PY501 - Mathematical Physics

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1 Calculus of Variations

References: Stone & Goldbart (SG) Chapter 1; Byron & Fuller (BF) Chapter 2; Arfken, Weber & Harris (AWH) Chapter 22.

Finding where the minimum or maximum value of some quantity occurs is an extremely common task. For example, you might want to know where the highest point on a map is, or when you had the highest heartrate throughout the day. Mathematically, we have a function $f(x)$, and we want to find the value of x which maximizes or minimizes $f(x)$. To do so, for a differentiable function f, we simply take the derivative and set it to zero

$$f'(x) = 0 \tag{1.1}$$

and solve for x to find the stationary points of the function.

Often though, we run into problems where we want to find the *function* at which the minimum or maximum value of some function occurs. Some examples include:

1. What is the shortest path to take between points A and B?
2. What closed curve of fixed length encloses the maximum possible area?
3. What form does a hanging heavy chain of fixed length take, so as to minimize its potential energy?

To answer these questions mathematically, we need an object called a **functional** $J[y]$, which maps smooth[1] functions y (e.g. a path, a curve) to a real number (e.g. a distance, an area). This is just another map, like a function is. But now, we want to develop the tools required to define a **functional derivative** such that setting

$$\frac{\delta J}{\delta y(x)} = 0 \tag{1.2}$$

will allow us to find a function $y(x)$ (e.g. a path, a curve) that maximizes $J[y]$ (e.g. a distance, an area).

[1] "Smooth" means that all derivatives of the function exists. We won't ever be interested in subtleties involving continuity and differentiability in this course.

1.1 Functionals

What does a functional look like? For our purposes, we will be dealing with functionals that have the following form:

$$J[y] = \int_{x_1}^{x_2} dx\, f(x, y, y', y'', \cdots, y^{(n)})\,, \tag{1.3}$$

where f is a function of the real numbers x, y, $y' \cdots$, *independently*.[2] We call these kinds of functionals **local** in x. As you can see, J takes in a function y, performs an integral, and returns a real number, which is exactly what a functional should do.

[2] This is a cause of endless confusion, so pay attention! From the perspective of f, y and y' are *independent variables*.

1.1.1 The functional derivative

Let us work out the functional derivative for the case where

$$J[y] = \int_{x_1}^{x_2} dx\, f(x, y, y')\,. \tag{1.4}$$

To do this, suppose we make an infinitesimal shift $y(x) \to y(x) + \varepsilon\eta(x)$, where ε is an infinitesimally small constant,[3] and $\eta(x)$ is some arbitrary function. Then

[3] i.e. with ε^2 and higher powers of ε all being zero.

the shift in J is

$$\delta J = J[y+\varepsilon\eta] - J[y] = \int_{x_1}^{x_2} dx\ \left[f(x, y+\varepsilon\eta, y'+\varepsilon\eta') - f(x,y,y')\right]\ . \quad (1.5)$$

Since ε is an infinitesimal quantity, we can perform a Taylor expansion up to first order about y and y' to find

$$\delta J = \int_{x_1}^{x_2} dx\ \left[\varepsilon\eta\frac{\partial f}{\partial y} + \varepsilon\eta'\frac{\partial f}{\partial y'}\right]\ . \quad (1.6)$$

To make further progress, we integrate the second term by parts, giving

$$\delta J = \left[\varepsilon\eta\frac{\partial f}{\partial y'}\right]_{x_1}^{x_2} + \int_{x_1}^{x_2} dx\ \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\varepsilon\eta\,. \quad (1.7)$$

We are frequently—but not always!—concerned with finding functions of y with fixed endpoints;[4] in that case, $\eta(x_1) = \eta(x_2) = 0$, and the boundary terms in the first term on the right vanishes, leaving

$$\delta J = \int_{x_1}^{x_2} dx\ \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\varepsilon\eta\,. \quad (1.8)$$

This can be written suggestively as

$$\delta J = \int_{x_1}^{x_2} dx\,\delta y(x)\left(\frac{\delta J}{\delta y(x)}\right)\ , \quad (1.9)$$

where $\delta y(x) \equiv \varepsilon\eta(x)$, and

$$\frac{\delta J}{\delta y(x)} \equiv \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \quad (1.10)$$

is the **functional derivative** of J with respect to $y(x)$.

To aid our understanding, it can be helpful to think discretely. We can discretize x between x_1 and x_2 into N discrete steps, so that the function y takes up values $y_i = y(x_i)$, where $i = 1, 2\cdots, N$. A choice of the function y corresponds in this discrete picture to a choice of $\{y_i\}$, which is a single point in an N-dimensional space. Fig. 1 has a visualization of this. At every point in this N-dimensional space, we can assign a value to J. The small variation $\varepsilon\eta$ can likewise be discretized, so that $\delta y_i = \varepsilon\eta_i$, which can be thought of as a step in a particular direction in the same N-dimensional space. In this discrete picture,

$$\delta J = \sum_{i=1}^{N}\frac{\partial J}{\partial y_i}\delta y_i\,, \quad (1.11)$$

just as one might expect for a function J defined in the N-dimensional space indexed by y_i. In the continuous limit, we need to trade the summation over discrete i to an integral over the continuous label x, leading to Eq. (1.9).[5]

1.1.2 The Euler-Lagrange equation

Now, to find the **stationary points**—maxima, minima or saddle points—of J, we want to set $\delta J = 0$ for any arbitrary variation $\varepsilon\eta$, just like for a function g

[4] For example, if we are interested in finding the path with the short distance between two fixed points.

Figure 1: A discretized visualization of varying over functions. (**HL**: To be completed, but not difficult to imagine!)

[5] In this picture, a choice of the function y is a point in an uncountably infinite dimensional space, and $\epsilon\eta$ is a step in some arbitrary direction, and J is function that returns a real number at every point in this space.

on $\mathbb{R}^n$, we want δg to be zero for a step in any direction at a stationary point. Referring to Eq. (1.8), we require

$$\int_{x_1}^{x_2} dx \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] \varepsilon\eta = 0\,. \tag{1.12}$$

Since this applies for *any* $\eta(x)$, the term in the square brackets $[\cdots]$ must vanish.[6] we must have

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0} \tag{1.13}$$

for a stationary point for J. This is the famous **Euler-Lagrange equation**.

Through derivations similar to what we saw above, we can get generalized Euler-Lagrange equations for more complicated versions of J. If J depends on more than one function y_i, for example, the stationary points are given by

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right) = 0\,, \tag{1.14}$$

which is one equation for each variable y_i. If on the other hand, f depends on higher derivatives y'', y''' and so on, then the generalized Euler-Lagrange equation we get is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) - \frac{d^3}{dx^3}\left(\frac{\partial f}{\partial y'''}\right) + \cdots = 0\,. \tag{1.15}$$

[6] We can prove that this is true rigorously, given various conditions on η and f. This is often known as the **fundamental lemma of the calculus of variations**. For further discussion, see SG 1.2.2 and Wikipedia.

1.1.3 Applications

Time to apply what we've learnt! We'll apply the Euler-Lagrange equations to two examples.

1.1.3.1 Soap film supported by a pair of coaxial rings. Consider Fig. 2, where a pair of co-axial rings support a soap film. The energy associated with the configuration is directly proportional to the area, and hence the soap film tends to minimize this energy by minimizing its area, subject to the constraint that the soap film has to end on the rings at either end. The area of associated with a segment of the film of width dx is

$$dA = 2\pi y(x)\sqrt{dx^2 + dy^2} = 2\pi y(x)\sqrt{1 + y'^2}\,dx\,, \tag{1.16}$$

and so the functional that we want to minimize is

$$J[y] = \int_{x_1}^{x_2} dx\, f(y, y')\,, \quad f(y, y') \equiv y\sqrt{1 + y'^2}\,, \tag{1.17}$$

with the endpoint values fixed at $y(x_1)$ and $y(x_2)$. The minimum for this functional can therefore be found by applying the Euler-Lagrange equations. The partial derivatives that we need for the Euler-Lagrange equation are

$$\frac{\partial f}{\partial y} = \sqrt{1 + y'^2}\,, \quad \frac{\partial f}{\partial y'} = \frac{yy'}{\sqrt{1 + y'^2}}\,, \tag{1.18}$$

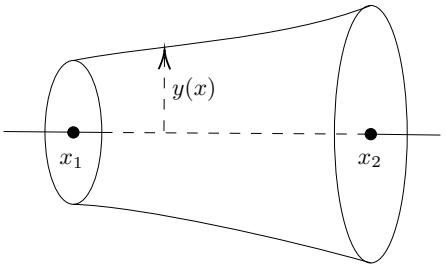

Figure 2: Soap film between two rings, centered at x_1 and x_2, with radii $y(x_1)$ and $y(x_2)$.

and so the Euler-Lagrange equation says that the minimal surface area profile $y(x)$ must satisfy

$$\sqrt{1+y'^2} - \frac{d}{dx}\left(\frac{yy'}{\sqrt{1+y'^2}}\right) = 0$$
$$\implies \sqrt{1+y'^2} - \frac{y'^2}{\sqrt{1+y'^2}} - \frac{yy''}{\sqrt{1+y'^2}} + \frac{yy'^2y''}{(1+y'^2)^{3/2}} = 0$$
$$\implies \frac{1}{\sqrt{1+y'^2}} - \frac{yy''}{(1+y'^2)^{3/2}} = 0\,. \qquad (1.19)$$

This differential equation looks difficult to solve, but fortunately there's a neat little trick to do so. Multiplying by y' on both sides gives

$$0 = \frac{y'}{\sqrt{1+y'^2}} - \frac{yy'y''}{(1+y'^2)^{3/2}} = \frac{d}{dx}\left(\frac{y}{\sqrt{1+y'^2}}\right)\,. \qquad (1.20)$$

We'll return to how we knew this trick would work later on. In the mean time, the solution is

$$\frac{y}{\sqrt{1+y'^2}} = \kappa \qquad (1.21)$$

for some constant κ. Rewriting this as

$$\frac{dy}{dx} = \sqrt{\frac{y^2}{\kappa^2} - 1} \implies \int \frac{dy}{\sqrt{y^2/\kappa^2 - 1}} = \int dx\,, \qquad (1.22)$$

we can integrate this first-order ordinary differential equation by substituting $y = \kappa\cosh t$ and $dy = \kappa\sinh t$ to find

$$\kappa\int dt = \int dx \implies \kappa t = x + C \implies y = \kappa\cosh\left(\frac{x+C}{\kappa}\right) \qquad (1.23)$$

for some constants κ and C. These can be determined by enforcing the two boundary conditions—the radii of the two rings, $y(x_1)$ and $y(x_2)$.

1.1.3.2 The brachistochrone The next problem we will consider is a famous one, posed by Johann Bernoulli in 1696. What shape should a wire with endpoints $(0,0)$ and (a,b) take, in order that a frictionless bead will slide from rest down the wire in the shortest possible time?

First, the total time T taken down a given path can be written as

$$T = \int_0^T dt = \int_0^L \frac{ds}{v}\,, \qquad (1.24)$$

where v is the speed of the bead, and s is the distance along the path, with a total length L. However, we can once again write $ds^2 = dx^2 + dy^2$ so that $ds = \sqrt{1+y'^2}\,dx$, and apply conservation of energy to find $v = \sqrt{2gy}$. Thus, we can define a functional T

$$T[y] = \int_0^a dx\,\sqrt{\frac{1+y'^2}{2gy}} \qquad (1.25)$$

that we want to minimize with respect to y, again with fixed end points. We can therefore apply the Euler-Lagrange equation, which gives after some algebra

$$yy'' + \frac{1}{2}(1+y'^2) = 0\,. \qquad (1.26)$$

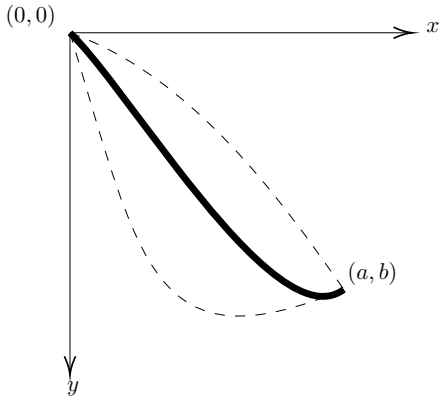

Figure 3: Possible shapes of a wire for a frictionless bead to travel from the origin $(0,0)$ to a point (a,b).

Once again, we can use the trick of multiplying by y' to find that

$$y'\left(yy'' + \frac{1}{2}(1+y'^2)\right) = \frac{1}{2}\frac{d}{dx}\left(y(1+y'^2)\right) = 0\,, \tag{1.27}$$

or

$$y(1+y'^2) = 2C \tag{1.28}$$

for some constant C. From this point, one can check that the following parametrization $(x(t), y(t))$ is indeed a solution to the differential equation above:

$$\begin{aligned} x &= C(\theta - \sin\theta) \\ y &= C(1-\cos\theta)\,, \end{aligned} \tag{1.29}$$

although it is surprisingly hard to pin down the details regarding this solution.[7] This parametric curve is known as the **cycloid**, which is the curve traced out by a fixed point on the rim of a wheel that is rolling without slipping along a flat surface.

[7] For example, is there a unique solution, and does the solution always occur with $\theta \in [0, 2\pi)$ for every point (a, b)? See for example Ref. [1] for more details.

1.1.4 First integral

In both applications discussed in Sec. 1.1.3, we were able to rephrase the differential equation to be solved as $dI/dx = 0$, implying that I is some constant associated with the problem. This quirk, which somewhat resembles the conservation of quantities like energy, is something we will revisit in much greater depth later: it is far deeper than just a mathematical coincidence. For now, let's just take a quick look at where it comes from. In both cases, the function inside the integral was $f(y, y')$, with no explicit dependence on x, implying that

$$\frac{df}{dx} = \cancelto{0}{\frac{\partial f}{\partial x}} + y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'}\,. \tag{1.30}$$

We define the **first integral** of the Euler-Lagrange equation as

$$I \equiv f - y'\frac{\partial f}{\partial y'}\,, \tag{1.31}$$

from which we can check that

$$\begin{aligned} \frac{dI}{dx} &= y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} - y''\frac{\partial f}{\partial y'} - y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \\ &= y'\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\,. \end{aligned} \tag{1.32}$$

Thus, $dI/dx = 0$ if the Euler-Lagrange equation is satisfied.

You can show that if f depends on more than one variable, so that we have a functional of the form

$$J[y_1, y_2, \cdots, y_n] = \int dx\, f(y_1, y_2, \cdots, y_n; y_1', y_2', \cdots, y_n')\,, \tag{1.33}$$

the first integral is of the form

$$I = f - \sum_i y_i'\frac{\partial f}{\partial y_i'}\,. \tag{1.34}$$

Note that there is only one first integral, even when there are multiple dependent variables y_i.

1.2 Lagrangian Mechanics

It turns out that classical mechanics can be reformulated as a problem of finding the stationary function of some functional. Given some system, we first define the **Lagrangian** function $L = T - V$, where T and V are the kinetic and potential energy functions of the system. We can make any choice of coordinates we would like to describe T and V; let's say we choose some set of **generalized coordinates** q with components q^i and time derivatives $\dot{q}^i$.[8]

The equations of motion governing the system between times t_i and t_f can then be obtained by finding the stationary function $q(t)$ of the **action functional**,

$$S[q] = \int_{t_i}^{t_f} dt\, L(t; q^i, \dot{q}^i)\,. \tag{1.35}$$

This is known as the **principle of least action**. It is no exaggeration to say that a lot of theoretical physics basically involves finding the appropriate action that describes the system of interest, once the principle of least action is applied.

(**End of Lecture**: Wednesday Sep 4 2024)

1.2.1 The central force problem

We'll now turn our attention to an important problem in mechanics—a particle of mass m moving in a potential $V(r)$ which depends only on the distance r from the origin, with the radial component of the force being $F_r = -\partial_r V$. We'll compute this two ways: the first using Newtonian mechanics, and the second using Lagrangian mechanics. The coordinate system we'll use is shown in Fig. 4.

In Newtonian mechanics, we would write down $-\partial_r V = m a_r$, and $-\partial_\theta V/r = 0 = m a_\theta$, where a_r and a_θ are the radial and tangential accelerations respectively. But to solve this equation, we need to find the acceleration in polar coordinates. To do so, we'll use a neat trick: take the particle to be traveling in the complex plane, with coordinate given by the complex number $z = re^{i\theta}$. Then

$$\begin{aligned} \dot{z} &= \dot{r}e^{i\theta} + i\dot{\theta}re^{i\theta} \\ \ddot{z} &= (\ddot{r} - r\dot{\theta}^2)e^{i\theta} + (2\dot{r}\dot{\theta} + r\ddot{\theta})(ie^{i\theta})\,. \end{aligned} \tag{1.36}$$

On the complex plane, the first term is a complex number that is represented by a vector that is parallel to $re^{i\theta}$, while the second term is represented by a vector that is perpendicular to $re^{i\theta}$. We can therefore conclude that the acceleration in polar coordinates is

$$a_r = \ddot{r} - r\dot{\theta}^2\,, \qquad a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta}\,. \tag{1.37}$$

Newton's laws therefore read

$$m(\ddot{r} - r\dot{\theta}^2) = -\partial_r V \tag{1.38}$$

$$m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0 \implies \frac{d}{dt}(mr^2\dot{\theta}) = 0\,. \tag{1.39}$$

You should recognize the second equation as expressing the conservation of angular momentum, with $l = mr^2\dot{\theta}$. Subtituting this expression into Eq. (1.38), we find

$$m\ddot{r} - \frac{l^2}{mr^3} = -\frac{\partial V}{\partial r}\,, \tag{1.40}$$

[8] This can be the usual x, y and z in 3D space, r, θ and ϕ in 3D spherical coordinates, or something even more abstract than these choices, it doesn't matter.

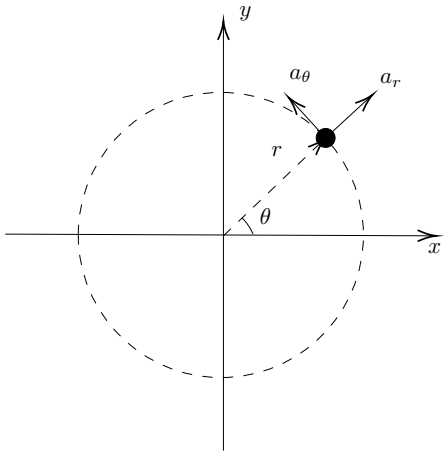

Figure 4: Coordinate system for the central force problem.

the equation of motion governing all central force problems.

Now that was a bit of hike, and again, a lot of issue was that we had to deal with vectors. Let's see how the Lagrangian approach works out. First, we write down the Lagrangian for the system,

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)\,. \tag{1.41}$$

This depends on two coordinates, but is independent of t, and therefore we can write down two Euler-Lagrange equations—one for each coordinate. These are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \implies m\ddot{r} - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0 \tag{1.42}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \implies \frac{d}{dt}(mr^2\dot{\theta}) = 0\,. \tag{1.43}$$

Notice that we obtain the conservation of angular momentum immediately from the second equation involving θ. Substituting once again $l = mr^2\dot{\theta}$, we get the same expression as before in Eq. (1.40).[9]

Let's also compute the first integral, defined for this problem as

$$I = L - \dot{r}\frac{\partial L}{\partial \dot{r}} - \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} = -\frac{1}{2}m\dot{r}^2 - \frac{1}{2}mr^2\dot{\theta}^2 - V(r) = -(T+V)\,, \tag{1.44}$$

which is simply (the negative of) the total energy. Thus, the fact that the first integral is constant is just a statement about conservation of energy in this problem.

[9] You may be tempted to plug l directly into the Lagrangian, which looks like it would remove one coordinate from the Lagrangian, simplifying matters significantly. *But this is pure folly*, and gets you the wrong answer. The coordinates and their derivatives are—each and every one of them—*independent variables of the Lagrangian*. The Lagrangian's only aim in life is to take in these coordinates, their derivatives, and the time coordinate—not knowing that they're related in any way—and spit out a number. The Lagrangian *knows absolutely nothing about trajectories involving these coordinates as a function of time*.

1.3 Variable Endpoints

So far, we've been dealing with situations where we want the endpoints to be fixed. Now let's see what happens when we relax that assumption. Consider the problem of constructing a railway between two ports, located across a strip of land with straight, parallel sides, illustrated in Fig. 5. Suppose that the cost of construction is proportional to the length of the track, but the cost of sea transport is negligible, and so that the ports can be wherever we want. We therefore want to minimize the total length given once again by

$$L[y] = \int_{x_1}^{x_2} dx\, \sqrt{1 + y'^2}\,. \tag{1.45}$$

This time however, we want to allow variations at the endpoints too. Considering a small perturbation, we see that

$$\delta L = \int_{x_1}^{x_2} dx\, \frac{y'}{\sqrt{1+y'^2}}\delta y' \tag{1.46}$$

By now you should be familiar with what to do next: integrate by parts! Don't forget, however, the boundary terms. This gives

$$\delta L = \frac{y'(x_2)}{\sqrt{1+y'(x_2)^2}}\delta y(x_2) - \frac{y'(x_1)}{\sqrt{1+y'(x_1)^2}}\delta y(x_1) - \int_{x_1}^{x_2} dx \left[\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right)\right]\delta y(x)\,. \tag{1.47}$$

The extremum for L is achieved when $\delta L = 0$. But since we have complete freedom to choose $y(x)$, including $y(x_2)$ and $y(x_1)$, the coefficients to each of

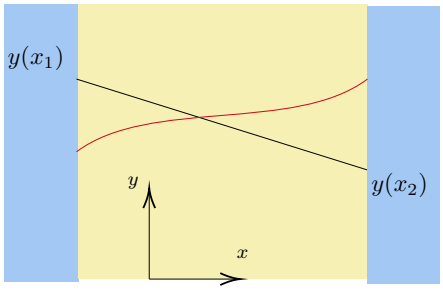

Figure 5: Railway across a strip of land with straight, parallel sides.

$\delta y(x_2)$, $\delta y(x_1)$ and δy inside the integral must be zero. Let's first consider the coefficient for $\delta y(x)$. This just says that

$$\frac{y'}{\sqrt{1+y'^2}} = C \implies y' = \text{constant}\,, \tag{1.48}$$

and so we'd better be building a straight track, as expected. In addition however, the other two terms now enforce

$$\frac{y'(x_2)}{\sqrt{1+y'(x_2)^2}} = \frac{y'(x_1)}{\sqrt{1+y'(x_1)^2}} = 0 \implies y'(x_1) = y'(x_2) = 0\,. \tag{1.49}$$

We therefore want to build the railway exactly perpendicular to the sea. All very reasonable!

What you might have noticed is that by allowing the endpoints to float, you ended up getting a boundary term that enforces some boundary conditions at the two endpoints. Boundary conditions obtained through variation are known as **natural boundary conditions**. We'll come back to more examples with natural boundary conditions later on.

1.4 Noether's Theorem

We now come to one of the most profound results in mathematical physics. We've seen in a few of the previous examples above that when we apply the principle of least action, we can sometimes obtain equations of the form $d(\cdots)/dt = 0$ when the Euler-Lagrange equation is satisfied. For example, for a Lagrangian of the form $L = L(q, \dot{q})$ with no explicit dependence on t, we know from Sec. 1.1.4 that the time derivative of the first integral is conserved:

$$\frac{d}{dt}\left(\dot{q}\frac{\partial L}{\partial \dot{q}} - L\right) = 0\,. \tag{1.50}$$

We also saw that in the central force problem in Sec. 1.2.1, we can show that the angular momentum is conserved, i.e.

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0\,, \tag{1.51}$$

and in that case, the Lagrangian had not explicit dependence on θ. Equations of this form are known as **conservation laws**, and you have, up to this point in your physics career, already encountered many of them. But you may now be seeing a pattern emerge: whenever there is no explicit dependence of L on some quantity, you get conservation laws. And indeed this is true! These are all manifestations of **Noether's theorem**,[10] which says that

> Any continuous symmetry of the action corresponds to a conservation law.

[10] This is actually Noether's *first* theorem; the second theorem is much more obscure, and we won't discuss it in this course.

Now, your first thought might be, it's a little strange that we have a theorem that's just written out in words. That's really because you can prove many different mathematical statements that are described by those words, with various levels of generality and formality. Because Noether's theorem is so important, I want to walk you through a proof of it at a relatively general level. Your second thought might be that there are a lot of terms in there that I haven't carefully defined. But I think in this case, it's best just to define things as we go along. The discussion that follows below is mostly based on Ref. [2].

So let's start with the classical action, written as

$$S[q^i] = \int_{t_i}^{t_f} dt\, L(t; q^i, \dot{q}^i)\,, \tag{1.52}$$

where I remind you that q is some generalized coordinate with components $q^i(t)$ and time derivatives $\dot{q}^i(t)$. We say that the action is **invariant up to a boundary term** under a transformation $q^i(t) \to q^i(t) + \varepsilon\eta^i(t)$, where ε is taken to be a small, time-independent quantity, if

$$\delta S \equiv S[q^i + \varepsilon\eta^i] - S[q^i] = \varepsilon \int_{t_i}^{t_f} dt\, \frac{dK}{dt}\,, \tag{1.53}$$

for all $q^i(t)$, and for some **boundary term** K.

If this is the first time you're seeing this, you might be wondering why even bother with K at all—after all, that's what the word *invariant* should mean. But in fact, a lot of extremely interesting models (in both particle physics and condensed matter physics) have actions which are invariant up to boundary terms, and we would like to apply Noether's theorem to them. An equivalent way to say the same thing is that the action is invariant up to a boundary term if the Lagrangian transforms by a **total derivative** dK/dt.

Any transformation which leaves the action invariant up to a boundary term, or transforms the Lagrangian by a total derivative, is called a **symmetry**.[11] The first question you should ask when you see a Lagrangian is always, "what are its symmetries?" This is the organizing principle behind all of the models that we study in physics. So let's check out some examples. Consider once again the central force problem,

$$S = \int_{t_i}^{t_f} dt\, \left[\frac{1}{2}m\dot{\vec{r}}^2 - V(r)\right]\,, \tag{1.54}$$

The action is invariant under rotations, i.e. perform the transformation $\vec{r} \to \mathsf{R}\vec{r}$, where R is constant, orthogonal matrix with $\mathsf{R}^\intercal = \mathsf{R}^{-1}$, even for very large rotations, since $\dot{\vec{r}}^2 \to (\mathsf{R}\dot{\vec{r}})^\intercal \mathsf{R}\dot{\vec{r}} = \dot{\vec{r}}^\intercal \mathsf{R}^\intercal \mathsf{R}\dot{\vec{r}} = \dot{\vec{r}}^2$, in addition to the fact that rotations preserve length, i.e. $|\mathsf{R}\vec{r}| = r$.

Let's consider another transformation to the action

$$\vec{r}(t) \to \vec{r}(t) - \varepsilon\dot{\vec{r}}(t)\,, \tag{1.55}$$

again with ε being small. Under this transformation, we find[12]

$$\begin{aligned} S[r^i] \to S[r^i - \varepsilon\dot{r}^i] &= \int dt\, \left[\frac{1}{2}m(\dot{\vec{r}} - \varepsilon\ddot{\vec{r}})^2 - V(\vec{r} - \varepsilon\dot{\vec{r}})\right] \\ &= \int dt\, \left[\frac{1}{2}m\dot{\vec{r}}^2 - \varepsilon m\dot{\vec{r}}\ddot{\vec{r}} - V(\vec{r}) + \varepsilon\dot{\vec{r}}^i\partial_i V\right] \\ &= S[r^i] - \varepsilon\int dt\, \frac{d}{dt}\left[\frac{1}{2}m\dot{\vec{r}}^2 - V(r)\right]\,. \end{aligned} \tag{1.56}$$

We can therefore conclude that the action is invariant under this transformation up to a boundary term, $K = L$. What is this strange transformation? Well, it really isn't that mysterious if you just think of it as relabeling time by the coordinate τ, which is just a constant shift from t, i.e. $\tau = t + \varepsilon$. With this transformation, $\vec{r}(t) = \vec{r}(\tau) - \varepsilon\dot{\vec{r}}(\tau)$, and it is easy to see that you get the same transformation as in Eq. (1.56).

Let's now consider an *arbitrary* transformation, one that isn't necessarily a symmetry. It also doesn't necessarily have to leave the endpoints fixed. Denoting this transformation as $q^i \to q^i + \varepsilon\zeta^i(t)$ and apply this to the classical action

[11] Sometimes you will hear people say that a transformation is only a symmetry if $K = 0$, and should be called a *quasisymmetry* otherwise. We won't make this distinction.

[12] For notation convenience, I'll drop the limits of the integral whenever nothing much ever happens to it.

Eq. (1.52). This gives (using index notation)

$$\begin{aligned}\delta S &= \int_{t_i}^{t_f} dt\, \frac{\partial L}{\partial q^i}\varepsilon\zeta^i(t) + \frac{\partial L}{\partial \dot{q}^i}\varepsilon\dot{\zeta}^i(t) \\ &= \int_{t_i}^{t_f} dt\, \frac{\partial L}{\partial q^i} + \frac{\partial L}{\partial \dot{q}^i}\varepsilon\zeta^i\Big|_{t_i}^{t_f} - \int_{t_i}^{t_f} dt\, \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right)\varepsilon\zeta^i \\ &= \int_{t_i}^{t_f} dt\, \left[\frac{\partial L}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right)\right]\varepsilon\zeta^i + \int_{t_i}^{t_f} dt\, \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\varepsilon\zeta^i\right), \end{aligned} \tag{1.57}$$

where we have kept the boundary term after performing integration by parts.

Now, let's choose the arbitrary transformation $\zeta^i(t)$ to be one that leaves the action invariant. Then we have the following relation, after dividing throughout by ε,

$$\int_{t_i}^{t_f} dt\, \frac{d}{dt}\left(K - \frac{\partial L}{\partial \dot{q}^i}\zeta^i\right) = \int_{t_i}^{t_f} dt\, \left[\frac{\partial L}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right)\right]\zeta^i\,. \tag{1.58}$$

In classical mechanics, we're often interested in the behavior of a system at arbitrary times, and indeed, it doesn't really matter what the precise values of t_i and t_f are. Because of that, we need the integrand themselves to be equal, i.e.

$$\frac{d}{dt}\left(K - \frac{\partial L}{\partial \dot{q}^i}\zeta^i\right) = \left[\frac{\partial L}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right)\right]\zeta^i\,. \tag{1.59}$$

This relation is known as the **Noether identity**, and is true for any ζ^i that is a symmetry. Note that at this point, we *have not even mentioned the Euler-Lagrange equation*. This relation is true for *arbitrary* $q^i(t)$, whether or not it corresponds to a the physical solution. When an equation is true before having enforced the Euler-Lagrange equation, we call in an **off-shell** relation, a term that you're bound to run into again.

Now let's take the relation **on-shell**, so that we are only considering $q^i(t)$ that satisfies the Euler-Lagrange equation. For such $q^i(t)$, the right-hand side of the previous equation vanishes, and we find

$$\frac{dQ}{dt} \overset{\text{E-L}}{=} 0\,, \qquad Q = K - \frac{\partial L}{\partial \dot{q}^i}\zeta^i\,. \tag{1.60}$$

This is our big result, **Noether's theorem**: given a symmetry of the action ζ^i, we can define a quantity called a **Noether charge** (or just "charge") that is conserved on-shell, i.e. for physical trajectories.

That was a lot of abstract discussion; let's now take a look at some examples.

(**End of Lecture**: Monday Sep 9 2024)

1.4.1 The central force problem revisited

Let's return once again to the to the central force problem, with

$$S[\vec{r}] = \int_{t_i}^{t_f} dt\, \left[\frac{1}{2}m\dot{\vec{r}}^2 - V(r)\right]. \tag{1.61}$$

We saw earlier that the action was invariant under rotations. You can convince yourself (see Fig. 6 that an infinitesimal rotation about some arbitrary axis with

unit vector $\hat{\alpha}$ can be written as $\vec{r} \to \vec{r} + \varepsilon\hat{\alpha} \times \vec{r}$. We also checked that the rotation leaves the action invariant with $K = 0$. Putting this altogether, we find that the Noether charge associated with rotational symmetry is (using index notation)

$$Q = -\frac{\partial L}{\partial \dot{r}^i}(\hat{\alpha} \times \vec{r})^i = -(m\dot{\vec{r}}_i)(\hat{\alpha} \times \vec{r})^i = \hat{\alpha} \cdot (\vec{r} \times \vec{p}) \,, \tag{1.62}$$

where $\vec{p} \equiv m\dot{\vec{r}}$ is the momentum. You should recognize immediately that $\vec{L} = \vec{r} \times \vec{p}$ is the **angular momentum**, the Noether charge corresponding to rotational symmetry, and since $\hat{\alpha}$ is arbitrary, $\vec{L}$ is conserved.

We also considered another transformation $\vec{r} \to \vec{r} - \varepsilon\dot{\vec{r}}$, which left the action invariant up to a boundary term, $K = L$, the Lagrangian. We also saw that this transformation was equivalent to time translation, $t \to t + \varepsilon$. Once again, the Noether charge associated with this symmetry is

$$Q = L - \frac{\partial L}{\partial \dot{r}^i}\dot{\vec{r}}^i = \frac{1}{2}m\dot{\vec{r}}^2 - V(r) - m\dot{\vec{r}}^2 = -\left[\frac{1}{2}m\dot{\vec{r}}^2 + V(r)\right] \,, \tag{1.63}$$

which is simply the statement that the Noether charge is the **total energy**, and is conserved. More generally, for any action that has time-translation as a symmetry, e.g. $L = L(q^i, \dot{q}^i)$, the total energy is

$$E = \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L \,, \tag{1.64}$$

and is conserved. You should recognize that E is nothing but the first integral that we described in Eq. (1.31).

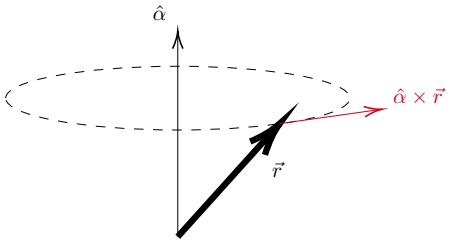

Figure 6: Infinitesimal rotations of $\vec{r}$ about axis $\hat{\alpha}$ are in the direction $\hat{\alpha} \times \vec{r}$.

1.4.2 Common symmetries

Before we move on from Noether's theorem, let's examine some common symmetries that you've probably already encountered in classical mechanics, and how they usually arise from Noether's theorem.

1.4.2.1 Spatial translation invariance implies momentum conservation.

If a Lagrangian is invariant under a small, constant, spatial translation ε of some coordinate q, i.e. $L(t; q + \varepsilon, \dot{q}) = L(t; q, \dot{q})$.[13] Noether's theorem (Eq. (1.60)) tells us that the associated charge is

$$Q = p \equiv \frac{\partial L}{\partial \dot{q}} \text{ and } \frac{dp}{dt} = 0 \,, \tag{1.65}$$

where p is the momentum conjugate to q. One simple scenario where momentum conservation holds is when the Lagrangian simply doesn't depend explicitly on q at all; in this case, q is known as a **cyclic coordinate**. This is what happens in systems with cylindrical symmetry, for example, and therefore have Lagrangians that do not depend on the azimuthal angle, leading to an associated conserved angular momentum.

[13] Since ε is constant, $\dot{q}$ remains unchanged.

1.4.2.2 Time translation invariance implies energy conservation.

Suppose we performed the translation $q(t) \to q(t + \varepsilon)$, where ε is once again small and constant. You can imagine this as taking a trajectory that previously started

at time t, and now starting it at $t-\varepsilon$ instead. Then the action transforms as

$$\begin{aligned} S[q] &\to \int_{t_i}^{t_f} dt\, L(t; q(t+\varepsilon), \dot{q}(t+\varepsilon)) \\ &= \int_{t_i+\varepsilon}^{t_f+\varepsilon} d\tau\, L(\tau-\varepsilon; q(\tau), \dot{q}(\tau)) \\ &= S[q] + \varepsilon\left[L(t_f, q(t_f), \dot{q}(t_f)) - L(t_i, q(t_i), \dot{q}(t_i))\right] - \varepsilon \int_{t_i}^{t_f} d\tau\, \frac{\partial L}{\partial \tau}\,. \end{aligned} \tag{1.66}$$

Here, the second term is a boundary term, but the third term might not be: we can therefore see that time translation is a symmetry of the action if the Lagrangian does not depend explicitly on t, with the conserved charge being energy, i.e.

$$\frac{\partial L}{\partial t} = 0 \implies \text{Energy is conserved.} \tag{1.67}$$

We have derived the expression for energy several times already, for example in Eq. (1.64): we do this by noting that $q(t+\varepsilon) = q(t) + \varepsilon\dot{q}(t)$, and then applying Noether's theorem.

1.5 Continuous Systems

So far, we have been looking at Lagrangians that have only a single particle. For multiple particles, we can similarly write down Lagrangians involving $\{q_i, \dot{q}_i\}$ for each particle. Frequently, however, we're interested in *fields*, like the electromagnetic field, and we want to be able to write down *field theories* by similarly writing down an action, extremizing it, and finding the equations of motion. Suppose we have a field with value $\varphi(x)$, where $x \equiv x^\mu$, $\mu = 0, 1, \cdots, d$ is the coordinate of a $(d+1)$-dimensional space (or spacetime). We can write down actions that look like

$$S[\varphi] = \int dt\, L = \int d^{d+1}x\, \mathcal{L}(x, \varphi, \partial_\mu \varphi)\,, \tag{1.68}$$

where $\mathcal{L}$ is the **Lagrangian density**,

$$L \equiv \int d^d x\, \mathcal{L}\,, \tag{1.69}$$

and

$$\partial_\mu \varphi \equiv \frac{\partial \varphi}{\partial x^\mu}\,. \tag{1.70}$$

We're going to follow a similar path to what we discussed in Sec. 1.1.1. We'll take a variation over the action, $\varphi(x) \to \varphi(x) + \varepsilon\eta(x)$, and so correspondingly $\partial_\mu\varphi(x) \to \partial_\mu\varphi(x) + \varepsilon\partial_\mu\eta(x)$, with no variation on the d-dimensional boundary of our $(d+1)$-dimensional space. We get

$$\delta S = \int d^{d+1}x \left[\frac{\partial \mathcal{L}}{\partial \varphi}\varepsilon\eta + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)}\varepsilon\partial_\mu\eta\right]\,. \tag{1.71}$$

At this point, we want to integrate by parts, which we can do using the **divergence theorem**,

$$\int_\Omega d^{n+1}x\,\partial_\mu f^\mu = \int_{\partial\Omega} dS\, n_\mu f^\mu\,, \tag{1.72}$$

where $\partial\Omega$ is the boundary of Ω, dS is an element of are on the boundary, and n_μ is the outward pointing normal. Much more on this later on in the course, but for now, we see that

$$\begin{aligned}\int_\Omega d^{n+1}x\,\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\epsilon\eta\right) &= \int d^{n+1}x\,\left(\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\varepsilon\eta + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\varepsilon\partial_\mu\eta\right)\\ &= \int_{\partial\Omega} dS\, n^\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right)\epsilon\eta\\ &= 0\,,\end{aligned} \tag{1.73}$$

since η does not vary on the boundary, allowing us to write

$$\delta S = \int d^{d+1}x\,\left[\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right]\varepsilon\eta\,. \tag{1.74}$$

Effectively, if we are only allowing for variations that are zero on the boundary, we can simply integrate by parts by switch the position of ∂_μ. Extremizing the action therefore means that

$$\frac{\delta S}{\delta\varphi} \equiv \frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} = 0\,, \tag{1.75}$$

which is the Euler-Lagrange equation in continuous form.

1.5.1 The vibrating string

Our first example of a continuous system is a string, shown in Fig. 7. The string has fixed ends, a mass per unit length of ρ, and is under tension T. If we assume only small displacements from equilibrium, the Lagrangian is

$$L = \int_0^L dx\,\left(\frac{1}{2}\rho\dot{y}^2 - \frac{1}{2}Ty'^2\right)\,, \tag{1.76}$$

where the dot denotes a partial derivative with respect to t, and the prime a partial derivative with respect to x.

The variation of the action is

$$\begin{aligned}\delta S &= \int dt\int_0^L dx\,\left(\rho\dot{y}\delta\dot{y} - Ty'\delta y'\right)\\ &= \int dt\int_0^L dx\,\left(-\rho\ddot{y} + Ty''\right)\delta y\,,\end{aligned} \tag{1.77}$$

where to reach the second line, we perform integration by parts, and because the endpoints are fixed, $\delta y = 0$ when $x = 0$ and $x = L$. Extremizing the action $\delta S = 0$ gives the equation of motion

$$\rho\ddot{y} - Ty'' = 0\,, \tag{1.78}$$

which is the wave equation with transverse waves propagating with speed $c = \sqrt{T/\rho}$.

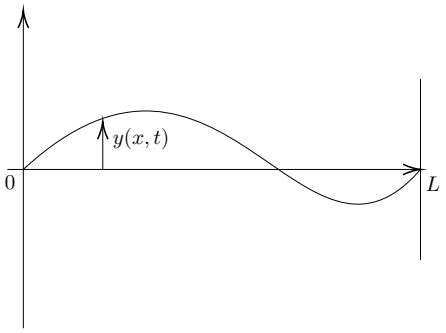

Figure 7: Set up for the vibrating string.

(**End of Lecture**: Wednesday Sep 11 2024)

1.5.2 The canonical energy-momentum tensor

Just as in the case of a single-particle Lagrangian, with conservation of momentum and energy, if the Lagrangian density $\mathcal{L} \equiv \mathcal{L}(\varphi, \partial_\mu \varphi)$ does not depend explicitly on x^μ, then we should expect a conservation law as well.

We'll go through the derivation of the Noether charge here again, as the result is presented in often confusing ways in many textbooks. Suppose our action is of the form

$$S = \int d^{d+1}x \, \mathcal{L}(\varphi, \partial_\mu \varphi) \,. \tag{1.79}$$

Then, we know that under the transformation $x^\mu \to x^\mu + \varepsilon^\mu$, where ε^μ is constant, we can expand $\mathcal{L}$ to give

$$\delta S = \int d^{d+1}x \, \varepsilon^\mu \partial_\mu \mathcal{L} \,, \tag{1.80}$$

where

$$\partial_\mu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \varphi} \partial_\mu \varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)} \partial_\mu (\partial_\nu \varphi) \,. \tag{1.81}$$

Note that in this transformation is a symmetry of the action, since $\varepsilon^\mu \partial_\mu \mathcal{L}$ is a total derivative. However, if we promote $\varepsilon^\mu \to \varepsilon^\mu(x^\mu)$ from a *global* to a local transformation, this stops being true.

Now, let's consider the *arbitrary* transformation, $x^\mu \to x^\mu + \zeta^\mu(x^\mu)$ (not necessarily constant, and not necessarily with no variation on the boundary!). This gives

$$\varphi \to \varphi + \zeta^\mu \partial_\mu \varphi \,, \quad \partial_\mu \varphi \to \partial_\mu \varphi + \partial_\mu (\zeta^\nu \partial_\nu \varphi) \,. \tag{1.82}$$

$$\begin{aligned} \delta S &= \int d^{d+1}x \left[\frac{\partial \mathcal{L}}{\partial \varphi} \zeta^\mu \partial_\mu \varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)} \partial_\nu (\zeta^\mu \partial_\mu \varphi) \right] \\ &= \int d^{d+1}x \left[\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)} \right] \zeta^\mu \partial_\mu \varphi + \int d^{d+1}x \, \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)} \zeta^\mu \partial_\mu \varphi \right) . \end{aligned} \tag{1.83}$$

Now, setting $\zeta^\mu = \varepsilon^\mu$, and going on-shell, the first term goes to zero as the Euler-Lagrange equations are satisfied, giving

$$\begin{aligned} \int d^{d+1}x \, \varepsilon^\mu \partial_\mu \mathcal{L} &= \int d^{d+1}x \, \varepsilon^\mu \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)} \partial_\mu \varphi \right) \\ &\implies \int d^{d+1}x \, \varepsilon^\mu \partial_\nu \left(\mathcal{L} \delta^\nu_\mu - \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)} \partial_\mu \varphi \right) = 0 \,. \end{aligned} \tag{1.84}$$

We call the Noether charge here the **canonical energy-momentum tensor**,[14]

$$T^\nu{}_\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)} \partial_\mu \varphi - \delta^\nu_\mu \mathcal{L} \,, \tag{1.85}$$

with the statement

$$\partial_\nu T^\nu{}_\mu = 0 \tag{1.86}$$

denoting *local* energy-momentum conservation. This can be extended to the case where there are multiple fields, in which case the first term on the right-hand side of Eq. (1.84) picks up a sum over all fields.

[14] The reason its called "canonical" and not just "the" energy-momentum tensor is because it's often the case that this isn't quite what you want to work with. Instead, you often want to work with $T^{\mu\nu}$ shifted by some other object whose total derivative is zero. In electromagnetism, this procedure of "fixing" $T^{\mu\nu}$ is often called the *Belinfante improvement procedure*.

1.5.3 The vibrating string revisited

We now return to the vibrating string, with Lagrangian given in Eq. (1.76). We can now compute the canonical energy-momentum tensor for this system. For $\mu = 0$,

$$T^{\nu}{}_{0} = \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)} \partial_0 \varphi - \delta^\nu_0 \mathcal{L} \,, \tag{1.87}$$

and so

$$T^{0}{}_{0} = \rho \dot{y}^2 - \left(\frac{1}{2}\rho \dot{y}^2 - \frac{1}{2} T y'^2 \right) = \frac{1}{2}\rho \dot{y}^2 + \frac{1}{2} T y'^2 \,, \tag{1.88}$$

$$T^{1}{}_{0} = -T y' \dot{y} \,. \tag{1.89}$$

In addition, the conservation laws read

$$\begin{aligned} 0 = \partial_\nu T^{\nu}{}_{0} &= \partial_\nu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)} \partial_0 \varphi \right] - \partial_\nu \delta^\nu_0 \mathcal{L} \\ &= \partial_t (\rho \dot{y}^2 - \mathcal{L}) + \partial_x (-T y' \dot{y}) \\ &= \partial_t \left(\frac{1}{2}\rho \dot{y}^2 + \frac{1}{2} T y'^2 \right) - \partial_x (T y' \dot{y}) \,, \end{aligned} \tag{1.90}$$

$$\begin{aligned} 0 = \partial_\nu T^{\nu}{}_{1} &= \partial_\nu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)} \partial_1 \varphi \right] - \partial_\nu \delta^\nu_1 \mathcal{L} \\ &= \partial_t (\rho \dot{y} y') + \partial_x (-T y'^2 - \mathcal{L}) \\ &= \partial_t (\rho \dot{y} y') - \partial_x \left(\frac{1}{2}\rho \dot{y}^2 + \frac{1}{2} T y'^2 \right) \,. \end{aligned} \tag{1.91}$$

These are local conservation laws of the same form as the *continuity equation*,

$$\frac{\partial q}{\partial t} + \nabla \cdot \vec{J} = 0 \,, \tag{1.92}$$

where q should be thought of as a local density (e.g. charge density of fluid density) with a globally conserved quantity $Q = \int d^d x \, q$, and $\vec{J}$ is a flux. For $\partial_\nu T^{\nu}{}_{0} = 0$, the local density is

$$T^{0}{}_{0} = \frac{1}{2}\rho \dot{y}^2 + \frac{1}{2} T y'^2 \,, \tag{1.93}$$

which is the energy density. The energy flux is $T^{1}{}_{0} = -T \dot{y} y'$, which is the rate that a segment of string is doing work on its neighbor to the right. Let's check that the total energy is globally conserved:

$$\begin{aligned} \frac{d}{dt} \int_0^L dx \left(\frac{1}{2}\rho \dot{y}^2 + \frac{1}{2} T y'^2 \right) &= \int_0^L dx \, \frac{\partial}{\partial t} \left(\frac{1}{2}\rho \dot{y}^2 + \frac{1}{2} T y'^2 \right) \\ &= \int_0^L dx \, \frac{\partial}{\partial x} (T y' \dot{y}) \\ &= T y' \dot{y} \big|_0^L = 0 \,, \end{aligned} \tag{1.94}$$

since the string is clamped at the ends, as expected.

That's really the last we'll say about continuous systems, but of course, there is so much more to say. You'll learn more about these systems in classes like quantum field theory and statistical physics.

1.6 Constraints

When we want to find the extremum of some function or functional, we often want to impose some constraints on that extremum. Here are some classic examples of problems that involve finding extrema under some constraint:

1. Given a curve a fixed length on a plane, what is the maximum area that it can enclose? What is the shape of the curve in that case?

2. Consider a chain with fixed length suspended between two points, both a fixed height above the ground. What is the shape of the chain that minimizes the potential energy?

The strategy that we are going to adopt here is identical to what you may have seen in vector calculus, with the use of **Lagrange multipliers**. Suppose you have a functional $J[y]$ that you want to extremize, subject to some constraint given by $K[y] = 0$. Consider the modified functional $\tilde{J}[y, \lambda] = J[y] - \lambda K[y]$, for some new parameter λ. First, we note that if vary the new parameter λ, we find

$$\tilde{J}[y, \lambda + \delta\lambda] - \tilde{J}[y, \lambda] = -\delta\lambda \cdot K[y]\,, \tag{1.95}$$

and so when we extremize $\tilde{J}$, including over possible variations in λ, we require $K[y] = 0$, the precise constraint that we wanted to impose!

Under a variation of y, on the other hand, we obtain

$$\tilde{J}[y + \delta y, \lambda] - \tilde{J}[y, \lambda] = \delta J[y, \delta y] - \lambda \cdot \delta K[y, \delta y]\,, \tag{1.96}$$

where $\delta J[y, \delta y] = \delta J[y + \delta y] - \delta J[y]$, and likewise for δK. At the extremum $\delta\tilde{J} = 0$ for some $y = y^*$ then, we must necessarily have

$$\delta J[y^*, \delta y] = \lambda \cdot \delta K[y^*, \delta y] \tag{1.97}$$

for all variations δy away from y^*. What's going on here? First of all, $K[y^*] = 0$, and so y^* corresponds to a point where the constraint is satisfied. Now, if the variation δy preserves this constraint, then $\delta K[y^*, y + \delta y] = 0$; in that case, the previous equation says that $\delta J[y^*, y + \delta y] = 0$ as well. In other words, J *itself is also extremized, as long as we limit ourselves to* δy *that preserves the constraint*.

Fig. 8 shows an illustration of how this method works. The equation of motion for λ enforces the constraint, and fixes the trajectory to lie along the blue line, with $K[y] = 0$. We can see that if we choose variations along the blue trajectory so that the constraint is always satisfied, $\delta K = 0$, but not necessarily δJ, *except* at the extremum of J along the trajectory (marked in red). At this point, in the direction $\delta K = 0$, $\delta J = 0$ as well, since it is a local extremum. Our clever choice of $\tilde{J}$ gave us *1)* an equation of motion for λ that constrained us to the blue trajectory, and *2)* an equation of motion for y that is satisfied for points where J is extremized along the trajectory.

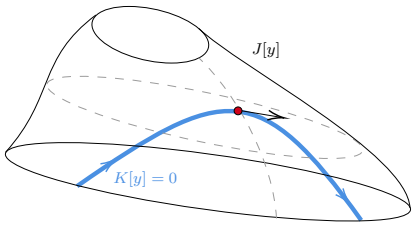

Figure 8: An illustration of the Lagrange multiplier method.

Another common situation where Lagrange multipliers can be used is in classical mechanics, when you want to extremize an action given by $S = \int dt\, L(t; q, \dot{q})$, subject to some constraint on the *coordinate* q, e.g. $g(t; q, \dot{q}) = 0$. Generic constraints are actually surprisingly difficult to handle, and aren't always amenable to the Lagrange multiplier method. However, for constraints of the form $g(t; q)$, which we call **holonomic constraints**, we can consider the modified action $\tilde{S}$, where

$$\tilde{S}[q, \lambda] = \int dt\, [L - \lambda(t) g(t; q)]\,. \tag{1.98}$$

Once again, extremizing $\tilde{S}$ simultaneously enforces the desired constraint, and extremizes L when subject to the constraint. For a much more detailed discussion of holonomic and nonholonomic constraints, and when the Lagrange multiplier method fails, see Ref. [3] for an excellent discussion of this topic.

1.6.1 Maximum entropy distribution

Our first example of how to use this comes from statistical mechanics. Let Γ denote the classical phase space of a mechanical system of N particles governed by a Hamiltonian $H(p_n^i, q_n^i)$, with $n = 1, \cdots, N$, $i = 1, 2, 3$ and $d\Gamma \equiv \prod_{n=1}^{N} (d^3\vec{p}_n \, d^3\vec{q}_n)$. We define the phase space density $\rho(p_n^i, q_n^i)$ such that $\rho(p_n^i, q_n^i) d\Gamma$ is the probability of the system being in some state given by p_n^i and q_n^i in the small region $d\Gamma$.

The entropy related to this probability distribution can be defined as the functional

$$S[\rho] = -\int_\Gamma d\Gamma \, \rho \log \rho \, . \tag{1.99}$$

We now want to find ρ that maximizes the entropy for a given mean energy,

$$\langle E \rangle = \int_\Gamma d\Gamma \, \rho H \, . \tag{1.100}$$

However, ρ is subject to the constraint that it is a probability density function, and so we must enforce the constraint

$$\int_\Gamma d\Gamma \, \rho = 1 \, . \tag{1.101}$$

We can solve this problem by defining a new functional $\tilde{S}[\rho, \alpha, \beta]$, where

$$\tilde{S}[\rho, \alpha, \beta] = S[\rho] + \alpha \left(\int_\Gamma d\Gamma \, \rho - 1 \right) + \beta \left(\int_\Gamma d\Gamma \, \rho H - \langle E \rangle \right) . \tag{1.102}$$

The equations of motion for the real-number[15] Lagrange multipliers α and β enforce the two constraints that we mentioned above. Now for ρ, we have

$$\rho \log \rho \to (\rho + \delta\rho) \log(\rho + \delta\rho) = \rho \log \rho + \delta\rho \log \rho + \rho \cdot \frac{\delta\rho}{\rho} \tag{1.103}$$

and so

$$\delta\tilde{S} = \int_\Gamma d\Gamma \, (1 + \log \rho + \alpha + \beta H) \, \delta\rho \, , \tag{1.104}$$

and setting the term in parentheses to zero to extremize $\delta\tilde{S}$ gives

$$\rho(p_n^i, q_n^i) = e^{-1-\alpha-\beta H(p_n^i, q_n^i)} \, . \tag{1.105}$$

At this point, α and β can be determined from the normalization and energy constraints detailed above, with the exact answer being determined by the expect form of H. This probability density is the usual *canonical distribution*. Note that you can see from this procedure that the temperature $T \equiv 1/\beta$ appears as a Lagrange multiplier in this procedure.

[15] They're not functions! In particular, they can live *outside* the integral over Γ. There's nothing wrong with having a functional depend on real numbers. You can also use Lagrange multipliers that are functions, if for example the constraint you are imposing is on ρ itself, and not the integral of ρ, as we are in this case.

(**End of Lecture**: Monday Sep 16 2024)

1.6.2 The catenary

We are now ready to tackle another classic calculus of variations problem: given a chain of fixed length and constant linear density suspended between two poles situated at $x = -R/2$ and $x = R/2$ (the height of these poles are fixed as well), what is the shape of the chain that minimizes the potential energy?

The total potential energy of the chain is the functional

$$E[y] = \int_{-R/2}^{R/2} d\ell\, y = \int_{-L}^{L} dx\, y\sqrt{1+y'^2}\,, \tag{1.106}$$

where I've dropped the density and the acceleration due to gravity for simplicity: they enter only as an overall multiplicative factor to the energy. I would like to extremize $E[y]$ subject to the constraint that the length of the chain is some constant L, i.e.

$$\int_{-R/2}^{R/2} dx\, \sqrt{1+y'^2} = L\,. \tag{1.107}$$

As before, we define a new functional $\tilde{E}[y,\lambda]$, where λ is a real number Lagrange multiplier, and

$$\begin{aligned}\tilde{E}[y,\lambda] &= E[y] - \lambda\left(\int_{-R/2}^{R/2} dx\, \sqrt{1+y'^2} - L\right) \\ &= \int_{-R/2}^{R/2} dx\, \left[(y-\lambda)\sqrt{1+y'^2} + \frac{\lambda L}{R}\right]\,. \end{aligned} \tag{1.108}$$

The equation of motion for y is then given by extremizing the first term in the equation above. We can just focus on the part of the action given by the integral since the other term doesn't depend on y, and use the Euler-Lagrange equations. Or, if we are a little bit cleverer, we can use the fact that the first integral is constant, since there is no explicit x dependence, i.e.

$$\begin{aligned}(y-\lambda)\sqrt{1+y'^2} - \frac{y'^2(y-\lambda)}{\sqrt{1+y'^2}} = C \implies 1+y'^2 - y'^2 &= \frac{C}{y-\lambda}\sqrt{1+y'^2} \\ \implies \frac{(y-\lambda)^2}{C^2} - y'^2 &= 1\,. \end{aligned} \tag{1.109}$$

This equation should remind you of $\cosh^2 t - \sinh^2 t = 1$, and so the general solution is

$$y - \lambda = C\cosh\left(\frac{x+D}{C}\right)\,, \tag{1.110}$$

i.e. the minimum energy curve should look like a hyperbola. This shape is also known as the **catenary**, and the constants are fixed by the heights of the two poles, and the requirement that the total length of the curve is L.

2 Calculus on Manifolds

References: Stone & Goldbart (SG) Chapter 10, Appendix A;

We're all very familiar by now with how to do calculus in $\mathbb{R}^n$. But often in physics, we want to be able to do calculus in spaces that don't quite look like $\mathbb{R}^n$. One example you've probably already seen is doing calculus in *Minkowski space* in special relativity. Another perhaps more mundane example is doing calculus on a sphere. Most of our intuition, however, is grounded in the logic and structures of $\mathbb{R}^n$, which we sometimes take for granted. And so it will help for us to think very carefully about how $\mathbb{R}^n$ works in a more formal language, in order for us to see how to generalize to other spaces.

2.1 Some Facts from Linear Algebra

The study of spaces like $\vec{R}^n$ falls under the subject of **linear algebra**. While a course in mathematical physics might feel a little incomplete without covering this topic in detail, linear algebra is generally well-covered in undergraduate curricula. We'll content ourselves with a lightning review of some key facts.

2.1.1 Vector spaces and inner products

So what are the structures of $\mathbb{R}^n$ that we take for granted? First, the fundamental quantities that we deal with in real space are **vectors**. A collection of objects that live in real space is simply a set, but the interesting thing about real space is that there are relations between the objects in the space. In fact, real space is an example of a **vector space**, a structure which is defined as follows:

A **vector space** V over a field $\mathbb{F}$ is a set equipped with two operations: a binary operation called **vector addition** which assigns to each pair of elements $\vec{x}, \vec{y} \in V$ a third element denoted $\vec{x} + \vec{y}$, and **scalar multiplication** which assigns to an element $\vec{x} \in V$ and $\lambda \in \mathbb{F}$ a new element $\lambda\vec{x} \in V$. There is also a distinguished element $\vec{0} \in V$ such that the follow axioms are obeyed:

1. Vector addition is commutative: $\vec{x} + \vec{y} = \vec{y} + \vec{x}$;
2. Vector addition is associative: $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$;
3. Additive identity: $\vec{0} + \vec{x} = \vec{x}$;
4. Existence of an additive inverse: for any $\vec{x} \in V$, there is an element $-\vec{x} \in V$ such that $\vec{x} + (-\vec{x}) = \vec{0}$;
5. Scalar distributive law: $\lambda(\vec{x}+\vec{y}) = \lambda\vec{x}+\lambda\vec{y}$, as well as $(\lambda+\mu)\vec{x} = \lambda\vec{x} + \mu\vec{x}$;
6. Scalar multiplication is associative: $(\lambda\mu)\vec{x} = \lambda(\mu\vec{x})$, and
7. Multiplicative identity: $1\vec{x} = \vec{x}$.

A lot of that just seems very natural, and so it might seem like a lot of useless abstraction. But the point is to be clear about what a vector in the abstract actually is, so that when we're in much less familiar settings, these formal structures are going to help us cut through the confusion. Furthermore, we can study properties of all vector spaces that would apply equally well to $\mathbb{R}^n$ as it does to any other vector space.

Here, you can see that $\mathbb{R}^n$ is a vector space over the field $\mathbb{R}$. However, you're also familiar with vector spaces over the complex numbers $\mathbb{C}$: one example is the *Hilbert space*, which underpins *quantum mechanics*. The states of a system are described as vectors $|\psi\rangle$ in a Hilbert space. The results of linear algebra apply equally well to both $\mathbb{R}^n$ and Hilbert spaces.

In addition to being able to add vectors, or multiply vectors by real numbers, another important thing you can do in $\mathbb{R}^n$ is talk about distances: you can take the dot product or **inner product** of a vector with itself to talk about length, or take the inner product of two different vectors and talk about angles. Formally, vector spaces with this additional structure are called **inner product spaces**. Inner product spaces are defined as follows:[16]

An **inner product space** is a vector space V over a field $\mathbb{F}$, together with an **inner product**, which is a map

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}\,, \tag{2.1}$$

that satisfies the following properties for all $\vec{x}, \vec{y}, \vec{z} \in V$ and $\lambda, \mu \in \mathbb{F}$:

1. Conjugate symmetry: $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle^*$, where $*$ denotes complex conjugation. If $\mathbb{F}$ is real, then this just means that the inner product should be symmetric;
2. Linearity in the second argument, i.e. $\langle \vec{x}, \lambda\vec{y} + \mu\vec{z} \rangle = \lambda \langle \vec{x}, \vec{y} \rangle + \mu \langle \vec{x}, \vec{z} \rangle$. Note that this together with conjugate symmetry implies that $\langle \lambda\vec{x} + \mu\vec{y}, z \rangle = \lambda^* \langle \vec{x}, \vec{z} \rangle + \mu^* \langle \vec{y}, \vec{z} \rangle$. The inner product is only linear in both arguments when $\mathbb{F} = \mathbb{R}$, and
3. Nondegenerate, i.e. if $\langle \vec{x}, \vec{y} \rangle = 0$ for all $\vec{y}$, then $\vec{x} = 0$.

The inner product on $\mathbb{R}^n$ is the dot product; in Hilbert space, it is denoted $\langle \psi' | \psi \rangle$; in Minkowski space, we have the metric tensor. We'll go into a lot more detail on this in just a bit.

The last thing that we'll talk about are **linear transformations** (also known as linear operators or linear maps), which are functions that take us between vector spaces. Let V and W be vector spaces with dimensions n and m respectively; $A : V \to W$ is a linear transformation if

$$A(\lambda\vec{x} + \mu\vec{y}) = \lambda A(\vec{x}) + \mu A(\vec{y})\,. \tag{2.2}$$

2.1.2 Bases and components

At this point, the vectors on a vector space are still abstract objects. In order to make contact with our usual representation of vectors as a column of numbers, we need to define a **basis** for the vector space. This is something that you've probably seen in linear algebra, but we'll state some facts and definitions that all are somewhat intuitive:

1. A set of vectors $\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n\}$ is **linearly dependent** if there exist $\lambda^1, \cdots, \lambda^n \in \mathbb{F}$, written as λ^μ for $\mu = 1, \cdots, n$, not all zero, such that
$$\lambda^1\vec{e}_1 + \lambda^2\vec{e}_2 + \cdots \lambda^n\vec{e}_n = \vec{0}\,. \tag{2.3}$$
2. If it is not linearly dependent, a set of vectors $\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n\}$ is **linearly independent**. For a linearly independent set, the relation
$$\lambda^1\vec{e}_1 + \lambda^2\vec{e}_2 + \cdots \lambda^n\vec{e}_n = \vec{0} \tag{2.4}$$
holds only if $\lambda^1 = \cdots = \lambda^n = 0$.

[16] There are a few differences here compared to the usual definition in mathematics. First, in mathematics, it is common to have linearity apply to the first argument. This is of course is entirely equivalent. We use this definition to conform with our usual intuition in braket notation. Second, the inner product space is usually defined as having a *positive definite* inner product; but this unfortunately excludes Minkowski space, which is more properly classified as a pseudo-inner product. We don't really care about these differences in physics though.

3. A set of vectors $\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n\}$ is said to **span** V if for any $\vec{x} \in V$, there are numbers x^μ such that $\vec{x}$ can be written (not necessarily uniquely) as

$$\vec{x} = x^1\vec{e}_1 + x^2\vec{e}_2 + \cdots + x^n\vec{e}_n \,. \tag{2.5}$$

   A vector space is **finite dimensional** if a finite spanning set exists.

4. A set of vectors $\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n\}$ is a **basis** if it is a *maximally linearly independent set*, i.e. introducing any additional vector makes the set linearly dependent. Equivalently, a basis is a *minimal spanning set*, i.e. deleting any of the $\vec{e}_i$ destroys the spanning property.

5. If $\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n\}$ is a basis, then any $\vec{x} \in V$ can be written

$$\vec{x} = x^1\vec{e}_1 + x^2\vec{e}_2 + \cdots + x^n\vec{e}_n \,, \tag{2.6}$$

   where the x^μ, known as the **components** of the vector with respect to this basis, are unique in that two vectors coincide if and only if they have the same components.

6. If the sets $\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n\}$ and $\{\vec{f}_1, \vec{f}_2, \cdots, \vec{f}_m\}$ are both bases for the space V, then $m = n$. This invariant integer is the **dimension**, $\dim(V)$, of the space.

At this point, you may be looking at the notation above and wondering about the placement of indices: when are indices placed above, and when are they placed below? This will made clear in the next part of our discussion.

2.2 Change of Bases, Covariant and Contravariant Transformations

Having defined the concepts of a basis, and the components of a vector with respect to a basis, we now want to understand how these components change as we choose different bases, since bases are not unique.

Suppose a vector space V has two different bases given by $\{\vec{e}_1, \cdots, \vec{e}_n\}$ and $\{\vec{e}'_1, \cdots, \vec{e}'_n\}$. Since both sets span V, every vector in $\{\vec{e}_1, \cdots, \vec{e}_n\}$ can be written as a sum of $\{\vec{e}'_1, \cdots, \vec{e}'_n\}$, and we can define a set of n^2 numbers ${a^\mu}_\nu$ which maps

$$\vec{e}_\nu = \sum_{\mu=1}^{n} {a^\mu}_\nu \vec{e}'_\mu \equiv {a^\mu}_\nu \vec{e}'_\mu = \vec{e}'_\mu {a^\mu}_\nu \,. \tag{2.7}$$

At this point, we've introduced the famous **Einstein notation**, which just says that every repeated index should be regarded as being summed over all possible values. Again, you may be worried about the placement of the indices, but all be clear as we go along. The final expression is helpful in helping you visualize the object $\vec{e}'_\mu$ as a row vector, multiplied by the **matrix** ${a^\mu}_\nu$, where μ indexes its rows, and ν indexes its columns.

${a^\mu}_\nu$ is clearly **invertible**: every vector has a unique representation in each basis, and the map takes the coordinates of any vector in one basis to another, and so it is certainly a bijective map. We can therefore define ${(a^{-1})^\mu}_\nu$ as the inverse map,

$$\vec{e}'_\nu = {(a^{-1})^\mu}_\nu \vec{e}_\mu \,, \tag{2.8}$$

with

$${(a^{-1})^\mu}_\nu {a^\nu}_\sigma = \delta^\mu_\sigma \,, \tag{2.9}$$

where δ^μ_σ is the Kronecker delta or the identity matrix.

So far, we have dealt with the transformation of the basis. But how does a general vector transform? Given the transformation between bases above, we see that for any arbitrary vector $\vec{x}$, which can be written as $x^\nu \vec{e}_\nu$ in one basis and $x'^\mu \vec{e}'_\mu$ in the other, are related by

$$\vec{x} = x'^\mu \vec{e}'_\mu = x^\nu \vec{e}_\nu = x^\nu (a^\mu{}_\nu \vec{e}'_\mu) = (a^\mu{}_\nu x^\nu)\vec{e}'_\mu \,, \tag{2.10}$$

or in other words,[17]

$$x'^\mu = a^\mu{}_\nu x^\nu \,. \tag{2.11}$$

One thing you should notice immediately is that the basis and the coordinates transform in the opposite way:

$$x'^\mu = a^\mu{}_\nu x^\nu \,, \qquad \vec{e}'_\mu = (a^{-1})^\sigma{}_\mu \vec{e}_\sigma \,, \tag{2.12}$$

because of course the vector itself, $x'^\mu \vec{e}'_\mu = x^\mu \vec{e}_\mu$, doesn't transform under a coordinate change at all! Any quantity that transforms under a change of basis like the basis itself is said to transform **covariantly**, while any quantity that transforms like the coordinates, i.e. in the opposite manner as the basis, is said to transform **contravariantly**. *We will always use indices on the top to indicate a quantity that transforms contravariantly, and indices on the bottom to indicate a quantity that transform covariantly.*

The best intuition for this comes from imagining a change of basis via rotation in $\mathbb{R}^2$, as shown in Fig. 9. Either we can imagine the basis vectors actually rotating counterclockwise and defining a new set of axes, as we would do in taking $x'^\nu \vec{e}'_\nu = x'^\nu (a^\mu{}_\nu \vec{e}_\mu)$, or equivalently, we can think of the components of the vector themselves rotating clockwise, with the axes just being relabeled, which corresponds to $x'^\nu \vec{e}'_\nu = (a^\mu{}_\nu x'^\nu)\vec{e}_\mu$.

(**End of Lecture**: Wednesday 18 Sep 2024)

2.3 The Dual Space

For every vector space V, we can define a **dual space** V^*, which is a set of linear transformations $f : V \to \mathbb{F}$, each of which takes in a vector and returns a number. The functions f are called **covectors** or **one-forms**, and you can convince yourself that V^* is also a vector space. Since these functions are linear, we have

$$f(\vec{x}) = f(x^\mu \vec{e}_\mu) = x^\mu f(\vec{e}_\mu) \equiv x^\mu f_\mu \,, \tag{2.13}$$

where in the last equality I have defined the set of numbers $f_\mu \equiv f(\vec{e}_\mu)$, which I can construct given the basis $\{\vec{e}_\mu\}$ in V. Under a change of basis in V,

$$f_\mu = f(\vec{e}_\mu) = f(a^\nu{}_\mu \vec{e}'_\nu) = a^\nu{}_\mu f(\vec{e}'_\nu) \equiv a^\nu{}_\mu f'_\nu \,, \tag{2.14}$$

where $f'_\nu \equiv f(\vec{e}'_\nu)$ are again a set of numbers that we can construct given the basis $\{\vec{e}'_\nu\}$ in V. Notice that under a change of basis in V, f_μ transforms *covariantly*, i.e. in the same manner as the change of basis in V.

Given a basis $\vec{e}_\mu$ of V, we can define a **dual basis** for V^*, which is the set of covectors $\vec{e}^{*\mu} \in V^*$ such that

$$\vec{e}^{*\mu}(\vec{e}_\nu) = \delta^\mu_\nu \,. \tag{2.15}$$

This is clearly a basis, since for any $f \in V^*$,

$$f(\vec{x}) = x^\mu f_\mu = x^\mu f_\nu \delta^\nu_\mu = x^\mu f_\nu \vec{e}^{*\nu}(\vec{e}_\mu) = f_\nu \vec{e}^{*\nu}(x^\mu \vec{e}_\mu) \,, \tag{2.16}$$

[17] Very often, you will see the notation $a^\mu{}_\nu \equiv \partial x'^\mu / \partial x^\nu$, which makes total sense if you look at Eq. (2.11). In fact, the advantage of writing it this way tells you how to obtain the matrix $a^\mu{}_\nu$.

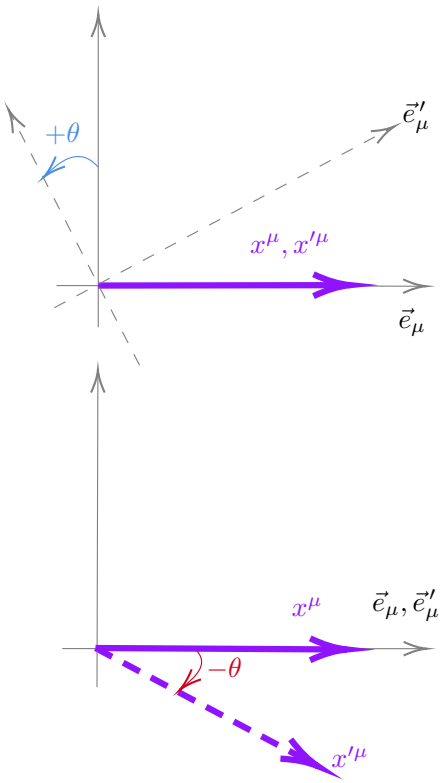

Figure 9: Two equivalent ways to think of a change of basis. (Top) the basis vectors themselves are transformed, or (bottom) the coordinates are transformed in the opposite direction. These pictures are equivalent.

or in other words,

$$f = f_\mu \vec{e}^{*\mu} \,. \tag{2.17}$$

We should therefore view $f_\mu = f(\vec{e}_\mu)$ as the *components* of f under the induced dual basis $\vec{e}^{*\mu}$.

You should already have a sense that V and V^* are very closely related; in fact, the map $\vec{e}_\mu \mapsto \vec{e}^{*\mu}$ is an **isomorphism**, i.e. a map of every element in V to another in V^* that preserves their respective relation to each other under addition and scalar multiplication.

2.4 The Metric

So far, everything we have discussed has been about vector spaces. We are now going to turn our attention to inner product spaces over $\mathbb{R}$, where the additional inner product structure is defined, giving us a way of talking about distances and angles.

As a reminder, the inner product is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ that takes two vectors in V, and spits out a number. Having chosen a basis $\{\vec{e}_\mu\}$ for our vector space V, we can now define a quantity known as the **metric** or **metric tensor** $g_{\mu\nu}$,

$$g_{\mu\nu} \equiv \langle \vec{e}_\mu, \vec{e}_\nu \rangle \,. \tag{2.18}$$

For $\mathbb{R}^n$, for example, with the inner product given by the dot product, we have simply $g_{\mu\nu} = \delta_{\mu\nu}$, while for Minkowski space, the metric is given by[18]

$$\eta_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \,. \tag{2.19}$$

[18] I will generally stick with the mostly minus convention for the Minkowski metric. My apologies to mostly plus aficionados, but I'm just slightly more comfortable with the mostly minus convention at this point.

Knowing the metric fully defines the inner product, since

$$\langle \vec{x}, \vec{y} \rangle = \langle x^\mu \vec{e}_\mu, y^\nu \vec{e}_\nu \rangle = g_{\mu\nu} x^\mu y^\nu \,. \tag{2.20}$$

Moreover, the structure of the inner product also guarantees that $g_{\mu\nu} = g_{\nu\mu}$, i.e. $g_{\mu\nu}$ is **symmetric**. Another thing we can note is that as a matrix, $g_{\mu\nu} x^\nu = 0$ only if $x^\nu = 0$ by the definition of the inner product; this means that $g_{\mu\nu}$ is invertible. We can therefore define the **inverse of the metric**, which we denote $g^{\mu\nu}$, with

$$g_{\mu\nu} g^{\nu\sigma} = g^{\mu\nu} g_{\nu\sigma} = \delta^\mu_\sigma \,. \tag{2.21}$$

For now, this is just a relationship between matrices; we'll come back and revisit the metric when we have discussed tensors later on.

2.4.1 Raising and lowering indices

Let's look at the expression in Eq. (2.20) more closely. We can reinterpret $\langle \vec{x}, \vec{y} \rangle = g_{\mu\nu} x^\mu y^\nu$ as $(g_{\mu\nu} x^\mu) y^\nu$. With this rewriting, can think of $(g_{\mu\nu} x^\mu)$ as being the components of the object $\langle \vec{x}, \cdot \rangle$, which takes in a vector $\vec{y}$ and returns $\langle \vec{x}, \vec{y} \rangle$. In fact, $\langle \vec{x}, \cdot \rangle$ is an object in V^*, mapping vectors in V to real numbers, and can be written as $\langle \vec{x}, \cdot \rangle = g_{\mu\nu} x^\mu \vec{e}^{*\nu}$, so that

$$g_{\mu\nu} x^\mu \vec{e}^{*\nu} (y^\sigma \vec{e}_\sigma) = g_{\mu\nu} x^\mu y^\sigma \delta^\nu_\sigma = g_{\mu\nu} x^\mu y^\nu = \langle \vec{x}, \vec{y} \rangle \,. \tag{2.22}$$

Clearly then, $g_{\mu\nu}x^\mu$ are indeed the components of $\langle\vec{x},\cdot\rangle$ in the basis of V^* induced by our chosen basis of V, and therefore transforms *covariantly*.

If all that was a bit dense, the upshot is that, starting from a contravariant quantity x^μ, we can **lower its index** by defining

$$x_\nu \equiv g_{\mu\nu}x^\mu\,, \tag{2.23}$$

which 1) is a quantity that transforms covariantly (and so has a lower index), and 2) can be **contracted** with a contravariant quantity to form a real number, or a **scalar** or an **invariant**. Intuitively, it is one half of the inner product: you need to put together a covariant and contravariant piece to obtain a scalar.

Multiplying Eq. (2.23) by $g^{\sigma\nu}$ on both sides, we also find

$$g^{\sigma\nu}x_\nu \equiv g^{\sigma\nu}g_{\mu\nu}x^\mu = g^{\sigma\nu}g_{\nu\mu}x^\mu = \delta^\sigma_\mu x^\mu = x^\sigma\,, \tag{2.24}$$

which shows that I can also **raise an index** by multiplying by the inverse tensor. Ultimately, all I'm doing is switching between the components of the two objects

$$\langle\vec{x},\cdot\rangle \leftrightarrow \vec{x}\,, \tag{2.25}$$

which are in 1-to-1 correspondence with each other between the isomorphic vector spaces V and V^*.

Finally, notice that every time we perform a contraction, we sum over one upper and one lower index. This is because every contraction represents the pairing of a function in V^*, with a vector in V, and results in a scalar. Another way of understanding this is that you want to pair up a contravariant with a covariant quantity, so that you end up with a quantity that doesn't transform, i.e. a scalar. I have never encountered a situation where you want to sum over the components of two objects which transform in the same way.

2.4.2 Example: Some Common Metrics

Let's pause for a moment and take a look at some important examples.

To digest all of this information, let's revisit $\mathbb{R}^2$ with all of this technology. $\mathbb{R}^2$ is a 2D vector space, with vectors $x^i\vec{e}_i$ that look like, for example, $3\vec{e}_x+2\vec{e}_y$, where 3 and 2 are the components of the vector, and $\{\vec{e}_x,\vec{e}_y\}$ is a chosen basis for the space. $\mathbb{R}^2$ also comes with an inner product, which is the usual dot product. In $\mathbb{R}^2$, we can choose a basis that is **orthonormal**, i.e. with a metric given by

$$g_{ij} = \langle\vec{e}_i,\vec{e}_j\rangle = \vec{e}_i\cdot\vec{e}_j = \delta_{ij}\,. \tag{2.26}$$

I can use this metric to raise and lower indices of covariant or contravariant quantities, so for example

$$g_{ij}x^j = \delta_{ij}x^j = x_i\,, \tag{2.27}$$

but if you explicitly plug in the indices, you can see that $x^0 = x_0$ and $x^1 = x_1$, which shows that *the position of indices doesn't matter in* $\mathbb{R}^n$. Inner products between two vectors can be written in component form as

$$g_{ij}x^iy^j = x_jy^j\,, \tag{2.28}$$

i.e. the sum of the product of individual components, as in the usual dot product. You can think of x_j as the components of $\langle\vec{x},\cdot\rangle$. In $\mathbb{R}^2$, you can also think of

x_j as a row matrix, which maps column vectors (which contain components of a vector) to numbers by matrix multiplication.

Now let's consider a basis change, given by $\vec{e}'_i = a^j{}_i \vec{e}_j$, where

$$a^j{}_i = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{2.29}$$

where j indexes the row and i indexes the column. This is a clockwise rotation of the basis vectors by some constant angle θ. Explicitly,

$$\begin{aligned} \vec{e}'_1 &= \cos\theta\, \vec{e}_1 - \sin\theta\, \vec{e}_2\,, \\ \vec{e}'_2 &= \sin\theta\, \vec{e}_1 + \cos\theta\, \vec{e}_2\,. \end{aligned} \tag{2.30}$$

At the same time, for any vector $\vec{x} = x^i \vec{e}_i$, the coordinates x^i transforms in the opposite sense, i.e. $x'^i = (a^{-1})^i{}_j x^j$, where

$$(a^{-1})^i{}_j = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{2.31}$$

so that

$$\begin{aligned} x'^1 &= \cos\theta\, x^1 - \sin\theta\, x^2 \\ x'^2 &= \sin\theta\, x^1 + \cos\theta\, x^2\,, \end{aligned} \tag{2.32}$$

which is instead a *counterclockwise* rotation of the components.[19] You can check that the vector itself, $x^i \vec{e}_i$, remains unchanged. This is the same intuition we had from Fig. 9.

[19] What may be confusing is that Eq. (2.30) and Eq. (2.32) look identical! But how we interpret what's happening is different. In the first, each line tells us how each basis vector is separately rotated, so you're looking out for $\vec{e}_1$ transforming into something else. But in the second, it is the transformation of the arrow denoted by (x^1, x^2) going into (x'^1, x'^2) that we are interested in.

We now graduate to something hopefully still familiar, but a little more nontrivial: 4D Minkowski space, where the 0-dimension is time, and dimensions 1,2,3 are spatial dimensions. The vectors that live in this space are called **4-vectors**, and they are of the form $x^\mu \vec{e}_\mu$, where $\{\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ forms a basis. Once again, we have an inner product and an associated metric; we can choose a basis such that the metric is the **Minkowski metric**

$$\eta_{\mu\nu} = \langle \vec{e}_i, \vec{e}_j \rangle\,, \tag{2.33}$$

where

$$\eta_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2.34}$$

You can verify for yourself that $\eta^{\mu\nu}$, the inverse of the matrix, has the same entries as $\eta_{\mu\nu}$.

Once again, I can lower indices by hitting a contravariant quantity with the metric, e.g. $x_\mu = \eta_{\mu\nu} x^\nu$, but this time, you can see that $x_0 = x^0$, and $x_i = -x^i$. Therefore, *the position of the indices does matter in Minkowski space*, and we need to be a little more careful. You can still think of x_μ as the components of $\langle \vec{x}, \cdot \rangle$, but x_μ is now no longer just a simple transposition (i.e. a row matrix) relative to x^μ (which we can view as a column matrix); you also need to change the sign of the spatial components. Inner products can be written, as before, as

$$g_{\mu\nu} x^\mu y^\nu = x_\nu y^\nu\,, \tag{2.35}$$

but note that because of the negative signs in the metric, *you are no longer guaranteed that the inner product is positive*.

We can again consider basis changes such as the **Lorentz boost**, given by $\vec{e}'_\nu = \Lambda^\mu{}_\nu \vec{e}_\mu$, where for example

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.36}$$

where $|\beta| < 1$ and $\gamma = (1-\beta^2)^{-1/2}$. Under this transformation,

$$\begin{aligned} \vec{e}'_0 &= \gamma\vec{e}_0 + \beta\gamma\vec{e}_1 \\ \vec{e}'_1 &= \beta\gamma\vec{e}_0 + \gamma\vec{e}_1 \\ \vec{e}'_2 &= \vec{e}_2 \\ \vec{e}'_3 &= \vec{e}_3 \,. \end{aligned} \tag{2.37}$$

On the other hand, the coordinates transform as $x'^\nu = (\Lambda^{-1})^\nu{}_\mu x^\mu$, where

$$(\Lambda^{-1})^\nu{}_\mu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.38}$$

i.e.

$$\begin{aligned} x'^0 &= \gamma x^0 - \beta\gamma x^1 \\ x'^1 &= -\beta\gamma x^0 + \gamma x^1 \\ x'^2 &= x^2 \\ x'^3 &= x^3 \,. \end{aligned} \tag{2.39}$$

2.5 Tensors

We have now seen vector spaces and their dual spaces. We can now start defining even more general objects by putting vector spaces and dual spaces together!

Consider three vector spaces U, V and W over $\mathbb{F}$. We can define the **tensor product** of spaces such as $V \otimes W$, or even $U \otimes V \otimes W$.

1. It is distributive, i.e. for $\vec{a} \in U$ and $\vec{x} \in V$,

$$\begin{aligned} \vec{a} \otimes (\vec{x} + \vec{y}) &= \vec{a} \otimes \vec{x} + \vec{a} \otimes \vec{y}\,, \\ (\vec{a} + \vec{b}) \otimes \vec{x} &= \vec{a} \otimes \vec{x} + \vec{b} \otimes \vec{x}\,; \end{aligned} \tag{2.40}$$

2. It is associative, so that we can chain together three vector spaces like $U \otimes V \otimes W$ without worrying about whether it's $(U \otimes V) \otimes W$ or $U \otimes (V \otimes W)$;

3. It commutes with $\mathbb{F}$, i.e.

$$\lambda(\vec{a} \otimes \vec{x}) = (\lambda\vec{a}) \otimes \vec{x} = \vec{a} \otimes (\lambda\vec{x})\,, \tag{2.41}$$

but it is *not* commutative over the vectors, i.e. $\vec{a} \otimes \vec{b} \neq \vec{b} \otimes \vec{a}$ in general.

Consider a vector space V with a basis $\{\vec{e}_\mu\}$. This basis induces a basis in tensor products of V and V^* in the natural way; for example, in $V^* \otimes V^*$, this

induces the basis $\vec{e}^{*\mu} \otimes \vec{e}^{*\nu}$. These basis vectors act on pairs of basis vectors, $(\vec{e}_\alpha, \vec{e}_\beta)$, in the expected way, so for our $V^* \otimes V^*$ example, we get

$$\vec{e}^{*\mu} \otimes \vec{e}^{*\nu}(\vec{e}_\alpha, \vec{e}_\beta) = \delta^\mu_\alpha \delta^\nu_\beta \,. \tag{2.42}$$

It can also act on elements of $V \otimes V$ in a manner that you might also expect:

$$\vec{e}^{*\mu} \otimes \vec{e}^{*\nu}(\vec{e}_\alpha \otimes \vec{e}_\beta) = \delta^\mu_\alpha \delta^\nu_\beta \,. \tag{2.43}$$

A good example of what a tensor is and what it does is the metric itself, which we often refer to as **metric tensor**. It is a tensor in $V^* \otimes V^*$,

$$\mathbf{g} = g_{\mu\nu} \vec{e}^{*\mu} \otimes \vec{e}^{*\nu} \,. \tag{2.44}$$

It acts on pairs of vectors $\vec{x}$ and $\vec{y}$ and returns a number in $\mathbb{F}$:

$$\mathbf{g}(\vec{x}, \vec{y}) = g_{\mu\nu} \vec{e}^{*\mu} \otimes \vec{e}^{*\nu}(x^\alpha \vec{e}_\alpha, y^\beta \vec{e}_\beta) = g_{\mu\nu} x^\alpha y^\beta \delta^\mu_\alpha \delta^\nu_\beta = g_{\mu\nu} x^\mu y^\nu = x^\mu y_\mu \,. \tag{2.45}$$

As with vectors and covectors, once we've picked a basis, we will only need to worry about the components, with the understanding that the object itself is specified by both the components and the basis, and that the spaces act in the natural way that you expect.

Under a change of basis $\{\vec{e}_\mu\} \mapsto \{\vec{e}'_\mu\}$, the metric itself undergoes a transformation:

$$g_{\mu\nu} \mapsto g'_{\mu\nu} = \langle \vec{e}'_\mu, \vec{e}'_\nu \rangle = \langle a^\sigma{}_\mu \vec{e}_\sigma, a^\lambda{}_\nu \vec{e}_\lambda \rangle = a^\sigma{}_\mu a^\lambda{}_\nu g_{\sigma\lambda} \,. \tag{2.46}$$

Each lower index is acted on by the change-of-basis transformation, with each transformation given by a covariant transformation. We say therefore that $g_{\mu\nu}$ a doubly covariant **tensor**, which explains why we often also refer to $g_{\mu\nu}$ as the **metric tensor**. You can see now that we can look into objects with more general number of indices, say $Q^{\alpha\beta}{}_{\gamma\delta\epsilon}$, which transforms as

$$Q'^{\alpha\beta}{}_{\gamma\delta\epsilon} = (a^{-1})^\alpha{}_{\alpha'} (a^{-1})^\beta{}_{\beta'} a^{\gamma'}{}_\gamma a^{\delta'}{}_\delta a^{\epsilon'}{}_\epsilon Q^{\alpha'\beta'}{}_{\gamma'\delta'\epsilon'} \,, \tag{2.47}$$

which is a doubly contravariant, triply covariant tensor, or a type $(2,3)$ tensor. The total number of indices is what we call the **rank** of the tensor. Notice how when we were writing down the transformation of $Q^{\alpha\beta}{}_{\gamma\delta\epsilon}$, the indices lined up: we contracted upper indices with lower indices, so that under each application of the change-of-basis transformation, upper indices remain upper indices.

Another thing you will notice is that I have been very careful with the relative positions of the tensors. This is good practice, but very often you'll find people get sloppy and collapse all the indices when they think the notation is obvious. The one tensor where this is always okay is the Kronecker delta δ^μ_ν, since we always have this tensor returning 1 if $\mu = \nu$ and 0 if $\mu \neq \nu$ regardless of the position of the indices.

(**End of Lecture**: Monday Sep 23 2024)

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