve those unknowns by applying the initial/ boundary conditions.

To apply the following boundary conditions, differentiation of the PDE solution is needed.

Boundary condition (BC) #1: $\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0$ , BC #2: $\left.\frac{\partial u}{\partial x}\right|_{x=a} = 0$

| Case | Differentiation of $u(x,y) = X(x)Y(y)$ wrt $x$ |
|---|---|
| Case #1:<br>(λ=0) | $u_1 = X_1(x)Y_1(y)$<br>$= \left(c_1 + c_2 y\right)\left(c_3 + c_4 x\right)$<br><br>$\frac{\partial u_1}{\partial x} = \left(c_1 + c_2 y\right)(c_4)$<br>Applying BC #1 or BC #2, we get $\left(c_1 + c_2 y\right)(c_4) = 0$ |

| | |
|---|---|
| | Since $\left(c_1 + c_2 y\right) \neq 0,\, c_4 = 0$<br><br>$\therefore u_1(x,y) = (c_1 + c_2 y)(c_3) = (A_1 + B_1 y)$ |
| Case #2:<br>($\lambda = -\alpha^2$)<br>$\alpha > 0$ | $u_2 = X_2(x)Y_2(y)$<br>$= \left(c_5 cos(\alpha y) + c_6 sin(\alpha y)\right)(c_7 cosh(\alpha x) + c_8 sinh(\alpha x))$<br><br>$\frac{\partial u_2}{\partial x} = \left(c_5 cos(\alpha y) + c_6 sin(\alpha y)\right)(c_7 \alpha sinh(\alpha x) + c_8 \alpha cosh(\alpha x))$<br><br>Applying BC #1: $(c_5 cos(\alpha y) + c_6 sin(\alpha y))\,(c_8 \alpha) = 0$<br>Since $(c_5 cos(\alpha y) + c_6 sin(\alpha y)) \neq 0, \alpha \neq 0$, thus $c_8 = 0$<br><br>$\rightarrow u_2(x,y) = \left(c_5 cos(\alpha y) + c_6 sin(\alpha y)\right)(c_7 cosh(\alpha x))$<br>$\rightarrow \frac{\partial u_2}{\partial x} = \left(c_5 cos(\alpha y) + c_6 sin(\alpha y)\right)(c_7 \alpha sinh(\alpha x))$<br><br>Applying BC #2: $\left(c_5 cos(\alpha y) + c_6 sin(\alpha y)\right)(c_7 \alpha sinh(\alpha a)) = 0$<br>Since $(c_5 cos(\alpha y) + c_6 sin(\alpha y)) \neq 0, \alpha \neq 0, sinh(\alpha a) \neq 0$ for $\alpha a > 0$<br>Hence, $c_7 = 0$<br><br>$\therefore u_2(x,y) = 0$ (No solution) |
| Case #3:<br>($\lambda = +\alpha^2$)<br>$\alpha > 0$ | $\therefore u_3 = X_3(x)Y_3(y)$<br><br>$= \left(c_9 cosh(\alpha y) + c_{10} sinh(\alpha y)\right)\left(c_{11} cos(\alpha x) + c_{12} sin(\alpha x)\right)$<br><br>$\frac{\partial u_3}{\partial x} = \left(c_9 cosh(\alpha y) + c_{10} sinh(\alpha y)\right)\left(-c_{11} \alpha sin(\alpha x) + c_{12} \alpha cos(\alpha x)\right)$<br><br>Applying BC #1: $\left(c_9 cosh(\alpha y) + c_{10} sinh(\alpha y)\right)\left(c_{12} \alpha\right) = 0$<br>Since $\left(c_9 cosh(\alpha y) + c_{10} sinh(\alpha y)\right) \neq 0$ & $\alpha \neq 0$<br>Hence, $c_{12} = 0$<br><br>$\rightarrow u_3 = \left(c_9 cosh(\alpha y) + c_{10} sinh(\alpha y)\right)(c_{11} cos(\alpha x))$<br>$\rightarrow \frac{\partial u_3}{\partial x} = \left(c_9 cosh(\alpha y) + c_{10} sinh(\alpha y)\right)(-c_{11} \alpha sin(\alpha x))$<br><br>Applying BC #2: $\left(c_9 cosh(\alpha y) + c_{10} sinh(\alpha y)\right)(-c_{11} \alpha sin(\alpha a)) = 0$<br>Since $\left(c_9 cosh(\alpha y) + c_{10} sinh(\alpha y)\right) \neq 0$ & $\alpha \neq 0$<br>$c_{11} \neq 0$ *when* $sin(\alpha a) = 0$ for $\alpha a = n\pi$, where $\alpha = \frac{n\pi}{a}, n = 1,2,3 \ldots$<br><br>There are infinite solutions in Case #3:<br>$u_{3,n} = \left(c_{9,n} cosh(\frac{n\pi}{a} y) + c_{10,n} sinh(\frac{n\pi}{a} y)\right)\left(c_{11,n} cos(\frac{n\pi}{a} x)\right)$<br>$u_{3,n} = \left(A_{3,n} cosh(\frac{n\pi}{a} y) + B_{3,n} sinh(\frac{n\pi}{a} y)\right)\left(cos(\frac{n\pi}{a} x)\right)$ *where* $n = 1,2,3,\ldots$ |

In summary, the eigenvalue and eigenfunction of the PDE for each case are listed below:

| Case | PDE solution | Eigenvalue and eigenfunction of PDE |
|---|---|---|
| Case #1: | $u_1(x,y) = A_1 + B_1 y$ | Eigenvalue, λ=0 |

| | | |
|---|---|---|
| (λ=0) | | Eigenfunction $= A_1 + B_1 y$ |
| Case #2: (λ$= -\alpha^2$) $\alpha > 0$ | $u_2 = 0$ | No solution<br>hence no eigenvalue and no eigenfunction |
| Case #3: (λ$= +\alpha^2$) $\alpha > 0$ | $u_{3,n} = \left(A_{3,n}cosh(\frac{n\pi}{a}y) + B_{3,n}sinh(\frac{n\pi}{a}y)\right)\left(cos(\frac{n\pi}{a}x)\right))$<br>*where* $n = 1,2,3,\dots$ | *Eigenvalue,* $\lambda_n = +{\alpha_n}^2 = \left(\frac{n\pi}{a}\right)^2$<br>*Eigenfunction* $u_{3,n} = \left(A_{3,n}cosh(\frac{n\pi}{a}y) + B_{3,n}sinh(\frac{n\pi}{a}y)\right)\left(cos(\frac{n\pi}{a}x)\right)$ |

***Step 4:*** Superposition Principle to find $u_{total}(x,y) = X_1Y_1 + X_2Y_2 + X_3Y_3$

$$u_{total}(x,y) = \underbrace{(A_1 + B_1 y)}_{solution\ 1\ from\ Case\ 1} + \underbrace{\sum_{n=1}^{\infty}\left(A_{3,n}cosh(\frac{n\pi}{a}y) + B_{3,n}sinh(\frac{n\pi}{a}y)\right)\left(cos(\frac{n\pi}{a}x)\right)}_{solution\ 2\ from\ Case\ 3}$$

Expanding it, we obtain

$$u_{total}(x,y) = (A_1 + B_1 y) + \sum_{n=1}^{\infty}\left(A_{3,n}cosh(\tfrac{n\pi}{a}y)\left(cos(\tfrac{n\pi}{a}x)\right)\right) + \sum_{n=1}^{\infty}\left(B_{3,n}sinh(\tfrac{n\pi}{a}y)\left(cos(\tfrac{n\pi}{a}x)\right)\right)$$

where there are 4 remaining unknowns (i.e. $A_1, B_1, A_{3,\mathrm{n}}$ & $B_{3,\mathrm{n}}$).

***Step 5:*** Continue to apply the remaining BC & Fourier series expansion.

**BC #3:** $u(x,\ 0) = 0$ for $0 < x < a$

$$u_{total}(x,0) = A_1 + \sum_{n=1}^{\infty}\left(A_{3,n}cosh(\tfrac{n\pi}{a}(0)) + B_{3,n}sinh(\tfrac{n\pi}{a}(0))\right)\left(cos(\tfrac{n\pi}{a}x)\right) = 0$$

$$= A_1 + \sum_{n=1}^{\infty}(A_{3,n})\left(cos(\tfrac{n\pi}{a}x)\right) = 0$$

Recall Half-range Fourier Cosine Series Expansion:

| |
|---|
| $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega x)$<br>*where* $a_0 = \frac{1}{L}\int_0^{\tau} f(x)\,dx;$<br>$a_n = \frac{2}{L}\int_0^{\tau} f(x)\cos n\omega x\ dx;$ |

We notice $f(x) = 0$ ; $A_1 = \frac{1}{L}\int_0^{\tau} f(x)\,dx = 0$ ; $A_{3,\mathrm{n}} = \frac{2}{L}\int_0^{\tau} f(x)\cos n\omega x\ dx = 0$

$$\rightarrow u_{total}(x,y) = (B_1 y) + \sum_{n=1}^{\infty}\left(B_{3,n}sinh(\tfrac{n\pi}{a}y)\left(cos(\tfrac{n\pi}{a}x)\right)\right)$$

***Step 5:*** Continue to apply the remaining BC & Fourier series expansion.

**BC #4:** $u(x,\ b) = f(x)$ for $0 < x < a$

$u_{total}(x,b) = B_1(b) + \sum_{n=1}^{\infty}\left(B_{3,n} sinh(\frac{n\pi}{a}b)\right)\ \left(cos(\frac{n\pi}{a}x)\right) = f(x)$

Recall Half-range Fourier Cosine Series Expansion:

| |
|---|
| $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega x)$<br>*where* $a_0 = \frac{1}{L}\int_0^{\tau} f(x)\, dx;$<br>$a_n = \frac{2}{L}\int_0^{\tau} f(x) \cos n\omega x\ dx;$ |

We notice $B_1 b = \frac{1}{L}\int_0^{\tau} f(x)\, dx$ ; $\left(B_{3,n} sinh(\frac{n\pi}{a}b)\right) = \frac{2}{L}\int_0^{\tau} f(x) \cos n\omega x\ dx$

where angular frequency, $\omega = \frac{\pi}{a}$, period, $p = \frac{2\pi}{\omega} = 2a$

Finite interval, $\tau$=$a$, half period, $L = \frac{p}{2} = \frac{2a}{2} = a$

| | |
|---|---|
| $B_1 b = \frac{1}{a}\int_0^{a} f(x)\, dx$<br>$B_1 = \frac{\int_0^a f(x)dx}{ab}$ | $B_{3,n} sinh(\frac{n\pi}{a}b) = \frac{2}{a}\int_0^{a} f(x) \cos n\frac{\pi}{a}x\, dx$<br>$B_{3,n} = \frac{2}{a sinh(\frac{n\pi}{a}b)}\int_0^{a} f(x) \cos n\frac{\pi}{a}x\, dx$ |

Thus, we have solved all the unknowns and obtain the particular PDE solution:

$$\therefore u_{total}(x,y) = B_1 y + \sum_{n=1}^{\infty}\left(B_{3,n} sinh(\frac{n\pi}{a}y)\right)\ \left(cos(\frac{n\pi}{a}x)\right)$$

$$u_{total}(x,t) = \frac{\int_0^a f(x)dx}{ab} y + \sum_{n=1}^{\infty}\left(\frac{2}{a sinh(\frac{n\pi}{a}b)}\int_0^{a} f(x) \cos n\frac{\pi}{a}x\, dx\, sinh(\frac{n\pi}{a}y)\right)\ \left(cos(\frac{n\pi}{a}x)\right)$$

Example: Let the temperature at the top end, $f(x) = 100$ , dimension, $a = b = 1$ for the previous problem.

$$B_1 = \frac{\int_0^a f(x)dx}{ab}$$

$$B_1 = \frac{\int_0^1 100dx}{(1)(1)} = 100$$

$$B_{3,n} = \frac{2}{a\sinh(\frac{n\pi}{a}b)} \int_0^a f(x) \cos n\frac{\pi}{a}x\, dx$$

$$= \frac{2}{(1)\sinh(\frac{n\pi}{1}1)} \int_0^a 100 \cos n\frac{\pi}{1}x\, dx$$

$$= \frac{2}{\sinh(n\pi)} \int_0^1 100 \cos n\pi x\, dx$$

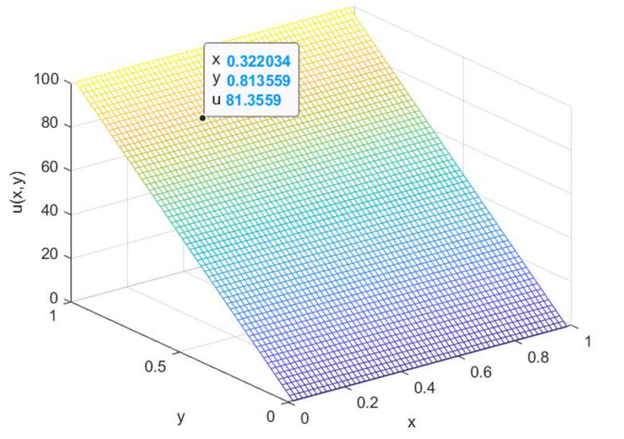
$$= \frac{2}{\sinh(n\pi)} \left[\frac{100 \sin n\pi x}{n\pi}\right]_0^1$$

$$= \frac{2}{\sinh(n\pi)} \left(\frac{100 \sin n\pi}{n\pi}\right)$$

$$\therefore u_{total}(x,y) = 100y + \sum_{n=1}^{\infty} \left(\frac{200}{\sinh(n\pi)} \left(\frac{\sin n\pi}{n\pi}\right) \sinh(n\pi y)\right) (\cos(n\pi x))$$

We can use the PDE solution to estimate the temperature distribution at any point on the heated plate. Example: The temperature results at $60 \times 60$ points of the $(x, y)$ locations have been plotted below:

**3D plot of $u_{total}(x,y)$ wrt $x$- & $y$- axes**



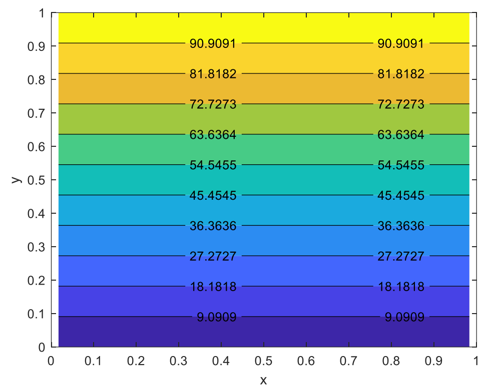

-Due to boundary conditions on 4 sides of the plates, the temperature becomes stable after certain period.

For example, $u_{total}(0.322,0.814) \approx 81\ ^oC$. Note that 20 terms are used for plotting the graphs. For higher accuracy, more terms & more grids can be included but computational time will be increased.

Try to verify the answer:

$$u_{total}(0.322,0.814)$$

**2D contour plot**



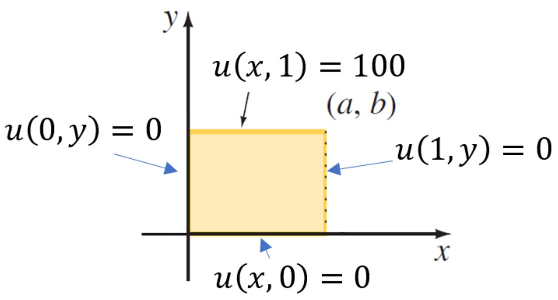

-Top view of the 3D plot with the contour, i.e. line with same magnitude.

Try to find the location of the plate that has temperature around 80 $^oC$.

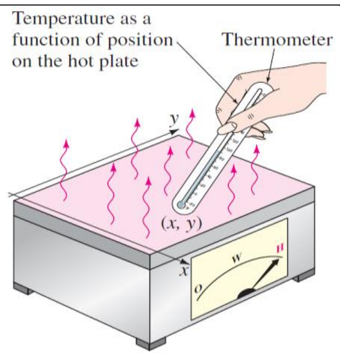
Hint: Orange color

| | |
|---|---|
| $\approx \Big(100(0.814) + \sum_{n=1}^{20}\left(\frac{200}{\sinh(n\pi)}\left(\frac{\sin n\pi}{n\pi}\right)\sinh(0.814n\pi)\right)\ (\cos(0.322n\pi))\Big)$ | |

*Consider a hot place of area (xy), find the* <u>*steady state temperature distribution over the x and y location*</u>*, i.e.* $u(x,y)$.



Top view of the 2D heated plate



3D view of the heated plate

- **Governing equation for the 2D Laplace equation**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

- **Boundary condition**: $u(0,y) = 0, u(1,y) = 0$ for $0 < y < 1$

- **Boundary condition:** $u(x,\ 0) = 0\ , u(x,\ 1) = 100$ for $0 < x < 1$

Solution:

Note that the PDE equation remains the same while the boundary conditions are changing. Thus,

The general PDE solution remains the same as below:

$$u(x,y) = \underbrace{(c_1 + c_2 y)(c_3 + c_4 x)}_{Solution\ of\ Case\ 1} + \underbrace{(c_5\cos(\alpha y) + c_6\sin(\alpha y))\,(c_7\cosh(\alpha x) + c_8\sinh(\alpha x))}_{Solution\ of\ Case\ 2}$$

$$+ \underbrace{(c_9\cosh(\alpha y) + c_{10}\sinh(\alpha y))\,(c_{11}\cos(\alpha x) + c_{12}\sin(\alpha x))}_{Solution\ of\ Case\ 3}$$

where there are 12 unknown coefficients $(c_1 - c_{12})$.

To apply the following boundary conditions, differentiation of the PDE solution is no needed.

Boundary condition (BC) #1: $u(0, y) = 0$, BC #2: $u(1, y) = 0$

| Case | Applying BC #1 & BC #2 |
|---|---|
| Case #1: (λ=0) | $u_1 = X_1(x)Y_1(y)$<br>$= (c_1 + c_2 y)(c_3 + c_4 x)$<br><br>Applying BC #1, $(c_1 + c_2 y)(c_3) = 0$<br>Since $(c_1 + c_2 y) \neq 0$, $c_3 = 0$<br>$\rightarrow u_1 = (c_1 + c_2 y)(c_4 x)$<br><br>Applying BC #2, $(c_1 + c_2 y)(c_4) = 0$<br>Since $(c_1 + c_2 y) \neq 0$, $c_4 = 0$<br><br>$\therefore u_1(x, y) = 0$ (No Solution) |
| Case #2: (λ$= -$α$^2$) α $> 0$ | $u_2 = X_2(x)Y_2(y)$<br>$= (c_5 cos(\alpha y) + c_6 sin(\alpha y))(c_7 cosh(\alpha x) + c_8 sinh(\alpha x))$<br><br>Applying BC #1: $(c_5 cos(\alpha y) + c_6 sin(\alpha y))(c_7) = 0$<br>*Since* $(c_5 cos(\alpha y) + c_6 sin(\alpha y)) \neq 0$, *thus* $c_7 = 0$<br>$\rightarrow u_2(x, y) = (c_5 cos(\alpha y) + c_6 sin(\alpha y))(c_8 sinh(\alpha x))$<br><br>Applying BC #2: $(c_5 cos(\alpha y) + c_6 sin(\alpha y))(c_8 sinh(\alpha)) = 0$<br>*Since* $(c_5 cos(\alpha y) + c_6 sin(\alpha y)) \neq 0$, $sinh(\alpha) \neq 0$ *for* $\alpha > 0$<br>Hence, $c_8 = 0$<br><br>$\therefore u_2(x, y) = 0$ (No solution) |
| Case #3: (λ$= +$α$^2$) α $> 0$ | $\therefore u_3 = X_3(x)Y_3(y)$<br>$= (c_9 cosh(\alpha y) + c_{10} sinh(\alpha y))(c_{11} cos(\alpha x) + c_{12} sin(\alpha x))$<br><br>Applying BC #1: $(c_9 cosh(\alpha y) + c_{10} sinh(\alpha y))(c_{11}) = 0$<br>Since $(c_9 cosh(\alpha y) + c_{10} sinh(\alpha y)) \neq 0$, hence, $c_{11} = 0$<br><br>$\rightarrow u_3 = (c_9 cosh(\alpha y) + c_{10} sinh(\alpha y))(c_{12} sin(\alpha x))$<br><br>Applying BC #2: $(c_9 cosh(\alpha y) + c_{10} sinh(\alpha y))(c_{12} sin(\alpha)) = 0$<br>Since $(c_9 cosh(\alpha y) + c_{10} sinh(\alpha y)) \neq 0$,<br>$c_{12} \neq 0$ *when* $sin(\alpha) = 0$ for α $= n\pi$, where $n = 1,2,3 \ldots$<br><br>There are infinite solutions in Case 3:<br>$u_{3,n} = (c_{9,n} cosh(n\pi y) + c_{10,n} sinh(n\pi y))(c_{12,n} sin(n\pi x))$<br>$u_{3,n} = (A_{3,n} cosh(n\pi y) + B_{3,n} sinh(n\pi y))(sin(n\pi x))$ *where* $n = 1,2,3,\ldots$ |

In summary, the eigenvalue and eigenfunction of the PDE for each case are listed below:

| Case | PDE solution | Eigenvalue and eigenfunction of PDE |
|---|---|---|
| Case #1: | $u_1 = 0$ | No solution |

| | | |
|---|---|---|
| (λ=0) | | hence no eigenvalue and no eigenfunction |
| Case #2: (λ$= -\alpha^2$) $\alpha > 0$ | $u_2 = 0$ | No solution<br>hence no eigenvalue and no eigenfunction |
| Case #3: (λ$= +\alpha^2$) $\alpha > 0$ | $u_{3,n} = \left(A_{3,n}cosh(n\pi y) + B_{3,n}sinh(n\pi y)\right)(sin(n\pi x))$<br>*where* $n = 1,2,3,\ldots$ | *Eigenvalue,* $\lambda_n = +{\alpha_n}^2 = (n\pi)^2$<br>*Eigenfunction* $u_{3,n} = \left(A_{3,n}cosh(n\pi y) + B_{3,n}sinh(n\pi y)\right)(sin(n\pi x))$ |

***Step 4:*** Superposition Principle to find $u_{total}(x, y) = X_1Y_1 + X_2Y_2 + X_3Y_3$

$$u_{total}(x, y) = \underbrace{\sum_{n=1}^{\infty}\left(A_{3,n}cosh(n\pi y) + B_{3,n}sinh(n\pi y)\right)(sin(n\pi x))}_{solution\ from\ Case\ 3}$$

Expanding it, we obtain

$$u_{total}(x, y) = \sum_{n=1}^{\infty}\left(A_{3,n}cosh(n\pi y)(sin(n\pi x))\right) + \sum_{n=1}^{\infty}\left(B_{3,n}sinh(n\pi y)\ (sin(n\pi x))\right)$$

where there are 2 remaining unknowns ($A_{3,n}$ & $B_{3,n}$).

***Step 5:*** Continue to apply the remaining BC & Fourier series expansion.

**BC #3:** $u(x,\ 0) = 0$ for $0 < x < a$

$$u_{total}(x, 0) = \sum_{n=1}^{\infty}\left(A_{3,n}(sin(n\pi x))\right) = 0$$

Recall Half-range Fourier Sine Series Expansion:

| |
|---|
| $f(x) = \sum_{n=1}^{\infty}(b_n\ sin\ n\omega x)$<br>*where* $b_n = \frac{2}{L}\int_0^{\tau} f(x)\ sin\ n\omega x\ dx$ |

We notice $f(x) = 0$ ; $A_{3,\mathrm{n}} = \frac{2}{L}\int_0^{\tau} f(x)\ sin\ n\omega x\ dx = 0$

$$\rightarrow u_{total}(x, y) = \sum_{n=1}^{\infty}\left(B_{3,n}sinh(n\pi y)\ (sin(n\pi x))\right)$$

**BC #4:** $u(x,\ 1) = 100$ for $0 < x < 1$

$$u_{total}(x, 1) = \sum_{n=1}^{\infty}\left(B_{3,n}sinh(n\pi)\ (sin(n\pi x))\right) = 100$$

Recall Half-range Fourier Sine Series Expansion:

$$f(x) = \sum_{n=1}^{\infty}(b_n \sin n\omega x)$$
$$where\ b_n = \frac{2}{L}\int_0^{\tau} f(x)\, sin\, n\omega x\, dx$$

We notice $B_{3,n} sinh(n\pi) = \frac{2}{L}\int_0^{\tau} f(x)\, sin\, n\omega x\ dx$

where angular frequency, $\omega = \pi$ , period, $p = \frac{2\pi}{\omega} = 2$ , $f(x) = 100$

Finite interval, $\tau$=1, half period, $L = \frac{p}{2} = \frac{2}{2} = 1$

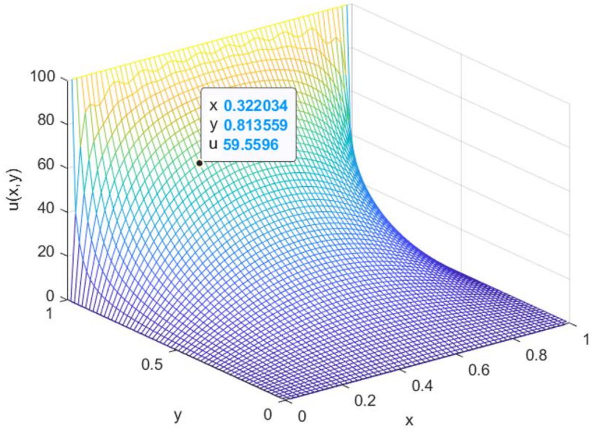
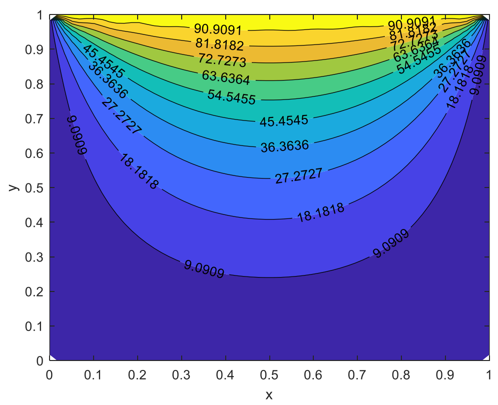
$$B_{3,n} sinh(n\pi) = \frac{2}{1}\int_0^1 100\, sin\, n\omega x\, dx$$
$$B_{3,n} = \frac{200}{sinh(n\pi)}\int_0^1 sin\, n\omega x\, dx = \frac{200}{sinh(n\pi)}\left[\frac{-cos\, n\pi - (-1)}{n\pi}\right] = \frac{200}{n\pi sinh(n\pi)}[1 - (-1)^n]$$

Thus, we have solved all the unknowns and obtain the particular PDE solution:

$$\therefore u_{total}(x,y) = \sum_{n=1}^{\infty}\left(B_{3,n} sinh(n\pi y)\ (sin(n\pi x))\right)$$

$$u_{total}(x,y) = \sum_{n=1}^{\infty}\left(\frac{200}{n\pi sinh(n\pi)}[1 - (-1)^n] sinh(n\pi y)\ (sin(n\pi x))\right)$$

We can use the PDE solution to estimate the temperature distribution at any point. Example: The temperature results at $60 \times 60$ points of the $(x, y)$ locations have been plotted below:

| 3D plot of $u_{total}(x,y)$ wrt $x$- & $y$- axis | 2D plot contour plot |
| --- | --- |
| <br><br>-Due to boundary conditions on 4 sides of the plates, the temperature becomes stable after certain period. | <br><br>-Top view of the 3D plot with the contour, i.e. line with same magnitude.<br><br>Try to find the location of the plate that has temperature around 60 $^oC$. |

| | |
|---|---|
| For example, $u_{total}(0.322, 0.814) \approx 60\ ^{o}C$. Note that 20 terms are used for plotting the graphs. For higher accuracy, more terms & more grids can be included but computational time will be increased.<br><br>Try to verify the answer:<br>$u_{total}(0.322, 0.814)$<br>$\approx \sum_{n=1}^{20}\left(\frac{200}{n\pi sinh(n\pi)}[1-(-1)^n]sinh(0.814n\pi)\,(sin(0.322n\pi))\right)$ | Hint: Green color |

## 13.3 SOLVING NON-HOMOGENEOUS BOUNDARY CONDITION VIA SUPERPOSITION PRINCIPLE

A Dirichlet problem for a rectangle can be readily solved by separation of variables when homogeneous boundary conditions are specified on two parallel boundaries. However, the method of separation variables is not applicable to a Dirichlet problem when the boundary conditions on all four sides of the rectangle are non-homogeneous. For example,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, \qquad 0<x<a, \qquad 0<y<b$$

$$u(0,y)=F(y), \qquad u(a,y)=G(y), \qquad 0<y<b$$

$$u(x,0)=f(x), \qquad u(x,b)=g(x), \qquad 0<x<a$$

The general PDE solution remains the same as below:

$$u(x,y)=\underbrace{(c_1+c_2y)(c_3+c_4x)}_{Solution\ of\ Case\ 1}+\underbrace{(c_5cos(\alpha y)+c_6sin(\alpha y))\,(c_7cosh(\alpha x)+c_8sinh(\alpha x))}_{Solution\ of\ Case\ 2}$$

$$+\underbrace{(c_9cosh(\alpha y)+c_{10}sinh(\alpha y))\,(c_{11}cos(\alpha x)+c_{12}sin(\alpha x))}_{Solution\ of\ Case\ 3}$$

To apply the following boundary conditions:

Boundary condition (BC) #1: $u(0,y)=F(y)$, BC #2: $u(a,y)=G(y)$

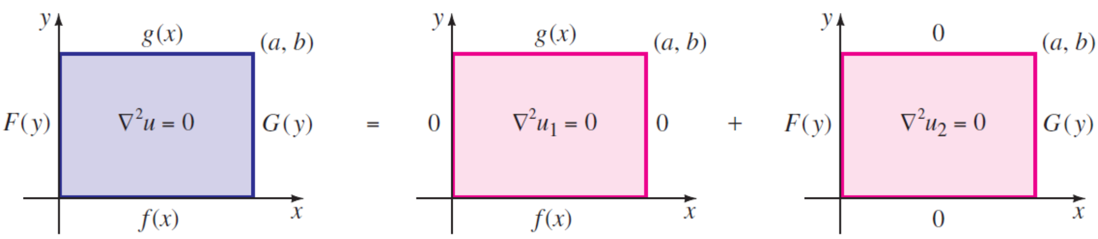
| Case | Applying BC #1 & BC #2 or (BC #3 & BC #4 return same result) |
|---|---|
| Case #1:<br>(λ=0) | $u_1=X_1(x)Y_1(y)$<br>$=(c_1+c_2y)(c_3+c_4x)$<br><br>Applying BC #1, $(c_1+c_2y)(c_3)=F(y)$ |

| | |
|---|---|
| | Applying BC #2, $\left(c_1 + c_2 y\right)(c_4 a) = G(y)$<br><br>Since no unique $c_1 - c_4$ can be obtained via the BC #1-#4, no particular solution can be obtained. This is due to non-homogeneous BC. |
| Case #2:<br>$(\lambda = -\alpha^2)$<br>$\alpha > 0$ | $u_2 = X_2(x)Y_2(y)$<br>$= \left(c_5 cos(\alpha y) + c_6 sin(\alpha y)\right)\left(c_7 cosh(\alpha x) + c_8 sinh(\alpha x)\right)$<br><br>Applying BC #1: $(c_5 cos(\alpha y) + c_6 sin(\alpha y))\,(c_7) = F(y)$<br>Applying BC #2:<br>$\left(c_5 cos(\alpha y) + c_6 sin(\alpha y)\right)\left(c_7 \cosh(\alpha a) + c_8 \sinh(\alpha a)\right) = G(y)$<br><br>Since no unique $c_5 - c_8$ can be obtained via the BC #1-#4, no particular solution can be obtained. This is due to non-homogeneous BC. |
| Case #3:<br>$(\lambda = +\alpha^2)$<br>$\alpha > 0$ | $\therefore u_3 = X_3(x)Y_3(y)$<br>$= \left(c_9 cosh(\alpha y) + c_{10} sinh(\alpha y)\right)\left(c_{11} cos(\alpha x) + c_{12} sin(\alpha x)\right)$<br><br>Applying BC #1: $\left(c_9 cosh(\alpha y) + c_{10} sinh(\alpha y)\right)\left(c_{11}\right) = F(y)$<br>Applying BC #2:<br>$\left(c_9 cosh(\alpha y) + c_{10} sinh(\alpha y)\right)\left(c_{11} cos(\alpha a) + c_{12} sin(\alpha a)\right) = G(y)$<br><br>Since no unique $c_9 - c_{12}$ can be obtained via the BC #1-#4, no particular solution can be obtained. This is due to non-homogeneous BC. |

In the previous examples, homogenous BC can ensure the unique particular solution of a boundary value problem to exist. However, it is difficulty to get the solution directly if non-homogeneous BC is encountered. The PDE problem with non-homogeneous can be solved if it can be separated into sub-problems with homogeneous BC. For example,

| Sub-problem #1 with homogeneous BC: | Sub-problem #2 with homogeneous BC: |
|---|---|
| $\dfrac{\partial^2 u_1}{\partial x^2} + \dfrac{\partial^2 u_1}{\partial y^2} = 0, \quad 0 < x < a, 0 < y < b$<br><br>$u_1(0,y) = 0, u_1(a,y) = 0, \quad 0 < y < b$<br>$u_1(x,0) = f(x), u_1(x,b) = g(x), 0 < x < a$ | $\dfrac{\partial^2 u_2}{\partial x^2} + \dfrac{\partial^2 u_2}{\partial y^2} = 0, \quad 0 < x < a, 0 < y < b$<br><br>$u_2(0,y) = F(y), u_2(a,y) = G(y), 0 < y < b$<br>$u_2(x,0) = 0, u_2(x,b) = 0, \quad 0 < x < a$ |

As shown in the figure below, PDE due to non-homogeneous PDE can be solved by separating it into two sub-problems, where the solutions of sub-problem #1, $u_1(x,y)$ and sub-problem #2, $u_2(x,y)$ can be added in the superposition manner to obtain the total solution, $u(x,y)$.




Note: $\nabla^2$ is called Laplacian or Laplace operator. For 2D problem, $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$

In this way, $u$ satisfies all boundary conditions in the original problem:

$$u(x,y) = u_1(x,y) + u_2(x,y)$$

For example,

***Solution of sub-problem 1:***

$$u_1(x,y) = \sum_{n=1}^{\infty}\left\{A_n \cosh\frac{n\pi}{a}y + B_n \sinh\frac{n\pi}{a}y\right\}\sin\frac{n\pi}{a}x$$

where

$$A_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi}{a}x\,dx \quad ; \quad B_n = \frac{1}{\sinh\frac{n\pi}{a}b}\left(\frac{2}{a}\int_0^a g(x)\sin\frac{n\pi}{a}x\,dx - A_n\cosh\frac{n\pi}{a}b\right)$$

***Solution of sub-problem 2:***

$$u_2(x,y) = \sum_{n=1}^{\infty}\left\{C_n \cosh\frac{n\pi}{b}x + D_n \sinh\frac{n\pi}{b}x\right\}\sin\frac{n\pi}{b}y$$

where

$$C_n = \frac{2}{b}\int_0^a F(y)\sin\frac{n\pi}{b}y\,dy \quad ; \quad D_n = \frac{1}{\sinh\frac{n\pi}{b}a}\left(\frac{2}{b}\int_0^a G(y)\sin\frac{n\pi}{b}y\,dy - C_n\cosh\frac{n\pi}{b}a\right)$$

***Total Solution of original problem:***

$$u(x,y) = \sum_{n=1}^{\infty}\left\{A_n \cosh\frac{n\pi}{a}y + B_n \sinh\frac{n\pi}{a}y\right\}\sin\frac{n\pi}{a}x + \sum_{n=1}^{\infty}\left\{C_n \cosh\frac{n\pi}{b}x + D_n \sinh\frac{n\pi}{b}x\right\}\sin\frac{n\pi}{b}y$$

where

$$A_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi}{a}x\,dx \quad ; \quad B_n = \frac{1}{\sinh\frac{n\pi}{a}b}\left(\frac{2}{a}\int_0^a g(x)\sin\frac{n\pi}{a}x\,dx - A_n\cosh\frac{n\pi}{a}b\right)$$

$$C_n = \frac{2}{b}\int_0^a F(y)\sin\frac{n\pi}{b}y\,dy \quad ; \quad D_n = \frac{1}{\sinh\frac{n\pi}{b}a}\left(\frac{2}{b}\int_0^a G(y)\sin\frac{n\pi}{b}y\,dy - C_n\cosh\frac{n\pi}{b}a\right)$$

# SOLVING PARTICULAR SOLUTION OF HEAT EQUATION & WAVE EQUATION

## WEEK 14: SOLVING PARTICULAR SOLUTION OF HEAT EQUATION & WAVE EQUATION

### 14.1 STRATEGY TO SOLVE HOMOGENEOUS PDE PROBLEM VIA SEPARABLE OF VARIABLE

Previously we have learned how to apply separation of variable method to solve the **Laplace equation**, then we formed the boundary conditions of the problem and apply it together with the Fourier series expansion to obtain the particular PDE solution. Same strategy is used to solve the **heat equation** and **wave equation** in this chapter, as summarized below:

Let $u =$ dependent variable, $x, t =$ independent variables

**Step 1:** $u(x,t) = X(x)T(t)$

**Step 2:** Obtains 2 ODE equations using separation constant, $-\lambda$. Let the coefficient of numerator to be 1 for easier calculation.

**Step 3:** Consider 3 cases: $\lambda=0$ ; $\lambda=-\alpha^2$ ; $\lambda=\alpha^2$, where $\alpha > 0$

Then, we can obtain all possible solutions, $u_1, u_2$, & $u_3$ respectively for each case.

**Step 4.1:** If initial/ boundary conditions can't be formed/ obtained,

**General PDE solution** via superposition principle, $u(x,y) = c_1u_1 + c_2u_2 + c_3u_3$,

where $c_1, c_2, c_3$ are unknowns.

**Step 4.2:** If initial/ boundary conditions can be formed/ obtained,

Then, we proceed to apply the homogeneous BC to solve the particular solution, $u_1, u_2$, & $u_3$ for each case. Then, eigenvalue and eigenfunction can be identified for case with solution and they can be combined to form the total solution.

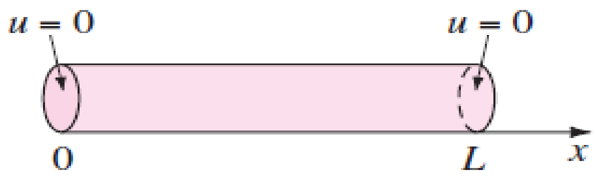
**Step 5:** Continue to apply the remaining initial/ boundary conditions & Fourier series expansion to solve the remaining unknown.

**Particular PDE solution** via superposition principle, $u(x,y) = u_1 + u_2 + u_3$,

where $c_1, c_2, c_3$ are found.

## 14.2 SOLVING PARTICULAR SOLUTION OF PARABOLIC PDE (HEAT EQUATION)

*Consider a thin rod of length L with an initial temperature f(x) throughout and whose ends are held at temperature zero for all time t > 0. Given these initial/boundary conditions, find the change of the temperature over the time and x location, i.e.* $u(x, t)$.




*1D rod with boundary conditions on both ends and initial temperature of the bar.*

- **Governing equation for the 1D heat equation**

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

- **Boundary condition 1 & 2**: $u(0,\ t) = 0\,, u(L,\ t) = 0$ for $t > 0$
- **Initial condition** : $u(x,\ 0) = f(x)$ for $0 < x < L$

Solution:

***Step 1:*** *Using separation of variable method:* Let $u(x, t) = X(x)T(t)$

$$kX''T = XT'$$

***Step 2:*** *Obtain 2 ODE equations*

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda$$

$T' + k\lambda T = 0$ --- (ODE #1)

$X'' + \lambda X = 0$ --- (ODE #2)

| Case | ODE #1 | ODE #2 | $u(x, y) = X(x)T(t)$ |
|---|---|---|---|
| Case #1: (λ=0) | $T' = 0$<br>Let $r$ =root<br>Characteristic equation: $r = 0$<br><br>$T(t) = c_1 e^{0t}$<br>$\therefore T(t) = c_1$ | $X'' = 0$<br>Let $r$ =root<br>Characteristic equation: $r^2 = 0$<br>Repeated root: $r_1 = 0,\ r_2 = 0$<br><br>$X(x) = c_2 e^{0x} + c_3 x e^{0x}$<br>$\therefore X(x) = c_2 + c_3 x$ | $\therefore u_1 = X_1(x)T_1(t)$<br>$= (c_1)(c_2 + c_3 x)$<br>$= A_1 x + B_1$ |

| | | | |
|---|---|---|---|
| Case #2:<br>(λ= $-\alpha^2$)<br>$\alpha > 0$ | $T' - (\alpha^2 k)T = 0$<br>Let $r$ =root<br>Characteristic equation:<br>$r - \alpha^2 k = 0$<br>$r = \alpha^2 k$<br><br>$\therefore T(t) = c_4 e^{\alpha^2 kt}$ | $X'' - \alpha^2 X = 0$<br>Let $r$ =root<br>Characteristic equation:<br>$r^2 - \alpha^2 = 0$<br>$r = \pm\sqrt{\alpha^2} = \pm\alpha$<br><br>Distinct roots:<br>$r_1 = \alpha, \quad r_2 = -\alpha$<br><br>$\therefore X(x) = c_5 cosh(\alpha x) + c_6 sinh(\alpha x)$ | $\therefore u_2 = X_2(x)T_2(t)$<br>$= c_4 e^{\alpha^2 kt}\,(c_5 cosh(\alpha x) + c_6 sinh(\alpha x))$<br>$= e^{\alpha^2 kt}\,(A_2 cosh(\alpha x) + B_2 sinh(\alpha x))$ |
| Case #3:<br>(λ= $+\alpha^2$)<br>$\alpha > 0$ | $T' + \alpha^2 kT = 0$<br>Let $r$ =root<br>Characteristic equation:<br>$r^2 + \alpha^2 k = 0$<br>$r = -\alpha^2 k$<br><br>$\therefore T(t) = c_7 e^{-\alpha^2 kt}$ | $X'' + (\alpha^2)X = 0$<br>Let $r$ =root<br>Characteristic equation:<br>$r^2 + \alpha^2 = 0$<br>$r = \pm\sqrt{-\alpha^2} = \pm\alpha i$<br><br>Complex conjugate roots:<br>$r_1 = \alpha i,\ r_2 = -\alpha i$<br><br>$\therefore X(x) = c_8 cos(\alpha x) + c_9 sin(\alpha x)$ | $\therefore u_3 = X_3(x)T_3(t)$<br><br>$= \left(c_7 e^{-\alpha^2 kt}\right)\left(c_8 cos(\alpha x) + c_9 sin(\alpha x)\right)$<br>$= e^{-\alpha^2 kt}\left(A_3 cos(\alpha x) + B_3 sin(\alpha x)\right)$ |

In fact, we can find the general PDE solution to the problem by using superposition principle:

$$u(x,t) = \underbrace{A_1 x + B_1}_{Solution\ of\ Case\ 1} + \underbrace{e^{\alpha^2 kt}\,(A_2 cosh(\alpha x) + B_2 sinh(\alpha x))}_{Solution\ of\ Case\ 2}$$
$$+ \underbrace{e^{-\alpha^2 kt}\left(A_3 cos(\alpha x) + B_3 sin(\alpha x)\right)}_{Solution\ of\ Case\ 3}$$

where there are 6 unknown coefficients ($A_1 - B_3$). Next, we will continue to solve those unknowns by applying the initial/ boundary conditions.

To apply the following boundary conditions.

Boundary condition (BC) #1: $u(0,\ t) = 0$ , BC #2: $u(L,\ t) = 0$

| Case | Applying BC #1 & BC #2 |
|---|---|
| Case #1:<br>(λ=0) | $u_1 = X_1(x)T_1(t)$<br>$= A_1 x + B_1$<br>Applying BC #1: $u_1(0,t) = \mathrm{A}_1(0) + \mathrm{B}_1 = 0$<br>Thus, $\mathrm{B}_1 = 0$<br>$\rightarrow u_1 = A_1 x$<br><br>Applying BC #2, we get $u_1(L,t) = A_1 L = 0$<br>Since $L \neq 0$, $A_1 = 0$<br><br>$\therefore u_1(x,t) = 0$ (No solution) |

| | |
|---|---|
| Case #2:<br>(λ= $-\alpha^2$)<br>$\alpha > 0$ | $u_2 = X_2(x)T_2(t)$<br>$= e^{\alpha^2 kt}\,(A_2 cosh(\alpha x) + B_2 sinh(\alpha x))$<br><br>Applying BC #1: $u_2(0,t) = e^{-\alpha^2 kt}\,\left(A_2\right) = 0$<br>Comparing coefficient $e^{-\alpha^2 kt}$: thus $A_2 = 0$<br>(Note: $e^{-\alpha^2 kt} \neq 0$ as the temperature changes over time, else no solution)<br>$\rightarrow u_2(x,t) = B_2 sinh(\alpha x)$<br><br>Applying BC #2: $u_2(L,t) = e^{-\alpha^2 kt}\,\left(B_2 sinh(\alpha L)\right) = 0$<br>Comparing coefficient $e^{-\alpha^2 kt}$: $\left(B_2 sinh(\alpha L)\right) = 0$<br>Since $sinh(\alpha L)$ will not be zero for $\alpha > 0$, thus $B_2 = 0$<br>Hint: $\alpha L > 0$<br><br>$\therefore u_2(x,t) = 0$ (No solution) |
| Case #3:<br>(λ= $+\alpha^2$)<br>$\alpha > 0$ | $\therefore u_3 = X_3(x)T_3(t)$<br>$= e^{-\alpha^2 kt}\,\left(A_3 cos(\alpha x) + B_3 sin(\alpha x)\right)$<br><br>Applying BC #1: $u_3(0,t) = e^{-\alpha^2 kt}\,\left(A_3\right) = 0$<br>Comparing coefficient $e^{-\alpha^2 kt}$: $A_3 = 0$<br>$\rightarrow u_3 = e^{-\alpha^2 kt}\,\left(B_3 sin(\alpha x)\right)$<br><br>Applying BC #2: $u_3(L,t) = e^{-\alpha^2 kt}\,\left(B_3 sin(\alpha L)\right) = 0$<br>Comparing coefficient $e^{-\alpha^2 kt}$: $\left(B_3 sin(\alpha L)\right) = 0$<br><br>Since $B_3 \neq 0$ *when* $sin(\alpha L) = 0$ for $\alpha L = n\pi$, where $\alpha = \frac{n\pi}{L}$, $n = 1,2,3 \ldots$<br><br>There are infinite solutions in Case #3:<br>$u_{3,n} = e^{-\left(\frac{n\pi}{L}\right)^2 kt}\,\left(B_{3,n} sin(\frac{n\pi}{L}x)\right)$ *where n =1,2,3,…* |

In summary, the eigenvalue and eigenfunction of the PDE for each case are listed below:

| Case | PDE solution | Eigenvalue and eigenfunction of PDE |
|---|---|---|
| Case #1:<br>(λ=0) | $u_1(x,t) = 0$ | No solution<br>hence no eigenvalue and no eigenfunction |
| Case #2:<br>(λ= $-\alpha^2$)<br>$\alpha > 0$ | $u_2(x,t) = 0$ | No solution<br>hence no eigenvalue and no eigenfunction |
| Case #3:<br>(λ= $+\alpha^2$)<br>$\alpha > 0$ | $u_{3,n} = e^{-\left(\frac{n\pi}{L}\right)^2 kt}\,\left(B_{3,n} sin(\frac{n\pi}{L}x)\right)$ | *Eigenvalue,* $\lambda_n = +{\alpha_n}^2 = \left(\frac{n\pi}{L}\right)^2$<br>*Eigenfunction* $u_{3,n}$<br>$= e^{-\left(\frac{n\pi}{L}\right)^2 kt}\,\left(B_{3,n} sin(\frac{n\pi}{L}x)\right)$ |

***Step 4:*** Superposition Principle to find $u_{total}(x,t) = X_1T_1 + X_2T_2 + X_3T_3$

$$u_{total}(x,t) = \underbrace{\sum_{n=1}^{\infty} e^{-\left(\frac{n\pi}{L}\right)^2 kt}\left(B_{3,n} sin(\frac{n\pi}{L}x)\right)}_{solution\ from\ Case\ 3}$$

where there are 1 unknown remaining (i.e. $B_{3,n}$).

***Step 5:*** Continue to apply the remaining BC & Fourier series expansion.

**BC #3:** $u(x,\ 0) = f(x)$ for $0 < x < L$

$u_{total}(x,0) = \sum_{n=1}^{\infty}\left(B_{3,n} sin(\frac{n\pi}{L}x)\right) = f(x)$

Recall Half-range Fourier Sine Series Expansion:

| |
|---|
| $f(x) = \sum_{n=1}^{\infty}(b_n \sin n\omega x)$<br>*where* $b_n = \frac{2}{L}\int_0^{\tau} f(x) \sin n\omega x\, dx$<br><br>Precaution: $L$ in the formula indicates the half period, i.e. $L = \frac{p}{2} = \frac{\pi}{\omega}$. Do not mix it with the length of the 1D bar, which is using the same symbol, $L$ as well.<br><br>Note that for (i) Half-range expansion: Finite interval, $\tau = half\ period, L$<br>(ii) Full-range expansion: Finite interval, $\tau = full\ period, 2L$ |

We notice $B_{3,n} = b_n = \frac{2}{L}\int_0^{\tau} f(x) \sin n\omega x\, dx$ ,

where $\omega = \frac{\pi}{L}$ &

From $0 < x < L, \tau = length, L$. For half-range expansion, $\tau = half\ period, L$. Thus, in this case it happens to have $\tau = half\ period, L = length, L$ in this special case.

Precaution: Note that it would be different for full-range expansion case.

$\rightarrow B_{3,n} = \frac{2}{L}\int_0^{L} f(x) \sin n\frac{\pi}{L}x\, dx$

Thus, we have solved all the unknowns and obtain the particular PDE solution:

$$\therefore u_{total}(x,t) = \sum_{n=1}^{\infty} e^{-\left(\frac{n\pi}{L}\right)^2 kt}\left(B_{3,n} sin(\frac{n\pi}{L}x)\right)$$

$$u_{total}(x,t) = \sum_{n=1}^{\infty} e^{-\left(\frac{n\pi}{L}\right)^2 kt} \left( \frac{2}{L}\int_0^L f(x)\, sin\, n\frac{\pi}{L}x\, dx\, sin(\frac{n\pi}{L}x) \right)$$

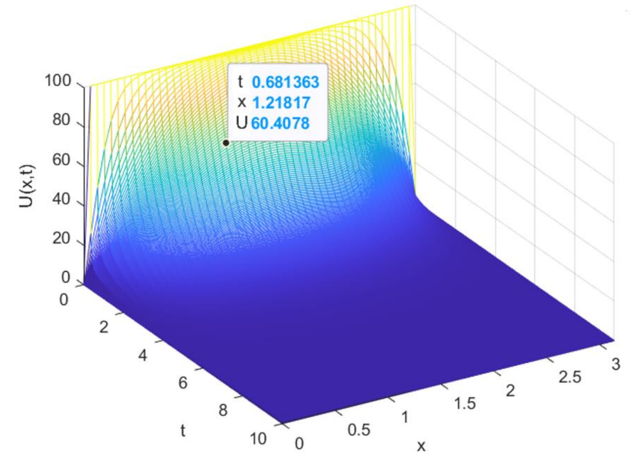
Example: Let the $f(x) = 100$ , dimension, $length, L = \pi$, PDE coefficient, $k = 1$ for the previous problem.

| |
|---|
| $B_{3,n} = \frac{2}{L}\int_0^L f(x)\, sin\, n\frac{\pi}{L}x\, dx$ <br> $= \frac{2}{\pi}\int_0^\pi 100\, sin\, nx\, dx$ <br> $= \frac{200}{\pi}\left[\frac{-cosnx}{n}\right]_0^\pi$ <br> $= \frac{200}{\pi}\left(\frac{-cosn\pi}{n} - \frac{-1}{n}\right)$ <br> $= \frac{200}{\pi}\left(\frac{1-(-1)^n}{n}\right)$ |

$$\therefore u_{total}(x,t) = \sum_{n=1}^{\infty} e^{-\left(\frac{n\pi}{\pi}\right)^2 (1)t} \left(\frac{200}{\pi}\left(\frac{1-(-1)^n}{n}\right) sin(\frac{n\pi}{\pi}x) \right)$$
$$= \sum_{n=1}^{\infty} e^{-n^2 t} \left(\frac{200}{\pi}\left(\frac{1-(-1)^n}{n}\right) sin(nx) \right)$$

We can use the PDE solution to estimate the temperature distribution at any point on the cooled rod. Example: The temperature results at $50 \times 500$ points of the $(x,t)$ locations for a duration of 10s have been plotted below:

| 3D plot of $u_{total}(x,t)$ wrt $x$- & $t$- axes | 2D contour plot |
|---|---|

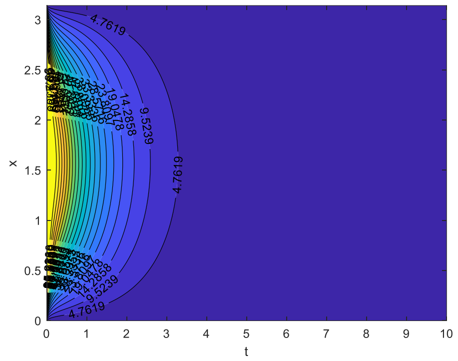



-Due to boundary conditions on both sides of the 1D rods and the initial temperature of the rod, the temperature of the rod changes over time, and it is converging to the BC temperature, i.e. $0\,^oC$.

For example, $u_{total}(1.218, 0.681) \approx 60\,^oC$. Note that 20 terms are used for plotting the graphs. For higher accuracy, more terms & more grids can be included but computational time will be increased.

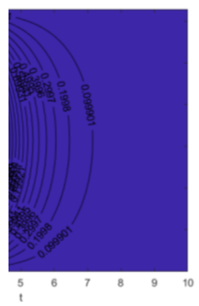
Try to verify the answer:

$$u_{total}(1.218, 0.681)$$
$$\approx \sum_{n=1}^{20} e^{-n^2(0.681)} \left(\frac{200}{\pi}\left(\frac{1-(-1)^n}{n}\right) sin(n(1.218))\right)$$




-Top view of the 3D plot with the contour, i.e. line with same magnitude.

Try to find the location of the plate that drops to $4.7619\,^oC$ last.
Hint: max of $u_{total}$ at fixed time ; sin characteristic.




By increasing the contours, we can observe that the temperatures of the whole bar takes around 7s, in order to drop to less than $0.1\,^oC$ due to BC on both ends.

**Relationship between Laplace equation and heat equation:**

In the heat equation example:

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

We observe that the temperature results become stable/ no change after some durations. This means that $\frac{\partial u}{\partial t} = 0$ for $t \to \infty$ (i.e. change of temperature, $u$ over time is zero for sufficient large duration, $t$).

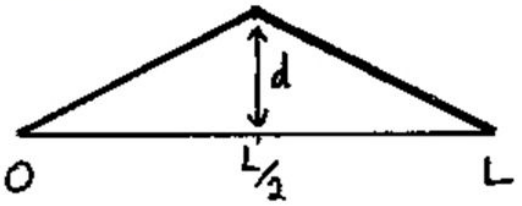
Depending on our application, we will go for

(i) Solving heat equation, $k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ if we are interested in finding out the change of the temperature, $u$ over time.
Note: The solution $u(x,t)$ contains the transient solution at beginning and steady state solution when $t \to \infty$ .

(ii) Solving Laplace equation, $k\frac{\partial^2 u}{\partial x^2} = 0$ by let $\frac{\partial u}{\partial t} = 0$ only if we are interested in finding out the stable temperature without changes over time.
Note: The solution $u(x)$ contains the steady state solution only.

## 14.3 SOLVING PARTICULAR SOLUTION OF HYPERBOLIC PDE (WAVE EQUATION)

*Consider a string of length L, stretched taut between 2 points on x-axis (e.g. x=0 and x=L)., find the change of vertical displacement with respect to time and x location, i.e.* $u(x,t)$*.*




*Transverse vibration u(x, t) in rod of length L*


*The string is fixed at both ends like guitar string.*

- **Governing equation for the 1D wave equation**

$$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

- **Boundary condition #1 & #2**: $u(0,\ t) = 0\ ,u(L,\ t) = 0$ for $t > 0$

- **Initial condition #1 & #2** : $u(x,\ 0) = f(x),\ \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$ for $0 < x < L$

  Note: For the string's vibration, $u(x,\ 0) =$ initial displacement, while $u_t(x,\ 0) =$ initial velocity.

Solution:

***Step 1:*** *Using separation of variable method:* Let $u(x,t) = X(x)T(t)$

$$a^2X''T = XT''$$

***Step 2:*** *Obtain 2 ODE equations*

$$\frac{X''}{X} = \frac{T''}{a^2T} = -\lambda$$

$$T'' + a^2\lambda T = 0 \text{ --- (ODE \#1)}$$

$$X'' + \lambda X = 0 \text{ --- (ODE \#2)}$$

| Case | ODE #1 | ODE #2 | $u(x,y) = X(x)T(t)$ |
|---|---|---|---|
| Case #1:<br>(λ=0) | $T'' = 0$<br>Let $r$ =root<br>Characteristic equation:<br>$r^2 = 0$<br>Repeated root: | $X'' = 0$<br>Let $r$ =root<br>Characteristic equation: $r^2 =$<br>$0$<br>Repeated root: | $\therefore u_1 = X_1(x)T_1(t)$<br>$= (c_1 + c_2t)(c_3$<br>$+ c_4x)$ |

| | | | |
|---|---|---|---|
| | $r_1 = 0,\ r_2 = 0$<br><br>$T(t) = c_1 e^{0t} + c_2 t e^{0t}$<br>$\therefore T(t) = c_1 + c_2 t$ | $r_1 = 0,\ r_2 = 0$<br><br>$X(x) = c_3 e^{0x} + c_4 x e^{0x}$<br>$\therefore X(x) = c_3 + c_4 x$ | |
| Case #2:<br>($\lambda = -\alpha^2$)<br>$\alpha > 0$ | $T'' - (\alpha^2 a^2)T = 0$<br>Let $r$ =root<br>Characteristic equation:<br>$r^2 - \alpha^2 a^2 = 0$<br>Distinct roots:<br>$r_1 = \sqrt{\alpha^2 a^2} = \alpha a,$<br>$r_2 = -\sqrt{\alpha^2 a^2} = \alpha a$<br><br>$\therefore T(t) = c_5 cosh(\alpha at) + c_6 sinh(\alpha at)$ | $X'' - \alpha^2 X = 0$<br>Let $r$ =root<br>Characteristic equation:<br>$r^2 - \alpha^2 = 0$<br>$r = \pm\sqrt{\alpha^2} = \pm\alpha$<br><br>Distinct roots:<br>$r_1 = \alpha, \quad r_2 = -\alpha$<br><br>$\therefore X(x) = c_7 cosh(\alpha x) + c_8 sinh(\alpha x)$ | $\therefore u_2 = X_2(x)T_2(t)$<br>$= \big(c_5\, cosh(\alpha at)$<br>$+ c_6\, sinh(\alpha at)\ \big).$<br>$(c_7 cosh(\alpha x)$<br>$+ c_8 sinh(\alpha x))$ |
| Case #3:<br>($\lambda = +\alpha^2$)<br>$\alpha > 0$ | $T'' + \alpha^2 a^2 T = 0$<br>Let $r$ =root<br>Characteristic equation:<br>$r^2 + \alpha^2 a^2 = 0$<br><br>Complex conjugate roots:<br>$r_1 = \alpha ai,\ r_2 = -\alpha ai$<br><br>$\therefore T(t) = c_9 cos(\alpha at) + c_{10} sin(\alpha at)$ | $X'' + (\alpha^2)X = 0$<br>Let $r$ =root<br>Characteristic equation:<br>$r^2 + \alpha^2 = 0$<br>$r = \pm\sqrt{-\alpha^2} = \pm\alpha i$<br><br>Complex conjugate roots:<br>$r_1 = \alpha i,\ r_2 = -\alpha i$<br><br>$\therefore X(x) = c_{11} cos(\alpha x) + c_{12} sin(\alpha x)$ | $\therefore u_3 = X_3(x)T_3(t)$<br><br>$= \big(c_9\, cos(\alpha at)$<br>$+ c_{10}\, sin(\alpha at)\ \big).$<br>$\big(c_{11} cos(\alpha x)$<br>$+ c_{12} sin(\alpha x)\ \big)$ |

In fact, we can find the general PDE solution to the problem by using superposition principle:

$$u(x,t) = \underbrace{\big(c_1 + c_2 t\ \big)\big(c_3 + c_4 x\ \big)}_{Solution\ of\ Case\ 1} + \underbrace{\big(c_5\, cosh(\alpha at) + c_6\, sinh(\alpha at)\big)\big(c_7 cosh(\alpha x) + c_8 sinh(\alpha x)\big)}_{Solution\ of\ Case\ 2}$$

$$+ \underbrace{\big(c_9\, cos(\alpha at) + c_{10}\, sin(\alpha at)\big)\big(c_{11} cos(\alpha x) + c_{12} sin(\alpha x)\ \big)}_{Solution\ of\ Case\ 3}$$

where there are 12 unknown coefficients ($c_1 - c_{12}$). Next, we will continue to solve those unknowns by applying the initial/ boundary conditions.

To apply the following boundary conditions.

Boundary condition (BC) #1: $u(0,\ t) = 0$ , BC #2: $u(L,\ t) = 0$

| Case | Applying BC #1 & BC #2 |
|---|---|
| Case #1:<br>($\lambda$=0) | $u_1 = X_1(x)T_1(t)$<br>$= \big(c_1 + c_2 t\ \big)\big(c_3 + c_4 x\ \big)$<br>Applying BC #1: $u_1(0,t) = \big(c_1 + c_2 t\ \big)(c_3) = 0$<br>Note: vibration is changing wrt time, thus $T(t) \neq 0$ for non-trivial solution.<br>Since $\big(c_1 + c_2 t\ \big) \neq 0$ , thus $c_3 = 0$<br>$\rightarrow u_1 = \big(c_1 + c_2 t\ \big)\big(c_4 x\ \big)$ |

| | |
|---|---|
| | Applying BC #2, we get $u_1(L,t) = (c_1 + c_2 t)(c_4 L) = 0$<br>Since $(c_1 + c_2 t) \neq 0$, $L \neq 0$, thus $c_4 = 0$<br><br>$\therefore u_1(x,t) = 0$ (No solution) |
| Case #2:<br>($\lambda = -\alpha^2$)<br>$\alpha > 0$ | $u_2 = X_2(x)T_2(t)$<br>$= (c_5\, cosh(\alpha at) + c_6\, sinh(\alpha at))(c_7 cosh(\alpha x) + c_8 sinh(\alpha x))$<br><br>Applying BC #1: $u_2(0,t) = (c_5 cosh(\alpha at) + c_6 sinh(\alpha at))(c_7) = 0$<br>Since $(c_5 cosh(\alpha at) + c_6 sinh(\alpha at)) \neq 0$, thus $c_7 = 0$<br><br>$\rightarrow u_2(x,t) = (c_5\, cosh(\alpha at) + c_6\, sinh(\alpha at))(c_8 sinh(\alpha x))$<br><br>Applying BC #2: $u_2(L,t) = (c_5\, cosh(\alpha at) + c_6\, sinh(\alpha at))(c_8 sinh(\alpha L)) = 0$<br>Since $(c_5\, cosh(\alpha at) + c_6\, sinh(\alpha at)) \neq 0, sinh(\alpha L) \neq 0$ for $\alpha L > 0$,<br>thus $c_8 = 0$<br>$\therefore u_2(x,t) = 0$ (No solution) |
| Case #3:<br>($\lambda = +\alpha^2$)<br>$\alpha > 0$ | $\therefore u_3 = X_3(x)T_3(t)$<br>$= (c_9\, cos(\alpha at) + c_{10}\, sin(\alpha at))(c_{11} cos(\alpha x) + c_{12} sin(\alpha x))$<br><br>Applying BC #1: $u_3(0,t) = (c_9\, cos(\alpha at) + c_{10}\, sin(\alpha at))(c_{11}) = 0$<br>Since $(c_9\, cos(\alpha at) + c_{10}\, sin(\alpha at)) \neq 0$, thus $c_{11} = 0$<br>$\rightarrow u_3 = (c_9\, cos(\alpha at) + c_{10}\, sin(\alpha at))(c_{12} sin(\alpha x))$<br><br>Applying BC #2: $u_3(L,t) = (c_9 cos(\alpha at) + c_{10} sin(\alpha at))(c_{12} sin(\alpha L)) = 0$<br>Since $(c_9\, cos(\alpha at) + c_{10}\, sin(\alpha at)) \neq 0$ and $c_{12} \neq 0$ *when* $sin(\alpha L) = 0$ for $\alpha L = n\pi$, where $\alpha = \frac{n\pi}{L}$, $n = 1,2,3 \ldots$<br><br>There are infinite solutions in Case #3:<br>$u_{3,n} = \left(c_{9,n} \cos\left(\frac{n\pi a}{L} t\right) + c_{10,n} \sin\left(\frac{n\pi a}{L} t\right)\right)\left(c_{12,n} sin(\frac{n\pi}{L} x)\right)$<br>*,where n =1,2,3,…* |

In summary, the eigenvalue and eigenfunction of the PDE for each case are listed below:

| Case | PDE solution | Eigenvalue and eigenfunction of PDE |
|---|---|---|
| Case #1:<br>($\lambda$=0) | $u_1(x,t) = 0$ | No solution<br>hence no eigenvalue and no eigenfunction |
| Case #2:<br>($\lambda = -\alpha^2$)<br>$\alpha > 0$ | $u_2(x,t) = 0$ | No solution<br>hence no eigenvalue and no eigenfunction |
| Case #3:<br>($\lambda = +\alpha^2$)<br>$\alpha > 0$ | $u_{3,n} = \left(c_{9,n} \cos\left(\frac{n\pi a}{L} t\right) + c_{10,n} \sin\left(\frac{n\pi a}{L} t\right)\right)\left(c_{12,n} sin(\frac{n\pi}{L} x)\right)$ | *Eigenvalue,* $\lambda_n = +{\alpha_n}^2 = \left(\frac{n\pi}{L}\right)^2$<br>*Eigenfunction* $u_{3,n} = \left(c_{9,n} \cos\left(\frac{n\pi a}{L} t\right) + c_{10,n} \sin\left(\frac{n\pi a}{L} t\right)\right)\left(c_{12,n} sin(\frac{n\pi}{L} x)\right)$ |

***Step 4:*** Superposition Principle to find $u_{total}(x,t) = X_1T_1 + X_2T_2 + X_3T_3$

$$u_{total}(x,t) = \underbrace{\sum_{n=1}^{\infty}\left(c_{9,n}\cos\left(\frac{n\pi a}{L}t\right) + c_{10,n}\sin\left(\frac{n\pi a}{L}t\right)\right)\left(c_{12,n}\sin\left(\frac{n\pi}{L}x\right)\right)}_{solution\ from\ Case\ 3}$$

where there are 3 remaining unknowns (i.e. $c_{9,n}, c_{10,n}, \&\ c_{12,n}$).

By expanding it, we can reduce the unknowns into 2 (i.e. $A_{3,n}, B_{3,n}$), as shown in displacement solution.

$$u_{total}(x,y) = \sum_{n=1}^{\infty} A_{3,n}\cos\left(\frac{n\pi a}{L}t\right)\left(\sin\left(\frac{n\pi}{L}x\right)\right) + \left(B_{3,n}\sin\left(\frac{n\pi a}{L}t\right)\left(\sin\left(\frac{n\pi}{L}x\right)\right)\right)$$

*Differentiate the displacement solution wrt t, then we obtain the velocity solution.*

$$\frac{\partial u(x,t)}{\partial t} = \sum_{n=1}^{\infty}\left(-A_{3,n}\frac{n\pi a}{L}\sin\left(\frac{n\pi a}{L}t\right)\left(\sin\left(\frac{n\pi}{L}x\right)\right)\right) + \sum_{n=1}^{\infty}\left(B_{3,n}\frac{n\pi a}{L}\cos\left(\frac{n\pi a}{L}t\right)\left(\sin\left(\frac{n\pi}{L}x\right)\right)\right)$$

***Step 5:*** Continue to apply the remaining IC & Fourier series expansion.

**IC #1:** $u(x,\ 0) = f(x)$ for $0 < x < L$

$u_{total}(x,0) = \sum_{n=1}^{\infty}\left(A_{3,n}\left(\sin\left(\frac{n\pi}{L}x\right)\right)\right) = f(x)$

Recall Half-range Fourier Sine Series Expansion:

> $f(x) = \sum_{n=1}^{\infty}(b_n \sin n\omega x)$
> where $b_n = \frac{2}{L}\int_0^{\tau} f(x)\sin n\omega x\, dx$
>
> Precaution: $L$ in the formula indicates the half period, i.e. $L = \frac{p}{2} = \frac{\pi}{\omega}$. Do not mix it with the length of the 1D string, which is using the same symbol, $L$ as well.
>
> Note that for (i) Half-range expansion: Finite interval, $\tau = half\ period, L$
> (ii) Full-range expansion: Finite interval, $\tau = full\ period, 2L$

We notice $A_{3,n} = b_n = \frac{2}{L}\int_0^{\tau} f(x)\sin n\omega x\, dx$ ,

where $\omega = \frac{\pi}{L}$ &

From $0 < x < L, \tau =\ length, L$. For half-range expansion, $\tau =\ half\ period, L$. Thus, in this case it happens to have finite interval, $\tau = half\ period, L = length, L$ in this special case.

Precaution: Note that it would be different for full-range expansion case.

$\rightarrow A_{3,n} = \frac{2}{L}\int_0^L f(x)\, sin\, n\frac{\pi}{L}x\, dx$

***Step 5:*** Continue to apply the remaining IC & Fourier series expansion.

**IC #2:** $u_t(x,\ 0) = g(x)$ for $0 < x < L$

$\frac{\partial u(x,0)}{\partial t} = \sum_{n=1}^{\infty}\left(B_{3,n}\frac{n\pi a}{L}\left(sin(\frac{n\pi}{L}x)\right)\right) = g(x)$

Recall Half-range Fourier Sine Series Expansion:

| |
|---|
| $g(x) = \sum_{n=1}^{\infty}(b_n\, sin\, n\omega x)$<br>*where* $b_n = \frac{2}{L}\int_0^{\tau} g(x)\, sin\, n\omega x\, dx$ |

We notice $B_{3,n}\frac{n\pi a}{L} = b_n = \frac{2}{L}\int_0^{\tau} f(x)\, sin\, n\omega x\, dx$ ,

where $\omega = \frac{\pi}{L}$ ;

From $0 < x < L, \tau = \ length, L$. For half-range expansion, $\tau = \ half\ period, L$. Thus, in this case it happens to have finite interval, $\tau = half\ period, L = length, L$ in this special case.

$\rightarrow B_{3,n}\frac{n\pi a}{L} = \frac{2}{L}\int_0^L g(x)\, sin\, n\frac{\pi}{L}x\, dx$

$\rightarrow B_{3,n} = \frac{2}{n\pi a}\int_0^L g(x)\, sin\, n\frac{\pi}{L}x\, dx$

Thus, we have solved all the unknowns and obtain the particular PDE solution:

$$\therefore u_{total}(x,t) = \sum_{n=1}^{\infty} A_{3,n}\cos\left(\frac{n\pi a}{L}t\right)\left(sin(\frac{n\pi}{L}x)\right) + \left(B_{3,n}sin(\frac{n\pi a}{L}t)\left(sin(\frac{n\pi}{L}x)\right)\right)$$

$$u_{total}(x,t) = \sum_{n=1}^{\infty}\frac{2}{L}\int_0^L f(x)\, sin\, n\frac{\pi}{L}x\, dx\cos\left(\frac{n\pi a}{L}t\right)\left(sin(\frac{n\pi}{L}x)\right) + \left(\frac{2}{n\pi a}\int_0^L g(x)\, sin\, n\frac{\pi}{L}x\, dx\, sin(\frac{n\pi a}{L}t)\left(sin(\frac{n\pi}{L}x)\right)\right)$$

Example: Let the initial displacement, $f(x) = x(L - x)$ , initial velocity, $g(x) = 0$, dimension, $length, L = 1$, PDE coefficient, $a = 1$ for the previous problem.

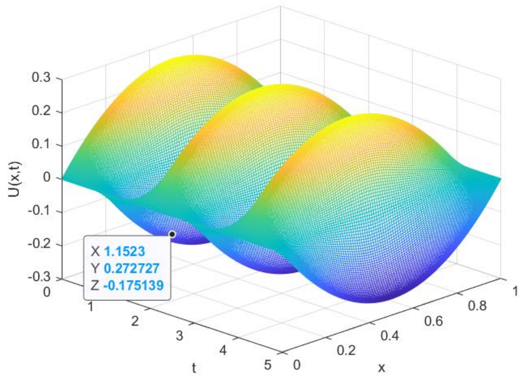
| | |
|---|---|
| $A_{3,n} = \frac{2}{L}\int_0^L f(x)\, sin\, n\frac{\pi}{L}x\, dx$ | $B_{3,n} = \frac{2}{n\pi a}\int_0^L g(x)\, sin\, n\frac{\pi}{L}x\, dx$ |

| | |
|---|---|
| $$= \frac{2}{1}\int_0^1 x(1-x)\, sin\, n\frac{\pi}{1} x\, dx$$ $$= 2\left[\int_0^1 x\, sin\, n\pi x\, dx - \int_0^1 x^2\, sin\, n\pi x\, dx\right]$$ $$= 2\left[\frac{\sin n\pi - n\pi cos n\pi}{n^2\pi^2} - \frac{2n\pi sin n\pi + (2-n^2\pi^2)cos n\pi - 2}{n^3\pi^3}\right]$$ $$= \left[-\frac{2n\pi sin n\pi + 4cos n\pi - 4}{n^3\pi^3}\right]$$ | $$= \frac{2}{n\pi(1)}\int_0^1 (0)\, sin\, n\frac{\pi}{L} x\, dx$$ $$= 0$$ |

$$\therefore u_{total}(x,t) = \sum_{n=1}^{\infty} A_{3,\mathrm{n}}\, cos\left(\frac{n\pi a}{L}t\right)\left(sin(\frac{n\pi}{L}x)\right) + \left(B_{3,n} sin(\frac{n\pi a}{L}t)\left(sin(\frac{n\pi}{L}x)\right)\right)$$
$$= \sum_{n=1}^{\infty} -\frac{2n\pi sin n\pi + 4cos n\pi - 4}{n^3\pi^3} cos(n\pi t)\left(sin(n\pi x)\right)$$

We can use the PDE solution to estimate the vibration at any point on the string. Example: The vibration results at $100 \times 500$ points of the $(x,t)$ locations for a duration of 5s have been plotted below:

| 3D plot of $u_{total}(x,t)$ wrt $x$- & $t$- axes | 2D plot |
|---|---|




-Due to boundary conditions on both sides of the 1D string and the initial displacement of the rod, the vertical displacement of the string changes over time.

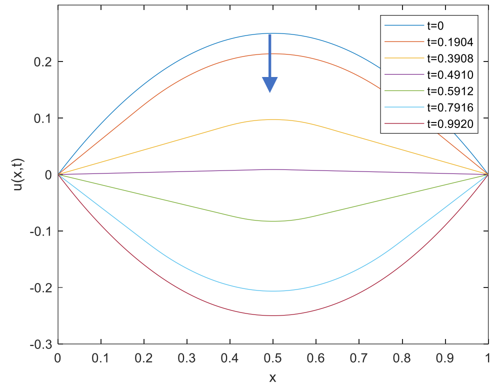
For example, $u_{total}(0.2727, 1.1523) \approx -0.1751$. Note that 20 terms are used for plotting the graphs. For higher accuracy, more terms & more grid can be included but computational time will be increased.

Try to verify the answer:

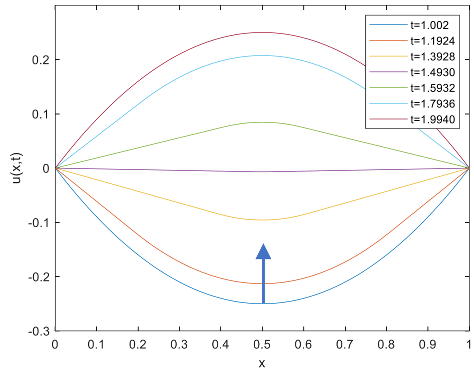
$$u_{total}(0.2727, 1.1523)$$

$$\approx \sum_{n=1}^{20}\left[-\frac{2n\pi sin n\pi + 4cos n\pi - 4}{n^3\pi^3} \cdot cos(1.1523n\pi)\left(sin(0.2727n\pi)\right)\right]$$

From $t = 0$ to $t = 0.992$




From $t = 1.002$ to $t = 1.994$




Note that the transverse vibration solution, $u_{total}(x,t)$ due to the initial displacement does not diminish over time, this is because the original PDE equation is excluding the damping component for an ideal case with no energy loss.

$$\text{Wave equation without damping component: } a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

To represent the actual system with friction/ energy loss, damping component, $k$ can be included as such

$$\text{Wave equation with damping component: } a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + k\frac{\partial u}{\partial t}$$

Same separation of variable method can be used to solve the damped case, thus the steps are excluded for brevity.

## APPENDIX 12.1 SOLVE THE PDE LIKE ODE – EXTRA INFO

We can solve the PDE like the ODE when there is only one-independent-variable derivative in the equation. For example:

$$\frac{\partial^2}{\partial t^2}\{u(x,t)\} - u(x,t) = 0$$

$$\frac{\partial^2}{\partial x^2}\{u(x,t)\} - u(x,t) = 0$$

$$\frac{\partial^2}{\partial x^2}\{u(x,t)\} + \frac{\partial}{\partial x}\{u(x,t)\} - u(x,t) = 0$$

There are similarity and differences between the ODE and PDE. For example:

**Case #1:** 2 Distinct Real Roots (Let dependent variable $= u$ ; independent variables $= x, t$ )

| **Solution for linear homogeneous ODE** | **Solution for linear homogeneous PDE** |
|---|---|
| Solve $\frac{d^2}{dt^2}\{u(t)\} - u(t) = 0$<br>$Let\, u(t) = e^{rt}$<br>$r^2 e^{rt} - e^{rt} = 0$<br>$(r^2 - 1)e^{rt} = 0$<br>The solution $e^{rt} \neq 0$<br>Hence, Characteristic equation: $(r^2 - 1) = 0$<br>$r^2 = 1$<br>$r = \pm 1$<br>We have 2 independent solutions, i.e. $e^t, e^{-t}$<br>Using linear superposition:<br>$\therefore u(t) = c_1 e^{-t} + c_2 e^t$<br>Boundary conditions: $u(0) = 1, u(1) = 0$<br>$\therefore u(t) = 1.157e^{-t} - 0.157e^t$ | Solve $\frac{\partial^2}{\partial t^2}\{u(x,t)\} - u(x,t) = 0$<br>$Let\, u(x,t) = e^{rt}$<br>$r^2 e^{rt} - e^{rt} = 0$<br>$(r^2 - 1)e^{rt} = 0$<br>The solution $e^{rt} \neq 0$<br>Hence, Characteristic equation: $(r^2 - 1) = 0$<br>$r^2 = 1$<br>$r = \pm 1$<br>We have 2 independent solutions, i.e. $e^t, e^{-t}$<br>Using linear superposition:<br>$\therefore u(x,t) = c_1(x)e^{-t} + c_2(x)e^t$<br>Boundary conditions: $u(x,0) = x, u(x,1) = 0$<br>$\therefore u(x,t) = (1.157x)e^{-t} - (0.157x)e^t$ |

**Note:** ODE has arbitrary constant (e.g. $c_1$) while PDE has arbitrary function (e.g. $c_1$(x))

**Case #2:** 2 Distinct Complex Roots (Let dependent variable $= u$ ; independent variables $= x, t$ )

| Solution for linear homogeneous ODE | Solution for linear homogeneous PDE |
|---|---|
| Solve $\frac{d^2}{dt^2}\{u(t)\} + u(t) = 0$<br>$Let\ u(t) = e^{rt}$<br>Hence, Characteristic equation: $(r^2 + 1) = 0$<br>$r^2 = -1$<br>$r = \pm\sqrt{1} = \pm i$<br>We have 2 independent solutions, i.e. $e^{it}, e^{-it}$<br>Using linear superposition:<br>$\therefore u(t) = c_1 e^{-it} + c_2 e^{it}$<br>$= A_1 cost + A_2 sint$ | Solve $\frac{\partial^2}{\partial t^2}\{u(x,t)\} + u(x,t) = 0$<br>$Let\ u(x,t) = e^{rt}$<br>Hence, Characteristic equation: $(r^2 + 1) = 0$<br>$r^2 = -1$<br>$r = \pm\sqrt{1} = \pm i$<br>We have 2 independent solutions, i.e. $e^{it}, e^{-it}$<br>Using linear superposition:<br>$\therefore u(x,t) = c_1(x) e^{-it} + c_2(x) e^{it}$<br>$= A_1(x) cost + A_2(x) sint$ |

**Note:** ODE has arbitrary constant (e.g. $c_1$) while PDE has arbitrary function (e.g. $c_1(x)$)

**Case #3:** 2 Identical Roots (Let dependent variable $= u$ ; independent variables $= x, t$ )

| Solution for linear homogeneous ODE | Solution for linear homogeneous PDE |
|---|---|
| Solve $\frac{d^2}{dt^2}\{u(t)\} + 2\frac{d}{dt}\{u(t)\} + u(t) = 0$<br>$Let\ u(t) = e^{rt}$<br>Characteristic equation: $(r^2 + 2r + 1) = 0$<br>$(r+1)(r+1) = 0$<br>$r = -1$<br>We have 2 dependent solutions, i.e. $e^{-t}, e^{-t}$<br>Treatment: Multiply its independent variable<br>New solutions: $e^{-t}, te^{-t}$<br>Using linear superposition:<br>$\therefore u(t) = c_1 e^{-t} + c_2 te^{-t}$ | Solve $\frac{\partial^2}{\partial t^2}\{u(x,t)\} + 2\frac{\partial}{\partial t}\{u(x,t)\} + u(x,t) = 0$<br>$Let\ u(x,t) = e^{rt}$<br>Characteristic equation: $(r^2 + 2r + 1) = 0$<br>$(r+1)(r+1) = 0$<br>$r = -1$<br>We have 2 dependent solutions, i.e. $e^{-t}, e^{-t}$<br>Treatment: Multiply its independent variable<br>New solutions: $e^{-t}, te^{-t}$<br>Using linear superposition:<br>$\therefore u(x,t) = c_1(x) e^{-t} + c_2(x) te^{-t}$ |

**Note:** ODE has arbitrary constant (e.g. $c_1$) while PDE has arbitrary function (e.g. $c_1(x)$)

More examples:

Solve $u_{xx} - u = 0$, where $u = u(x, y)$

Solution:

Since $u = u(x, y)$
Dependent variable: $u$
Independent variable: $x, y$

One-independent-variable derivative, i.e. $x$ –derivative, where $x$ as the variable while $y$ as the constant, thus we can solve the PDE like ODE.

$u_{xx} = \frac{\partial^2}{\partial x^2}\{u(x, y)\}$
$u_{xx} - u = \frac{\partial^2}{\partial x^2}\{u(x, y)\} - u(x, y) = 0$

Similar to ODE, $u''(x) - u(x) = 0$ where $u(x) = e^{rx}$
Characteristic equation, $r^2 - 1 = 0$
2 real roots: $r_1 = 1$ , $r_2 = -1$
Solution of PDE: $u(x, y) = c_1(y)e^{x} + c_2(y)e^{-x}$ , where $c_1(y), c_2(y)$= arbitrary functions

Solve $u_{yy} - u = 0$, where $u = u(x, y)$

Solution:

Since $u = u(x, y)$
Dependent variable: $u$
Independent variable: $x, y$

One-independent-variable derivative, i.e. $y$ –derivative, where $y$ as the variable while $x$ as the constant, thus we can solve the PDE like ODE.

$u_{xx} = \frac{\partial^2}{\partial x^2}\{u(x, y)\}$
$u_{xx} - u = \frac{\partial^2}{\partial x^2}\{u(x, y)\} - u(x, y) = 0$

Similar to ODE, $u''(y) - u(y) = 0, where\, u(y) = e^{ry}$
Characteristic equation, $r^2 - 1 = 0$
2 real roots: $r_1 = 1$ , $r_2 = -1$
Solution of PDE: $u(x, y) = c_1(x)e^{x} + c_2(x)e^{-x}$ , where $c_1(x), c_2(x)$= arbitrary functions

Note that this approach can't solve the PDE problems if there are two-independent-variable derivative.

For example:

$$\frac{\partial^2}{\partial x \partial y}\{u(x, t)\} + \frac{\partial}{\partial x}\{u(x, t)\} - u(x, t) = 0$$

$$\frac{\partial^2}{\partial x^2}\{u(x, t)\} + \frac{\partial}{\partial y}\{u(x, t)\} - u(x, t) = 0$$

## APPENDIX 12.2 SOLVE THE PDE BY DIRECT INTEGRATION– EXTRA INFO

We can solve the PDE by direct integration when there is only one derivative component in the equation. For example:

$$\frac{\partial^2}{\partial t^2}\{u(x,t)\} = 5xe^{-10t}$$

$$\frac{\partial}{\partial t}\{u(x,t)\} = 5xe^{-10t}$$

$$\frac{\partial^2}{\partial t \partial x}\{u(x,t)\} = 5xe^{-10t}$$

- Using Direct integration on ODE vs PDE

| Integration in ODE (Arbitrary Constants) | Integration in PDE (Arbitrary Functions) |
|---|---|
| Solve $\frac{d^2}{dt^2}\{u(t)\} = 0$<br><br>Integrate both sides,<br><br>$\int \frac{d^2}{dt^2}\{u(t)\}dt = \int 0dt$<br><br>$\frac{d}{dt}\{u(t)\} = 0t + c_1$<br><br>Integrate both sides again,<br><br>$\int \frac{d}{dt}\{u(t)\}dt = \int c_1 dt$<br><br>$\therefore u(t) = c_1 t + c_2$<br><br>Where $c_1$ and $c_2$ are 2 arbitrary constants. These constants can be solved if 2 initial conditions or boundary conditions are provided.<br><br>$\boldsymbol{Note}$: $n^{\text{th}}$ order ODE will have $n$ constants to be solved. (e.g. $2^{\text{nd}}$ order ODE have 2 arbitrary constants) | Solve $\frac{\partial^2}{\partial t^2}\{u(x,t)\} = 0$<br><br>Integrate both sides,<br><br>$\int \frac{\partial^2}{\partial t^2}\{u(x,t)\}dt = \int 0dt$<br><br>$\frac{\partial}{\partial t}\{u(x,t)\} = 0t + c_1(x)$<br><br>Integrate both sides again,<br><br>$\int \frac{\partial}{\partial t}\{u(x,t)\}dt = \int c_1(x)dt$<br><br>$\therefore u(x,t) = c_1(x)t + c_2(x)$<br><br>Where $c_1(x)$and $c_2(x)$are 2 arbitrary functions of variable $x$. These functions can be solved if the initial conditions or boundary conditions are provided.<br><br>$\boldsymbol{Note}$: $n^{\text{th}}$ order PDE may need more than $n$ arbitrary functions to be solved |

- More examples:

Solve $\frac{\partial^2}{\partial x \partial y}\{u(x,y)\} = 0$

| **Solution for linear homogeneous PDE** |
|---|
| Integrate both sides with respect to variable $x$,<br>$\int \frac{\partial^2}{\partial x \partial y}\{u(x,y)\}dx = \int 0dx$<br>$\frac{\partial}{\partial y}\{u(x,y)\} = 0x + c_1(y)$ |

Integrate both sides with respect to variable $y$,

$\int \frac{\partial}{\partial y}\{u(x,y)\}dy = \int c_1(y)dy$

$\therefore u(x,y) = \int c_1(y)dy$

where $c_1(y)$ is the arbitrary function of variable $y$.

Solve $u_{xx} = 6xe^{-t}$ where $u_{xx} = \frac{\partial^2}{\partial x^2}\{u(x,t)\}$ ; BC: $u(0,t) = t\ and\ u_x(0,t) = e^{-t}$

Solution:

- Dependent variable: $u$
- Independent variable: $x, t$

$$u_{xx} = \frac{\partial^2}{\partial x^2}\{u(x,t)\} = 6xe^{-t}$$

Note: One derivative component $\frac{\partial^2}{\partial x^2}$ and thus we can use direct integration

- Integrate the PDE with respect to variable $x$ (Hence, variable $t$ is constant)

$$\int \frac{\partial^2}{\partial x^2}\{u(x,t)\}dx = \int 6xe^{-t}dx$$

$$\frac{\partial}{\partial x}\{u(x,t)\} = \underbrace{6e^{-t}}_{\substack{treated\ as\ constant \\ when\ we\ integrated \\ wrt\ the\ variable\ x}} \int xdx = 6e^{-t}\frac{x^2}{2} + c_1(t)$$

- Integrate the PDE with respect to variable $x$ (Hence, variable $t$ is constant)

$$\int \frac{\partial}{\partial x}\{u(x,t)\}dx = \int 3e^{-t}x^2 + c_1(t)\ dx$$

**General PDE solution:** $u(x,t) = e^{-t}x^3 + xc_1(t) + c_2(t)$ ,

where the unknown arbitrary functions are $c_1(t)$ & $c_2(t)$.

Next, we continue to apply the boundary condition to solve the particular PDE solution.

$u(0,t) = t$

**For $x = 0$:** $u(x,t) = e^{-t}(0) + (0)c_1(t) + c_2(t) = t$

$\therefore c_2(t) = t$

$u_x(x,t) = \frac{\partial}{\partial x}[e^{-t}x^3 + xc_1(t) + c_2(t)] = 3e^{-t}x^2 + c_1(t)$

$u_x(0,t) = e^{-t}$

**For $x = 0$:** $u_x(x,t) = 3e^{-t}(0) + c_1(t) = e^{-t}$

$\therefore c_1(t) = e^{-t}$

**Particular PDE solution:** $u(x,t) = e^{-t}x^3 + xe^{-t} + t$

Solve $u_{xy} = sinxcosy$ where the boundary conditions are given:

When $y = \frac{\pi}{2}$, $u_x = 2x$

When $x = \pi$, $u = 2siny$

Solution:

- ***Dependent variable:*** $u$
- ***Independent variable:*** $x$ & $y$

$u_{xy} = \frac{\partial^2}{\partial x \partial y}\{u(x,y)\} = sinxcosy$

Note: One derivative component $\frac{\partial^2}{\partial x \partial y}$ and thus we can use direct integration

- Integrate the PDE with respect to variable $y$ (Hence, variable $x$ is constant)

  $\int \frac{\partial^2}{\partial x \partial y}\{u(x,y)\}dy = \int sinxcosydy$

  $\frac{\partial}{\partial x}\{u(x,y)\} = sinx \int cosydy = sinxsiny + c_1(x)$

- Integrate the PDE with respect to variable $x$ (Hence, variable $y$ is constant)

  $\int \frac{\partial}{\partial x}\{u(x,y)\}dx = \int sinxsiny + c_1(x)dx$

  **General PDE solution:** $u(x,y) = -cosxsiny + \int c_1(x)dx + c_2(y)$

  where the unknown arbitrary functions are $c_1(x)$ & $c_2(y)$.

Next, we continue to apply the boundary condition to solve the particular PDE solution.

$u(\pi, y) = 2siny$

**For $x = \pi$:** $u(x,y) = -cos\pi siny + \int c_1(x)dx + c_2(y) = 2siny$

$\int c_1(x)dx + c_2(y) = siny$

$\therefore c_2(y) = siny - \int c_1(x)dx$ (Note: $c_2(y)$ has unknown $c_1(x)$ to be solved)

$u_x(x,y) = \frac{\partial}{\partial x}[-cosxsiny + \int c_1(x)dx + c_2(y)] = sinxsiny + c_1(x)$

$u_x\left(x, \frac{\pi}{2}\right) = 2x$

**For $y = \frac{\pi}{2}$:** $u_x(x,y) = sinxsin\frac{\pi}{2} + c_1(x) = 2x$

$\therefore c_1(x) = 2x - sinx$

Note: $c_1(x)$ is expressed in the variable $x$ only

Substitute $c_1(x)$ into $c_2(y)$ equation where $u(\pi, y) = 2siny$

$$\begin{aligned} c_2(y) &= siny - \int 2x - sinxdx \\ &= siny - (x^2 + cosx) \\ &= siny - (\pi^2 + cos\pi) \\ &= siny + 1 - \pi^2 \end{aligned}$$

Note: $c_2(y)$ is expressed in the variable $y$ only

**Particular PDE solution:** $$\begin{aligned} u(x,y) &= -cosxsiny + \int 2x - sinxdx + siny + 1 - \pi^2 \\ &= -cosxsiny + x^2 + cosx + siny + 1 - \pi^2 \end{aligned}$$

## APPENDIX 12.3 SOLVE THE PDE BY REDUCTION OF ORDER METHOD– EXTRA INFO

We can solve the PDE by reduction of order method when the order can be reduced by proper substitution.

For example:

$$\frac{\partial^2}{\partial x^2}\{u(x,t)\} + \frac{\partial}{\partial x}\{u(x,t)\} = 0$$

Order can be reduced by let $p(x,t) = \frac{\partial}{\partial x}\{u(x,t)\}$

$$\rightarrow \frac{\partial}{\partial x}\{p(x,t)\} + p(x,t) = 0$$

$$\frac{\partial^2}{\partial x \partial y}\{u(x,y)\} + \frac{\partial}{\partial x}\{u(x,y)\} = 0$$

Order can be reduced by let $g(x,y) = \frac{\partial}{\partial x}\{u(x,y)\}$

$$\rightarrow \frac{\partial}{\partial y}\{g(x,y)\} + g(x,y) = 0$$

Hence, we can solve the problem by using the integration, solve PDE like ode approach, etc.

For example, repeating the problem in Appendix 12.2:

Solve $u_{xx} = 6xe^{-t}$ where $u_{xx} = \frac{\partial^2}{\partial x^2}\{u(x,t)\}$ ; BC: $u(0,t) = t \; and \; u_x(0,t) = e^{-t}$

Order can be reduced by let $p(x,t) = \frac{\partial}{\partial x}\{u(x,t)\}$

$u_{xx} = \frac{\partial^2}{\partial x^2}\{u(x,t)\} = 6xe^{-t} = \frac{\partial}{\partial x}\{p(x,t)\}$

- *Integrate the PDE with respect to variable x (Hence, variable t is constant)*

$\int \frac{\partial}{\partial x}\{p(x,t)\}dx = \int 6xe^{-t}dx$

$p(x,t) = 6e^{-t}\int x dx = 6e^{-t}\frac{x^2}{2} + c_1(t)$

- *Back substitution the $p(x,t) = \frac{\partial}{\partial x}\{u(x,t)\}$. Hence, Integrate the PDE with respect to variable x (Note: variable t is constant in this case)*

  $\int \frac{\partial}{\partial x}\{u(x,t)\}dx = \int 3e^{-t}x^2 + c_1(t)\, dx$

  $\therefore u(x,t) = e^{-t}x^3 + xc_1(t) + c_2(t)$