PY501 - Mathematical Physics

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1 Calculus of Variations

References: Stone & Goldbart (SG) Chapter 1; Byron & Fuller (BF) Chapter 2; Arfken, Weber & Harris (AWH) Chapter 22.

Finding where the minimum or maximum value of some quantity occurs is an extremely common task. For example, you might want to know where the highest point on a map is, or when you had the highest heartrate throughout the day. Mathematically, we have a function $f(x)$, and we want to find the value of x which maximizes or minimizes $f(x)$. To do so, for a differentiable function f, we simply take the derivative and set it to zero

$$f'(x) = 0 \tag{1.1}$$

and solve for x to find the stationary points of the function.

Often though, we run into problems where we want to find the *function* at which the minimum or maximum value of some function occurs. Some examples include:

1. What is the shortest path to take between points A and B?
2. What closed curve of fixed length encloses the maximum possible area?
3. What form does a hanging heavy chain of fixed length take, so as to minimize its potential energy?

To answer these questions mathematically, we need an object called a **functional** $J[y]$, which maps smooth[1] functions y (e.g. a path, a curve) to a real number (e.g. a distance, an area). This is just another map, like a function is. But now, we want to develop the tools required to define a **functional derivative** such that setting

$$\frac{\delta J}{\delta y(x)} = 0 \tag{1.2}$$

will allow us to find a function $y(x)$ (e.g. a path, a curve) that maximizes $J[y]$ (e.g. a distance, an area).

[1] "Smooth" means that all derivatives of the function exists. We won't ever be interested in subtleties involving continuity and differentiability in this course.

1.1 Functionals

What does a functional look like? For our purposes, we will be dealing with functionals that have the following form:

$$J[y] = \int_{x_1}^{x_2} dx\, f(x, y, y', y'', \cdots, y^{(n)})\,, \tag{1.3}$$

where f is a function of the real numbers x, y, $y' \cdots$, *independently*.[2] We call these kinds of functionals **local** in x. As you can see, J takes in a function y, performs an integral, and returns a real number, which is exactly what a functional should do.

[2] This is a cause of endless confusion, so pay attention! From the perspective of f, y and y' are *independent variables*.

1.1.1 The functional derivative

Let us work out the functional derivative for the case where

$$J[y] = \int_{x_1}^{x_2} dx\, f(x, y, y')\,. \tag{1.4}$$

To do this, suppose we make an infinitesimal shift $y(x) \to y(x) + \varepsilon\eta(x)$, where ε is an infinitesimally small constant,[3] and $\eta(x)$ is some arbitrary function. Then

[3] i.e. with ε^2 and higher powers of ε all being zero.

the shift in J is

$$\delta J = J[y+\varepsilon\eta] - J[y] = \int_{x_1}^{x_2} dx\ \left[f(x, y+\varepsilon\eta, y'+\varepsilon\eta') - f(x,y,y')\right]\ . \quad (1.5)$$

Since ε is an infinitesimal quantity, we can perform a Taylor expansion up to first order about y and y' to find

$$\delta J = \int_{x_1}^{x_2} dx\ \left[\varepsilon\eta\frac{\partial f}{\partial y} + \varepsilon\eta'\frac{\partial f}{\partial y'}\right]\ . \quad (1.6)$$

To make further progress, we integrate the second term by parts, giving

$$\delta J = \left[\varepsilon\eta\frac{\partial f}{\partial y'}\right]_{x_1}^{x_2} + \int_{x_1}^{x_2} dx\ \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\varepsilon\eta\,. \quad (1.7)$$

We are frequently—but not always!—concerned with finding functions of y with fixed endpoints;[4] in that case, $\eta(x_1) = \eta(x_2) = 0$, and the boundary terms in the first term on the right vanishes, leaving

$$\delta J = \int_{x_1}^{x_2} dx\ \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\varepsilon\eta\,. \quad (1.8)$$

This can be written suggestively as

$$\delta J = \int_{x_1}^{x_2} dx\,\delta y(x)\left(\frac{\delta J}{\delta y(x)}\right)\ , \quad (1.9)$$

where $\delta y(x) \equiv \varepsilon\eta(x)$, and

$$\frac{\delta J}{\delta y(x)} \equiv \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \quad (1.10)$$

is the **functional derivative** of J with respect to $y(x)$.

To aid our understanding, it can be helpful to think discretely. We can discretize x between x_1 and x_2 into N discrete steps, so that the function y takes up values $y_i = y(x_i)$, where $i = 1, 2\cdots, N$. A choice of the function y corresponds in this discrete picture to a choice of $\{y_i\}$, which is a single point in an N-dimensional space. Fig. 1 has a visualization of this. At every point in this N-dimensional space, we can assign a value to J. The small variation $\varepsilon\eta$ can likewise be discretized, so that $\delta y_i = \varepsilon\eta_i$, which can be thought of as a step in a particular direction in the same N-dimensional space. In this discrete picture,

$$\delta J = \sum_{i=1}^{N}\frac{\partial J}{\partial y_i}\delta y_i\,, \quad (1.11)$$

just as one might expect for a function J defined in the N-dimensional space indexed by y_i. In the continuous limit, we need to trade the summation over discrete i to an integral over the continuous label x, leading to Eq. (1.9).[5]

1.1.2 The Euler-Lagrange equation

Now, to find the **stationary points**—maxima, minima or saddle points—of J, we want to set $\delta J = 0$ for any arbitrary variation $\varepsilon\eta$, just like for a function g

[4] For example, if we are interested in finding the path with the short distance between two fixed points.

Figure 1: A discretized visualization of varying over functions. (**HL**: To be completed, but not difficult to imagine!)

[5] In this picture, a choice of the function y is a point in an uncountably infinite dimensional space, and $\epsilon\eta$ is a step in some arbitrary direction, and J is function that returns a real number at every point in this space.

on $\mathbb{R}^n$, we want δg to be zero for a step in any direction at a stationary point. Referring to Eq. (1.8), we require

$$\int_{x_1}^{x_2} dx \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] \varepsilon\eta = 0\,. \tag{1.12}$$

Since this applies for *any* $\eta(x)$, the term in the square brackets $[\cdots]$ must vanish.[6] we must have

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0} \tag{1.13}$$

for a stationary point for J. This is the famous **Euler-Lagrange equation**.

Through derivations similar to what we saw above, we can get generalized Euler-Lagrange equations for more complicated versions of J. If J depends on more than one function y_i, for example, the stationary points are given by

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right) = 0\,, \tag{1.14}$$

which is one equation for each variable y_i. If on the other hand, f depends on higher derivatives y'', y''' and so on, then the generalized Euler-Lagrange equation we get is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) - \frac{d^3}{dx^3}\left(\frac{\partial f}{\partial y'''}\right) + \cdots = 0\,. \tag{1.15}$$

[6] We can prove that this is true rigorously, given various conditions on η and f. This is often known as the **fundamental lemma of the calculus of variations**. For further discussion, see SG 1.2.2 and Wikipedia.

1.1.3 Applications

Time to apply what we've learnt! We'll apply the Euler-Lagrange equations to two examples.

1.1.3.1 Soap film supported by a pair of coaxial rings. Consider Fig. 2, where a pair of co-axial rings support a soap film. The energy associated with the configuration is directly proportional to the area, and hence the soap film tends to minimize this energy by minimizing its area, subject to the constraint that the soap film has to end on the rings at either end. The area of associated with a segment of the film of width dx is

$$dA = 2\pi y(x)\sqrt{dx^2 + dy^2} = 2\pi y(x)\sqrt{1 + y'^2}\,dx\,, \tag{1.16}$$

and so the functional that we want to minimize is

$$J[y] = \int_{x_1}^{x_2} dx\, f(y, y')\,, \quad f(y, y') \equiv y\sqrt{1 + y'^2}\,, \tag{1.17}$$

with the endpoint values fixed at $y(x_1)$ and $y(x_2)$. The minimum for this functional can therefore be found by applying the Euler-Lagrange equations. The partial derivatives that we need for the Euler-Lagrange equation are

$$\frac{\partial f}{\partial y} = \sqrt{1 + y'^2}\,, \quad \frac{\partial f}{\partial y'} = \frac{yy'}{\sqrt{1 + y'^2}}\,, \tag{1.18}$$

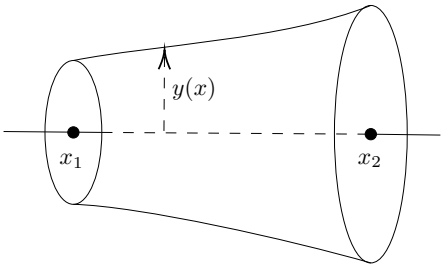

Figure 2: Soap film between two rings, centered at x_1 and x_2, with radii $y(x_1)$ and $y(x_2)$.

and so the Euler-Lagrange equation says that the minimal surface area profile $y(x)$ must satisfy

$$\sqrt{1+y'^2} - \frac{d}{dx}\left(\frac{yy'}{\sqrt{1+y'^2}}\right) = 0$$
$$\implies \sqrt{1+y'^2} - \frac{y'^2}{\sqrt{1+y'^2}} - \frac{yy''}{\sqrt{1+y'^2}} + \frac{yy'^2y''}{(1+y'^2)^{3/2}} = 0$$
$$\implies \frac{1}{\sqrt{1+y'^2}} - \frac{yy''}{(1+y'^2)^{3/2}} = 0\,. \qquad (1.19)$$

This differential equation looks difficult to solve, but fortunately there's a neat little trick to do so. Multiplying by y' on both sides gives

$$0 = \frac{y'}{\sqrt{1+y'^2}} - \frac{yy'y''}{(1+y'^2)^{3/2}} = \frac{d}{dx}\left(\frac{y}{\sqrt{1+y'^2}}\right)\,. \qquad (1.20)$$

We'll return to how we knew this trick would work later on. In the mean time, the solution is

$$\frac{y}{\sqrt{1+y'^2}} = \kappa \qquad (1.21)$$

for some constant κ. Rewriting this as

$$\frac{dy}{dx} = \sqrt{\frac{y^2}{\kappa^2} - 1} \implies \int \frac{dy}{\sqrt{y^2/\kappa^2 - 1}} = \int dx\,, \qquad (1.22)$$

we can integrate this first-order ordinary differential equation by substituting $y = \kappa\cosh t$ and $dy = \kappa\sinh t$ to find

$$\kappa\int dt = \int dx \implies \kappa t = x + C \implies y = \kappa\cosh\left(\frac{x+C}{\kappa}\right) \qquad (1.23)$$

for some constants κ and C. These can be determined by enforcing the two boundary conditions—the radii of the two rings, $y(x_1)$ and $y(x_2)$.

1.1.3.2 The brachistochrone The next problem we will consider is a famous one, posed by Johann Bernoulli in 1696. What shape should a wire with endpoints $(0,0)$ and (a,b) take, in order that a frictionless bead will slide from rest down the wire in the shortest possible time?

First, the total time T taken down a given path can be written as

$$T = \int_0^T dt = \int_0^L \frac{ds}{v}\,, \qquad (1.24)$$

where v is the speed of the bead, and s is the distance along the path, with a total length L. However, we can once again write $ds^2 = dx^2 + dy^2$ so that $ds = \sqrt{1+y'^2}\,dx$, and apply conservation of energy to find $v = \sqrt{2gy}$. Thus, we can define a functional T

$$T[y] = \int_0^a dx\,\sqrt{\frac{1+y'^2}{2gy}} \qquad (1.25)$$

that we want to minimize with respect to y, again with fixed end points. We can therefore apply the Euler-Lagrange equation, which gives after some algebra

$$yy'' + \frac{1}{2}(1+y'^2) = 0\,. \qquad (1.26)$$

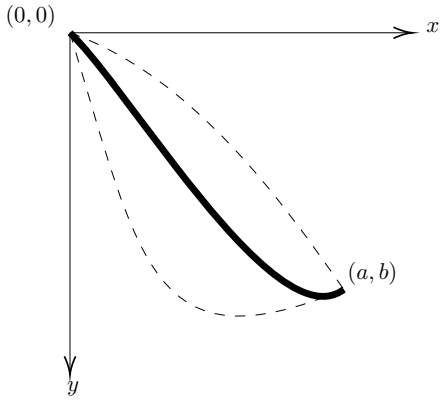

Figure 3: Possible shapes of a wire for a frictionless bead to travel from the origin $(0,0)$ to a point (a,b).

Once again, we can use the trick of multiplying by y' to find that

$$y'\left(yy'' + \frac{1}{2}(1+y'^2)\right) = \frac{1}{2}\frac{d}{dx}\left(y(1+y'^2)\right) = 0\,, \tag{1.27}$$

or

$$y(1+y'^2) = 2C \tag{1.28}$$

for some constant C. From this point, one can check that the following parametrization $(x(t), y(t))$ is indeed a solution to the differential equation above:

$$\begin{aligned} x &= C(\theta - \sin\theta) \\ y &= C(1-\cos\theta)\,, \end{aligned} \tag{1.29}$$

although it is surprisingly hard to pin down the details regarding this solution.[7] This parametric curve is known as the **cycloid**, which is the curve traced out by a fixed point on the rim of a wheel that is rolling without slipping along a flat surface.

[7] For example, is there a unique solution, and does the solution always occur with $\theta \in [0, 2\pi)$ for every point (a, b)? See for example Ref. [1] for more details.

1.1.4 First integral

In both applications discussed in Sec. 1.1.3, we were able to rephrase the differential equation to be solved as $dI/dx = 0$, implying that I is some constant associated with the problem. This quirk, which somewhat resembles the conservation of quantities like energy, is something we will revisit in much greater depth later: it is far deeper than just a mathematical coincidence. For now, let's just take a quick look at where it comes from. In both cases, the function inside the integral was $f(y, y')$, with no explicit dependence on x, implying that

$$\frac{df}{dx} = \cancelto{0}{\frac{\partial f}{\partial x}} + y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'}\,. \tag{1.30}$$

We define the **first integral** of the Euler-Lagrange equation as

$$I \equiv f - y'\frac{\partial f}{\partial y'}\,, \tag{1.31}$$

from which we can check that

$$\begin{aligned} \frac{dI}{dx} &= y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} - y''\frac{\partial f}{\partial y'} - y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \\ &= y'\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\,. \end{aligned} \tag{1.32}$$

Thus, $dI/dx = 0$ if the Euler-Lagrange equation is satisfied.

You can show that if f depends on more than one variable, so that we have a functional of the form

$$J[y_1, y_2, \cdots, y_n] = \int dx\, f(y_1, y_2, \cdots, y_n; y_1', y_2', \cdots, y_n')\,, \tag{1.33}$$

the first integral is of the form

$$I = f - \sum_i y_i'\frac{\partial f}{\partial y_i'}\,. \tag{1.34}$$

Note that there is only one first integral, even when there are multiple dependent variables y_i.

1.2 Lagrangian Mechanics

It turns out that classical mechanics can be reformulated as a problem of finding the stationary function of some functional. Given some system, we first define the **Lagrangian** function $L = T - V$, where T and V are the kinetic and potential energy functions of the system. We can make any choice of coordinates we would like to describe T and V; let's say we choose some set of **generalized coordinates** q with components q^i and time derivatives $\dot{q}^i$.[8]

The equations of motion governing the system between times t_i and t_f can then be obtained by finding the stationary function $q(t)$ of the **action functional**,

$$S[q] = \int_{t_i}^{t_f} dt\, L(t; q^i, \dot{q}^i)\,. \tag{1.35}$$

This is known as the **principle of least action**. It is no exaggeration to say that a lot of theoretical physics basically involves finding the appropriate action that describes the system of interest, once the principle of least action is applied.

(**End of Lecture**: Wednesday Sep 4 2024)

1.2.1 The central force problem

We'll now turn our attention to an important problem in mechanics—a particle of mass m moving in a potential $V(r)$ which depends only on the distance r from the origin, with the radial component of the force being $F_r = -\partial_r V$. We'll compute this two ways: the first using Newtonian mechanics, and the second using Lagrangian mechanics. The coordinate system we'll use is shown in Fig. 4.

In Newtonian mechanics, we would write down $-\partial_r V = m a_r$, and $-\partial_\theta V/r = 0 = m a_\theta$, where a_r and a_θ are the radial and tangential accelerations respectively. But to solve this equation, we need to find the acceleration in polar coordinates. To do so, we'll use a neat trick: take the particle to be traveling in the complex plane, with coordinate given by the complex number $z = re^{i\theta}$. Then

$$\begin{aligned} \dot{z} &= \dot{r}e^{i\theta} + i\dot{\theta}re^{i\theta} \\ \ddot{z} &= (\ddot{r} - r\dot{\theta}^2)e^{i\theta} + (2\dot{r}\dot{\theta} + r\ddot{\theta})(ie^{i\theta})\,. \end{aligned} \tag{1.36}$$

On the complex plane, the first term is a complex number that is represented by a vector that is parallel to $re^{i\theta}$, while the second term is represented by a vector that is perpendicular to $re^{i\theta}$. We can therefore conclude that the acceleration in polar coordinates is

$$a_r = \ddot{r} - r\dot{\theta}^2\,, \qquad a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta}\,. \tag{1.37}$$

Newton's laws therefore read

$$m(\ddot{r} - r\dot{\theta}^2) = -\partial_r V \tag{1.38}$$

$$m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0 \implies \frac{d}{dt}(mr^2\dot{\theta}) = 0\,. \tag{1.39}$$

You should recognize the second equation as expressing the conservation of angular momentum, with $l = mr^2\dot{\theta}$. Subtituting this expression into Eq. (1.38), we find

$$m\ddot{r} - \frac{l^2}{mr^3} = -\frac{\partial V}{\partial r}\,, \tag{1.40}$$

[8] This can be the usual x, y and z in 3D space, r, θ and ϕ in 3D spherical coordinates, or something even more abstract than these choices, it doesn't matter.

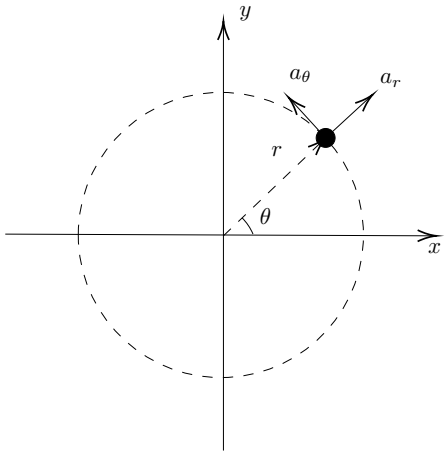

Figure 4: Coordinate system for the central force problem.

the equation of motion governing all central force problems.

Now that was a bit of hike, and again, a lot of issue was that we had to deal with vectors. Let's see how the Lagrangian approach works out. First, we write down the Lagrangian for the system,

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)\,. \tag{1.41}$$

This depends on two coordinates, but is independent of t, and therefore we can write down two Euler-Lagrange equations—one for each coordinate. These are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \implies m\ddot{r} - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0 \tag{1.42}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \implies \frac{d}{dt}(mr^2\dot{\theta}) = 0\,. \tag{1.43}$$

Notice that we obtain the conservation of angular momentum immediately from the second equation involving θ. Substituting once again $l = mr^2\dot{\theta}$, we get the same expression as before in Eq. (1.40).[9]

Let's also compute the first integral, defined for this problem as

$$I = L - \dot{r}\frac{\partial L}{\partial \dot{r}} - \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} = -\frac{1}{2}m\dot{r}^2 - \frac{1}{2}mr^2\dot{\theta}^2 - V(r) = -(T+V)\,, \tag{1.44}$$

which is simply (the negative of) the total energy. Thus, the fact that the first integral is constant is just a statement about conservation of energy in this problem.

[9] You may be tempted to plug l directly into the Lagrangian, which looks like it would remove one coordinate from the Lagrangian, simplifying matters significantly. *But this is pure folly*, and gets you the wrong answer. The coordinates and their derivatives are—each and every one of them—*independent variables of the Lagrangian*. The Lagrangian's only aim in life is to take in these coordinates, their derivatives, and the time coordinate—not knowing that they're related in any way—and spit out a number. The Lagrangian *knows absolutely nothing about trajectories involving these coordinates as a function of time*.

1.3 Variable Endpoints

So far, we've been dealing with situations where we want the endpoints to be fixed. Now let's see what happens when we relax that assumption. Consider the problem of constructing a railway between two ports, located across a strip of land with straight, parallel sides, illustrated in Fig. 5. Suppose that the cost of construction is proportional to the length of the track, but the cost of sea transport is negligible, and so that the ports can be wherever we want. We therefore want to minimize the total length given once again by

$$L[y] = \int_{x_1}^{x_2} dx\, \sqrt{1 + y'^2}\,. \tag{1.45}$$

This time however, we want to allow variations at the endpoints too. Considering a small perturbation, we see that

$$\delta L = \int_{x_1}^{x_2} dx\, \frac{y'}{\sqrt{1+y'^2}}\delta y' \tag{1.46}$$

By now you should be familiar with what to do next: integrate by parts! Don't forget, however, the boundary terms. This gives

$$\delta L = \frac{y'(x_2)}{\sqrt{1+y'(x_2)^2}}\delta y(x_2) - \frac{y'(x_1)}{\sqrt{1+y'(x_1)^2}}\delta y(x_1) - \int_{x_1}^{x_2} dx \left[\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right)\right]\delta y(x)\,. \tag{1.47}$$

The extremum for L is achieved when $\delta L = 0$. But since we have complete freedom to choose $y(x)$, including $y(x_2)$ and $y(x_1)$, the coefficients to each of

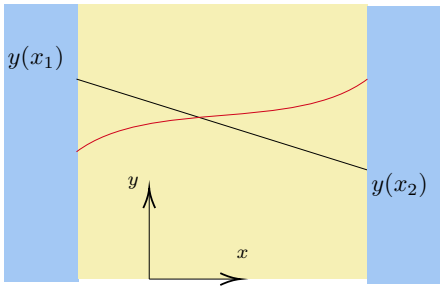

Figure 5: Railway across a strip of land with straight, parallel sides.

$\delta y(x_2)$, $\delta y(x_1)$ and δy inside the integral must be zero. Let's first consider the coefficient for $\delta y(x)$. This just says that

$$\frac{y'}{\sqrt{1+y'^2}} = C \implies y' = \text{constant}\,, \tag{1.48}$$

and so we'd better be building a straight track, as expected. In addition however, the other two terms now enforce

$$\frac{y'(x_2)}{\sqrt{1+y'(x_2)^2}} = \frac{y'(x_1)}{\sqrt{1+y'(x_1)^2}} = 0 \implies y'(x_1) = y'(x_2) = 0\,. \tag{1.49}$$

We therefore want to build the railway exactly perpendicular to the sea. All very reasonable!

What you might have noticed is that by allowing the endpoints to float, you ended up getting a boundary term that enforces some boundary conditions at the two endpoints. Boundary conditions obtained through variation are known as **natural boundary conditions**. We'll come back to more examples with natural boundary conditions later on.

1.4 Noether's Theorem

We now come to one of the most profound results in mathematical physics. We've seen in a few of the previous examples above that when we apply the principle of least action, we can sometimes obtain equations of the form $d(\cdots)/dt = 0$ when the Euler-Lagrange equation is satisfied. For example, for a Lagrangian of the form $L = L(q, \dot{q})$ with no explicit dependence on t, we know from Sec. 1.1.4 that the time derivative of the first integral is conserved:

$$\frac{d}{dt}\left(\dot{q}\frac{\partial L}{\partial \dot{q}} - L\right) = 0\,. \tag{1.50}$$

We also saw that in the central force problem in Sec. 1.2.1, we can show that the angular momentum is conserved, i.e.

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0\,, \tag{1.51}$$

and in that case, the Lagrangian had not explicit dependence on θ. Equations of this form are known as **conservation laws**, and you have, up to this point in your physics career, already encountered many of them. But you may now be seeing a pattern emerge: whenever there is no explicit dependence of L on some quantity, you get conservation laws. And indeed this is true! These are all manifestations of **Noether's theorem**,[10] which says that

> Any continuous symmetry of the action corresponds to a conservation law.

[10] This is actually Noether's *first* theorem; the second theorem is much more obscure, and we won't discuss it in this course.

Now, your first thought might be, it's a little strange that we have a theorem that's just written out in words. That's really because you can prove many different mathematical statements that are described by those words, with various levels of generality and formality. Because Noether's theorem is so important, I want to walk you through a proof of it at a relatively general level. Your second thought might be that there are a lot of terms in there that I haven't carefully defined. But I think in this case, it's best just to define things as we go along. The discussion that follows below is mostly based on Ref. [2].

So let's start with the classical action, written as

$$S[q^i] = \int_{t_i}^{t_f} dt\, L(t; q^i, \dot{q}^i)\,, \tag{1.52}$$

where I remind you that q is some generalized coordinate with components $q^i(t)$ and time derivatives $\dot{q}^i(t)$. We say that the action is **invariant up to a boundary term** under a transformation $q^i(t) \to q^i(t) + \varepsilon\eta^i(t)$, where ε is taken to be a small, time-independent quantity, if

$$\delta S \equiv S[q^i + \varepsilon\eta^i] - S[q^i] = \varepsilon \int_{t_i}^{t_f} dt\, \frac{dK}{dt}\,, \tag{1.53}$$

for all $q^i(t)$, and for some **boundary term** K.

If this is the first time you're seeing this, you might be wondering why even bother with K at all—after all, that's what the word *invariant* should mean. But in fact, a lot of extremely interesting models (in both particle physics and condensed matter physics) have actions which are invariant up to boundary terms, and we would like to apply Noether's theorem to them. An equivalent way to say the same thing is that the action is invariant up to a boundary term if the Lagrangian transforms by a **total derivative** dK/dt.

Any transformation which leaves the action invariant up to a boundary term, or transforms the Lagrangian by a total derivative, is called a **symmetry**.[11] The first question you should ask when you see a Lagrangian is always, "what are its symmetries?" This is the organizing principle behind all of the models that we study in physics. So let's check out some examples. Consider once again the central force problem,

$$S = \int_{t_i}^{t_f} dt\, \left[\frac{1}{2}m\dot{\vec{r}}^2 - V(r)\right]\,, \tag{1.54}$$

The action is invariant under rotations, i.e. perform the transformation $\vec{r} \to \mathsf{R}\vec{r}$, where R is constant, orthogonal matrix with $\mathsf{R}^\intercal = \mathsf{R}^{-1}$, even for very large rotations, since $\dot{\vec{r}}^2 \to (\mathsf{R}\dot{\vec{r}})^\intercal \mathsf{R}\dot{\vec{r}} = \dot{\vec{r}}^\intercal \mathsf{R}^\intercal \mathsf{R}\dot{\vec{r}} = \dot{\vec{r}}^2$, in addition to the fact that rotations preserve length, i.e. $|\mathsf{R}\vec{r}| = r$.

Let's consider another transformation to the action

$$\vec{r}(t) \to \vec{r}(t) - \varepsilon\dot{\vec{r}}(t)\,, \tag{1.55}$$

again with ε being small. Under this transformation, we find[12]

$$\begin{aligned} S[r^i] \to S[r^i - \varepsilon\dot{r}^i] &= \int dt\, \left[\frac{1}{2}m(\dot{\vec{r}} - \varepsilon\ddot{\vec{r}})^2 - V(\vec{r} - \varepsilon\dot{\vec{r}})\right] \\ &= \int dt\, \left[\frac{1}{2}m\dot{\vec{r}}^2 - \varepsilon m\dot{\vec{r}}\ddot{\vec{r}} - V(\vec{r}) + \varepsilon\dot{\vec{r}}^i\partial_i V\right] \\ &= S[r^i] - \varepsilon\int dt\, \frac{d}{dt}\left[\frac{1}{2}m\dot{\vec{r}}^2 - V(r)\right]\,. \end{aligned} \tag{1.56}$$

We can therefore conclude that the action is invariant under this transformation up to a boundary term, $K = L$. What is this strange transformation? Well, it really isn't that mysterious if you just think of it as relabeling time by the coordinate τ, which is just a constant shift from t, i.e. $\tau = t + \varepsilon$. With this transformation, $\vec{r}(t) = \vec{r}(\tau) - \varepsilon\dot{\vec{r}}(\tau)$, and it is easy to see that you get the same transformation as in Eq. (1.56).

Let's now consider an *arbitrary* transformation, one that isn't necessarily a symmetry. It also doesn't necessarily have to leave the endpoints fixed. Denoting this transformation as $q^i \to q^i + \varepsilon\zeta^i(t)$ and apply this to the classical action

[11] Sometimes you will hear people say that a transformation is only a symmetry if $K = 0$, and should be called a *quasisymmetry* otherwise. We won't make this distinction.

[12] For notation convenience, I'll drop the limits of the integral whenever nothing much ever happens to it.

Eq. (1.52). This gives (using index notation)

$$\begin{aligned}\delta S &= \int_{t_i}^{t_f} dt\, \frac{\partial L}{\partial q^i}\varepsilon\zeta^i(t) + \frac{\partial L}{\partial \dot{q}^i}\varepsilon\dot{\zeta}^i(t) \\ &= \int_{t_i}^{t_f} dt\, \frac{\partial L}{\partial q^i} + \frac{\partial L}{\partial \dot{q}^i}\varepsilon\zeta^i\Big|_{t_i}^{t_f} - \int_{t_i}^{t_f} dt\, \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right)\varepsilon\zeta^i \\ &= \int_{t_i}^{t_f} dt\, \left[\frac{\partial L}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right)\right]\varepsilon\zeta^i + \int_{t_i}^{t_f} dt\, \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\varepsilon\zeta^i\right), \end{aligned} \tag{1.57}$$

where we have kept the boundary term after performing integration by parts.

Now, let's choose the arbitrary transformation $\zeta^i(t)$ to be one that leaves the action invariant. Then we have the following relation, after dividing throughout by ε,

$$\int_{t_i}^{t_f} dt\, \frac{d}{dt}\left(K - \frac{\partial L}{\partial \dot{q}^i}\zeta^i\right) = \int_{t_i}^{t_f} dt\, \left[\frac{\partial L}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right)\right]\zeta^i\,. \tag{1.58}$$

In classical mechanics, we're often interested in the behavior of a system at arbitrary times, and indeed, it doesn't really matter what the precise values of t_i and t_f are. Because of that, we need the integrand themselves to be equal, i.e.

$$\frac{d}{dt}\left(K - \frac{\partial L}{\partial \dot{q}^i}\zeta^i\right) = \left[\frac{\partial L}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right)\right]\zeta^i\,. \tag{1.59}$$

This relation is known as the **Noether identity**, and is true for any ζ^i that is a symmetry. Note that at this point, we *have not even mentioned the Euler-Lagrange equation*. This relation is true for *arbitrary* $q^i(t)$, whether or not it corresponds to a the physical solution. When an equation is true before having enforced the Euler-Lagrange equation, we call in an **off-shell** relation, a term that you're bound to run into again.

Now let's take the relation **on-shell**, so that we are only considering $q^i(t)$ that satisfies the Euler-Lagrange equation. For such $q^i(t)$, the right-hand side of the previous equation vanishes, and we find

$$\frac{dQ}{dt} \overset{\text{E-L}}{=} 0\,, \qquad Q = K - \frac{\partial L}{\partial \dot{q}^i}\zeta^i\,. \tag{1.60}$$

This is our big result, **Noether's theorem**: given a symmetry of the action ζ^i, we can define a quantity called a **Noether charge** (or just "charge") that is conserved on-shell, i.e. for physical trajectories.

That was a lot of abstract discussion; let's now take a look at some examples.

(**End of Lecture**: Monday Sep 9 2024)

1.4.1 The central force problem revisited

Let's return once again to the to the central force problem, with

$$S[\vec{r}] = \int_{t_i}^{t_f} dt\, \left[\frac{1}{2}m\dot{\vec{r}}^2 - V(r)\right]. \tag{1.61}$$

We saw earlier that the action was invariant under rotations. You can convince yourself (see Fig. 6 that an infinitesimal rotation about some arbitrary axis with

unit vector $\hat{\alpha}$ can be written as $\vec{r} \to \vec{r} + \varepsilon\hat{\alpha} \times \vec{r}$. We also checked that the rotation leaves the action invariant with $K = 0$. Putting this altogether, we find that the Noether charge associated with rotational symmetry is (using index notation)

$$Q = -\frac{\partial L}{\partial \dot{r}^i}(\hat{\alpha} \times \vec{r})^i = -(m\dot{\vec{r}}_i)(\hat{\alpha} \times \vec{r})^i = \hat{\alpha} \cdot (\vec{r} \times \vec{p}) \,, \tag{1.62}$$

where $\vec{p} \equiv m\dot{\vec{r}}$ is the momentum. You should recognize immediately that $\vec{L} = \vec{r} \times \vec{p}$ is the **angular momentum**, the Noether charge corresponding to rotational symmetry, and since $\hat{\alpha}$ is arbitrary, $\vec{L}$ is conserved.

We also considered another transformation $\vec{r} \to \vec{r} - \varepsilon\dot{\vec{r}}$, which left the action invariant up to a boundary term, $K = L$, the Lagrangian. We also saw that this transformation was equivalent to time translation, $t \to t + \varepsilon$. Once again, the Noether charge associated with this symmetry is

$$Q = L - \frac{\partial L}{\partial \dot{r}^i}\dot{\vec{r}}^i = \frac{1}{2}m\dot{\vec{r}}^2 - V(r) - m\dot{\vec{r}}^2 = -\left[\frac{1}{2}m\dot{\vec{r}}^2 + V(r)\right] \,, \tag{1.63}$$

which is simply the statement that the Noether charge is the **total energy**, and is conserved. More generally, for any action that has time-translation as a symmetry, e.g. $L = L(q^i, \dot{q}^i)$, the total energy is

$$E = \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L \,, \tag{1.64}$$

and is conserved. You should recognize that E is nothing but the first integral that we described in Eq. (1.31).

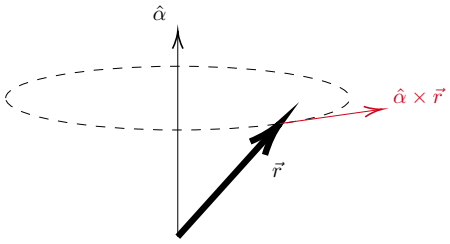

Figure 6: Infinitesimal rotations of $\vec{r}$ about axis $\hat{\alpha}$ are in the direction $\hat{\alpha} \times \vec{r}$.

1.4.2 Common symmetries

Before we move on from Noether's theorem, let's examine some common symmetries that you've probably already encountered in classical mechanics, and how they usually arise from Noether's theorem.

1.4.2.1 Spatial translation invariance implies momentum conservation.

If a Lagrangian is invariant under a small, constant, spatial translation ε of some coordinate q, i.e. $L(t; q + \varepsilon, \dot{q}) = L(t; q, \dot{q})$.[13] Noether's theorem (Eq. (1.60)) tells us that the associated charge is

$$Q = p \equiv \frac{\partial L}{\partial \dot{q}} \text{ and } \frac{dp}{dt} = 0 \,, \tag{1.65}$$

where p is the momentum conjugate to q. One simple scenario where momentum conservation holds is when the Lagrangian simply doesn't depend explicitly on q at all; in this case, q is known as a **cyclic coordinate**. This is what happens in systems with cylindrical symmetry, for example, and therefore have Lagrangians that do not depend on the azimuthal angle, leading to an associated conserved angular momentum.

[13] Since ε is constant, $\dot{q}$ remains unchanged.

1.4.2.2 Time translation invariance implies energy conservation.

Suppose we performed the translation $q(t) \to q(t + \varepsilon)$, where ε is once again small and constant. You can imagine this as taking a trajectory that previously started

at time t, and now starting it at $t-\varepsilon$ instead. Then the action transforms as

$$\begin{aligned} S[q] &\to \int_{t_i}^{t_f} dt\, L(t; q(t+\varepsilon), \dot{q}(t+\varepsilon)) \\ &= \int_{t_i+\varepsilon}^{t_f+\varepsilon} d\tau\, L(\tau-\varepsilon; q(\tau), \dot{q}(\tau)) \\ &= S[q] + \varepsilon\left[L(t_f, q(t_f), \dot{q}(t_f)) - L(t_i, q(t_i), \dot{q}(t_i))\right] - \varepsilon \int_{t_i}^{t_f} d\tau\, \frac{\partial L}{\partial \tau}\,. \end{aligned} \tag{1.66}$$

Here, the second term is a boundary term, but the third term might not be: we can therefore see that time translation is a symmetry of the action if the Lagrangian does not depend explicitly on t, with the conserved charge being energy, i.e.

$$\frac{\partial L}{\partial t} = 0 \implies \text{Energy is conserved.} \tag{1.67}$$

We have derived the expression for energy several times already, for example in Eq. (1.64): we do this by noting that $q(t+\varepsilon) = q(t) + \varepsilon\dot{q}(t)$, and then applying Noether's theorem.

1.5 Continuous Systems

So far, we have been looking at Lagrangians that have only a single particle. For multiple particles, we can similarly write down Lagrangians involving $\{q_i, \dot{q}_i\}$ for each particle. Frequently, however, we're interested in *fields*, like the electromagnetic field, and we want to be able to write down *field theories* by similarly writing down an action, extremizing it, and finding the equations of motion. Suppose we have a field with value $\varphi(x)$, where $x \equiv x^\mu$, $\mu = 0, 1, \cdots, d$ is the coordinate of a $(d+1)$-dimensional space (or spacetime). We can write down actions that look like

$$S[\varphi] = \int dt\, L = \int d^{d+1}x\, \mathcal{L}(x, \varphi, \partial_\mu \varphi)\,, \tag{1.68}$$

where $\mathcal{L}$ is the **Lagrangian density**,

$$L \equiv \int d^d x\, \mathcal{L}\,, \tag{1.69}$$

and

$$\partial_\mu \varphi \equiv \frac{\partial \varphi}{\partial x^\mu}\,. \tag{1.70}$$

We're going to follow a similar path to what we discussed in Sec. 1.1.1. We'll take a variation over the action, $\varphi(x) \to \varphi(x) + \varepsilon\eta(x)$, and so correspondingly $\partial_\mu\varphi(x) \to \partial_\mu\varphi(x) + \varepsilon\partial_\mu\eta(x)$, with no variation on the d-dimensional boundary of our $(d+1)$-dimensional space. We get

$$\delta S = \int d^{d+1}x \left[\frac{\partial \mathcal{L}}{\partial \varphi}\varepsilon\eta + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)}\varepsilon\partial_\mu\eta\right]\,. \tag{1.71}$$

At this point, we want to integrate by parts, which we can do using the **divergence theorem**,

$$\int_\Omega d^{n+1}x\,\partial_\mu f^\mu = \int_{\partial\Omega} dS\, n_\mu f^\mu\,, \tag{1.72}$$

where $\partial\Omega$ is the boundary of Ω, dS is an element of are on the boundary, and n_μ is the outward pointing normal. Much more on this later on in the course, but for now, we see that

$$\begin{aligned}\int_\Omega d^{n+1}x\,\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\epsilon\eta\right) &= \int d^{n+1}x\,\left(\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\varepsilon\eta + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\varepsilon\partial_\mu\eta\right)\\ &= \int_{\partial\Omega} dS\, n^\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right)\epsilon\eta\\ &= 0\,,\end{aligned} \tag{1.73}$$

since η does not vary on the boundary, allowing us to write

$$\delta S = \int d^{d+1}x\,\left[\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right]\varepsilon\eta\,. \tag{1.74}$$

Effectively, if we are only allowing for variations that are zero on the boundary, we can simply integrate by parts by switch the position of ∂_μ. Extremizing the action therefore means that

$$\frac{\delta S}{\delta\varphi} \equiv \frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} = 0\,, \tag{1.75}$$

which is the Euler-Lagrange equation in continuous form.

1.5.1 The vibrating string

Our first example of a continuous system is a string, shown in Fig. 7. The string has fixed ends, a mass per unit length of ρ, and is under tension T. If we assume only small displacements from equilibrium, the Lagrangian is

$$L = \int_0^L dx\,\left(\frac{1}{2}\rho\dot{y}^2 - \frac{1}{2}Ty'^2\right)\,, \tag{1.76}$$

where the dot denotes a partial derivative with respect to t, and the prime a partial derivative with respect to x.

The variation of the action is

$$\begin{aligned}\delta S &= \int dt\int_0^L dx\,\left(\rho\dot{y}\delta\dot{y} - Ty'\delta y'\right)\\ &= \int dt\int_0^L dx\,\left(-\rho\ddot{y} + Ty''\right)\delta y\,,\end{aligned} \tag{1.77}$$

where to reach the second line, we perform integration by parts, and because the endpoints are fixed, $\delta y = 0$ when $x = 0$ and $x = L$. Extremizing the action $\delta S = 0$ gives the equation of motion

$$\rho\ddot{y} - Ty'' = 0\,, \tag{1.78}$$

which is the wave equation with transverse waves propagating with speed $c = \sqrt{T/\rho}$.

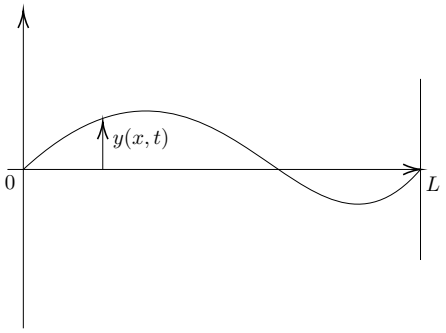

Figure 7: Set up for the vibrating string.

(**End of Lecture**: Wednesday Sep 11 2024)

1.5.2 The canonical energy-momentum tensor

Just as in the case of a single-particle Lagrangian, with conservation of momentum and energy, if the Lagrangian density $\mathcal{L} \equiv \mathcal{L}(\varphi, \partial_\mu \varphi)$ does not depend explicitly on x^μ, then we should expect a conservation law as well.

We'll go through the derivation of the Noether charge here again, as the result is presented in often confusing ways in many textbooks. Suppose our action is of the form

$$S = \int d^{d+1}x \, \mathcal{L}(\varphi, \partial_\mu \varphi) \,. \tag{1.79}$$

Then, we know that under the transformation $x^\mu \to x^\mu + \varepsilon^\mu$, where ε^μ is constant, we can expand $\mathcal{L}$ to give

$$\delta S = \int d^{d+1}x \, \varepsilon^\mu \partial_\mu \mathcal{L} \,, \tag{1.80}$$

where

$$\partial_\mu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \varphi} \partial_\mu \varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)} \partial_\mu (\partial_\nu \varphi) \,. \tag{1.81}$$

Note that in this transformation is a symmetry of the action, since $\varepsilon^\mu \partial_\mu \mathcal{L}$ is a total derivative. However, if we promote $\varepsilon^\mu \to \varepsilon^\mu(x^\mu)$ from a *global* to a local transformation, this stops being true.

Now, let's consider the *arbitrary* transformation, $x^\mu \to x^\mu + \zeta^\mu(x^\mu)$ (not necessarily constant, and not necessarily with no variation on the boundary!). This gives

$$\varphi \to \varphi + \zeta^\mu \partial_\mu \varphi \,, \quad \partial_\mu \varphi \to \partial_\mu \varphi + \partial_\mu (\zeta^\nu \partial_\nu \varphi) \,. \tag{1.82}$$

$$\begin{aligned} \delta S &= \int d^{d+1}x \left[\frac{\partial \mathcal{L}}{\partial \varphi} \zeta^\mu \partial_\mu \varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)} \partial_\nu (\zeta^\mu \partial_\mu \varphi) \right] \\ &= \int d^{d+1}x \left[\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)} \right] \zeta^\mu \partial_\mu \varphi + \int d^{d+1}x \, \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)} \zeta^\mu \partial_\mu \varphi \right) . \end{aligned} \tag{1.83}$$

Now, setting $\zeta^\mu = \varepsilon^\mu$, and going on-shell, the first term goes to zero as the Euler-Lagrange equations are satisfied, giving

$$\begin{aligned} \int d^{d+1}x \, \varepsilon^\mu \partial_\mu \mathcal{L} &= \int d^{d+1}x \, \varepsilon^\mu \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)} \partial_\mu \varphi \right) \\ &\implies \int d^{d+1}x \, \varepsilon^\mu \partial_\nu \left(\mathcal{L} \delta^\nu_\mu - \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)} \partial_\mu \varphi \right) = 0 \,. \end{aligned} \tag{1.84}$$

We call the Noether charge here the **canonical energy-momentum tensor**,[14]

$$T^\nu{}_\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)} \partial_\mu \varphi - \delta^\nu_\mu \mathcal{L} \,, \tag{1.85}$$

with the statement

$$\partial_\nu T^\nu{}_\mu = 0 \tag{1.86}$$

denoting *local* energy-momentum conservation. This can be extended to the case where there are multiple fields, in which case the first term on the right-hand side of Eq. (1.84) picks up a sum over all fields.

[14] The reason its called "canonical" and not just "the" energy-momentum tensor is because it's often the case that this isn't quite what you want to work with. Instead, you often want to work with $T^{\mu\nu}$ shifted by some other object whose total derivative is zero. In electromagnetism, this procedure of "fixing" $T^{\mu\nu}$ is often called the *Belinfante improvement procedure*.

1.5.3 The vibrating string revisited

We now return to the vibrating string, with Lagrangian given in Eq. (1.76). We can now compute the canonical energy-momentum tensor for this system. For $\mu = 0$,

$$T^{\nu}{}_{0} = \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)} \partial_0 \varphi - \delta^\nu_0 \mathcal{L} \,, \tag{1.87}$$

and so

$$T^{0}{}_{0} = \rho \dot{y}^2 - \left(\frac{1}{2}\rho \dot{y}^2 - \frac{1}{2} T y'^2 \right) = \frac{1}{2}\rho \dot{y}^2 + \frac{1}{2} T y'^2 \,, \tag{1.88}$$

$$T^{1}{}_{0} = -T y' \dot{y} \,. \tag{1.89}$$

In addition, the conservation laws read

$$\begin{aligned} 0 = \partial_\nu T^{\nu}{}_{0} &= \partial_\nu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)} \partial_0 \varphi \right] - \partial_\nu \delta^\nu_0 \mathcal{L} \\ &= \partial_t (\rho \dot{y}^2 - \mathcal{L}) + \partial_x (-T y' \dot{y}) \\ &= \partial_t \left(\frac{1}{2}\rho \dot{y}^2 + \frac{1}{2} T y'^2 \right) - \partial_x (T y' \dot{y}) \,, \end{aligned} \tag{1.90}$$

$$\begin{aligned} 0 = \partial_\nu T^{\nu}{}_{1} &= \partial_\nu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)} \partial_1 \varphi \right] - \partial_\nu \delta^\nu_1 \mathcal{L} \\ &= \partial_t (\rho \dot{y} y') + \partial_x (-T y'^2 - \mathcal{L}) \\ &= \partial_t (\rho \dot{y} y') - \partial_x \left(\frac{1}{2}\rho \dot{y}^2 + \frac{1}{2} T y'^2 \right) \,. \end{aligned} \tag{1.91}$$

These are local conservation laws of the same form as the *continuity equation*,

$$\frac{\partial q}{\partial t} + \nabla \cdot \vec{J} = 0 \,, \tag{1.92}$$

where q should be thought of as a local density (e.g. charge density of fluid density) with a globally conserved quantity $Q = \int d^d x \, q$, and $\vec{J}$ is a flux. For $\partial_\nu T^{\nu}{}_{0} = 0$, the local density is

$$T^{0}{}_{0} = \frac{1}{2}\rho \dot{y}^2 + \frac{1}{2} T y'^2 \,, \tag{1.93}$$

which is the energy density. The energy flux is $T^{1}{}_{0} = -T \dot{y} y'$, which is the rate that a segment of string is doing work on its neighbor to the right. Let's check that the total energy is globally conserved:

$$\begin{aligned} \frac{d}{dt} \int_0^L dx \left(\frac{1}{2}\rho \dot{y}^2 + \frac{1}{2} T y'^2 \right) &= \int_0^L dx \, \frac{\partial}{\partial t} \left(\frac{1}{2}\rho \dot{y}^2 + \frac{1}{2} T y'^2 \right) \\ &= \int_0^L dx \, \frac{\partial}{\partial x} (T y' \dot{y}) \\ &= T y' \dot{y} \big|_0^L = 0 \,, \end{aligned} \tag{1.94}$$

since the string is clamped at the ends, as expected.

That's really the last we'll say about continuous systems, but of course, there is so much more to say. You'll learn more about these systems in classes like quantum field theory and statistical physics.

1.6 Constraints

When we want to find the extremum of some function or functional, we often want to impose some constraints on that extremum. Here are some classic examples of problems that involve finding extrema under some constraint:

1. Given a curve a fixed length on a plane, what is the maximum area that it can enclose? What is the shape of the curve in that case?

2. Consider a chain with fixed length suspended between two points, both a fixed height above the ground. What is the shape of the chain that minimizes the potential energy?

The strategy that we are going to adopt here is identical to what you may have seen in vector calculus, with the use of **Lagrange multipliers**. Suppose you have a functional $J[y]$ that you want to extremize, subject to some constraint given by $K[y] = 0$. Consider the modified functional $\tilde{J}[y, \lambda] = J[y] - \lambda K[y]$, for some new parameter λ. First, we note that if vary the new parameter λ, we find

$$\tilde{J}[y, \lambda + \delta\lambda] - \tilde{J}[y, \lambda] = -\delta\lambda \cdot K[y]\,, \tag{1.95}$$

and so when we extremize $\tilde{J}$, including over possible variations in λ, we require $K[y] = 0$, the precise constraint that we wanted to impose!

Under a variation of y, on the other hand, we obtain

$$\tilde{J}[y + \delta y, \lambda] - \tilde{J}[y, \lambda] = \delta J[y, \delta y] - \lambda \cdot \delta K[y, \delta y]\,, \tag{1.96}$$

where $\delta J[y, \delta y] = \delta J[y + \delta y] - \delta J[y]$, and likewise for δK. At the extremum $\delta\tilde{J} = 0$ for some $y = y^*$ then, we must necessarily have

$$\delta J[y^*, \delta y] = \lambda \cdot \delta K[y^*, \delta y] \tag{1.97}$$

for all variations δy away from y^*. What's going on here? First of all, $K[y^*] = 0$, and so y^* corresponds to a point where the constraint is satisfied. Now, if the variation δy preserves this constraint, then $\delta K[y^*, y + \delta y] = 0$; in that case, the previous equation says that $\delta J[y^*, y + \delta y] = 0$ as well. In other words, J *itself is also extremized, as long as we limit ourselves to* δy *that preserves the constraint*. The trick of considering $\tilde{J}$ in place of J therefore does exactly what we want!

Fig. 8 shows an illustration of how this method works. The equation of motion for λ enforces the constraint, and fixes the trajectory to lie along the blue line, with $K[y] = 0$. We can see that if we choose variations along the blue trajectory so that the constraint is always satisfied, $\delta K = 0$, but not necessarily δJ, *except* at the extremum of J along the trajectory (marked in red). At this point, in the direction $\delta K = 0$, $\delta J = 0$ as well, since it is a local extremum. Our clever choice of $\tilde{J}$ gave us *1)* an equation of motion for λ that constrained us to the blue trajectory, and *2)* an equation of motion for y that is satisfied for points where J is extremized along the trajectory.

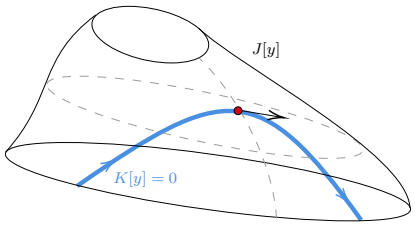

Figure 8: An illustration of the Lagrange multiplier method.

1.6.1 Maximum entropy distribution

Our first example of how to use this comes from statistical mechanics. Let Γ denote the classical phase space of a mechanical system of N particles governed by a Hamiltonian $H(p_n^i, q_n^i)$, with $n = 1, \cdots, N$, $i = 1, 2, 3$ and $d\Gamma \equiv \prod_{n=1}^{N} (d^3\vec{p}_n\, d^3\vec{q}_n)$. We define the phase space density $\rho(p_n^i, q_n^i)$ such that $\rho(p_n^i, q_n^i) d\Gamma$ is the probability of the system being in some state given by p_n^i and q_n^i in the small region $d\Gamma$.

The entropy related to this probability distribution can be defined as the functional

$$S[\rho] = -\int_\Gamma d\Gamma\, \rho \log \rho\,. \tag{1.98}$$

We now want to find ρ that maximizes the entropy for a given mean energy,

$$\langle E\rangle = \int_\Gamma d\Gamma\, \rho H\,. \tag{1.99}$$

However, ρ is subject to the constraint that it is a probability density function, and so we must enforce the constraint

$$\int_\Gamma d\Gamma\, \rho = 1\,. \tag{1.100}$$

We can solve this problem by defining a new functional $\tilde{S}[\rho,\alpha,\beta]$, where

$$\tilde{S}[\rho,\alpha,\beta] = S[\rho] + \alpha\left(\int_\Gamma d\Gamma\,\rho - 1\right) + \beta\left(\int_\Gamma d\Gamma\,\rho H - \langle E\rangle\right)\,. \tag{1.101}$$

The equations of motion for the real-number[15] Lagrange multipliers α and β enforce the two constraints that we mentioned above. Now for ρ, we have

$$\rho\log\rho \to (\rho+\delta\rho)\log(\rho+\delta\rho) = \rho\log\rho + \delta\rho\log\rho + \rho\cdot\frac{\delta\rho}{\rho} \tag{1.102}$$

and so

$$\delta\tilde{S} = \int_\Gamma d\Gamma\,(1+\log\rho+\alpha+\beta H)\,\delta\rho\,, \tag{1.103}$$

and setting the term in parentheses to zero to extremize $\delta\tilde{S}$ gives

$$\rho(p_n^i, q_n^i) = e^{-1-\alpha-\beta H(p_n^i,q_n^i)}\,. \tag{1.104}$$

At this point, α and β can be determined from the normalization and energy constraints detailed above, with the exact answer being determined by the expect form of H. This probability density is the usual *canonical distribution*. Note that you can see from this procedure that the temperature $T \equiv 1/\beta$ appears as a Lagrange multiplier in this procedure.

[15] They're not functions! In particular, they can live *outside* the integral over Γ. There's nothing wrong with having a functional depend on real numbers. You can also use Lagrange multipliers that are functions, if for example the constraint you are imposing is on ρ itself, and not the integral of ρ, as we are in this case.

(**End of Lecture**: Monday Sep 16 2024)

References

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