PY501 – Mathematical Physics

Hongwan Liu

Table of Contents

1 Calculus of Variations

References: Stone & Goldbart (SG) Chapter 1; Byron & Fuller (BF) Chapter 2; Arfken, Weber & Harris (AWH) Chapter 22.

Finding where the minimum or maximum value of some quantity occurs is an extremely common task. For example, you might want to know where the highest point on a map is, or when you had the highest heartrate throughout the day. Mathematically, we have a function $f(x)$, and we want to find the value of x which maximizes or minimizes $f(x)$. To do so, for a differentiable function f, we simply take the derivative and set it to zero

$$f'(x) = 0 \tag{1.1}$$

and solve for x to find the stationary points of the function.

Often though, we run into problems where we want to find the *function* at which the minimum or maximum value of some function occurs. Some examples include:

1. What is the shortest path to take between points A and B?

2. What closed curve of fixed length encloses the maximum possible area?

3. What form does a hanging heavy chain of fixed length take, so as to minimize its potential energy?

To answer these questions mathematically, we need an object called a **functional** $J[y]$, which maps smooth[1] functions y (e.g. a path, a curve) to a real number (e.g. a distance, an area). This is just another map, like a function is. But now, we want to develop the tools required to define a **functional derivative** such that setting

$$\frac{\delta J}{\delta y(x)} = 0 \tag{1.2}$$

will allow us to find a function $y(x)$ (e.g. a path, a curve) that maximizes $J[y]$ (e.g. a distance, an area).

1.1 Functionals

What does a functional look like? For our purposes, we will be dealing with functionals that have the following form:

$$J[y] = \int_{x_1}^{x_2} dx \, f(x, y, y', y'', \cdots, y^{(n)}), \tag{1.3}$$

where f is a function of the real numbers x, y, $y' \cdots$, *independently*.[2] We call these kinds of functionals **local** in x. As you can see, J takes in a function y, performs an integral, and returns a real number, which is exactly what a functional should do.

1.1.1 The functional derivative

Let us work out the functional derivative for the case where

$$J[y] = \int_{x_1}^{x_2} dx \, f(x, y, y'). \tag{1.4}$$

To do this, suppose we make an infinitesimal shift $y(x) \to y(x) + \varepsilon \eta(x)$, where ε is an infinitesimally small constant,[3] and $\eta(x)$ is some arbitrary function. Then

---

[1] "Smooth" means that all derivatives of the function exists. We won't ever be interested in subtleties involving continuity and differentiability in this course.

[2] This is a cause of endless confusion, so pay attention! From the perspective of f, y and y' are *independent variables*.

[3] i.e. with ε^2 and higher powers of ε all being zero.

the shift in J is

$$\delta J = J[y + \varepsilon\eta] - J[y] = \int_{x_1}^{x_2} dx \, [f(x, y + \varepsilon\eta, y' + \varepsilon\eta') - f(x, y, y')] \,. \quad (1.5)$$

Since ε is an infinitesimal quantity, we can perform a Taylor expansion up to first order about y and y' to find

$$\delta J = \int_{x_1}^{x_2} dx \, \left[\varepsilon\eta \frac{\partial f}{\partial y} + \varepsilon\eta' \frac{\partial f}{\partial y'} \right] \,. \quad (1.6)$$

To make further progress, we integrate the second term by parts, giving

$$\delta J = \left[\varepsilon\eta \frac{\partial f}{\partial y'} \right]_{x_1}^{x_2} + \int_{x_1}^{x_2} dx \, \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \right] \varepsilon\eta \,. \quad (1.7)$$

We are frequently—but not always!—concerned with finding functions of y with fixed endpoints;[4] in that case, $\eta(x_1) = \eta(x_2) = 0$, and the boundary terms in the first term on the right vanishes, leaving

$$\delta J = \int_{x_1}^{x_2} dx \, \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \right] \varepsilon\eta \,. \quad (1.8)$$

This can be written suggestively as

$$\delta J = \int_{x_1}^{x_2} dx \, \delta y(x) \left(\frac{\delta J}{\delta y(x)} \right) \,, \quad (1.9)$$

where $\delta y(x) \equiv \varepsilon\eta(x)$, and

$$\boxed{\frac{\delta J}{\delta y(x)} \equiv \frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \quad (1.10)}$$

is the **functional derivative** of J with respect to $y(x)$.

To aid our understanding, it can be helpful to think discretely. We can discretize x between x_1 and x_2 into N discrete steps, so that the function y takes up values $y_i = y(x_i)$, where $i = 1, 2 \cdots, N$. A choice of the function y corresponds in this discrete picture to a choice of $\{y_i\}$, which is a single point in an N-dimensional space. Fig. 1 has a visualization of this. At every point in this N-dimensional space, we can assign a value to J. The small variation $\varepsilon\eta$ can likewise be discretized, so that $\delta y_i = \varepsilon\eta_i$, which can be thought of as a step in a particular direction in the same N-dimensional space. In this discrete picture,

$$\delta J = \sum_{i=1}^{N} \frac{\partial J}{\partial y_i} \delta y_i \,, \quad (1.11)$$

just as one might expect for a function J defined in the N-dimensional space indexed by y_i. In the continuous limit, we need to trade the summation over discrete i to an integral over the continuous label x, leading to Eq. (1.9).[5]

1.1.2 The Euler-Lagrange equation

Now, to find the **stationary points**—maxima, minima or saddle points—of J, we want to set $\delta J = 0$ for any arbitrary variation $\varepsilon\eta$, just like for a function g

[4] For example, if we are interested in finding the path with the short distance between two fixed points.

Figure 1: A discretized visualization of varying over functions. (**HL**: To be completed, but not difficult to imagine!)

[5] In this picture, a choice of the function y is a point in an uncountably infinite dimensional space, and $\epsilon\eta$ is a step in some arbitrary direction, and J is function that returns a real number at every point in this space.

on $\mathbb{R}^n$, we want δg to be zero for a step in any direction at a stationary point. Referring to Eq. (1.8), we require

$$\int_{x_1}^{x_2} dx \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \right] \varepsilon \eta = 0 \,. \tag{1.12}$$

Since this applies for *any* $\eta(x)$, the term in the square brackets $[\cdots]$ must vanish.[6] we must have

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0 \tag{1.13}$$

[6] We can prove that this is true rigorously, given various conditions on η and f. This is often known as the **fundamental lemma of the calculus of variations**. For further discussion, see SG 1.2.2 and Wikipedia.

for a stationary point for J. This is the famous **Euler-Lagrange equation**.

Through derivations similar to what we saw above, we can get generalized Euler-Lagrange equations for more complicated versions of J. If J depends on more than one function y_i, for example, the stationary points are given by

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx} \left(\frac{\partial f}{\partial y_i'} \right) = 0 \,, \tag{1.14}$$

which is one equation for each variable y_i. If on the other hand, f depends on higher derivatives y'', y''' and so on, then the generalized Euler-Lagrange equation we get is

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial f}{\partial y''} \right) - \frac{d^3}{dx^3} \left(\frac{\partial f}{\partial y'''} \right) + \cdots = 0 \,. \tag{1.15}$$

1.1.3 Applications

Time to apply what we've learnt! We'll apply the Euler-Lagrange equations to two examples.

1.1.3.1 Soap film supported by a pair of coaxial rings. Consider Fig. 2, where a pair of co-axial rings support a soap film. The energy associated with the configuration is directly proportional to the area, and hence the soap film tends to minimize this energy by minimizing its area, subject to the constraint that the soap film has to end on the rings at either end. The area of associated with a segment of the film of width dx is

Figure 2: Soap film between two rings, centered at x_1 and x_2, with radii $y(x_1)$ and $y(x_2)$.

$$dA = 2\pi y(x) \sqrt{dx^2 + dy^2} = 2\pi y(x) \sqrt{1 + y'^2} \, dx \,, \tag{1.16}$$

and so the functional that we want to minimize is

$$J[y] = \int_{x_1}^{x_2} dx \, f(y, y') \,, \quad f(y, y') \equiv y \sqrt{1 + y'^2} \,, \tag{1.17}$$

with the endpoint values fixed at $y(x_1)$ and $y(x_2)$. The minimum for this functional can therefore be found by applying the Euler-Lagrange equations. The partial derivatives that we need for the Euler-Lagrange equation are

$$\frac{\partial f}{\partial y} = \sqrt{1 + y'^2} \,, \quad \frac{\partial f}{\partial y'} = \frac{yy'}{\sqrt{1 + y'^2}} \,, \tag{1.18}$$

3

and so the Euler-Lagrange equation says that the minimal surface area profile $y(x)$ must satisfy

$$\sqrt{1+y'^2} - \frac{d}{dx}\left(\frac{yy'}{\sqrt{1+y'^2}}\right) = 0$$

$$\implies \sqrt{1+y'^2} - \frac{y'^2}{\sqrt{1+y'^2}} - \frac{yy''}{\sqrt{1+y'^2}} + \frac{yy'^2 y''}{(1+y'^2)^{3/2}} = 0$$

$$\implies \frac{1}{\sqrt{1+y'^2}} - \frac{yy''}{(1+y'^2)^{3/2}} = 0. \qquad (1.19)$$

This differential equation looks difficult to solve, but fortunately there's a neat little trick to do so. Multiplying by y' on both sides gives

$$0 = \frac{y'}{\sqrt{1+y'^2}} - \frac{yy'y''}{(1+y'^2)^{3/2}} = \frac{d}{dx}\left(\frac{y}{\sqrt{1+y'^2}}\right). \qquad (1.20)$$

We'll return to how we knew this trick would work later on. In the mean time, the solution is

$$\frac{y}{\sqrt{1+y'^2}} = \kappa \qquad (1.21)$$

for some constant κ. Rewriting this as

$$\frac{dy}{dx} = \sqrt{\frac{y^2}{\kappa^2} - 1} \implies \int \frac{dy}{\sqrt{y^2/\kappa^2 - 1}} = \int dx, \qquad (1.22)$$

we can integrate this first-order ordinary differential equation by substituting $y = \kappa \cosh t$ and $dy = \kappa \sinh t$ to find

$$\kappa \int dt = \int dx \implies \kappa t = x + C \implies y = \kappa \cosh\left(\frac{x+C}{\kappa}\right) \qquad (1.23)$$

for some constants κ and C. These can be determined by enforcing the two boundary conditions—the radii of the two rings, $y(x_1)$ and $y(x_2)$.

1.1.3.2   The brachistochrone   The next problem we will consider is a famous one, posed by Johann Bernoulli in 1696. What shape should a wire with endpoints $(0,0)$ and (a,b) take, in order that a frictionless bead will slide from rest down the wire in the shortest possible time?

First, the total time T taken down a given path can be written as

$$T = \int_0^T dt = \int_0^L \frac{ds}{v}, \qquad (1.24)$$

where v is the speed of the bead, and s is the distance along the path, with a total length L. However, we can once again write $ds^2 = dx^2 + dy^2$ so that $ds = \sqrt{1+y'^2}\, dx$, and apply conservation of energy to find $v = \sqrt{2gy}$. Thus, we can define a functional T

$$T[y] = \int_0^a dx\, \sqrt{\frac{1+y'^2}{2gy}} \qquad (1.25)$$

that we want to minimize with respect to y, again with fixed end points. We can therefore apply the Euler-Lagrange equation, which gives after some algebra

$$yy'' + \frac{1}{2}(1+y'^2) = 0. \qquad (1.26)$$

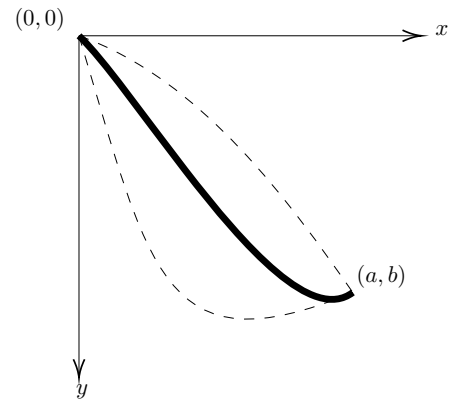Figure 3: Possible shapes of a wire for a frictionless bead to travel from the origin $(0,0)$ to a point (a,b).

Once again, we can use the trick of multiplying by y' to find that

$$y'\left(yy'' + \frac{1}{2}(1 + y'^2)\right) = \frac{1}{2}\frac{d}{dx}\left(y(1 + y'^2)\right) = 0, \qquad (1.27)$$

or

$$y(1 + y'^2) = 2C \qquad (1.28)$$

for some constant C. From this point, one can check that the following parametrization $(x(t), y(t))$ is indeed a solution to the differential equation above:

$$x = C(\theta - \sin\theta)$$
$$y = C(1 - \cos\theta), \qquad (1.29)$$

although it is surprisingly hard to pin down the details regarding this solution.[7] This parametric curve is known as the **cycloid**, which is the curve traced out by a fixed point on the rim of a wheel that is rolling without slipping along a flat surface.

1.1.4 First integral

In both applications discussed in Sec. 1.1.3, we were able to rephrase the differential equation to be solved as $dI/dx = 0$, implying that I is some constant associated with the problem. This quirk, which somewhat resembles the conservation of quantities like energy, is something we will revisit in much greater depth later: it is far deeper than just a mathematical coincidence. For now, let's just take a quick look at where it comes from. In both cases, the function inside the integral was $f(y, y')$, with no explicit dependence on x, implying that

$$\frac{df}{dx} = \overset{0}{\cancel{\frac{\partial f}{\partial x}}} + y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'}. \qquad (1.30)$$

We define the **first integral** of the Euler-Lagrange equation as

$$I \equiv f - y'\frac{\partial f}{\partial y'}, \qquad (1.31)$$

from which we can check that

$$\frac{dI}{dx} = y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} - y''\frac{\partial f}{\partial y'} - y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$$
$$= y'\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]. \qquad (1.32)$$

Thus, $dI/dx = 0$ if the Euler-Lagrange equation is satisfied.

You can show that if f depends on more than one variable, so that we have a functional of the form

$$J[y_1, y_2, \cdots, y_n] = \int dx\, f(y_1, y_2, \cdots, y_n; y_1', y_2', \cdots, y_n'), \qquad (1.33)$$

the first integral is of the form

$$I = f - \sum_i y_i'\frac{\partial f}{\partial y_i'}. \qquad (1.34)$$

Note that there is only one first integral, even when there are multiple dependent variables y_i.

[7] For example, is there a unique solution, and does the solution always occur with $\theta \in [0, 2\pi)$ for every point (a, b)? See for example Ref. [1] for more details.

5

1.2 Lagrangian Mechanics

It turns out that classical mechanics can be reformulated as a problem of finding the stationary function of some functional. Given some system, we first define the **Lagrangian** function $L = T - V$, where T and V are the kinetic and potential energy functions of the system. We can make any choice of coordinates we would like to describe T and V; let's say we choose some set of **generalized coordinates** q with components q^i and time derivatives $\dot{q}^i$.[8]

The equations of motion governing the system between times t_i and t_f can then be obtained by finding the stationary function $q(t)$ of the **action functional**,

$$S[q] = \int_{t_i}^{t_f} dt\, L(t; q^i, \dot{q}^i)\,. \tag{1.35}$$

This is known as the **principle of least action**. It is no exaggeration to say that a lot of theoretical physics basically involves finding the appropriate action that describes the system of interest, once the principle of least action is applied.

(**End of Lecture**: Wednesday Sep 4 2024)

1.2.1 The central force problem

We'll now turn our attention to an important problem in mechanics—a particle of mass m moving in a potential $V(r)$ which depends only on the distance r from the origin, with the radial component of the force being $F_r = -\partial_r V$. We'll compute this two ways: the first using Newtonian mechanics, and the second using Lagrangian mechanics. The coordinate system we'll use is shown in Fig. 4.

In Newtonian mechanics, we would write down $-\partial_r V = ma_r$, and $-\partial_\theta V/r = 0 = ma_\theta$, where a_r and a_θ are the radial and tangential accelerations respectively. But to solve this equation, we need to find the acceleration in polar coordinates. To do so, we'll use a neat trick: take the particle to be traveling in the complex plane, with coordinate given by the complex number $z = re^{i\theta}$. Then

$$\dot{z} = \dot{r}e^{i\theta} + i\dot{\theta}re^{i\theta}$$
$$\ddot{z} = (\ddot{r} - r\dot{\theta}^2)e^{i\theta} + (2\dot{r}\dot{\theta} + r\ddot{\theta})(ie^{i\theta})\,. \tag{1.36}$$

On the complex plane, the first term is a complex number that is represented by a vector that is parallel to $re^{i\theta}$, while the second term is represented by a vector that is perpendicular to $re^{i\theta}$. We can therefore conclude that the acceleration in polar coordinates is

$$a_r = \ddot{r} - r\dot{\theta}^2\,, \qquad a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta}\,. \tag{1.37}$$

Newton's laws therefore read

$$m(\ddot{r} - r\dot{\theta}^2) = -\partial_r V \tag{1.38}$$

$$m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0 \implies \frac{d}{dt}(mr^2\dot{\theta}) = 0\,. \tag{1.39}$$

You should recognize the second equation as expressing the conservation of angular momentum, with $l = mr^2\dot{\theta}$. Substituting this expression into Eq. (1.38), we find

$$m\ddot{r} - \frac{l^2}{mr^3} = -\frac{\partial V}{\partial r}\,, \tag{1.40}$$

[8] This can be the usual x, y and z in 3D space, r, θ and ϕ in 3D spherical coordinates, or something even more abstract than these choices, it doesn't matter.
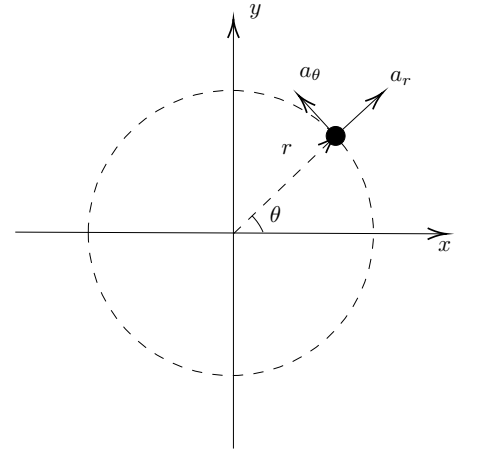

Figure 4: Coordinate system for the central force problem.

6

the equation of motion governing all central force problems.

Now that was a bit of hike, and again, a lot of issue was that we had to deal with vectors. Let's see how the Lagrangian approach works out. First, we write down the Lagrangian for the system,

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r). \tag{1.41}$$

This depends on two coordinates, but is independent of t, and therefore we can write down two Euler-Lagrange equations—one for each coordinate. These are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \implies m\ddot{r} - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0 \tag{1.42}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \implies \frac{d}{dt}(mr^2\dot{\theta}) = 0. \tag{1.43}$$

Notice that we obtain the conservation of angular momentum immediately from the second equation involving θ. Substituting once again $l = mr^2\dot{\theta}$, we get the same expression as before in Eq. (1.40).[9]

Let's also compute the first integral, defined for this problem as

$$I = L - \dot{r}\frac{\partial L}{\partial \dot{r}} - \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} = -\frac{1}{2}m\dot{r}^2 - \frac{1}{2}mr^2\dot{\theta}^2 - V(r) = -(T + V), \tag{1.44}$$

which is simply (the negative of) the total energy. Thus, the fact that the first integral is constant is just a statement about conservation of energy in this problem.

1.3 Variable Endpoints

So far, we've been dealing with situations where we want the endpoints to be fixed. Now let's see what happens when we relax that assumption. Consider the problem of constructing a railway between two ports, located across a strip of land with straight, parallel sides, illustrated in Fig. 5. Suppose that the cost of construction is proportional to the length of the track, but the cost of sea transport is negligible, and so that the ports can be wherever we want. We therefore want to minimize the total length given once again by

$$L[y] = \int_{x_1}^{x_2} dx \sqrt{1 + y'^2}. \tag{1.45}$$

This time however, we want to allow variations at the endpoints too. Considering a small perturbation, we see that

$$\delta L = \int_{x_1}^{x_2} dx \frac{y'}{\sqrt{1 + y'^2}}\delta y' \tag{1.46}$$

By now you should be familiar with what to do next: integrate by parts! Don't forget, however, the boundary terms. This gives

$$\delta L = \frac{y'(x_2)}{\sqrt{1 + y'(x_2)^2}}\delta y(x_2) - \frac{y'(x_1)}{\sqrt{1 + y'(x_1)^2}}\delta y(x_1)$$
$$- \int_{x_1}^{x_2} dx \left[\frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2}}\right)\right]\delta y(x). \tag{1.47}$$

The extremum for L is achieved when $\delta L = 0$. But since we have complete freedom to choose $y(x)$, including $y(x_2)$ and $y(x_1)$, the coefficients to each of

[9] You may be tempted to plug l directly into the Lagrangian, which looks like it would remove one coordinate from the Lagrangian, simplifying matters significantly. *But this is pure folly*, and gets you the wrong answer. The coordinates and their derivatives are—each and every one of them—*independent variables of the Lagrangian*. The Lagrangian's only aim in life is to take in these coordinates, their derivatives, and the time coordinate—not knowing that they're related in any way—and spit out a number. The Lagrangian *knows absolutely nothing about trajectories involving these coordinates as a function of time*.
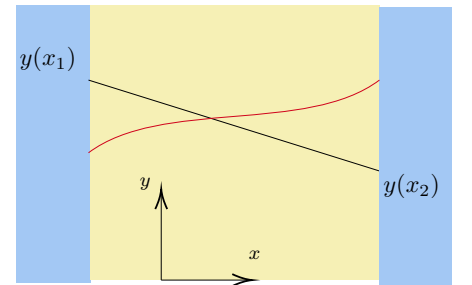
Figure 5: Railway across a strip of land with straight, parallel sides.

$\delta y(x_2)$, $\delta y(x_1)$ and δy inside the integral must be zero. Let's first consider the coefficient for $\delta y(x)$. This just says that

$$\frac{y'}{\sqrt{1 + y'^2}} = C \implies y' = \text{constant}\,, \tag{1.48}$$

and so we'd better be building a straight track, as expected. In addition however, the other two terms now enforce

$$\frac{y'(x_2)}{\sqrt{1 + y'(x_2)^2}} = \frac{y'(x_1)}{\sqrt{1 + y'(x_1)^2}} = 0 \implies y'(x_1) = y'(x_2) = 0\,. \tag{1.49}$$

We therefore want to build the railway exactly perpendicular to the sea. All very reasonable!

What you might have noticed is that by allowing the endpoints to float, you ended up getting a boundary term that enforces some boundary conditions at the two endpoints. Boundary conditions obtained through variation are known as **natural boundary conditions**. We'll come back to more examples with natural boundary conditions later on.

1.4 Noether's Theorem

We now come to one of the most profound results in mathematical physics. We've seen in a few of the previous examples above that when we apply the principle of least action, we can sometimes obtain equations of the form $d(\cdots)/dt = 0$ when the Euler-Lagrange equation is satisfied. For example, for a Lagrangian of the form $L = L(q, \dot{q})$ with no explicit dependence on t, we know from Sec. 1.1.4 that the time derivative of the first integral is conserved:

$$\frac{d}{dt}\left(\dot{q}\frac{\partial L}{\partial \dot{q}} - L\right) = 0\,. \tag{1.50}$$

We also saw that in the central force problem in Sec. 1.2.1, we can show that the angular momentum is conserved, i.e.

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0\,, \tag{1.51}$$

and in that case, the Lagrangian had not explicit dependence on θ. Equations of this form are known as **conservation laws**, and you have, up to this point in your physics career, already encountered many of them. But you may now be seeing a pattern emerge: whenever there is no explicit dependence of L on some quantity, you get conservation laws. And indeed this is true! These are all manifestations of **Noether's theorem**,[10] which says that

> Any continuous symmetry of the action corresponds to a conservation law.

[10] This is actually Noether's *first* theorem; the second theorem is much more obscure, and we won't discuss it in this course.

Now, your first thought might be, it's a little strange that we have a theorem that's just written out in words. That's really because you can prove many different mathematical statements that are described by those words, with various levels of generality and formality. Because Noether's theorem is so important, I want to walk you through a proof of it at a relatively general level. Your second thought might be that there are a lot of terms in there that I haven't carefully defined. But I think in this case, it's best just to define things as we go along. The discussion that follows below is mostly based on Ref. [2].

So let's start with the classical action, written as

$$S[q^i] = \int_{t_i}^{t_f} dt\, L(t; q^i, \dot{q}^i)\,, \tag{1.52}$$

where I remind you that q is some generalized coordinate with components $q^i(t)$ and time derivatives $\dot{q}^i(t)$. We say that the action is **invariant up to a boundary term** under a transformation $q^i(t) \to q^i(t) + \varepsilon \eta^i(t)$, where ε is taken to be a small, time-independent quantity, if

$$\delta S \equiv S[q^i + \varepsilon \eta^i] - S[q^i] = \varepsilon \int_{t_i}^{t_f} dt \, \frac{dK}{dt} \,, \tag{1.53}$$

for all $q^i(t)$, and for some **boundary term** K.

If this is the first time you're seeing this, you might be wondering why even bother with K at all—after all, that's what the word *invariant* should mean. But in fact, a lot of extremely interesting models (in both particle physics and condensed matter physics) have actions which are invariant up to boundary terms, and we would like to apply Noether's theorem to them. An equivalent way to say the same thing is that the action is invariant up to a boundary term if the Lagrangian transforms by a **total derivative** dK/dt.

Any transformation which leaves the action invariant up to a boundary term, or transforms the Lagrangian by a total derivative, is called a **symmetry**.[11] The first question you should ask when you see a Lagrangian is always, "what are its symmetries?" This is the organizing principle behind all of the models that we study in physics. So let's check out some examples. Consider once again the central force problem,

$$S = \int_{t_i}^{t_f} dt \, \left[\frac{1}{2} m \dot{\vec{r}}^2 - V(r) \right] \,, \tag{1.54}$$

The action is invariant under rotations, i.e. perform the transformation $\vec{r} \to \mathsf{R}\vec{r}$, where R is constant, orthogonal matrix with $\mathsf{R}^\mathsf{T} = \mathsf{R}^{-1}$, even for very large rotations, since $\dot{\vec{r}}^2 \to (\mathsf{R}\dot{\vec{r}})^\mathsf{T} \mathsf{R}\dot{\vec{r}} = \dot{\vec{r}}^\mathsf{T} \mathsf{R}^\mathsf{T} \mathsf{R}\dot{\vec{r}} = \dot{\vec{r}}^2$, in addition to the fact that rotations preserve length, i.e. $|\mathsf{R}\vec{r}| = r$.

Let's consider another transformation to the action

$$\vec{r}(t) \to \vec{r}(t) - \varepsilon \dot{\vec{r}}(t) \,, \tag{1.55}$$

again with ε being small. Under this transformation, we find[12]

$$
\begin{aligned}
S[r^i] \to S[r^i - \varepsilon \dot{r}^i] &= \int dt \, \left[\frac{1}{2} m (\dot{\vec{r}} - \varepsilon \ddot{\vec{r}})^2 - V(\vec{r} - \varepsilon \dot{\vec{r}}) \right] \\
&= \int dt \, \left[\frac{1}{2} m \dot{\vec{r}}^2 - \varepsilon m \dot{\vec{r}} \ddot{\vec{r}} - V(\vec{r}) + \varepsilon \dot{r}^i \partial_i V \right] \\
&= S[r^i] - \varepsilon \int dt \, \frac{d}{dt} \left[\frac{1}{2} m \dot{\vec{r}}^2 - V(r) \right] \,.
\end{aligned}
\tag{1.56}
$$

We can therefore conclude that the action is invariant under this transformation up to a boundary term, $K = L$. What is this strange transformation? Well, it really isn't that mysterious if you just think of it as relabeling time by the coordinate τ, which is just a constant shift from t, i.e. $\tau = t + \varepsilon$. With this transformation, $\vec{r}(t) = \vec{r}(\tau) - \varepsilon \dot{\vec{r}}(\tau)$, and it is easy to see that you get the same transformation as in Eq. (1.56).

Let's now consider an *arbitrary* transformation, one that isn't necessarily a symmetry. It also doesn't necessarily have to leave the endpoints fixed. Denoting this transformation as $q^i \to q^i + \varepsilon \zeta^i(t)$ and apply this to the classical action

[11] Sometimes you will hear people say that a transformation is only a symmetry if $K = 0$, and should be called a *quasisymmetry* otherwise. We won't make this distinction.

[12] For notation convenience, I'll drop the limits of the integral whenever nothing much ever happens to it.

Eq. (1.52). This gives (using index notation)

$$\delta S = \int_{t_i}^{t_f} dt \, \frac{\partial L}{\partial q^i} \varepsilon \zeta^i(t) + \frac{\partial L}{\partial \dot{q}^i} \varepsilon \dot{\zeta}^i(t)$$

$$= \int_{t_i}^{t_f} dt \, \frac{\partial L}{\partial q^i} + \frac{\partial L}{\partial \dot{q}^i} \varepsilon \zeta^i \Big|_{t_i}^{t_f} - \int_{t_i}^{t_f} dt \, \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) \varepsilon \zeta^i$$

$$= \int_{t_i}^{t_f} dt \, \left[\frac{\partial L}{\partial q^i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) \right] \varepsilon \zeta^i + \int_{t_i}^{t_f} dt \, \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \varepsilon \zeta^i \right), \qquad (1.57)$$

where we have kept the boundary term after performing integration by parts.

Now, let's choose the arbitrary transformation $\zeta^i(t)$ to be one that leaves the action invariant. Then we have the following relation, after dividing throughout by ε,

$$\int_{t_i}^{t_f} dt \, \frac{d}{dt} \left(K - \frac{\partial L}{\partial \dot{q}^i} \zeta^i \right) = \int_{t_i}^{t_f} dt \, \left[\frac{\partial L}{\partial q^i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) \right] \zeta^i. \qquad (1.58)$$

In classical mechanics, we're often interested in the behavior of a system at arbitrary times, and indeed, it doesn't really matter what the precise values of t_i and t_f are. Because of that, we need the integrand themselves to be equal, i.e.

$$\frac{d}{dt} \left(K - \frac{\partial L}{\partial \dot{q}^i} \zeta^i \right) = \left[\frac{\partial L}{\partial q^i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) \right] \zeta^i. \qquad (1.59)$$

This relation is known as the **Noether identity**, and is true for any ζ^i that is a symmetry. Note that at this point, we *have not even mentioned the Euler-Lagrange equation*. This relation is true for *arbitrary* $q^i(t)$, whether or not it corresponds to a the physical solution. When an equation is true before having enforced the Euler-Lagrange equation, we call in an **off-shell** relation, a term that you're bound to run into again.

Now let's take the relation **on-shell**, so that we are only considering $q^i(t)$ that satisfies the Euler-Lagrange equation. For such $q^i(t)$, the right-hand side of the previous equation vanishes, and we find

$$\frac{dQ}{dt} \overset{\text{E-L}}{=} 0, \qquad Q = K - \frac{\partial L}{\partial \dot{q}^i} \zeta^i. \qquad (1.60)$$

This is our big result, **Noether's theorem**: given a symmetry of the action ζ^i, we can define a quantity called a **Noether charge** (or just "charge") that is conserved on-shell, i.e. for physical trajectories.

That was a lot of abstract discussion; let's now take a look at some examples.

<div align="center">(End of Lecture: Monday Sep 9 2024)</div>

1.4.1 The central force problem revisited

Let's return once again to the to the central force problem, with

$$S[\vec{r}] = \int_{t_i}^{t_f} dt \, \left[\frac{1}{2} m \dot{\vec{r}}^2 - V(r) \right]. \qquad (1.61)$$

We saw earlier that the action was invariant under rotations. You can convince yourself (see Fig. 6 that an infinitesimal rotation about some arbitrary axis with

unit vector $\hat{\alpha}$ can be written as $\vec{r} \to \vec{r} + \varepsilon \hat{\alpha} \times \vec{r}$. We also checked that the rotation leaves the action invariant with $K = 0$. Putting this altogether, we find that the Noether charge associated with rotational symmetry is (using index notation)

$$Q = -\frac{\partial L}{\partial \dot{r}^i}(\hat{\alpha} \times \vec{r})^i = -(m\dot{r}_i)(\hat{\alpha} \times \vec{r})^i = \hat{\alpha} \cdot (\vec{r} \times \vec{p}), \qquad (1.62)$$

where $\vec{p} \equiv m\dot{\vec{r}}$ is the momentum. You should recognize immediately that $\vec{L} = \vec{r} \times \vec{p}$ is the **angular momentum**, the Noether charge corresponding to rotational symmetry, and since $\hat{\alpha}$ is arbitrary, $\vec{L}$ is conserved.

We also considered another transformation $\vec{r} \to \vec{r} - \varepsilon \dot{\vec{r}}$, which left the action invariant up to a boundary term, $K = L$, the Lagrangian. We also saw that this transformation was equivalent to time translation, $t \to t + \varepsilon$. Once again, the Noether charge associated with this symmetry is

$$Q = L - \frac{\partial L}{\partial \dot{r}^i}\dot{r}^i = \frac{1}{2}m\dot{\vec{r}}^2 - V(r) - m\dot{\vec{r}}^2 = -\left[\frac{1}{2}m\dot{\vec{r}}^2 + V(r)\right], \qquad (1.63)$$

which is simply the statement that the Noether charge is the **total energy**, and is conserved. More generally, for any action that has time-translation as a symmetry, e.g. $L = L(q^i, \dot{q}^i)$, the total energy is

$$E = \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L, \qquad (1.64)$$

and is conserved. You should recognize that E is nothing but the first integral that we described in Eq. (1.31).

Figure 6: Infinitesimal rotations of $\vec{r}$ about axis $\hat{\alpha}$ are in the direction $\hat{\alpha} \times \vec{r}$.

1.4.2 Common symmetries

Before we move on from Noether's theorem, let's examine some common symmetries that you've probably already encountered in classical mechanics, and how they usually arise from Noether's theorem.

1.4.2.1 Spatial translation invariance implies momentum conservation.
If a Lagrangian is invariant under a small, constant, spatial translation ε of some coordinate q, i.e. $L(t; q + \varepsilon, \dot{q}) = L(t; q, \dot{q})$.[13] Noether's theorem (Eq. (1.60)) tells us that the associated charge is

[13] Since ε is constant, $\dot{q}$ remains unchanged.

$$Q = p \equiv \frac{\partial L}{\partial \dot{q}} \text{ and } \frac{dp}{dt} = 0, \qquad (1.65)$$

where p is the momentum conjugate to q. One simple scenario where momentum conservation holds is when the Lagrangian simply doesn't depend explicitly on q at all; in this case, q is known as a **cyclic coordinate**. This is what happens in systems with cylindrical symmetry, for example, and therefore have Lagrangians that do not depend on the azimuthal angle, leading to an associated conserved angular momentum.

1.4.2.2 Time translation invariance implies energy conservation.
Suppose we performed the translation $q(t) \to q(t + \varepsilon)$, where ε is once again small and constant. You can imagine this as taking a trajectory that previously started
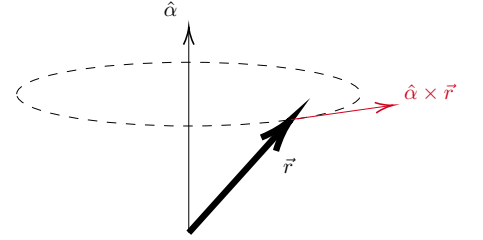
at time t, and now starting it at $t - \varepsilon$ instead. Then the action transforms as

$$
\begin{aligned}
S[q] &\to \int_{t_i}^{t_f} dt\, L(t; q(t+\varepsilon), \dot{q}(t+\varepsilon)) \\
&= \int_{t_i+\varepsilon}^{t_f+\varepsilon} d\tau\, L(\tau - \varepsilon; q(\tau), \dot{q}(\tau)) \\
&= S[q] + \varepsilon\left[L(t_f, q(t_f), \dot{q}(t_f)) - L(t_i, q(t_i), \dot{q}(t_i))\right] - \varepsilon \int_{t_i}^{t_f} d\tau\, \frac{\partial L}{\partial \tau}\,.
\end{aligned}
\tag{1.66}
$$

Here, the second term is a boundary term, but the third term might not be: we can therefore see that time translation is a symmetry of the action if the Lagrangian does not depend explicitly on t, with the conserved charge being energy, i.e.

$$
\frac{\partial L}{\partial t} = 0 \implies \text{Energy is conserved.} \tag{1.67}
$$

We have derived the expression for energy several times already, for example in Eq. (1.64): we do this by noting that $q(t+\varepsilon) = q(t) + \varepsilon \dot{q}(t)$, and then applying Noether's theorem.

1.5 Continuous Systems

So far, we have been looking at Lagrangians that have only a single particle. For multiple particles, we can similarly write down Lagrangians involving $\{q_i, \dot{q}_i\}$ for each particle. Frequently, however, we're interested in *fields*, like the electromagnetic field, and we want to be able to write down *field theories* by similarly writing down an action, extremizing it, and finding the equations of motion. Suppose we have a field with value $\varphi(x)$, where $x \equiv x^\mu$, $\mu = 0, 1, \cdots, d$ is the coordinate of a $(d+1)$-dimensional space (or spacetime). We can write down actions that look like

$$
S[\varphi] = \int dt\, L = \int d^{d+1}x\, \mathcal{L}(x, \varphi, \partial_\mu \varphi)\,, \tag{1.68}
$$

where $\mathcal{L}$ is the **Lagrangian density**,

$$
L \equiv \int d^d x\, \mathcal{L}\,, \tag{1.69}
$$

and

$$
\partial_\mu \varphi \equiv \frac{\partial \varphi}{\partial x^\mu}\,. \tag{1.70}
$$

We're going to follow a similar path to what we discussed in Sec. 1.1.1. We'll take a variation over the action, $\varphi(x) \to \varphi(x) + \varepsilon \eta(x)$, and so correspondingly $\partial_\mu \varphi(x) \to \partial_\mu \varphi(x) + \varepsilon \partial_\mu \eta(x)$, with no variation on the d-dimensional boundary of our $(d+1)$-dimensional space. We get

$$
\delta S = \int d^{d+1}x \left[\frac{\partial \mathcal{L}}{\partial \varphi} \varepsilon \eta + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \varepsilon \partial_\mu \eta \right]\,. \tag{1.71}
$$

At this point, we want to integrate by parts, which we can do using the **divergence theorem**,

$$\int_\Omega d^{n+1}x \, \partial_\mu f^\mu = \int_{\partial\Omega} dS \, n_\mu f^\mu \,, \qquad (1.72)$$

where $\partial\Omega$ is the boundary of Ω, dS is an element of area on the boundary, and n_μ is the outward-pointing normal vector. Much more on this later on in the course, but for now, we see that

$$\int_\Omega d^{n+1}x \, \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} \epsilon\eta \right) = \int d^{n+1}x \left(\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} \varepsilon\eta + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} \varepsilon\partial_\mu\eta \right)$$
$$= \int_{\partial\Omega} dS \, n^\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} \right) \epsilon\eta$$
$$= 0 \,, \qquad (1.73)$$

since η does not vary on the boundary, allowing us to write

$$\delta S = \int d^{d+1}x \left[\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} \right] \varepsilon\eta \,. \qquad (1.74)$$

Effectively, if we are only allowing for variations that are zero on the boundary, we can simply integrate by parts by switch the position of ∂_μ. Extremizing the action therefore means that

$$\frac{\delta S}{\delta\varphi} \equiv \frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} = 0 \,, \qquad (1.75)$$

which is the Euler-Lagrange equation in continuous form.

1.5.1 The vibrating string

Our first example of a continuous system is a string, shown in Fig. 7. The string has fixed ends, a mass per unit length of ρ, and is under tension T. If we assume only small displacements from equilibrium, the Lagrangian is

$$L = \int_0^L dx \left(\frac{1}{2}\rho\dot{y}^2 - \frac{1}{2}Ty'^2 \right) \,, \qquad (1.76)$$

where the dot denotes a partial derivative with respect to t, and the prime a partial derivative with respect to x.

The variation of the action is

$$\delta S = \int dt \int_0^L dx \, (\rho\dot{y}\delta\dot{y} - Ty'\delta y')$$
$$= \int dt \int_0^L dx \, (-\rho\ddot{y} + Ty'') \, \delta y \,, \qquad (1.77)$$

where to reach the second line, we perform integration by parts, and because the endpoints are fixed, $\delta y = 0$ when $x = 0$ and $x = L$. Extremizing the action $\delta S = 0$ gives the equation of motion

$$\rho\ddot{y} - Ty'' = 0 \,, \qquad (1.78)$$

which is the wave equation with transverse waves propagating with speed $c = \sqrt{T/\rho}$.
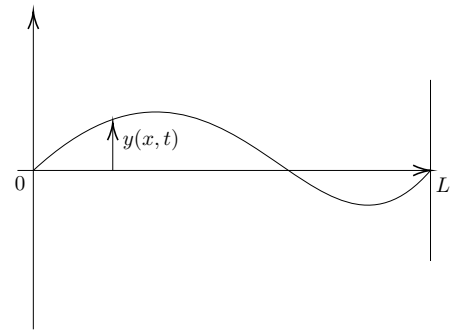

Figure 7: Set up for the vibrating string.

(**End of Lecture**: Wednesday Sep 11 2024)

1.5.2 The canonical energy-momentum tensor

Just as in the case of a single-particle Lagrangian, with conservation of momentum and energy, if the Lagrangian density $\mathcal{L} \equiv \mathcal{L}(\varphi, \partial_\mu \varphi)$ does not depend explicitly on x^μ, then we should expect a conservation law as well.

We'll go through the derivation of the Noether charge here again, as the result is presented in often confusing ways in many textbooks. Suppose our action is of the form

$$S = \int d^{d+1}x\, \mathcal{L}(\varphi, \partial_\mu \varphi)\,. \tag{1.79}$$

Then, we know that under the transformation $x^\mu \to x^\mu + \varepsilon^\mu$, where ε^μ is constant, we can expand $\mathcal{L}$ to give

$$\delta S = \int d^{d+1}x\, \varepsilon^\mu \partial_\mu \mathcal{L}\,, \tag{1.80}$$

where

$$\partial_\mu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \varphi}\partial_\mu \varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)}\partial_\mu(\partial_\nu \varphi)\,. \tag{1.81}$$

Note that in this transformation is a symmetry of the action, since $\varepsilon^\mu \partial_\mu \mathcal{L}$ is a total derivative. However, if we promote $\varepsilon^\mu \to \varepsilon^\mu(x^\mu)$ from a *global* to a local transformation, this stops being true.

Now, let's consider the *arbitrary* transformation, $x^\mu \to x^\mu + \zeta^\mu(x^\mu)$ (not necessarily constant, and not necessarily with no variation on the boundary!). This gives

$$\varphi \to \varphi + \zeta^\mu \partial_\mu \varphi\,, \quad \partial_\mu \varphi \to \partial_\mu \varphi + \partial_\mu(\zeta^\nu \partial_\nu \varphi)\,. \tag{1.82}$$

$$
\begin{aligned}
\delta S &= \int d^{d+1}x \left[\frac{\partial \mathcal{L}}{\partial \varphi}\zeta^\mu \partial_\mu \varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)}\partial_\nu(\zeta^\mu \partial_\mu \varphi) \right] \\
&= \int d^{d+1}x \left[\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)} \right]\zeta^\mu \partial_\mu \varphi + \int d^{d+1}x\, \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)}\zeta^\mu \partial_\mu \varphi \right)\,.
\end{aligned} \tag{1.83}
$$

Now, setting $\zeta^\mu = \varepsilon^\mu$, and going on-shell, the first term goes to zero as the Euler-Lagrange equations are satisfied, giving

$$\int d^{d+1}x\, \varepsilon^\mu \partial_\mu \mathcal{L} = \int d^{d+1}x\, \varepsilon^\mu \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)}\partial_\mu \varphi \right)$$

$$\implies \int d^{d+1}x\, \varepsilon^\mu \partial_\nu \left(\mathcal{L}\delta^\nu_\mu - \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)}\partial_\mu \varphi \right) = 0\,. \tag{1.84}$$

We call the Noether charge here the **canonical energy-momentum tensor**,[14]

$$T^\nu{}_\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)}\partial_\mu \varphi - \delta^\nu_\mu \mathcal{L}\,, \tag{1.85}$$

with the statement

$$\partial_\nu T^\nu{}_\mu = 0 \tag{1.86}$$

denoting *local* energy-momentum conservation. This can be extended to the case where there are multiple fields, in which case the first term on the right-hand side of Eq. (1.84) picks up a sum over all fields.

[14] The reason its called "canonical" and not just "the" energy-momentum tensor is because it's often the case that this isn't quite what you want to work with. Instead, you often want to work with $T^{\mu\nu}$ shifted by some other object whose total derivative is zero. In electromagnetism, this procedure of "fixing" $T^{\mu\nu}$ is often called the *Belinfante improvement procedure*.

1.5.3 The vibrating string revisited

We now return to the vibrating string, with Lagrangian given in Eq. (1.76). We can now compute the canonical energy-momentum tensor for this system. For $\mu = 0$,

$$T^\nu{}_0 = \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)} \partial_0 \varphi - \delta^\nu_0 \mathcal{L} , \tag{1.87}$$

and so

$$T^0{}_0 = \rho \dot{y}^2 - \left(\frac{1}{2} \rho \dot{y}^2 - \frac{1}{2} T y'^2 \right) = \frac{1}{2} \rho \dot{y}^2 + \frac{1}{2} T y'^2 , \tag{1.88}$$

$$T^1{}_0 = -T y' \dot{y} . \tag{1.89}$$

In addition, the conservation laws read

$$\begin{aligned} 0 = \partial_\nu T^\nu{}_0 &= \partial_\nu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)} \partial_0 \varphi \right] - \partial_\nu \delta^\nu_0 \mathcal{L} \\ &= \partial_t (\rho \dot{y}^2 - \mathcal{L}) + \partial_x (-T y' \dot{y}) \\ &= \partial_t \left(\frac{1}{2} \rho \dot{y}^2 + \frac{1}{2} T y'^2 \right) - \partial_x (T y' \dot{y}) , \end{aligned} \tag{1.90}$$

$$\begin{aligned} 0 = \partial_\nu T^\nu{}_1 &= \partial_\nu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)} \partial_1 \varphi \right] - \partial_\nu \delta^\nu_1 \mathcal{L} \\ &= \partial_t (\rho \dot{y} y') + \partial_x (-T y'^2 - \mathcal{L}) \\ &= \partial_t (\rho \dot{y} y') - \partial_x \left(\frac{1}{2} \rho \dot{y}^2 + \frac{1}{2} T y'^2 \right) . \end{aligned} \tag{1.91}$$

These are local conservation laws of the same form as the *continuity equation*,

$$\frac{\partial q}{\partial t} + \nabla \cdot \vec{J} = 0 , \tag{1.92}$$

where q should be thought of as a local density (e.g. charge density of fluid density) with a globally conserved quantity $Q = \int d^d x \, q$, and $\vec{J}$ is a flux. For $\partial_\nu T^\nu{}_0 = 0$, the local density is

$$T^0{}_0 = \frac{1}{2} \rho \dot{y}^2 + \frac{1}{2} T y'^2 , \tag{1.93}$$

which is the energy density. The energy flux is $T^1{}_0 = -T \dot{y} y'$, which is the rate that a segment of string is doing work on its neighbor to the right. Let's check that the total energy is globally conserved:

$$\begin{aligned} \frac{d}{dt} \int_0^L dx \left(\frac{1}{2} \rho \dot{y}^2 + \frac{1}{2} T y'^2 \right) &= \int_0^L dx \, \frac{\partial}{\partial t} \left(\frac{1}{2} \rho \dot{y}^2 + \frac{1}{2} T y'^2 \right) \\ &= \int_0^L dx \, \frac{\partial}{\partial x} (T y' \dot{y}) \\ &= T y' \dot{y} |_0^L = 0 , \end{aligned} \tag{1.94}$$

since the string is clamped at the ends, as expected.

That's really the last we'll say about continuous systems, but of course, there is so much more to say. You'll learn more about these systems in classes like quantum field theory and statistical physics.

1.6 Constraints

When we want to find the extremum of some function or functional, we often want to impose some constraints on that extremum. Here are some classic examples of problems that involve finding extrema under some constraint:

1. Given a curve a fixed length on a plane, what is the maximum area that it can enclose? What is the shape of the curve in that case?

2. Consider a chain with fixed length suspended between two points, both a fixed height above the ground. What is the shape of the chain that minimizes the potential energy?

The strategy that we are going to adopt here is identical to what you may have seen in vector calculus, with the use of **Lagrange multipliers**. Suppose you have a functional $J[y]$ that you want to extremize, subject to some constraint given by $K[y] = 0$. Consider the modified functional $\tilde{J}[y, \lambda] = J[y] - \lambda K[y]$, for some new parameter λ. First, we note that if vary the new parameter λ, we find

$$\tilde{J}[y, \lambda + \delta\lambda] - \tilde{J}[y, \lambda] = -\delta\lambda \cdot K[y], \tag{1.95}$$

and so when we extremize $\tilde{J}$, including over possible variations in λ, we require $K[y] = 0$, the precise constraint that we wanted to impose!

Under a variation of y, on the other hand, we obtain

$$\tilde{J}[y + \delta y, \lambda] - \tilde{J}[y, \lambda] = \delta J[y, \delta y] - \lambda \cdot \delta K[y, \delta y], \tag{1.96}$$

where $\delta J[y, \delta y] = \delta J[y + \delta y] - \delta J[y]$, and likewise for δK. At the extremum $\delta\tilde{J} = 0$ for some $y = y^*$ then, we must necessarily have

$$\delta J[y^*, \delta y] = \lambda \cdot \delta K[y^*, \delta y] \tag{1.97}$$

for all variations δy away from y^*. What's going on here? First of all, $K[y^*] = 0$, and so y^* corresponds to a point where the constraint is satisfied. Now, if the variation δy preserves this constraint, then $\delta K[y^*, y + \delta y] = 0$; in that case, the previous equation says that $\delta J[y^*, y + \delta y] = 0$ as well. In other words, J *itself is also extremized, as long as we limit ourselves to δy that preserves the constraint*.

Fig. 8 shows an illustration of how this method works. The equation of motion for λ enforces the constraint, and fixes the trajectory to lie along the blue line, with $K[y] = 0$. We can see that if we choose variations along the blue trajectory so that the constraint is always satisfied, $\delta K = 0$, but not necessarily δJ, *except* at the extremum of J along the trajectory (marked in red). At this point, in the direction $\delta K = 0$, $\delta J = 0$ as well, since it is a local extremum. Our clever choice of $\tilde{J}$ gave us *1)* an equation of motion for λ that constrained us to the blue trajectory, and *2)* an equation of motion for y that is satisfied for points where J is extremized along the trajectory.

Another common situation where Lagrange multipliers can be used is in classical mechanics, when you want to extremize an action given by $S = \int dt\, L(t; q, \dot{q})$, subject to some constraint on the *coordinate* q, e.g. $g(t; q, \dot{q}) = 0$. We can again consider the modified action $\tilde{S}$, where

$$\tilde{S}[q, \lambda] = \int dt\, [L - \lambda(t) g(t; q, \dot{q})], \tag{1.98}$$

where once again, extremizing $\tilde{S}$ over all q and λ simultaneously enforces the desired constraint, and extremizes L when subject to the constraint.[15]
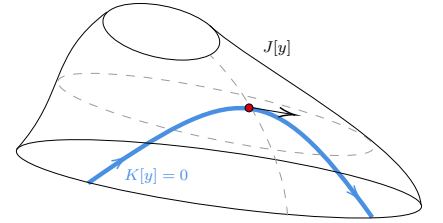
Figure 8: An illustration of the Lagrange multiplier method.

[15] One has to be careful when applying this to Newtonian mechanics, where the constraints should be applied *to the Euler-Lagrange equation themselves* in order to get the right dynamics. Equivalently, the extremization of $\tilde{S}$ cannot be over arbitrary q, but must also be over only those δq where the constraints are satisfied. Using our procedure for constraints of the form $g(t; q)$, with no dependence on $\dot{q}$, works; these are called *holonomic constraints*. However, what we have here does not work for general constraints. For a much more detailed discussion of holonomic and non-holonomic constraints, and when the Lagrange multiplier method fails, see Ref. [3] for an excellent discussion of this topic.

1.6.1 Maximum entropy distribution

Our first example of how to use this comes from statistical mechanics. Let Γ denote the classical phase space of a mechanical system of N particles governed by a Hamiltonian $H(p_n^i, q_n^i)$, with $n = 1, \cdots, N$, $i = 1, 2, 3$ and $d\Gamma \equiv \prod_{n=1}^{N} (d^3\vec{p}_n \, d^3\vec{q}_n)$. We define the phase space density $\rho(p_n^i, q_n^i)$ such that $\rho(p_n^i, q_n^i)d\Gamma$ is the probability of the system being in some state given by p_n^i and q_n^i in the small region $d\Gamma$.

The entropy related to this probability distribution can be defined as the functional

$$S[\rho] = -\int_\Gamma d\Gamma \, \rho \log \rho \,. \tag{1.99}$$

We now want to find ρ that maximizes the entropy for a given mean energy,

$$\langle E \rangle = \int_\Gamma d\Gamma \, \rho H \,. \tag{1.100}$$

However, ρ is subject to the constraint that it is a probability density function, and so we must enforce the constraint

$$\int_\Gamma d\Gamma \, \rho = 1 \,. \tag{1.101}$$

We can solve this problem by defining a new functional $\tilde{S}[\rho, \alpha, \beta]$, where

$$\tilde{S}[\rho, \alpha, \beta] = S[\rho] + \alpha \left(\int_\Gamma d\Gamma \, \rho - 1 \right) + \beta \left(\int_\Gamma d\Gamma \, \rho H - \langle E \rangle \right) \,. \tag{1.102}$$

The equations of motion for the real-number[16] Lagrange multipliers α and β enforce the two constraints that we mentioned above. Now for ρ, we have

$$\rho \log \rho \to (\rho + \delta\rho)\log(\rho + \delta\rho) = \rho \log \rho + \delta\rho \log \rho + \rho \cdot \frac{\delta\rho}{\rho} \tag{1.103}$$

and so

$$\delta\tilde{S} = \int_\Gamma d\Gamma \, (1 + \log\rho + \alpha + \beta H) \, \delta\rho \,, \tag{1.104}$$

and setting the term in parentheses to zero to extremize $\delta\tilde{S}$ gives

$$\rho(p_n^i, q_n^i) = e^{-1-\alpha-\beta H(p_n^i, q_n^i)} \,. \tag{1.105}$$

At this point, α and β can be determined from the normalization and energy constraints detailed above, with the exact answer being determined by the expect form of H. This probability density is the usual *canonical distribution*. Note that you can see from this procedure that the temperature $T \equiv 1/\beta$ appears as a Lagrange multiplier in this procedure.

(End of Lecture: Monday Sep 16 2024)

1.6.2 The catenary

We are now ready to tackle another classic calculus of variations problem: given a chain of fixed length and constant linear density suspended between two poles situated at $x = -R/2$ and $x = R/2$ (the height of these poles are fixed as well), what is the shape of the chain that minimizes the potential energy?

[16] They're not functions! In particular, they can live *outside* the integral over Γ. There's nothing wrong with having a functional depend on real numbers. You can also use Lagrange multipliers that are functions, if for example the constraint you are imposing is on ρ itself, and not the integral of ρ, as we are in this case.

The total potential energy of the chain is the functional

$$E[y] = \int_{-R/2}^{R/2} d\ell \, y = \int_{-L}^{L} dx \, y \sqrt{1 + y'^2} \,, \tag{1.106}$$

where I've dropped the density and the acceleration due to gravity for simplicity: they enter only as an overall multiplicative factor to the energy. I would like to extremize $E[y]$ subject to the constraint that the length of the chain is some constant L, i.e.

$$\int_{-R/2}^{R/2} dx \, \sqrt{1 + y'^2} = L \,. \tag{1.107}$$

As before, we define a new functional $\tilde{E}[y, \lambda]$, where λ is a real number Lagrange multiplier, and

$$\tilde{E}[y, \lambda] = E[y] - \lambda \left(\int_{-R/2}^{R/2} dx \, \sqrt{1 + y'^2} - L \right)$$
$$= \int_{-R/2}^{R/2} dx \, \left[(y - \lambda)\sqrt{1 + y'^2} + \frac{\lambda L}{R} \right] \,. \tag{1.108}$$

The equation of motion for y is then given by extremizing the first term in the equation above. We can just focus on the part of the action given by the integral since the other term doesn't depend on y, and use the Euler-Lagrange equations. Or, if we are a little bit cleverer, we can use the fact that the first integral is constant, since there is no explicit x dependence, i.e.

$$(y - \lambda)\sqrt{1 + y'^2} - \frac{y'^2(y - \lambda)}{\sqrt{1 + y'^2}} = C \implies 1 + y'^2 - y'^2 = \frac{C}{y - \lambda}\sqrt{1 + y'^2}$$
$$\implies \frac{(y - \lambda)^2}{C^2} - y'^2 = 1 \,. \tag{1.109}$$

This equation should remind you of $\cosh^2 t - \sinh^2 t = 1$, and so the general solution is

$$y - \lambda = C \cosh \left(\frac{x + D}{C} \right) \,, \tag{1.110}$$

i.e. the minimum energy curve should look like a hyperbola. This shape is also known as the **catenary**, and the constants are fixed by the heights of the two poles, and the requirement that the total length of the curve is L.

2 Calculus on Manifolds

References: Stone & Goldbart (SG) Chapter 10, Appendix A; Carroll Chapter 2

We're all very familiar by now with how to do calculus in $\mathbb{R}^n$. But often in physics, we want to be able to do calculus in spaces that don't quite look like $\mathbb{R}^n$. One example you've probably already seen is doing calculus in *Minkowski space* in special relativity. Another perhaps more mundane example is doing calculus on a sphere. Most of our intuition, however, is grounded in the logic and structures of $\mathbb{R}^n$, which we sometimes take for granted. And so it will help for us to think very carefully about how $\mathbb{R}^n$ works in a more formal language, in order for us to see how to generalize to other spaces.

2.1 Some Facts from Linear Algebra

The study of spaces like $\mathbb{R}^n$ falls under the subject of **linear algebra**. While a course in mathematical physics might feel a little incomplete without covering this topic in detail, linear algebra is generally well-covered in undergraduate curricula. We'll content ourselves with a lightning review of some key facts.

2.1.1 Vector spaces and inner products

So what are the structures of $\mathbb{R}^n$ that we take for granted? First, the fundamental quantities that we deal with in real space are **vectors**. A collection of objects that live in real space is simply a set, but the interesting thing about real space is that there are relations between the objects in the space. In fact, real space is an example of a **vector space**, a structure which is defined as follows:

A **vector space** V over a field $\mathbb{F}$ is a set equipped with two operations: a binary operation called **vector addition** which assigns to each pair of elements $\vec{x}, \vec{y} \in V$ a third element denoted $\vec{x} + \vec{y}$, and **scalar multiplication** which assigns to an element $\vec{x} \in V$ and $\lambda \in \mathbb{F}$ a new element $\lambda \vec{x} \in V$. There is also a distinguished element $\vec{0} \in V$ such that the follow axioms are obeyed:

1. Vector addition is commutative: $\vec{x} + \vec{y} = \vec{y} + \vec{x}$;

2. Vector addition is associative: $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$;

3. Additive identity: $\vec{0} + \vec{x} = \vec{x}$;

4. Existence of an additive inverse: for any $\vec{x} \in V$, there is an element $-\vec{x} \in V$ such that $\vec{x} + (-\vec{x}) = \vec{0}$;

5. Scalar distributive law: $\lambda(\vec{x} + \vec{y}) = \lambda\vec{x} + \lambda\vec{y}$, as well as $(\lambda + \mu)\vec{x} = \lambda\vec{x} + \mu\vec{x}$;

6. Scalar multiplication is associative: $(\lambda\mu)\vec{x} = \lambda(\mu\vec{x})$, and

7. Multiplicative identity: $1\vec{x} = \vec{x}$.

A lot of that just seems very natural, and so it might seem like a lot of useless abstraction. But the point is to be clear about what a vector in the abstract actually is, so that when we're in much less familiar settings, these formal structures are going to help us cut through the confusion. Furthermore, we can study properties of all vector spaces that would apply equally well to $\mathbb{R}^n$ as it does to any other vector space.

Here, you can see that $\mathbb{R}^n$ is a vector space over the field $\mathbb{R}$. However, you're also familiar with vector spaces over the complex numbers $\mathbb{C}$: one example is the *Hilbert space*, which underpins *quantum mechanics*. The states of a system are described as vectors $|\psi\rangle$ in a Hilbert space. The results of linear algebra apply equally well to both $\mathbb{R}^n$ and Hilbert spaces.

In addition to being able to add vectors, or multiply vectors by real numbers, another important thing you can do in $\mathbb{R}^n$ is talk about distances: you can take the dot product or **inner product** of a vector with itself to talk about length, or take the inner product of two different vectors and talk about angles. Formally, vector spaces with this additional structure are called **inner product spaces**. Inner product spaces are defined as follows:[17]

> An **inner product space** is a vector space V over a field $\mathbb{F}$, together with an **inner product**, which is a map
>
> $$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}, \tag{2.1}$$
>
> that satisfies the following properties for all $\vec{x}, \vec{y}, \vec{z} \in V$ and $\lambda, \mu \in \mathbb{F}$:
>
> 1. Conjugate symmetry: $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle^*$, where $*$ denotes complex conjugation. If $\mathbb{F}$ is real, then this just means that the inner product should be symmetric;
>
> 2. Linearity in the second argument, i.e. $\langle \vec{x}, \lambda \vec{y} + \mu \vec{z} \rangle = \lambda \langle \vec{x}, \vec{y} \rangle + \mu \langle \vec{x}, \vec{z} \rangle$. Note that this together with conjugate symmetry implies that $\langle \lambda \vec{x} + \mu \vec{y}, z \rangle = \lambda^* \langle \vec{x}, \vec{z} \rangle + \mu^* \langle \vec{y}, \vec{z} \rangle$. The inner product is only linear in both arguments when $\mathbb{F} = \mathbb{R}$, and
>
> 3. Nondegenerate, i.e. if $\langle \vec{x}, \vec{y} \rangle = 0$ for all $\vec{y}$, then $\vec{x} = 0$.

The inner product on $\mathbb{R}^n$ is the dot product; in Hilbert space, it is denoted $\langle \psi' | \psi \rangle$; in Minkowski space, we have the metric tensor. We'll go into a lot more detail on this in just a bit.

The last thing that we'll talk about are **linear transformations** (also known as linear operators or linear maps), which are functions that take us between vector spaces. Let V and W be vector spaces with dimensions n and m respectively; $A : V \to W$ is a linear transformation if

$$A(\lambda \vec{x} + \mu \vec{y}) = \lambda A(\vec{x}) + \mu A(\vec{y}). \tag{2.2}$$

2.1.2 Bases and components

At this point, the vectors on a vector space are still abstract objects. In order to make contact with our usual representation of vectors as a column of numbers, we need to define a **basis** for the vector space. This is something that you've probably seen in linear algebra, but we'll state some facts and definitions that all are somewhat intuitive:

1. A set of vectors $\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n\}$ is **linearly dependent** if there exist $\lambda^1, \cdots, \lambda^n \in \mathbb{F}$, written as λ^μ for $\mu = 1, \cdots, n$, not all zero, such that

$$\lambda^1 \vec{e}_1 + \lambda^2 \vec{e}_2 + \cdots \lambda^n \vec{e}_n = \vec{0}. \tag{2.3}$$

2. If it is not linearly dependent, a set of vectors $\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n\}$ is **linearly independent**. For a linearly independent set, the relation

$$\lambda^1 \vec{e}_1 + \lambda^2 \vec{e}_2 + \cdots \lambda^n \vec{e}_n = \vec{0} \tag{2.4}$$

holds only if $\lambda^1 = \cdots = \lambda^n = 0$.

[17] There are a few differences here compared to the usual definition in mathematics. First, in mathematics, it is common to have linearity apply to the first argument. This is of course is entirely equivalent. We use this definition to conform with our usual intuition in braket notation. Second, the inner product space is usually defined as having a *positive definite* inner product; but this unfortunately excludes Minkowski space, which is more properly classified as a pseudo-inner product. We don't really care about these differences in physics though.

3. A set of vectors $\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n\}$ is said to **span** V if for any $\vec{x} \in V$, there are numbers x^μ such that $\vec{x}$ can be written (not necessarily uniquely) as

$$\vec{x} = x^1 \vec{e}_1 + x^2 \vec{e}_2 + \cdots + x^n \vec{e}_n . \tag{2.5}$$

A vector space is **finite dimensional** if a finite spanning set exists.

4. A set of vectors $\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n\}$ is a **basis** if it is a *maximally linearly independent set*, i.e. introducing any additional vector makes the set linearly dependent. Equivalently, a basis is a *minimal spanning set*, i.e. deleting any of the $\vec{e}_i$ destroys the spanning property.

5. If $\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n\}$ is a basis, then any $\vec{x} \in V$ can be written

$$\vec{x} = x^1 \vec{e}_1 + x^2 \vec{e}_2 + \cdots + x^n \vec{e}_n , \tag{2.6}$$

where the x^μ, known as the **components** of the vector with respect to this basis, are unique in that two vectors coincide if and only if they have the same components.

6. If the sets $\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n\}$ and $\{\vec{f}_1, \vec{f}_2, \cdots, \vec{f}_m\}$ are both bases for the space V, then $m = n$. This invariant integer is the **dimension**, $\dim(V)$, of the space.

At this point, you may be looking at the notation above and wondering about the placement of indices: when are indices placed above, and when are they placed below? This will made clear in the next part of our discussion.

2.2 Change of Bases, Covariant and Contravariant Transformations

Having defined the concepts of a basis, and the components of a vector with respect to a basis, we now want to understand how these components change as we choose different bases, since bases are not unique.

Suppose a vector space V has two different bases given by $\{\vec{e}_1, \cdots, \vec{e}_n\}$ and $\{\vec{e}_1', \cdots, \vec{e}_n'\}$. Since both sets span V, every vector in $\{\vec{e}_1, \cdots, \vec{e}_n\}$ can be written as a sum of $\{\vec{e}_1', \cdots, \vec{e}_n'\}$, and we can define a set of n^2 numbers $a^\mu{}_\nu$ which maps

$$\vec{e}_\nu = \sum_{\mu=1}^{n} a^\mu{}_\nu \vec{e}_\mu' \equiv a^\mu{}_\nu \vec{e}_\mu' = \vec{e}_\mu' a^\mu{}_\nu . \tag{2.7}$$

At this point, we've introduced the famous **Einstein notation**, which just says that every repeated index should be regarded as being summed over all possible values. Again, you may be worried about the placement of the indices, but all be clear as we go along. The final expression is helpful in helping you visualize the object $\vec{e}_\mu'$ as a row vector, multiplied by the **matrix** $a^\mu{}_\nu$, where μ indexes its rows, and ν indexes its columns.

$a^\mu{}_\nu$ is clearly **invertible**: every vector has a unique representation in each basis, and the map takes the coordinates of any vector in one basis to another, and so it is certainly a bijective map. We can therefore define $(a^{-1})^\mu{}_\nu$ as the inverse map,

$$\vec{e}_\nu' = (a^{-1})^\mu{}_\nu \vec{e}_\mu , \tag{2.8}$$

with

$$(a^{-1})^\mu{}_\nu a^\nu{}_\sigma = \delta^\mu_\sigma , \tag{2.9}$$

where δ^μ_σ is the Kronecker delta or the identity matrix.

So far, we have dealt with the transformation of the basis. But how does a general vector transform? Given the transformation between bases above, we see that for any arbitrary vector $\vec{x}$, which can be written as $x^\nu \vec{e}_\nu$ in one basis and $x'^\mu \vec{e}'_\mu$ in the other, are related by

$$\vec{x} = x'^\mu \vec{e}'_\mu = x^\nu \vec{e}_\nu = x^\nu (a^\mu{}_\nu \vec{e}'_\mu) = (a^\mu{}_\nu x^\nu) \vec{e}'_\mu , \qquad (2.10)$$

or in other words,[18]

$$x'^\mu = a^\mu{}_\nu x^\nu . \qquad (2.11)$$

One thing you should notice immediately is that the basis and the coordinates transform in the opposite way:

$$x'^\mu = a^\mu{}_\nu x^\nu , \qquad \vec{e}'_\mu = (a^{-1})^\sigma{}_\mu \vec{e}_\sigma , \qquad (2.12)$$

because of course the vector itself, $x'^\mu \vec{e}'_\mu = x^\mu \vec{e}_\mu$, doesn't transform under a coordinate change at all! Any quantity that transforms under a change of basis like the basis itself is said to transform **covariantly**, while any quantity that transforms like the coordinates, i.e. in the opposite manner as the basis, is said to transform **contravariantly**. *We will always use indices on the top to indicate a quantity that transforms contravariantly, and indices on the bottom to indicate a quantity that transform covariantly.*

The best intuition for this comes from imagining a change of basis via rotation in $\mathbb{R}^2$, as shown in Fig. 9. Either we can imagine the basis vectors actually rotating counterclockwise and defining a new set of axes, as we would do in taking $x'^\nu \vec{e}'_\nu = x'^\nu (a^\mu{}_\nu \vec{e}_\mu)$, or equivalently, we can think of the components of the vector themselves rotating clockwise, with the axes just being relabeled, which corresponds to $x'^\nu \vec{e}'_\nu = (a^\mu{}_\nu x'^\nu) \vec{e}_\mu$.

(**End of Lecture**: Wednesday 18 Sep 2024)

2.3 The Dual Space

For every vector space V, we can define a **dual space** V^*, which is a set of linear transformations $f : V \to \mathbb{F}$, each of which takes in a vector and returns a number. The functions f are called **covectors** or **one-forms**, and you can convince yourself that V^* is also a vector space. Since these functions are linear, we have

$$f(\vec{x}) = f(x^\mu \vec{e}_\mu) = x^\mu f(\vec{e}_\mu) \equiv x^\mu f_\mu , \qquad (2.13)$$

where in the last equality I have defined the set of numbers $f_\mu \equiv f(\vec{e}_\mu)$, which I can construct given the basis $\{\vec{e}_\mu\}$ in V. Under a change of basis in V,

$$f_\mu = f(\vec{e}_\mu) = f(a^\nu{}_\mu \vec{e}'_\nu) = a^\nu{}_\mu f(\vec{e}'_\nu) \equiv a^\nu{}_\mu f'_\nu , \qquad (2.14)$$

where $f'_\nu \equiv f(\vec{e}'_\nu)$ are again a set of numbers that we can construct given the basis $\{\vec{e}'_\nu\}$ in V. Notice that under a change of basis in V, f_μ transforms *covariantly*, i.e. in the same manner as the change of basis in V.

Given a basis $\vec{e}_\mu$ of V, we can define a **dual basis** for V^*, which is the set of covectors $\vec{e}^{*\mu} \in V^*$ such that

$$\vec{e}^{*\mu}(\vec{e}_\nu) = \delta^\mu_\nu . \qquad (2.15)$$

This is clearly a basis, since for any $f \in V^*$,

$$f(\vec{x}) = x^\mu f_\mu = x^\mu f_\nu \delta^\nu_\mu = x^\mu f_\nu \vec{e}^{*\nu}(\vec{e}_\mu) = f_\nu \vec{e}^{*\nu}(x^\mu \vec{e}_\mu) , \qquad (2.16)$$

[18] Very often, you will see the notation $a^\mu{}_\nu \equiv \partial x'^\mu / \partial x^\nu$, which makes total sense if you look at Eq. (2.11). In fact, the advantage of writing it this way tells you how to obtain the matrix $a^\mu{}_\nu$.
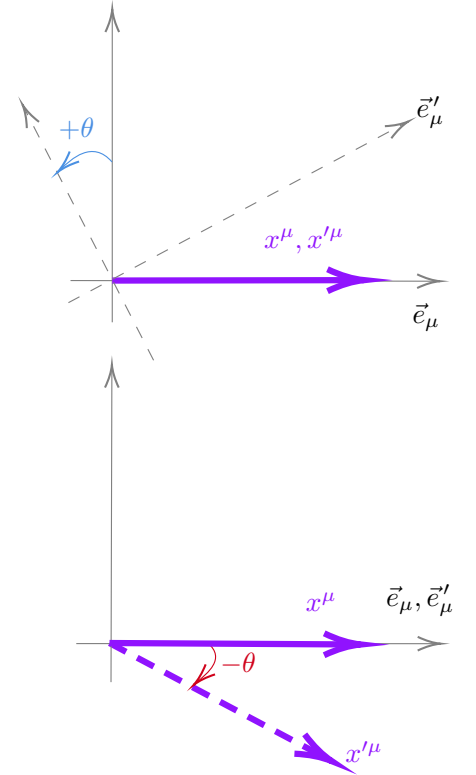
Figure 9: Two equivalent ways to think of a change of basis. (Top) the basis vectors themselves are transformed, or (bottom) the coordinates are transformed in the opposite direction. These pictures are equivalent.

or in other words,

$$f = f_\mu \vec{e}^{*\mu} \,. \tag{2.17}$$

We should therefore view $f_\mu = f(\vec{e}_\mu)$ as the *components* of f under the induced dual basis $\vec{e}^{*\mu}$.

You should already have a sense that V and V^* are very closely related; in fact, the map $\vec{e}_\mu \mapsto \vec{e}^{*\mu}$ is an **isomorphism**, i.e. a map of every element in V to another in V^* that preserves their respective relation to each other under addition and scalar multiplication.

2.4 The Metric

So far, everything we have discussed has been about vector spaces. We are now going to turn our attention to inner product spaces over $\mathbb{R}$, where the additional inner product structure is defined, giving us a way of talking about distances and angles.

As a reminder, the inner product is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ that takes two vectors in V, and spits out a number. Having chosen a basis $\{\vec{e}_\mu\}$ for our vector space V, we can now define a quantity known as the **metric** or **metric tensor** $g_{\mu\nu}$,

$$g_{\mu\nu} \equiv \langle \vec{e}_\mu, \vec{e}_\nu \rangle \,. \tag{2.18}$$

For $\mathbb{R}^n$, for example, with the inner product given by the dot product, we have simply $g_{\mu\nu} = \delta_{\mu\nu}$, while for Minkowski space, the metric is given by[19]

$$\eta_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \,. \tag{2.19}$$

[19] I will generally stick with the mostly minus convention for the Minkowski metric. My apologies to mostly plus aficionados, but I'm just slightly more comfortable with the mostly minus convention at this point.

Knowing the metric fully defines the inner product, since

$$\langle \vec{x}, \vec{y} \rangle = \langle x^\mu \vec{e}_\mu, y^\nu \vec{e}_\nu \rangle = g_{\mu\nu} x^\mu y^\nu \,. \tag{2.20}$$

Moreover, the structure of the inner product also guarantees that $g_{\mu\nu} = g_{\nu\mu}$, i.e. $g_{\mu\nu}$ is **symmetric**. Another thing we can note is that as a matrix, $g_{\mu\nu} x^\nu = 0$ only if $x^\nu = 0$ by the definition of the inner product; this means that $g_{\mu\nu}$ is invertible. We can therefore define the **inverse of the metric**, which we denote $g^{\mu\nu}$, with

$$g_{\mu\nu} g^{\nu\sigma} = g^{\mu\nu} g_{\nu\sigma} = \delta^\mu_\sigma \,. \tag{2.21}$$

For now, this is just a relationship between matrices; we'll come back and revisit the metric when we have discussed tensors later on.

2.4.1 Raising and lowering indices

Let's look at the expression in Eq. (2.20) more closely. We can reinterpret $\langle \vec{x}, \vec{y} \rangle = g_{\mu\nu} x^\mu y^\nu$ as $(g_{\mu\nu} x^\mu) y^\nu$. With this rewriting, can think of $(g_{\mu\nu} x^\mu)$ as being the components of the object $\langle \vec{x}, \cdot \rangle$, which takes in a vector $\vec{y}$ and returns $\langle \vec{x}, \vec{y} \rangle$. In fact, $\langle \vec{x}, \cdot \rangle$ is an object in V^*, mapping vectors in V to real numbers, and can be written as $\langle \vec{x}, \cdot \rangle = g_{\mu\nu} x^\mu \vec{e}^{*\nu}$, so that

$$g_{\mu\nu} x^\mu \vec{e}^{*\nu} (y^\sigma \vec{e}_\sigma) = g_{\mu\nu} x^\mu y^\sigma \delta^\nu_\sigma = g_{\mu\nu} x^\mu y^\nu = \langle \vec{x}, \vec{y} \rangle \,. \tag{2.22}$$

Clearly then, $g_{\mu\nu}x^\mu$ are indeed the components of $\langle \vec{x}, \cdot \rangle$ in the basis of V^* induced by our chosen basis of V, and therefore transforms *covariantly*.

If all that was a bit dense, the upshot is that, starting from a contravariant quantity x^μ, we can **lower its index** by defining

$$x_\nu \equiv g_{\mu\nu}x^\mu, \tag{2.23}$$

which 1) is a quantity that transforms covariantly (and so has a lower index), and 2) can be **contracted** with a contravariant quantity to form a real number, or a **scalar** or an **invariant**. Intuitively, it is one half of the inner product: you need to put together a covariant and contravariant piece to obtain a scalar.

Multiplying Eq. (2.23) by $g^{\sigma\nu}$ on both sides, we also find

$$g^{\sigma\nu}x_\nu \equiv g^{\sigma\nu}g_{\mu\nu}x^\mu = g^{\sigma\nu}g_{\nu\mu}x^\mu = \delta^\sigma_\mu x^\mu = x^\sigma, \tag{2.24}$$

which shows that I can also **raise an index** by multiplying by the inverse tensor. Ultimately, all I'm doing is switching between the components of the two objects

$$\langle \vec{x}, \cdot \rangle \leftrightarrow \vec{x}, \tag{2.25}$$

which are in 1-to-1 correspondence with each other between the isomorphic vector spaces V and V^*.

Finally, notice that every time we perform a contraction, we sum over one upper and one lower index. This is because every contraction represents the pairing of a function in V^*, with a vector in V, and results in a scalar. Another way of understanding this is that you want to pair up a contravariant with a covariant quantity, so that you end up with a quantity that doesn't transform, i.e. a scalar. I have never encountered a situation where you want to sum over the components of two objects which transform in the same way.

2.4.2 Example: Some Common Metrics

Let's pause for a moment and take a look at some important examples.

To digest all of this information, let's revisit $\mathbb{R}^2$ with all of this technology. $\mathbb{R}^2$ is a 2D vector space, with vectors $x^i \vec{e}_i$ that look like, for example, $3\vec{e}_x + 2\vec{e}_y$, where 3 and 2 are the components of the vector, and $\{\vec{e}_x, \vec{e}_y\}$ is a chosen basis for the space. $\mathbb{R}^2$ also comes with an inner product, which is the usual dot product. In $\mathbb{R}^2$, we can choose a basis that is **orthonormal**, i.e. with a metric given by

$$g_{ij} = \langle \vec{e}_i, \vec{e}_j \rangle = \vec{e}_i \cdot \vec{e}_j = \delta_{ij}. \tag{2.26}$$

I can use this metric to raise and lower indices of covariant or contravariant quantities, so for example

$$g_{ij}x^j = \delta_{ij}x^j = x_i, \tag{2.27}$$

but if you explicitly plug in the indices, you can see that $x^0 = x_0$ and $x^1 = x_1$, which shows that *the position of indices doesn't matter in $\mathbb{R}^n$*. Inner products between two vectors can be written in component form as

$$g_{ij}x^i y^j = x_j y^j, \tag{2.28}$$

i.e. the sum of the product of individual components, as in the usual dot product. You can think of x_j as the components of $\langle \vec{x}, \cdot \rangle$. In $\mathbb{R}^2$, you can also think of

x_j as a row matrix, which maps column vectors (which contain components of a vector) to numbers by matrix multiplication.

Now let's consider a basis change, given by $\vec{e}_i' = a^j{}_i \vec{e}_j$, where

$$a^j{}_i = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{2.29}$$

where j indexes the row and i indexes the column. This is a clockwise rotation of the basis vectors by some constant angle θ. Explicitly,

$$\begin{aligned} \vec{e}_1' &= \cos\theta\,\vec{e}_1 - \sin\theta\,\vec{e}_2\,, \\ \vec{e}_2' &= \sin\theta\,\vec{e}_1 + \cos\theta\,\vec{e}_2\,. \end{aligned} \tag{2.30}$$

At the same time, for any vector $\vec{x} = x^i \vec{e}_i$, the coordinates x^i transforms in the opposite sense, i.e. $x'^i = (a^{-1})^i{}_j x^j$, where

$$(a^{-1})^i{}_j = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{2.31}$$

so that

$$\begin{aligned} x'^1 &= \cos\theta\,x^1 - \sin\theta\,x^2 \\ x'^2 &= \sin\theta\,x^1 + \cos\theta\,x^2\,, \end{aligned} \tag{2.32}$$

which is instead a *counterclockwise* rotation of the components.[20] You can check that the vector itself, $x^i \vec{e}_i$, remains unchanged. This is the same intuition we had from Fig. 9.

We now graduate to something hopefully still familiar, but a little more nontrivial: 4D Minkowski space, where the 0-dimension is time, and dimensions 1,2,3 are spatial dimensions. The vectors that live in this space are called **4-vectors**, and they are of the form $x^\mu \vec{e}_\mu$, where $\{\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ forms a basis. Once again, we have an inner product and an associated metric; we can choose a basis such that the metric is the **Minkowski metric**

$$\eta_{\mu\nu} = \langle \vec{e}_i, \vec{e}_j \rangle\,, \tag{2.33}$$

where

$$\eta_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2.34}$$

You can verify for yourself that $\eta^{\mu\nu}$, the inverse of the matrix, has the same entries as $\eta_{\mu\nu}$.

Once again, I can lower indices by hitting a contravariant quantity with the metric, e.g. $x_\mu = \eta_{\mu\nu} x^\nu$, but this time, you can see that $x_0 = x^0$, and $x_i = -x^i$. Therefore, *the position of the indices does matter in Minkowski space*, and we need to be a little more careful. You can still think of x_μ as the components of $\langle \vec{x}, \cdot \rangle$, but x_μ is now no longer just a simple transposition (i.e. a row matrix) relative to x^μ (which we can view as a column matrix); you also need to change the sign of the spatial components. Inner products can be written, as before, as

$$g_{\mu\nu} x^\mu y^\nu = x_\nu y^\nu\,, \tag{2.35}$$

but note that because of the negative signs in the metric, *you are no longer guaranteed that the inner product is positive*.

[20] What may be confusing is that Eq. (2.30) and Eq. (2.32) look identical! But how we interpret what's happening is different. In the first, each line tells us how each basis vector is separately rotated, so you're looking out for $\vec{e}_1$ transforming into something else. But in the second, it is the transformation of the arrow denoted by (x^1, x^2) going into (x'^1, x'^2) that we are interested in.

We can again consider basis changes such as the **Lorentz boost**, given by $\vec{e}_\nu' = \Lambda^\mu{}_\nu \vec{e}_\mu$, where for example

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.36}$$

where $|\beta| < 1$ and $\gamma = (1 - \beta^2)^{-1/2}$. Under this transformation,

$$\begin{aligned} \vec{e}_0' &= \gamma\vec{e}_0 + \beta\gamma\vec{e}_1 \\ \vec{e}_1' &= \beta\gamma\vec{e}_0 + \gamma\vec{e}_1 \\ \vec{e}_2' &= \vec{e}_2 \\ \vec{e}_3' &= \vec{e}_3 \,. \end{aligned} \tag{2.37}$$

On the other hand, the coordinates transform as $x'^\nu = (\Lambda^{-1})^\nu{}_\mu x^\mu$, where

$$(\Lambda^{-1})^\nu{}_\mu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.38}$$

i.e.

$$\begin{aligned} x'^0 &= \gamma x^0 - \beta\gamma x^1 \\ x'^1 &= -\beta\gamma x^0 + \gamma x^1 \\ x'^2 &= x^2 \\ x'^3 &= x^3 \,. \end{aligned} \tag{2.39}$$

2.5 Tensors

We have now seen vector spaces and their dual spaces. We can now start defining even more general objects by putting vector spaces and dual spaces together!

Consider three vector spaces U, V and W over $\mathbb{F}$. We can define the **tensor product** of spaces such as $V \otimes W$, or even $U \otimes V \otimes W$.

1. It is distributive, i.e. for $\vec{a} \in U$ and $\vec{x} \in V$,

$$\begin{aligned} \vec{a} \otimes (\vec{x} + \vec{y}) &= \vec{a} \otimes \vec{x} + \vec{a} \otimes \vec{y}, \\ (\vec{a} + \vec{b}) \otimes \vec{x} &= \vec{a} \otimes \vec{x} + \vec{b} \otimes \vec{x}; \end{aligned} \tag{2.40}$$

2. It is associative, so that we can chain together three vector spaces like $U \otimes V \otimes W$ without worrying about whether it's $(U \otimes V) \otimes W$ or $U \otimes (V \otimes W)$;

3. It commutes with $\mathbb{F}$, i.e.

$$\lambda(\vec{a} \otimes \vec{x}) = (\lambda\vec{a}) \otimes \vec{x} = \vec{a} \otimes (\lambda\vec{x}), \tag{2.41}$$

but it is *not* commutative over the vectors, i.e. $\vec{a} \otimes \vec{b} \neq \vec{b} \otimes \vec{a}$ in general.

Consider a vector space V with a basis $\{\vec{e}_\mu\}$. This basis induces a basis in tensor products of V and V^* in the natural way; for example, in $V^* \otimes V^*$, this

induces the basis $\vec{e}^{*\mu} \otimes \vec{e}^{*\nu}$. These basis vectors act on pairs of basis vectors, $(\vec{e}_\alpha, \vec{e}_\beta)$, in the expected way, so for our $V^* \otimes V^*$ example, we get

$$\vec{e}^{*\mu} \otimes \vec{e}^{*\nu}(\vec{e}_\alpha, \vec{e}_\beta) = \delta^\mu_\alpha \delta^\nu_\beta \,. \qquad (2.42)$$

It can also act on elements of $V \otimes V$ in a manner that you might also expect:

$$\vec{e}^{*\mu} \otimes \vec{e}^{*\nu}(\vec{e}_\alpha \otimes \vec{e}_\beta) = \delta^\mu_\alpha \delta^\nu_\beta \,. \qquad (2.43)$$

A good example of what a tensor is and what it does is the metric itself, which we often refer to as **metric tensor**. It is a tensor in $V^* \otimes V^*$,

$$\mathbf{g} = g_{\mu\nu} \vec{e}^{*\mu} \otimes \vec{e}^{*\nu} \,. \qquad (2.44)$$

It acts on pairs of vectors $\vec{x}$ and $\vec{y}$ and returns a number in $\mathbb{F}$:

$$\mathbf{g}(\vec{x}, \vec{y}) = g_{\mu\nu} \vec{e}^{*\mu} \otimes \vec{e}^{*\nu}(x^\alpha \vec{e}_\alpha, y^\beta \vec{e}_\beta) = g_{\mu\nu} x^\alpha y^\beta \delta^\mu_\alpha \delta^\nu_\beta = g_{\mu\nu} x^\mu y^\nu = x^\mu y_\mu \,. \qquad (2.45)$$

As with vectors and covectors, once we've picked a basis, we will only need to worry about the components, with the understanding that the object itself is specified by both the components and the basis, and that the spaces act in the natural way that you expect.

Under a change of basis $\{\vec{e}_\mu\} \mapsto \{\vec{e}'_\mu\}$, the metric itself undergoes a transformation:

$$g_{\mu\nu} \mapsto g'_{\mu\nu} = \langle \vec{e}'_\mu, \vec{e}'_\nu \rangle = \langle a^\sigma{}_\mu \vec{e}_\sigma, a^\lambda{}_\nu \vec{e}_\lambda \rangle = a^\sigma{}_\mu a^\lambda{}_\nu g_{\sigma\lambda} \,. \qquad (2.46)$$

Each lower index is acted on by the change-of-basis transformation, with each transformation given by a covariant transformation. We say therefore that $g_{\mu\nu}$ a doubly covariant **tensor**, which explains why we often also refer to $g_{\mu\nu}$ as the **metric tensor**. You can see now that we can look into objects with more general number of indices, say $Q^{\alpha\beta}{}_{\gamma\delta\epsilon}$, which transforms as

$$Q'^{\alpha\beta}{}_{\gamma\delta\epsilon} = (a^{-1})^\alpha{}_{\alpha'} (a^{-1})^\beta{}_{\beta'} a^{\gamma'}{}_\gamma a^{\delta'}{}_\delta a^{\epsilon'}{}_\epsilon Q^{\alpha'\beta'}{}_{\gamma'\delta'\epsilon'} \,, \qquad (2.47)$$

which is a doubly contravariant, triply covariant tensor, or a type $(2,3)$ tensor. The total number of indices is what we call the **rank** of the tensor. Notice how when we were writing down the transformation of $Q^{\alpha\beta}{}_{\gamma\delta\epsilon}$, the indices lined up: we contracted upper indices with lower indices, so that under each application of the change-of-basis transformation, upper indices remain upper indices.

Another thing you will notice is that I have been very careful with the relative positions of the tensors. This is good practice, but very often you'll find people get sloppy and collapse all the indices when they think the notation is obvious. The one tensor where this is always okay is the Kronecker delta δ^μ_ν, since we always have this tensor returning 1 if $\mu = \nu$ and 0 if $\mu \neq \nu$ regardless of the position of the indices.

2.5.1   Tensor algebra

So what can we do with tensors? Well, we can add tensors together, but we have to ensure that you're adding things that are transforming with a change of bases in the same way. So for example

$$A^\mu{}_{\nu\lambda} = B^{\mu\tau}{}_{\nu\lambda\tau} + C^\mu{}_{\nu\lambda} \qquad (2.48)$$

is legal, but

$$A^{\mu}{}_{\nu\lambda} \overset{\text{wrong}}{=} B^{\nu}{}_{\mu\lambda} + C^{\mu}{}_{\nu\lambda\sigma\sigma} + D^{\mu}{}_{\nu\lambda\tau} \tag{2.49}$$

makes no sense.

We can multiply tensors together. Suppose we have tensors $A^{\mu}{}_{\nu\lambda}$ and $B^{\mu}{}_{\nu\lambda\tau}$, which are tensors of type $(1,2)$ and $(1,3)$ respectively. Then, we can multiply them together to get

$$C^{\alpha\beta}{}_{\nu\lambda\rho\sigma\tau} = A^{\alpha}{}_{\nu\lambda} B^{\beta}{}_{\rho\sigma\tau} , \tag{2.50}$$

which is a tensor of type $(2,5)$.

You can also contract indices, or equivalently, multiply tensors by the metric, so that for example

$$C^{\alpha\beta}{}_{\alpha\beta\rho\sigma\tau} = g^{\alpha\lambda} g^{\beta\mu} C_{\lambda\mu\alpha\beta\rho\sigma\tau} \tag{2.51}$$

is now a tensor of type $(0,3)$. Contracting two vectors is a special case that leads to a real number, also called a scalar or sometimes an **invariant**, which does not transform under a change of basis. Let me stress again: *you must contract one upper and one lower index!* You can write down objects like $B_{\alpha\beta\beta}$, but these objects are *not* tensors.

2.5.2 Example: Rotations and Lorentz transformations

Let's study the properties of basis transformations $a^{\mu}{}_{\nu}$ that leave the metric invariant. In $\mathbb{R}^n$, these are the transformations such that angles and lengths are all preserved. Under the transformation $a^{\mu}{}_{\nu}$, the metric tensor transforms as

$$g_{\mu\nu} \mapsto a^{\sigma}{}_{\mu} a^{\lambda}{}_{\nu} g_{\sigma\lambda} , \tag{2.52}$$

Transformations O that leave the metric invariant are therefore of the form

$$O^{\sigma}{}_{\mu} g_{\sigma\lambda} O^{\lambda}{}_{\nu} = g_{\mu\nu} . \tag{2.53}$$

In $\mathbb{R}^n$, if we start with the canonical metric $g_{\sigma\lambda} = \delta_{\sigma\lambda}$, then these transformations must satisfy

$$O^{\sigma}{}_{\mu} \delta_{\sigma\lambda} O^{\lambda}{}_{\nu} = \delta_{\mu\nu} \implies O^{\lambda}{}_{\mu} O_{\lambda\nu} = \delta_{\mu\nu} \implies (O^{-1})_{\mu}{}^{\lambda} = O^{\lambda}{}_{\mu} . \tag{2.54}$$

Therefore, as matrices, we must have $O^{-1} = O^{\mathsf{T}}$. The set of all such matrices is the **orthogonal matrices**, which can be thought of as a **group** called $O(n)$, called the **orthogonal group**.[21] This group is made up of matrices corresponding to *rotations* and *reflections*. The group containing matrices corresponding only to rotations is called the **special orthogonal group**, $SO(n)$.

In Minkowski space, we have instead

$$\Lambda^{\sigma}{}_{\mu} \eta_{\sigma\lambda} \Lambda^{\lambda}{}_{\nu} = \eta_{\mu\nu} . \tag{2.55}$$

The Λ matrices also form a group that we call the **Lorentz group**, $O(1,3)$ for spacetime. The set of all Lorentz transformations (both boosts and spatial rotations), as well as time reversal and reflections, make up this group. Again, the group with just boosts and rotations is called $SO(1,3)$.

2.5.3 Example: Linear transformations

In the crash course in linear algebra, I mentioned the concept of linear transformations briefly. Let's linear transformations $M : V \to V$, mapping vectors in a

[21] A group G is a set of elements with an operation $\cdot$, that obeys the following: 1) associativity, i.e. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in G$; 2) it contains an identity element e in G such that $a \cdot e = e \cdot a = a$, and 3) every element a in G has an inverse b, such that $a \cdot b = b \cdot a = e$.

vector space V over a field $\mathbb{F}$ to other vectors in V. The linear transformation must satisfy the following property:

$$M(\lambda\vec{x} + \mu\vec{y}) = \lambda M(\vec{x}) + \mu M(\vec{y}) \tag{2.56}$$

for all $\vec{x}, \vec{y} \in V$, and $\lambda, \mu \in \mathbb{F}$. This object exists independently of any basis, but given a basis, it can be represented by a matrix $M^\mu{}_\nu$, obtained by examining the action of the transformation on the basis vectors:

$$M(\vec{e}_\mu) = M^\nu{}_\mu \vec{e}_\nu \,. \tag{2.57}$$

Suppose M acting on some arbitrary vector $\vec{x} = x^\mu \vec{e}_\mu$ gives the result $\vec{y} = y^\nu \vec{e}_\nu$. We can see however that

$$\vec{y} = y^\nu \vec{e}_\nu = M(x^\mu \vec{e}_\mu) = x^\mu M(\vec{e}_\mu) = M^\nu{}_\mu x^\mu \vec{e}_\nu \,, \tag{2.58}$$

or in other words, the components transform as

$$y^\nu = M^\nu{}_\mu x^\mu \,. \tag{2.59}$$

We can therefore see that given a basis, M behaves just like matrix multiplication, as we already knew to be true from linear algebra.

In another basis related to the old one via $\vec{e}_\nu = a^\sigma{}_\nu \vec{e}'_\sigma$, we see that

$$M(\vec{e}_\mu) = M^\nu{}_\mu \vec{e}_\nu = M^\nu{}_\mu a^\sigma{}_\nu \vec{e}'_\sigma \,, \tag{2.60}$$

but also

$$M(\vec{e}_\mu) = M(a^\lambda{}_\mu \vec{e}'_\lambda) = a^\lambda{}_\mu M(\vec{e}'_\lambda) \implies M(\vec{e}'_\lambda) = (a^{-1})^\mu{}_\lambda M(\vec{e}_\mu) \,. \tag{2.61}$$

Therefore, in the new basis,

$$M(\vec{e}'_\lambda) = a^\sigma{}_\nu M^\nu{}_\mu (a^{-1})^\mu{}_\lambda \vec{e}'_\sigma \,, \tag{2.62}$$

i.e. under the change of basis,

$$M^\sigma{}_\lambda \mapsto a^\sigma{}_\nu M^\nu{}_\mu (a^{-1})^\mu{}_\lambda \,, \tag{2.63}$$

which is the transformation rule for type $(1,1)$ tensors, and also what you may be familiar with in linear algebra about the change of basis of linear transformations.

Compare this with the transformation rule for the metric tensor,

$$g_{\mu\nu} \mapsto a^\sigma{}_\mu a^\lambda{}_\nu g_{\sigma\lambda} \,, \tag{2.64}$$

and you'll notice a big difference! And that's because the metric tensor is a type $(0,2)$ tensor. The key lesson here is that *2D tensors are much more than just matrices*; while it can be useful to write out tensor components as matrices, one needs to keep in mind that tensors also come equipped with a transformation rule!

2.5.4 Invariants of 2D tensors

Invariants are extremely useful: they don't transform under a basis change, and so they're very easy to deal with. They're also giving you information about the tensor that is independent of the basis. The first commonly discussed invariant of 2D tensors are the **determinant**, which you can compute from the matrix representation of the tensor; we will discuss this in much greater detail after we've built up some machinery to discuss it. The second invariant is called the **trace**. In matrix language, this is simply the sum of the terms along the diagonal. It's hard to understand how this could be an invariant from the matrix perspective, but in terms of indices, it is simply given by

$$\mathrm{tr}(M) = g^{\mu\nu}M_{\mu\nu} = g_{\mu\nu}M^{\mu\nu} = \delta^\nu_\mu M^\mu{}_\nu = M^\mu{}_\mu \,, \qquad (2.65)$$

where I'm showing you how to obtain the trace for 2D tensors of all types.

2.5.5 Symmetric and antisymmetric tensors

A tensor is said to be **symmetric** in some indices if the tensor values are the same when the two indices are swapped. For example, we say that $S^{\mu\nu}$ is a symmetric tensor if $S^{\mu\nu} = S^{\nu\mu}$. Note that we have also

$$S^\mu{}_\nu = g_{\nu\alpha}S^{\mu\alpha} = g_{\nu\alpha}S^{\alpha\mu} = S_\nu{}^\mu \,, \qquad (2.66)$$

and so the position of the indices doesn't matter in determining whether a tensor is symmetric or not. Moreover, under a change of basis,

$$S'^{\alpha\beta} = a^\alpha{}_\mu a^\beta{}_\nu S^{\mu\nu} = a^\alpha{}_\mu a^\beta{}_\nu S^{\nu\mu} = S'^{\beta\alpha} \,, \qquad (2.67)$$

and so a symmetric matrix stays symmetric under a change of basis.

We can similarly define a tensor to be **antisymmetric** in some indices if, under a swap of the two indices, the tensor picks up a minus sign, e.g. $A^{\mu\nu} = -A^{\nu\mu}$. Again, a tensor that is antisymmetric when indices are in one position are still antisymmetric if the indices are raised or lowered; they also remain antisymmetric under an arbitrary change of basis.

The contraction of a symmetric and an antisymmetric tensor is always zero, since e.g.

$$S_{\mu\nu}A^{\mu\nu} = S_{\nu\mu}A^{\mu\nu} = -S_{\nu\mu}A^{\nu\mu} = -S_{\mu\nu}A^{\mu\nu} = 0 \,, \qquad (2.68)$$

where in the second last step I have simply relabeled the contracted indices (since they are dummy indices), and noted that $S_{\mu\nu}A^{\mu\nu}$ is equal to its negative, and therefore has to be zero.

Every tensor can always be decomposed into a symmetric piece and an antisymmetric piece. To see this, take an arbitrary tensor $B^{\mu\nu}$. We can always rewrite this as

$$B^{\mu\nu} = \frac{1}{2}(B^{\mu\nu} + B^{\nu\mu}) + \frac{1}{2}(B^{\mu\nu} - B^{\nu\mu}) \equiv S^{\mu\nu} + A^{\mu\nu} \,, \qquad (2.69)$$

where $S^{\mu\nu} = (B^{\mu\nu} + B^{\nu\mu})/2$ a symmetric tensor, and $A^{\mu\nu} = (B^{\mu\nu} - B^{\nu\mu})/2$ is antisymmetric.

2.5.6 Kronecker and Levi-Civita tensors

We'll now discuss two special tensors that arise very commonly in tensor algebra. The first of them, the **Kronecker delta** δ^μ_ν, is defined as a $(1,1)$ tensor that, in some basis, is unity if $\mu = \nu$ and zero otherwise. Let's check what happens under an arbitrary change of basis:

$$\delta^\mu_\nu \mapsto (a^{-1})^\lambda{}_\nu a^\mu{}_\sigma \delta^\sigma_\lambda = \delta^\mu_\nu \,. \qquad (2.70)$$

In other words, *the Kronecker delta always has the same numerical components in all coordinate systems.*[22] [23]

The **Levi-Civita symbol** $\epsilon_{\mu_1\mu_2\cdots\mu_n}$ is defined as an object with n indices such that $\epsilon_{12\ldots n} = 1$, and $\epsilon_{\ldots i_p \ldots i_q \ldots} = -\epsilon_{\ldots i_q \ldots i_p \ldots}$, i.e. every time two indices are exchanged, the result differs by a minus sign. This definition guarantees that when two indices are equal, the Levi-Civita symbol is zero. One particularly important use-case of the Levi-Civita symbol is in expressing antisymmetric

[22] This explains why the indices are almost never distinguished in the literature: both $\delta^\mu{}_\nu$ and $\delta_\nu{}^\mu$ act the same way on any object: just replace μ with ν or vice-versa.

[23] Sometimes, people talk about the $(0,2)$ tensor $\delta_{\mu\nu} = g_{\mu\lambda}\delta^\lambda_\nu = g_{\mu\nu}$, which as you can see is really the metric tensor (whose coordinates *can* change with a change of basis). I will usually avoid using $\delta_{\mu\nu}$, unless we are talking about $\mathbb{R}^n$, where the index position doesn't matter.

quantities in component form. For example, the **cross product** $\vec{a} \times \vec{b}$ can be written as

$$(\vec{a} \times \vec{b})_k = \epsilon_{ijk} a^i b^j \,. \tag{2.71}$$

The **determinant** of an $n \times n$ matrix M can likewise be written as (assuming summation over repeated indices)

$$\epsilon_{\mu_1 \cdots \mu_n} \det(M) = \epsilon_{\nu_1 \cdots \nu_n} M^{\nu_1}{}_{\mu_1} \cdots M^{\nu_n}{}_{\mu_n} \,. \tag{2.72}$$

Let's suppose there is an n-dimensional tensor $\eta_{\mu_1 \mu_2 \cdots \mu_n}$ whose components coincide with $\epsilon_{\mu_1 \mu_2 \cdots \mu_n}$ in one particular basis. Then under a change of basis,

$$\begin{aligned}
\eta_{\mu_1 \cdots \mu_n} &\mapsto a^{\nu_1}{}_{\mu_1} a^{\nu_2}{}_{\mu_2} \cdots a^{\nu_n}{}_{\mu_n} \epsilon_{\nu_1 \cdots \nu_n} \\
&= \epsilon_{\mu_1 \cdots \mu_n} \det(a) = \eta_{\mu_1 \cdots \mu_n} \det(a) \,.
\end{aligned} \tag{2.73}$$

We see that the Levi-Civita symbol is *almost* a tensor whose components do not transform, up to a pesky determinant.

At this point, let's examine the determinant of the metric tensor itself, $g \equiv \det(g_{\mu\nu})$. We see that under the same transformation, the determinant after the change of basis is

$$g' \equiv \det(g'_{\mu\nu}) = \det(a^\lambda{}_\mu a^\sigma{}_\nu g_{\lambda\sigma}) = (\det(a))^2 g \,, \tag{2.74}$$

and therefore the quantity $\sqrt{|g'|} = |\det(a)| \sqrt{|g|}$; note that the absolute value is important, since the metric tensor can have negative determinant (e.g. in Minkowski space!). Now let's consider the object

$$\varepsilon_{\mu_1 \mu_2 \cdots \mu_n} = \sqrt{|g|} \epsilon_{\mu_1 \mu_2 \cdots \mu_n} \tag{2.75}$$

Under the same transformation, we now see that provided $\det(a) > 0$, which are referred to as *orientation preserving changes of basis*,[24]

$$\varepsilon_{\mu_1 \cdots \mu_n} \mapsto \sqrt{|g|} \det(a) \epsilon_{\mu_1 \cdots \mu_n} = \sqrt{|g'|} \epsilon_{\mu_1 \cdots \mu_n} \,, \tag{2.76}$$

and so once again, we find a tensor known as the **Levi-Civita tensor** $\varepsilon_{\mu_1 \cdots \mu_n}$ that always has the same form in any basis (although we must evaluate the determinant of the transformed metric, which is in general different for each basis).

If we limit ourselves to rotations in $\mathbb{R}^n$ and boosts (and rotations) in Minkowski space, then $\varepsilon_{\mu_1 \cdots \mu_n} = \epsilon_{\mu_1 \cdots \mu_n}$ is a good old tensor. However, if we include transformations like reflections, for example, then we need a lot more care.[25]

2.5.7 Isotropic Cartesian tensors

Another special tensor that we'll now discuss is the isotropic, Cartesian tensor. Consider a Cartesian coordinate system with orthonormal basis vectors, so that $g_{ij} = \delta_{ij}$, the Kronecker delta function. We looked at the set of orthogonal matrices form what we called $O(n)$, which are matrices with the property that $O^{-1} = O^{\mathsf{T}}$. When we perform a change of basis under these matrices, we found that this leaves the metric invariant, since

$$g'_{kl} = O^i{}_k O^j{}_l \delta_{ij} = O_{jk} O^j{}_l = O^{\mathsf{T}}_{kj} O^j{}_l = \delta_{kl} \,. \tag{2.77}$$

You can check that the same thing is true for products of δ_{ij}, for example $T_{ijklmn} = \delta_{ij} \delta_{kl} \delta_{mn}$.

What is the most general form a tensor of rank m that is invariant under an $O(n)$ transformation? These are important questions that have tedious answers to them, and so we won't try to prove these results (see for example Ref. [4], but rather just state them. The most general $O(n)$ invariant tensor of rank 4 I_{ijkl} is

[24] An easy way to see that $\varepsilon_{\mu_1 \cdots \mu_n}$ needs to pick up a sign under non-orientation-preserving transformations is by considering what happens to a cross-product under reflection, where $\vec{x} \mapsto -\vec{x}$. With the sign flip, $\varepsilon^{ijk} x_j y_k \mapsto -\varepsilon^{ijk} x_j y_k$ as needed, since the final vector as a result of the cross product also needs to pick up a minus sign under reflection.

[25] You might have heard a joke about how 'a tensor is something that transforms like a tensor'. We can always exclude some class of basis changes from consideration to make a class of objects well-behaved and tensor-like.

$$I_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{lj} + \gamma\delta_{il}\delta_{jk} \qquad (2.78)$$

for some numbers α, β and γ.

How about for $SO(n)$, which is simply all the orthogonal matrices with determinant 1, corresponding to just rotations, and not reflections? We can check, for example, that ϵ_{ijk} is invariant under an $SO(3)$ transformation O, since

$$
\begin{aligned}
\epsilon'_{lmn} &= O^i{}_l O^j{}_m O^k{}_n \epsilon_{ijk} \\
&= \epsilon_{lmn}\det(O) \\
&= \epsilon_{lmn}\,,
\end{aligned}
\qquad (2.79)
$$

since by definition, an $SO(n)$ matrix has determinant 1. With this, it's not surprising that, for example, the most general $SO(4)$-invariant, rank-4 tensor J_{ijkl} is

$$J_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{il}\delta_{jk} + \gamma\delta_{il}\delta_{jk} + \lambda\epsilon_{ijkl}\,.$$

(**End of Lecture**: Wednesday Sep 25 2024)

2.6 Vectors and Tensors on Manifolds

We are now armed and ready to do calculus on surfaces that aren't just flat spaces like $\mathbb{R}^n$, or even flat spacetimes like Minkowski space. We want to build up the machinery to be able to do calculus on what are called **manifolds**.

So what are manifolds? There's a lot of technical set-up that Carroll goes through in his notes/textbook, while Stone and Goldbart basically condenses it down into a paragraph of dense text. But for our purposes the details don't really matter. A manifold is something that, if you zoom in enough, looks like $\mathbb{R}^n$ or Minkowski space, $\mathbb{R}^{1,3}$ (from here on, I will include Minkowski space when I say $\mathbb{R}^n$), in the sense that you can map patches of a manifold to these spaces in a completely invertible way. Spheres and tori are examples of manifolds; a line ending on a plane is an example that is *not* a manifold. The 2D surface of a cone is also a manifold, but there's something badly behaved about the tip of the cone: curves that go through it have to under a sudden change in direction, and so calculus on the cone sounds problematic near the tip. We will in general only care about **differentiable** manifolds, which do not have such troubling singularities.

Note that we only need to be able to map the manifold into $\mathbb{R}^n$ one patch at a time; manifolds in general do not usually allow for the entire manifold to be mapped at one go into $\mathbb{R}^n$. Take for example the sphere: any patch of the sphere can be mapped without trouble onto $\mathbb{R}^2$, but not the whole sphere at one go. This is why map making is so complicated!

So how do we set ourselves up to do calculus on manifolds? There are a few things to consider:

1. One thing we take for granted in $\mathbb{R}^n$, however, is that vectors can "slide around" on $\mathbb{R}^n$ trivially, allowing us to compare vectors from one point in $\mathbb{R}^n$ with another without much thought. This is very different on arbitrary manifolds however; on a sphere like to Earth, sliding a vector around in a closed loop causes the vector to rotate with respect to itself (see Fig. 10)! We therefore have to be extremely careful about sliding vectors around, or comparing vectors across different points. As such, vectors on general manifolds are therefore objects that belong to each particular *point* on the manifold.

2. We know that any manifold looks locally like $\mathbb{R}^n$. Picture a sphere for example: just like on Earth, every point locally looks like $\mathbb{R}^2$. Instinctively, we understand that at every point, we can set up an n-dimensional vector space with a metric. This is known as the **tangent space**: the space of all vectors at a point on a manifold. With the picture of the Earth in mind, the tangent space is a 2D vector space with vectors tangent to the surface of the Earth. But keep in mind that this picture is misleading, because the "arrows" that we have in our heads point out into outer space, whereas a general manifold doesn't have to embedded into a higher dimensional space at all.

3. Which brings us to our next point. We also want to find a way to describe the n-dimensional vector space that doesn't depend on a choice of basis or an embedding like the Earth into outer space. A vector in the tangent space should be an object that knows only about the manifold itself, and doesn't depend on the choice of basis. Of course, we will often want to choose a basis to calculate things, but the definition of the space should be basis independent.

2.6.1 Tangent spaces

We'll now try to put this intuition on solid mathematical ground. To properly define the tangent space at the point p on a manifold M, let's consider the set of all smooth functions $f : M \to \mathbb{R}$. Take any curve on the manifold passing through p parametrized by λ. Then at point p, each curve defines an operator $d/d\lambda$ that maps $f \mapsto df/d\lambda$, which is simply the **directional derivative** at the point of f along the curve. *The tangent space T_p at the point p is then the space of directional derivative operators $d/d\lambda$ through p.* This intuition is illustrated in Fig. 11.

We can check is that the directional derivatives do form a vector space. A lot of the desired properties in the definition in Sec. 2.1.1 follow from the properties of derivatives; you can check many of them yourself. Most importantly, the linear combination of any two directional derivatives, $a(d/d\lambda) + b(d/d\eta)$, is itself a perfectly good directional derivative, obeying all the expected rules of the derivative, e.g. the product rule, and so the vector space is closed on itself.

Let's move on next to constructing a basis for T_p. By definition of a manifold M, we can zoom in on a small patch around p, which looks like $\mathbb{R}^n$, where n is the **dimension** of M. On this patch, I can define coordinates x^μ on this $\mathbb{R}^n$-like patch, so that for any function $f : M \to \mathbb{R}$ on this patch, the directional derivative is simply

$$\frac{df}{d\lambda} = \frac{\partial f}{\partial x^\mu} \frac{dx^\mu}{d\lambda} \,, \tag{2.80}$$

as expected from the chain rule. This shows that any element of T_p can be written as

$$\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda} \partial_\mu \,, \tag{2.81}$$

where we've defined $\partial_\mu \equiv \partial/\partial x^\mu$. Notice that this has the form of $y^\mu \vec{e}_\mu$ as we have come to expect from vectors in a vector space; we should therefore interpret $dx^\mu/d\lambda$ as components, and ∂_μ as a basis. The physical intuition is very similar to what is shown in Fig. 11 once again, except now you should imagine throwing on a small $\mathbb{R}^2$ surface with a Cartesian grid, which locally
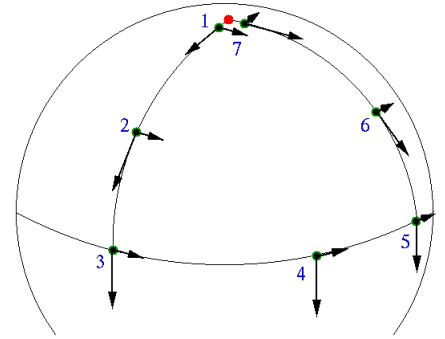

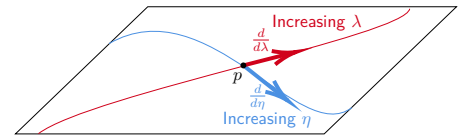Figure 10: Sliding a vector around on a sphere causes it to rotate with respect to itself!

Figure 11: Directional derivatives as elements of the tangent space T_p.

defines a coordinate system x^μ, and which then allows T_p to inherit the basis $\{\partial_\mu\}$. This is known as a **coordinate basis** for T_p. Keep in mind that you are *not limited to choosing a Cartesian basis!* You are free to choose, for example, polar coordinates (r, θ) in $\mathbb{R}^2$, in which case your basis is $\{\partial_r, \partial_\theta\}$; this basis is *neither normalized nor orthogonal*, but it is perfectly acceptable.

What happens under a change of coordinates, say switching between Cartesian and polar coordinates on $\mathbb{R}^2$? That induces also a change of basis. Suppose we perform a change of coordinates from x^μ to x'^μ; all of the derivatives then transform as[26]

$$\partial'_\nu \equiv \frac{\partial}{\partial x'^\nu} = \frac{\partial x^\mu}{\partial x'^\nu} \partial_\mu \, . \tag{2.82}$$

[26] Note that Carroll adopts the notation $x^{\mu'}$ instead, whereas SG avoids using x'^μ, using z^μ to denote the components in the new basis, but then using the notation ∂'_μ to denote differentiation in the new basis. Every notation is nonideal in its own way.

A simple example is the transformation from polar to Cartesian coordinates, given by $x = r\cos\theta$ and $y = r\sin\theta$, so that

$$\partial_r = \frac{\partial x}{\partial r}\partial_x + \frac{\partial y}{\partial r}\partial_y = \cos\theta \cdot \partial_x + \sin\theta \cdot \partial_y \, , \tag{2.83}$$

$$\partial_\theta = \frac{\partial x}{\partial \theta}\partial_x + \frac{\partial y}{\partial \theta}\partial_y = -r\sin\theta \cdot \partial_x + r\cos\theta \cdot \partial_y \, . \tag{2.84}$$

Given this, any arbitrary $V \in T_p$ can be written as $V = V^\mu \partial_\mu$ given any choice of coordinate system. Under a change of coordinates, becomes

$$V^\mu \partial_\mu = V'^\nu \partial'_\nu = V'^\nu \frac{\partial x^\mu}{\partial x'^\nu}\partial_\mu \, , \tag{2.85}$$

and therefore

$$V'^\nu = \frac{\partial x'^\nu}{\partial x^\mu} V^\mu \, . \tag{2.86}$$

Comparing Eqs. (2.82) and (2.86), we can see that the components of the vector transforms **contravariantly**, while the basis vectors transform **covariantly**. As before, we'll move away rapidly from the formal beginnings we are at here, and will almost entirely deal with the components of vector, leaving the basis implicit. It is therefore common to say that "vectors transform contravariantly" according to Eq. (2.86), even though we really just mean the components.

Finally, we call the collection of all the tangent spaces at every point p on M the **tangent bundle**.

(End of Lecture: Monday Sep 30 2024)

2.6.2 Cotangent spaces, tensors and the metric

As we did before, we can now proceed to look at the dual of the tangent space T_p^* at the point p on M, which we call the **cotangent space**. This is a vector space of linear maps $\omega : T_p \to \mathbb{R}$, and the elements that live in T_p^* are called **one-forms**. As before, a choice of basis $\{\partial_\mu\}$ in T_p induces a basis on T_p^*, which we write suggestively as $\{\mathrm{d}x^\mu\}$, such that

$$\mathrm{d}x^\mu(\partial_\nu) = \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu \, . \tag{2.87}$$

Any arbitrary $\omega \in T_p^*$ can therefore be written as $\omega = \omega_\mu \mathrm{d}x^\mu$. One way to think of elements in T_p^* is again to go back to the space of functions $f : M \to \mathbb{R}$ that

pass through the point p, and to view elements in T_p^* as $\mathrm{d}f$, the gradient of the function, so that given a coordinate system,

$$\mathrm{d}f = \frac{\partial f}{\partial x^\mu}\mathrm{d}x^\mu\,. \tag{2.88}$$

Under a change of basis, we can see that the basis vectors

$$\mathrm{d}x'^\nu = \frac{\partial x'^\nu}{\partial x^\mu}\mathrm{d}x^\mu\,, \tag{2.89}$$

which transform contravariantly, and the components

$$\omega'_\nu = \frac{\partial x^\mu}{\partial x'_\nu}\omega_\mu\,, \tag{2.90}$$

transform covariantly. As usual, we will only usually see the components discussed when talk about one-forms.

Similarly, the collection of all cotangent spaces at every point p on M is called the **cotangent bundle**.

We can also define **tensors** given the tangent space T_p and the cotangent space T_p^* by performing a series of tensor products. Given a coordinate system, we can define objects like

$$T = T^{\mu_1\cdots\mu_j}{}_{\nu_1\cdots\nu_k}\,\partial_{\mu_1}\otimes\cdots\otimes\partial_{\mu_j}\otimes\mathrm{d}x^{\nu_1}\otimes\cdots\otimes\mathrm{d}x^{\nu_k}\,, \tag{2.91}$$

that can act on some combination of vectors, one-forms or tensors in the same manner as before. Once again, the coordinates transform under a change of coordinate system as e.g.

$$T'^{\alpha_1\cdots\alpha_j}{}_{\beta_1\cdots\beta_k} = \frac{\partial x^{\alpha_1}}{\partial x^{\mu_1}}\cdots\frac{\partial x^{\alpha_j}}{\partial x^{\mu_j}}\frac{\partial x^{\nu_1}}{\partial x^{\beta_1}}\cdots\frac{\partial x^{\nu_k}}{\partial x^{\beta_k}}T^{\mu_1\cdots\mu_j}{}_{\nu_1\cdots\nu_k}\,. \tag{2.92}$$

Very often, however, it is better to take ∂_μ and $\mathrm{d}x^\mu$ and apply the change of coordinates we are familiar with from calculus, treating ∂_μ as a partial derivative and $\mathrm{d}x^\mu$ as an infinitesimal quantity respectively.

An extremely important rank-2 tensor that you're already familiar with is the **metric tensor**, which given some choice of coordinate system can be written as $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\otimes\mathrm{d}x^\nu$. The components of the metric must be symmetric (given the properties of the inner product), and so even though the tensor product does not commute, it is very common to see the metric written as $g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$; you should keep in mind that expressions like $2\mathrm{d}x\mathrm{d}y$ really should be thought of as $\mathrm{d}x\otimes\mathrm{d}y + \mathrm{d}y\otimes\mathrm{d}x$. Just like any $(0,2)$ tensor, $\mathrm{d}s^2$ takes in two vectors, and spits out a real number, with

$$\mathrm{d}s^2(V,W) = g_{\mu\nu}V^\mu W^\nu\,, \tag{2.93}$$

which should be familiar to you from our previous discussions about the metric.

A very simple example of a metric is the typical Euclidean $\mathbb{R}^3$ metric,

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\,, \tag{2.94}$$

which under a transformation from Cartesian to spherical coordinates gives

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2\,. \tag{2.95}$$

These two metrics are really exactly the same thing, referring to the same inner product space, just written in a different coordinate basis. Another slightly less trivial metric is the metric of a 2-sphere,

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\,,\tag{2.96}$$

which is emphatically *different* from the metric of flat, Euclidean $\mathbb{R}^2$. The metric contains all relevant information about the curvature of the manifold. While we will unfortunately not have time to discuss curvature in any detail, you are now armed with all the tools necessary to talk about curvature and general relativity.

2.7 Differentiation on Manifolds

We're now finally ready to start tackling the problem of doing calculus on manifolds, starting with differentiation. This is a vast area of knowledge that we do not have time to fully explore; instead, we will take a look at two different types of derivatives we can take on a manifold, and leave a more complete treatment of this topic to more advanced courses e.g. in general relativity.

Going forward, we will now turn our attention to tensor *fields* over a manifold M that we can use a coordinate system given by x^μ to describe. This defines a tensor at every point on the manifold, all describable by components and basis vectors given by the coordinate basis induced by x^μ.

We have already seen that given a scalar field f defined everywhere on some manifold M, on which we can assign a coordinate system given by x^μ, we can construct the gradient

$$\mathrm{d}f = \frac{\partial f}{\partial x^\mu}\mathrm{d}x^\mu\,,\tag{2.97}$$

which is an object that lives in T_p^*, and is therefore a well-behaved one-form. However, now suppose instead we have a *one-form field* V_ν defined everywhere on the same manifold M. If we take the gradient of this object, $\partial_\mu V_\nu$, we do *not* in general obtain a tensor. This is because under a change of coordinates, we find

$$
\begin{aligned}
\partial'_\alpha W'_\beta &= \frac{\partial x^\mu}{\partial x'^\alpha}\partial_\mu\left(\frac{\partial x^\nu}{\partial x'^\beta}W_\nu\right)\\
&= \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta}\partial_\mu W_\nu + W_\nu\frac{\partial x^\mu}{\partial x'^\alpha}\cdot\frac{\partial}{\partial x^\mu}\frac{\partial x^\nu}{\partial x'^\beta}\,.
\end{aligned}\tag{2.98}
$$

If the second term wasn't there, we would have obtained an honest-to-goodness $(0,2)$ tensor, but unfortunately that's not what we have. Objects like $\partial_\mu W_\nu$ are therefore not tensors in general, and are coordinate-dependent objects, and not the coordinate-independent objects that we want tensors to be. This shouldn't have come as too much of a surprise; we began our discussion of manifolds by emphasizing that vectors and one-forms are very much objects that are tied to the tangent space at point p, whereas the gradient of a vector or one-form naturally means subtracting vectors between nearby points, which we need to define carefully.

We therefore have to work a bit harder to define operations that look like derivatives on manifolds. It turns out that there are several operations that behave like derivatives on the manifold; in this course, we will only have time to look at two of them: the **Lie derivative**, and the **exterior derivative**. We will leave the **covariant derivative** to courses focused on classical and quantum field theories to explore.

36

2.7.1 The Lie derivative

We saw earlier that vector fields $X \equiv X^\mu \partial_\mu$ on a manifold M can be thought of as directional derivatives acting on functions on the manifold M, i.e. $X(f) = X^\mu \partial_\mu f$, the directional derivative of f in the direction of X^μ. Consider yet another vector field $Y \equiv Y^\mu \partial_\mu$, and define the **Lie bracket**

$$[X, Y](f) \equiv X(Y(f)) - Y(X(f)). \tag{2.99}$$

In your problem set, you will prove that the following statements about the Lie bracket are true:

1. The Lie bracket is linear, i.e. for f, g functions on M and a, b real numbers,

$$[X, Y](af + bg) = a[X, Y](f) + b[X, Y](g). \tag{2.100}$$

2. The Lie bracket obeys Leibniz's rule, i.e.

$$[X, Y](fg) = f[X, Y](g) + g[X, Y](f). \tag{2.101}$$

3. In terms of the components of X and Y, we can write

$$[X, Y]^\mu = X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu. \tag{2.102}$$

4. $[X, Y]^\mu$ transforms under a change of coordinates as a vector.

What does the Lie bracket represent? For a given vector field Y, we can define the **flow** associated with Y as the map that takes a point x_0^μ and maps it to $x(t)$ by solving

$$\frac{dx^\mu}{dt} = Y^\mu \tag{2.103}$$

with initial condition $x^\mu(0) = x_0^\mu$. In other words, the vector field Y can be thought of as a velocity field of a flowing fluid, with x riding along with the fluid. Now consider two vector fields X and Y. Starting at a point, flow along X for some small time t, and then along Y for some small time s. Now, try to undo the flow by now flowing along $-X$ for some time t again, and along $-Y$ for some time s. The Lie bracket tells you how far away you are after this procedure from the starting point: you will miss by $\delta x^\mu = st[X, Y]^\mu$ as s and t go to zero.

To get a better sense of what the Lie bracket is doing, let's consider $\mathbb{R}^2$ with polar coordinates (r, ϕ), with the vector fields

$$A = \frac{1}{r} \partial_\phi, \qquad B = \partial_r. \tag{2.104}$$

The flow lines on the 2D-plane corresponds to circular paths around the origin for A, and radial lines outward for B. The Lie bracket for these vector fields is

$$[A, B] = \frac{1}{r} \partial_\phi \partial_r - \partial_r \left(\frac{1}{r} \partial_\phi \right) = \frac{1}{r^2} \partial_\phi. \tag{2.105}$$

We can see that if we take a little circuit where we flow along A over some time t, followed by B over some time s, and then back along $-A$ over time t and then along $-B$ over time $-s$, we find that

$$\delta r = 0, \qquad \delta \phi = \frac{t}{r} - \frac{t}{r + s} = \frac{ts}{r^2} = st[A, B]^\theta. \tag{2.106}$$

We are now ready to define an operation called the **Lie derivative** $\mathcal{L}_X$ along some vector field X. It acts on a scalar function f through X itself, i.e.

$$\mathcal{L}_X f \equiv X(f) = X^\mu \partial_\mu f \,, \qquad (2.107)$$

which is, once again, the directional derivative of f along X. Next, on a vector field Y, it acts via the Lie bracket, i.e.

$$\mathcal{L}_X Y \equiv [X, Y] \,. \qquad (2.108)$$

We can now deduce the action of the Lie derivative on more complicated objects by ensuring that the Lie derivative respects *1) linearity*, with $\mathcal{L}_X(aT + bS) = a\mathcal{L}_X T + b\mathcal{L}_X S$, and *2) Leibniz's rule*, i.e. the product rule, $\mathcal{L}_X(T \otimes S) = (\mathcal{L}_X T) \otimes S + T \otimes (\mathcal{L}_X S)$.

For example, to see how the Lie derivative acts on a one-form ω_μ, we see that given a vector field V^μ, the Lie derivative should act on the object $\omega_\mu V^\mu$ in the following manner:

$$\mathcal{L}_X(\omega_\mu V^\mu) = X_\nu \partial^\nu (\omega_\mu V^\mu) = X_\nu (V^\mu \partial^\nu \omega_\mu + \omega_\mu \partial^\nu V^\mu) \,. \qquad (2.109)$$

On the other hand, Leibniz's rule requires

$$\mathcal{L}_X(\omega_\mu V^\mu) = V^\mu \mathcal{L}_X \omega_\mu + \omega_\mu \mathcal{L}_X V^\mu = V^\mu \mathcal{L}_X \omega_\mu + \omega_\mu [X, V]^\mu$$
$$= V^\mu \mathcal{L}_X \omega_\mu + \omega_\mu (X^\nu \partial_\nu V^\mu - V^\nu \partial_\nu X^\mu) \,. \qquad (2.110)$$

Comparing both expressions, we find

$$V^\mu \mathcal{L}_X \omega_\mu = X_\nu V^\mu \partial^\nu \omega_\mu + \omega_\mu V^\nu \partial_\nu X^\mu \implies \mathcal{L}_X \omega_\mu = X_\nu \partial^\nu \omega_\mu + \omega_\nu \partial_\mu X^\nu \,. \qquad (2.111)$$

You can check that $\mathcal{L}_X \omega_\mu$ transforms as a one-form; in fact, the Lie derivative of any (k, l)-tensor always transforms as a (k, l)-tensor.

Of particular importance is the Lie derivative of $(0, 2)$-tensors such as the metric, which reads (and you will prove in your problem set)

$$(\mathcal{L}_X g)_{\mu\nu} = X^\alpha \partial_\alpha g_{\mu\nu} + g_{\mu\alpha} \partial_\nu X^\alpha + g_{\alpha\nu} \partial_\mu X^\alpha \,. \qquad (2.112)$$

Let's compare this with what happens to the metric $g_{\mu\nu}(x)\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$ when we make a displacement $x^\alpha \mapsto x^\alpha + \epsilon X^\alpha$ in the direction of the vector field X. Under such a displacement, we have

$$\mathrm{d}x^\mu \mapsto \mathrm{d}x^\mu + \epsilon \partial_\beta X^\mu \cdot \mathrm{d}x^\beta \,,$$
$$g_{\mu\nu} \mapsto g_{\mu\nu} + \partial_\beta g_{\mu\nu} \cdot \epsilon X^\beta \,. \qquad (2.113)$$

Therefore,

$$g_{\mu\nu}(x)\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \mapsto g_{\mu\nu}(x)\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu + \epsilon X^\beta \partial_\beta g_{\mu\nu}\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$$
$$+ \epsilon g_{\mu\nu}\left(\partial_\beta X^\mu \,\mathrm{d}x^\beta \otimes \mathrm{d}x^\nu + \partial_\beta X^\nu \,\mathrm{d}x^\mu \otimes \mathrm{d}x^\beta\right)$$
$$= \left[g_{\mu\nu} + \epsilon\left(X^\beta \partial_\beta g_{\mu\nu} + g_{\beta\nu} \partial_\mu X^\beta + g_{\mu\beta} \partial_\nu X^\beta\right)\right] \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$$
$$= (g_{\mu\nu} + \epsilon(\mathcal{L}_X g)_{\mu\nu})\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \,. \qquad (2.114)$$

Therefore, for an infinitesimal displacement along a vector field X, the metric changes by $\epsilon \mathcal{L}_X g$. In particular, if $\mathcal{L}_X g = 0$, then the metric is preserved. X

is said to be a **Killing field**, and indicates an existence of an **isometry** of the manifold, a transformation of the manifold that preserves distances. You should have in your mind a field indicating how points transform if a sphere is rotated about any axis, for example, as an example of a Killing field.

<center>(**End of Lecture**: Monday Oct 7 2024)</center>

2.7.2 Differential forms and the wedge product

Before we can move on to the next derivative operation, we need to introduce a special class of tensors known as **differential forms**, or simply **forms**. These are simply $(0, p)$ tensors that are completely antisymmetric. Thus, a scalar is a 0-form, a co-vector is a one-form—a term we have already been consistently using—and, for example, the Levi-Civita tensor $\varepsilon_{\mu\nu\rho\sigma}$ is a 4-form.

To make differential forms starting from one-forms, we can define a new operator called the **wedge product**; we can then take two forms $\omega \in T_p^*$ and $\eta \in T_p^*$ and construct $\omega \wedge \eta$, a higher order form. Suppose in a coordinate basis, $\omega = \omega_{\mu_1 \cdots \mu_p} \mathrm{d}x^{\mu_1} \otimes \cdots \otimes \mathrm{d}x^{\mu_p}$ is a p-form, and $\eta = \eta_{\mu_{p+1} \cdots \mu_{p+q}} \mathrm{d}x^{\mu_{p+1}} \otimes \cdots \otimes \mathrm{d}x^{\mu_{p+q}}$ is a q-form. Then the wedge product is a $(p+q)$-form, defined as

$$
\begin{aligned}
\omega \wedge \eta &= \frac{(p+q)!}{p!q!} \omega_{[\mu_1 \cdots \mu_p} \eta_{\mu_{p+1} \cdots \mu_{p+q}]} \mathrm{d}x^{\mu_1} \otimes \cdots \otimes \mathrm{d}x^{\mu_{p+q}} \\
&= \frac{(p+q)!}{p!q!} \omega_{\mu_1 \cdots \mu_p} \eta_{\mu_{p+1} \cdots \mu_{p+q}} \mathrm{d}x^{[\mu_1} \otimes \cdots \otimes \mathrm{d}x^{\mu_{p+q}]} , \quad (2.115)
\end{aligned}
$$

where

$$
T_{[\mu_1 \mu_2 \cdots \mu_n]} = \frac{1}{n!} (T_{\mu_1 \mu_2 \cdots \mu_n} + \text{alternating sum over permutations of indices}) .
$$
$$(2.116)$$

So, for example, for two one-forms A and B,

$$
(A \wedge B)_{\mu\nu} = 2A_{[\mu} B_{\nu]} = A_\mu B_\nu - A_\nu B_\mu . \tag{2.117}
$$

From the definition of the wedge product, we can see that it inherits the distributivity, associativity, and commutativity with $\mathbb{R}$ from the tensor product; just to be very clear, this means that for any three forms A, B and C, and real number λ,

$$
\begin{aligned}
A \wedge (B + C) &= A \wedge B + A \wedge C , \\
(A \wedge B) \wedge C &= A \wedge (B \wedge C) = A \wedge B \wedge C , \\
A \wedge \lambda B &= \lambda A \wedge B = \lambda(A \wedge B) . \tag{2.118}
\end{aligned}
$$

In addition, from the definition, we can also show that for a p-form ω and a q-form η,

$$
\omega \wedge \eta = (-1)^{pq} \eta \wedge \omega , \tag{2.119}
$$

so one has to be extra careful in switching the order of forms in a wedge product! The set of all wedge products of one-forms from T_p^* forms a vector space, $T_p^* \wedge T_p^*$. You can of course perform arbitrarily many wedge products to form higher-order forms in spaces that we denote as $\bigwedge^k(T_p^*)$, which is a vector space of k-forms.

<center>39</center>

As before, once a coordinate system is chosen on the tangent space, this induces a basis on $\bigwedge^k(T_p^*)$, denoted by $dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}$. Two-forms are objects that act on a pair of vectors, and return a number. From the definition of the wedge product that the action of the basis two-forms is

$$dx^\mu \wedge dx^\nu(\partial_\alpha, \partial_\beta) = \delta_\alpha^\mu \delta_\beta^\nu - \delta_\alpha^\nu \delta_\beta^\mu. \tag{2.120}$$

The analogous expression for a p-term would have $p!$ terms, corresponding to all the different permutations with the appropriate sign depending on the number of swaps to get to the permutation at hand.

As an example, on a 3-dimensional manifold with some choice of coordinate basis, a one-form looks like

$$A = A_\mu dx^\mu = A_1 dx^1 + A_2 dx^2 + A_3 dx^3, \tag{2.121}$$

a two-form has the form

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = F_{12} dx^1 \wedge dx^2 + F_{23} dx^2 \wedge dx^3 + F_{31} dx^3 \wedge dx^1, \tag{2.122}$$

where $dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu$. A three-form looks like

$$\Omega = \frac{1}{3!} \Omega_{\mu\nu\sigma} dx^\mu \wedge dx^\nu \wedge dx^\sigma = \Omega_{123} dx^1 \wedge dx^2 \wedge dx^3. \tag{2.123}$$

All of the components have to be completely antisymmetric. Once again, it is unfortunately extremely common for people to simply drop the wedge product, writing $F_{\mu\nu} dx^\mu \wedge dx^\nu$ as $F_{\mu\nu} dx^\mu dx^\nu$, leaving the reader to remember that in fact F is a two-form.

In d-dimensions, the space of p-forms is $d!/[p!(d-p)!]$ dimensional, and all p-forms with $p > d$ vanish identically; the basis of the p-form would involve the wedge product of two basis one-forms with the same index, which is necessarily zero by the antisymmetrization of the indices in the definition of the wedge product.

2.7.3 The exterior derivative

The wonderful world of forms and the antisymmetry of swapping the order of the wedge product of two one-forms essentially gives us a nice safe space for us to define yet another useful derivative-like operator. This is known as the **exterior derivative**, which takes a p-form $\omega = \omega_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$ in the coordinate basis, and produces a $(p+1)$-form $d\omega$, given by

$$\begin{aligned}
d\omega &= (p+1)\partial_{[\mu_1} \omega_{\mu_2 \cdots \mu_{p+1}]} dx^{\mu_1} \otimes \cdots \otimes dx^{\mu_{p+1}} \\
&= (p+1)\partial_{\mu_1} \omega_{\mu_2 \cdots \mu_{p+1}} dx^{[\mu_1} \otimes \cdots \otimes dx^{\mu_{p+1}]} \\
&= \frac{p+1}{(p+1)!} p! \partial_{\mu_1} \omega_{\mu_2 \cdots \mu_{p+1}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p+1}} \\
&= \partial_{\mu_1} \omega_{\mu_2 \cdots \mu_{p+1}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p+1}}. \tag{2.124}
\end{aligned}$$

where in the second last step, I used the definition of the wedge product to rewrite tensor products into wedge products. The last line is typically the form that is most useful.

Let's warm up by doing some examples. Taking the exterior derivative of a zero-form, which is just a scalar field f on the manifold, gives

$$df = \partial_\mu f \, dx^\mu, \tag{2.125}$$

which once again is the gradient. Comparing this with the Lie derivative, you might start to see a pattern here: all derivatives acting on scalar fields on the manifold must return the same thing.

For a one-form A, define $F = \mathrm{d}A$; we find

$$F = \mathrm{d}A = 2\partial_{[\mu}A_{\nu]}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = \partial_\mu A_\nu \,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \frac{1}{2}F_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,,$$
(2.126)

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$; you should at this point be noticing a striking resemblance to the electromagnetic field tensor! Let's write this out explicitly in component form, assuming we're on a manifold in 3D:

$$F = \mathrm{d}A = \left(\frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2}\right)\mathrm{d}x^1 \wedge \mathrm{d}x^2 + \left(\frac{\partial A_3}{\partial x^2} - \frac{\partial A_2}{\partial x^3}\right)\mathrm{d}x^2 \wedge \mathrm{d}x^3$$
$$+ \left(\frac{\partial A_1}{\partial x^3} - \frac{\partial A_3}{\partial x^1}\right)\mathrm{d}x^3 \wedge \mathrm{d}x^1$$
(2.127)

What you should notice here is that the exterior of the one-form A gives something that resembles the curl, $\nabla \times$ on $\mathbb{R}^3$!

Finally, suppose we have a two-form $G = G_{12}\mathrm{d}x^1 \wedge \mathrm{d}x^2 + G_{23}\mathrm{d}x^2 \wedge \mathrm{d}x^3 + G_{31}\mathrm{d}x^3 \wedge \mathrm{d}x^1$. Then

$$\mathrm{d}G = \left(\frac{\partial G_{23}}{\partial x^1} + \frac{\partial G_{31}}{\partial x^2} + \frac{\partial G_{12}}{\partial x^3}\right)\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3\,,$$
(2.128)

which resembles the divergence, $\nabla \cdot$ in $\mathbb{R}^3$.

How is the exterior derivative a derivative? Remember that a derivative-like operator should do two things: *1)* it should be linear, i.e. $\mathrm{d}(a\omega + b\eta) = a\,\mathrm{d}\omega + b\,\mathrm{d}\eta$, which clearly holds given that the partial derivative itself is linear, and *2)* that the operator should obey Leibniz's rule. In fact, that exterior derivative obeys a slightly modified version of Leibniz's rule: given a p-form ω and a q-form η,

$$\mathrm{d}(\omega \wedge \eta) = (\mathrm{d}\omega) \wedge \eta + (-1)^p \omega \wedge (\mathrm{d}\eta)\,,$$
(2.129)

the proof of which I will not show here, but can be straightforwardly derived from all the definitions above.

Let's now consider what happens when you take the exterior derivative twice. We can observe from the definition that

$$\mathrm{d}^2 A = \partial_k \partial_l A_{\mu_1 \cdots \mu_p}\mathrm{d}x^k \wedge \mathrm{d}x^l \wedge \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_k}\,.$$
(2.130)

However, we should be able to commute the two partial derivatives, while k and l are supposed to be antisymmetric indices in the wedge product, implying that the expression is zero. This therefore implies the following extremely important result:

$$\mathrm{d}^2 = 0\,,$$
(2.131)

sometimes known as **Poincaré's lemma**. We say that a p-form A is **closed** if $\mathrm{d}A = 0$, and **exact** if $A = \mathrm{d}B$ for some $(p-1)$-form B. All exact forms are closed, but whether or not all closed forms are also exact depends on the topology of the manifold you are in; all closed p-forms where $p \geq 1$ are exact in Minkowski or real space.

What are we to make of all of this? By studying forms and the exterior derivative, we have developed machinery that is the generalization of ∇ to

arbitrary dimensions and manifolds with arbitrary coordinate basis. In fact, Poincaré's lemma itself tells us that if I take the curl of the grad, or the div of a curl, I get zero! The exterior derivative and the wedge product are the much-hoped-for generalization of vector calculus to manifolds, and are extremely powerful tools indeed.

<div align="center">(End of Lecture: Wednesday Oct 9 2024)</div>

2.7.4 Electromagnetism

As we already alluded to earlier, the language of differential forms is particularly suited to describing electromagnetism, especially given that Maxwell's equations are basically a bunch of relations involving divs and curls! The manifold that is relevant here is Minkowski space; we'll take the metric to be $(+1, -1, -1, -1)$, and we'll work everything out in natural units, where $c = \varepsilon_0 = \mu_0 = 1$.

The basic object of interest is the four-potential, $A^\mu = (\phi, \vec{A})$, where ϕ is the electric potential, and $\vec{A}$ is the vector potential. The one-form that we are interested in is therefore $A = A_\mu \, \mathrm{d}x^\mu$, where $A_\mu = (\phi, -\vec{A})$. As we saw earlier, I can define the two-form[27]

$$F = -\mathrm{d}A = -\frac{1}{2}F_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \,, \tag{2.132}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Using the standard relation that

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \,, \qquad \vec{B} = \nabla \times \vec{A} \,, \tag{2.133}$$

we can see that in fact the components of F are precisely the electromagnetic fields, e.g.

$$F_{01} = \partial_0 A_1 - \partial_1 A_0 = -\partial_t \vec{A}_x - \partial_x \phi = \vec{E}_x$$
$$F_{12} = \partial_1 A_2 - \partial_2 A_1 = -(\partial_x \vec{A}_y - \partial_y \vec{A}_x) = -\vec{B}_z \tag{2.134}$$

with

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \,. \tag{2.135}$$

Since F is exact, we arrive immediately at our first equation of motion:

$$\boxed{\mathrm{d}F = 0 \,, \tag{2.136}}$$

with almost no effort. You'll prove in the next problem set that this is equivalent to $\nabla \times \vec{E} = -\partial_t \vec{B}$ and $\nabla \cdot \vec{B} = 0$, which are the two source-free Maxwell equations.

The other two equations take a little more work to write down in form language. We first have to define the **Hodge star operator** on a manifold of dimension n, which maps p-forms to $(n-p)$-forms via the following transformation of the coordinate basis:

$$\star \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_{n-p}} = \frac{1}{p!}\varepsilon^{\mu_1\cdots\mu_{n-p}}{}_{\nu_1\cdots\nu_p}\mathrm{d}x^{\nu_1} \wedge \cdots \wedge \mathrm{d}x^{\nu_p} \,. \tag{2.137}$$

[27] Unfortunately, the choice of the mostly minus metric and sign conventions used in the literature have culminated in this horrendous minus sign, which I wouldn't personally introduce here, but will to be consistent with e.g. Ref. [5].

where $\varepsilon_{\mu_1\cdots\mu_{n-p}\nu_1\cdots\nu_p} = \sqrt{|g|}\epsilon_{\mu_1\cdots\mu_{n-p}\nu_1\cdots\nu_p}$ is the Levi-Civita tensor. This looks scary, but in an orthonormal Cartesian or Minkowski basis, this is quite straightforward. For 2-forms on Minkowski space, it is simply the map

$$\star(\mathrm{d}x \wedge \mathrm{d}y) = \mathrm{d}t \wedge \mathrm{d}z$$
$$\star(\mathrm{d}y \wedge \mathrm{d}z) = \mathrm{d}t \wedge \mathrm{d}x$$
$$\star(\mathrm{d}x \wedge \mathrm{d}z) = -\mathrm{d}t \wedge \mathrm{d}y$$
$$\star(\mathrm{d}t \wedge \mathrm{d}x) = -\mathrm{d}y \wedge \mathrm{d}z$$
$$\star(\mathrm{d}t \wedge \mathrm{d}y) = \mathrm{d}x \wedge \mathrm{d}z$$
$$\star(\mathrm{d}t \wedge \mathrm{d}z) = -\mathrm{d}x \wedge \mathrm{d}y$$

The rule of thumb is that for a given form ω, $\star\omega$ can be obtained by writing the components not involved in ω in an order such that $\omega \wedge (\star\omega) = \mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$, and adding an extra minus sign for every spatial basis component in ω.[28] The two Maxwell's equations with sources, $\nabla \cdot \vec{E} = \rho$ and $\nabla \times \vec{B} = \vec{j} + \partial_t \vec{E}$, can ultimately be written as

$$\mathrm{d}(\star F) = \star J\,, \qquad\qquad (2.138)$$

where $J = J_\mu \mathrm{d}x^\mu$ is the one-form corresponding to the four-current, $J_\mu = (\rho, -\vec{j})$, with ρ being the charge density, and $\vec{j}$ being the current density.

On your problem set, you will also see that charge conservation and gauge transformations fall out quite naturally from using the language of differential forms.

2.8 Integration on Manifolds

Having seen all the added work we needed to do to define operations that look like derivatives on a manifold, you might be worried that integration might be similarly cumbersome. It turns out, however, that integration on manifolds is a much more familiar beast: once we inroduce some formalities, you'll realize soon enough that you've always been comfortable integrating on manifolds!

2.8.1 Volume forms

Integrating a function over a manifold involves breaking up the manifold into little pieces, each with a **volume**[29] and a function value, and adding up all of the contributions from the pieces. To do this, we will have to start by understanding how to define volume locally. We begin by trying to understand how to define the **volume** of k vectors in a tangent space. For two vectors $\vec{v}_1, \vec{v}_2$ in 3D space, for example, we know that we can define a volume by taking the area of the parallelogram spanned by these vectors, i.e. $|\vec{v}_1 \times \vec{v}_2|$. Similarly, in 3D space, we can take three vectors and form them into a parallelopiped, More generally, the volume ω_k is therefore something that takes k vectors and returns a real number, and so ω_k is a $(0, k)$-object that lives in $\bigotimes^k T_p^*$. In addition, we would like the following properties for the volume form:

1. *Scaling*: If we double the length of one of the vectors, we expect the volume to double, i.e. given k vectors $\vec{v}_1, \cdots, \vec{v}_k \in T_p$, and $\lambda \in \mathbb{R}$,

$$\omega_k(\lambda\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k) = \omega_k(\vec{v}_1, \lambda\vec{v}_2, \cdots, \vec{v}_k) = \cdots = \omega_k(\vec{v}_1, \vec{v}_2, \cdots, \lambda\vec{v}_k)$$
$$= \lambda\omega_k(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k)\,. \qquad (2.139)$$

Note that this allows for negative values for the volume.

[28] This is true in the mostly minus metric convention we are using. In the mostly plus convention, you add a minus sign for every $\mathrm{d}t$ in ω. For $\mathbb{R}^n$, there is no need to worry about extra minus signs.

[29] When speaking generally, area will also be regarded as a volume

2. *Additivity*: For an additional vector $\vec{u} \in T_p$, we also want

$$\omega_k(\vec{v}_1 + \vec{u}, \vec{v}_2, \cdots, \vec{v}_k) = \omega_k(\vec{u}, \vec{v}_2, \cdots, \vec{v}_k) + \omega_k(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k),$$
(2.140)

and similarly for all the other slots.

3. *Degeneracy*: If two sides coincide, the volume should be zero, i.e. if any two vectors are equal, the form must be zero.

These properties combine to ensure that in fact, ω_k *has to be a k-form*. To see this, observe that

$$\omega_k(\cdots, \vec{u} + \vec{v}, \cdots, \vec{u} + \vec{v}, \cdots) = 0$$
$$\implies \omega_k(\cdots, \vec{u}, \cdots, \vec{v}, \cdots) + \omega_k(\cdots, \vec{v}, \cdots, \vec{u}, \cdots) = 0$$
$$\implies \omega_k(\cdots, \vec{u}, \cdots, \vec{v}, \cdots) = -\omega_k(\cdots, \vec{v}, \cdots, \vec{u}, \cdots),$$
(2.141)

i.e. ω_k is *completely antisymmetric with respect any pair of its arguments*, which is exactly what a k-form is.

2.8.2 Defining an integral

So we've got a notion of volume in the tangent space of every point on the manifold, and we now want to be able to add these volumes up, just like how an integral should work. The natural setting we have in mind is to have a p-form defined in $\mathbb{R}^n$, e.g.

$$\omega = \frac{1}{p!}\omega_{\mu_1 \cdots \mu_p} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p},$$
(2.142)

where x^μ is a choice of coordinate system for $\mathbb{R}^n$ (e.g. Cartesian coordinate system) and to integrate that form over some p-dimensional manifold Ω embedded in $\mathbb{R}^n$, to obtain a real number. Suppose Ω is parametrized by ξ^i, where

$$x^1 = x^1(\xi^1, \cdots, \xi^p), \cdots x^d = x^d(\xi^1, \cdots, \xi^p).$$
(2.143)

so that

$$\mathrm{d}x^\mu = \frac{\partial x^\mu}{\partial \xi^i}\mathrm{d}\xi^i$$
(2.144)

Then we can write, in this new parametrization,

$$\omega = \omega_{\mu_1\mu_2\cdots\mu_p}\frac{\partial x^{\mu_1}}{\partial \xi^1}\cdots\frac{\partial x^{\mu_p}}{\partial \xi^p}\mathrm{d}\xi^1 \wedge \cdots \wedge \mathrm{d}\xi^p,$$
(2.145)

which is a p-form on Ω.

We now define the integral of ω over the manifold Ω as

$$I = \int_\Omega \omega \equiv \int_\Omega d\xi^1 \cdots d\xi^p\, \omega_{\mu_1\mu_2\cdots\mu_p}\frac{\partial x^{\mu_1}}{\partial \xi^1}\cdots\frac{\partial x^{\mu_p}}{\partial \xi^p},$$
(2.146)

where $I \in \mathbb{R}$, and the object on the right is just the regular multidimensional integral. Note that if $\xi^i = x^i$, i.e. the parametrization of Ω is the same or some subset of $\mathbb{R}^n$, then this is just a straightforward integral over the component of the form; otherwise, because $\omega_{\mu_1\mu_2\cdots\mu_p}$ is antisymmetric in all indices, the contraction against x^μ is really the Jacobian that we're used to in regular

integration. The punchline really is that *you can simply treat forms as regular differentials*, remembering that a change of coordinates requires a Jacobian.

A few examples will help. Let's consider a 1-dimensional path Γ starting at some point P_0 and ending at point P_1, and a function f defined across the path. We can integrate the 1-form df, if we parametrize the path as $x(s)$, for $0 \leq s \leq 1$, so that $x(0) = P_0$ and $x(1) = P_1$, we have

$$I = \int_\Gamma df = \int_0^1 ds \frac{df}{ds} = f(P_1) - f(P_0) \,. \qquad (2.147)$$

Next, let's integrate the 2-form $\omega = x\,dy \wedge dz$ over the surface of a two-dimensional sphere S^2 of radius R. We can parametrize the sphere using polar angles, which are related to the usual $\mathbb{R}^3$ Cartesian coordinates via

$$x = R\cos\phi\sin\theta$$
$$y = R\sin\phi\sin\theta$$
$$z = R\cos\theta \,. \qquad (2.148)$$

To perform the integral, we need the Jacobian $J \equiv |\partial x^i / \partial \xi^j|$. We have

$$\frac{\partial y}{\partial \theta} = R\sin\phi\cos\theta \,, \quad \frac{\partial y}{\partial \phi} = R\cos\phi\sin\theta$$
$$\frac{\partial z}{\partial \theta} = -R\sin\theta \,, \quad \frac{\partial z}{\partial \phi} = 0 \,. \qquad (2.149)$$

Using the standard orientation with $\xi^i = \theta$ and $\xi^2 = \phi$ so that $\hat{\theta} \times \hat{\phi}$ points outward, $J = R^2\cos\phi\sin^2\theta$. Hence,

$$\int_{S^2} x\,dy \wedge dz = \int_0^{2\pi} d\phi \int_0^\pi d\theta (R\cos\phi\sin\theta)(R^2\cos\phi\sin^2\theta)$$
$$= \frac{4}{3}\pi R^3 \,. \qquad (2.150)$$

We could also have noted that We also have the relation

$$dy = R\sin\phi\cos\theta\,d\theta + R\cos\phi\sin\theta\,d\phi$$
$$dz = -R\sin\theta\,d\theta \,, \qquad (2.151)$$

so that the form on the sphere is

$$x\,dy \wedge dz = (R\cos\phi\sin\theta)(R^2\cos\phi\sin^2\theta)\,d\theta \wedge d\phi \,, \qquad (2.152)$$

and then proceeding directly with the integral. This picture is perhaps much more intuitive though: you can think of the sphere as being tiled by little patches formed by intersecting longitudinal and latitudinal lines, each bordered by a vector $\vec{v}_\theta$ in the $\hat{\theta}$ direction and $\vec{v}_\phi$ in the $\hat{\phi}$ direction. The integral is then just summing up $\omega(\vec{v}_\theta, \vec{v}_\phi)$ in all of the little squares (taking the limit as the squares become infinitesimally small).

(**End of Lecture**: Tuesday Oct 15 2024)

2.8.3 Orientability

You might have noticed that we had to make a choice about the **orientation** of the 2-sphere, i.e. pick an ordering to the basis elements $d\theta \wedge d\phi$ (as opposed to $d\phi \wedge d\theta$), so that the direction of $\hat{\theta} \times \hat{\phi}$ always points outward. This is possible on the sphere, making the sphere an **orientable** manifold. I won't go into a detailed discussion of the technical definition of orientable, but the picture you should

have in mind is that the basis vectors are changing as you move from point to point, but the transformation mapping basis vectors from one point to another must always have a positive determinant for an orientable manifold. One simple example of this failing is the Möbius strip, where you can see that one of the two basis vectors transported in a loop ends up in the opposite orientation when it comes back to itself. Integration as defined here really only makes sense on orientable manifolds.

2.8.4 The volume form

Note that so far, in discussing the integration of forms, we have made no mention whatsoever of the metric, and hence we do not need any notion of length to make sense of these integrals. However, when a manifold of dimension $p < d$ is embedded in $\mathbb{R}^d$, it *inherits* a metric from the usual metric of $\mathbb{R}^d$.

For a manifold Ω, given any two vectors $\vec{v}, \vec{w} \in \Omega$; if the manifold is embedded in $\mathbb{R}^d$, then we can also view these vectors as belonging to $\mathbb{R}^d$. The induced metric $g_{\mu\nu}$ on Ω, on which we make a choice of coordinates ξ^μ on the manifold, is simply the metric that, when acting on $\vec{v}$ and $\vec{w}$, gives the same result as the usual metric $\delta_{\alpha\beta}$ on $\mathbb{R}^d$ with coordinates x^α, i.e.

$$g_{\mu\nu} v'^\mu w'^\nu = \delta_{\alpha\beta} v^\alpha w^\beta \,, \tag{2.153}$$

where v^α and w^β are the components of v and w in the usual Cartesian coordinate basis respectively, while v'^μ and v'^ν are the components in coordinates parametrizing the manifold, ξ^μ. These are of course related to v^μ and w^ν through the usual change-of-coordinates formula, i.e.

$$g_{\mu\nu} v'^\mu w'^\nu = \delta_{\alpha\beta} \frac{\partial x^\alpha}{\partial \xi^\mu} \frac{\partial x^\beta}{\partial \xi^\nu} v'^\mu w'^\nu \,, \tag{2.154}$$

i.e. the induced metric on the manifold is

$$g_{\mu\nu} = \delta_{\alpha\beta} \frac{\partial x^\alpha}{\partial \xi^\mu} \frac{\partial x^\beta}{\partial \xi^\nu} = \sum_{i=1}^{d} \frac{\partial x^i}{\partial \xi^\mu} \frac{\partial x^i}{\partial \xi^\nu} \,. \tag{2.155}$$

For a p-dimensional manifold Ω, we can define the **volume form** associated with this induced metric,

$$\mathrm{d}\omega = \sqrt{g} \, \mathrm{d}\xi^1 \wedge \cdots \wedge \mathrm{d}\xi^p \,, \tag{2.156}$$

where you can see that $\sqrt{g}$ is the Jacobian between ξ^μ and x^α, the Cartesian coordinates on $\mathbb{R}^d$. Thus, integrating this form over the Ω gives the total p-dimensional volume of Ω.

Let's look at a simple example, the unit-radius two-sphere embedded in $\mathbb{R}^3$. Choosing polar angle coordinates on the sphere, we have

$$x = \cos\phi \sin\theta \,,$$
$$y = \sin\phi \sin\theta \,,$$
$$z = \cos\theta \,, \tag{2.157}$$

from which we see that the induced metric has components

$$g_{\phi\phi} = (-\sin\phi \sin\theta)^2 + (\cos\phi \sin\theta)^2 = \sin^2\theta \,,$$
$$g_{\phi\theta} = (-\sin\phi \sin\theta)(\cos\phi \cos\theta) + (\cos\phi \sin\theta)(\sin\phi \cos\theta) = 0 \,,$$
$$g_{\theta\theta} = (\cos\phi \cos\theta)^2 + (\sin\phi \cos\theta)^2 + (-\sin\theta)^2 = 1 \,, \tag{2.158}$$

i.e. the induced metric is

$$g = \mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2\theta\, \mathrm{d}\phi \otimes \mathrm{d}\phi\,. \tag{2.159}$$

The volume form is then

$$\sqrt{g}\, \mathrm{d}\theta \wedge \mathrm{d}\phi = \sin\theta\, \mathrm{d}\theta \wedge \mathrm{d}\phi\,, \tag{2.160}$$

which corresponds to the usual surface element on the two-sphere.

2.8.5 Stokes' Theorem

We finish our discussion of integration—and the calculus of manifolds!—with Stokes' theorem. You should be familiar with this result from 3D vector calculus: given a surface S in $\mathbb{R}^3$, with boundary Γ, for a smooth vector field $\vec{F}$ throughout $\mathbb{R}^3$ we have the relation

$$\int_S d\vec{S} \cdot (\nabla \times \vec{F}) = \oint_\Gamma d\vec{\Gamma} \cdot \vec{F}\,. \tag{2.161}$$

This allows us to move between differential and integral forms of Maxwell's equations, for example

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \implies \int_S d\vec{A} \cdot (\nabla \times \vec{E}) = -\frac{\partial}{\partial t} \int_S d\vec{A} \cdot \vec{B} = \oint_\Gamma d\vec{\ell} \cdot \vec{E}\,. \tag{2.162}$$

The generalized version of Stokes' theorem relates the integral of a exterior derivative $\mathrm{d}\omega$ of a form ω across a manifold Ω with an integral along the boundary $\partial\Omega$ of Ω:

$$\int_\Omega \mathrm{d}\omega = \int_{\partial\Omega} \omega\,. \tag{2.163}$$

This is an incredibly beautiful and useful result. We won't go through the formal proof, but the intuitive explanation for this is pretty straightforward. Notice that for a p-form,

$$\mathrm{d}(\omega_{\mu_1 \cdots \mu_p}\, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}) = \partial_\nu \omega_{\mu_1 \cdots \mu_p}\, \mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}\,. \tag{2.164}$$

Divide ω up into tiny little boxes, and perform the integral in each of these boxes. This gives you a bunch of terms that look like the sum of ω evaluated on one side of a p-dimensional box. However, the adjacent box shares one surface with the current box, with the opposite orientation, and therefore when we sum the effect of these two boxes together, the term corresponding to the shared surface cancels out. Therefore, when we sum over all of the small boxes, we are left with ω summed along the boundary of Ω.

Let's see a few simple examples:

1. If Ω is a region in $\mathbb{R}^2$, then since

$$\mathrm{d}\left[\frac{1}{2}(x\, \mathrm{d}y - y\, \mathrm{d}x)\right] = \mathrm{d}x \wedge \mathrm{d}y\,, \tag{2.165}$$

we can write

$$\mathrm{Area}(\Omega) = \int_\Omega \mathrm{d}x \wedge \mathrm{d}y = \frac{1}{2}\int_{\partial\Omega}(x\, \mathrm{d}y - y\, \mathrm{d}x)\,, \tag{2.166}$$

which is just Green's theorem.

2. For Ω a region in $\mathbb{R}^2$, we have

$$\mathrm{d}\left[\frac{1}{2}r^2\,\mathrm{d}\theta\right] = r\,\mathrm{d}r \wedge \mathrm{d}\theta\,, \qquad\qquad (2.167)$$

and so

$$\mathrm{Area}(\Omega) = \int_\Omega r\,\mathrm{d}r \wedge \mathrm{d}\theta = \frac{1}{2}\int_{\partial\Omega} r^2\,\mathrm{d}\theta\,, \qquad\qquad (2.168)$$

which are two equivalent ways of evaluating the area in polar coordinates.

3. If Ω is the interior of a sphere of radius R, then

$$\int_\Omega \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = \int_{\partial\Omega} x\,\mathrm{d}y \wedge \mathrm{d}z = \frac{4}{3}\pi R^3\,, \qquad\qquad (2.169)$$

a result which we obtained in Eq. (2.150).

3 Complex Analysis

References: Churchill and Brown (CB) Chapters 1–7

From doing calculus on general spaces but with a focus on real numbers, we'll now turn our attention to doing calculus with functions of complex numbers. From our experience with real analysis, one may have expected the calculus of complex numbers to be as complicated, varied and sometimes pathological as the calculus of real numbers; but we'll see that the structure of the complex plane makes life very simple and pretty, making complex analysis a very powerful tool, even when you're really only interested in the real line.

3.1 Preliminaries

We'll begin with a crash course on complex numbers and how to manipulate them, which should be familiar to you from your undergraduate experience. Complex numbers can be obtained by supplementing the real numbers with an additional element i, satisfying $i^2 = -1$. It turns out that with this new element, we can still sensibly define addition, subtraction, multiplication and division; they all work in a manner very similar to what happens in $\mathbb{R}$ and also for the rational numbers $\mathbb{Q}$. The mathematical jargon here is that the complex numbers form a **field**, with $\mathbb{R}$ as a subfield. With this new element, the bigger field $\mathbb{C}$ actually behaves even more nicely than $\mathbb{R}$; for example, every polynomial of degree n with complex coefficients always has n complex roots, while it certainly isn't the case polynomials with real coefficients of degree n have n real roots. A straightforward example is $x^2 + 1$, which has no real roots, but two complex roots i and $-i$.

A general complex number z can be written as $z = x + iy$, where $x, y \in \mathbb{R}$. We call x and y the real and imaginary part of z respectively, often written as $x = \text{Re}(z)$ and $y = \text{Im}(z)$.

In addition to the usual field operations, we have two other important operations acting on a general complex number $z = x + iy \in \mathbb{C}$:

1. **Complex conjugate**, written either as $z*$ or $\overline{z}$, defined as $z^* = x - iy$, and

2. **Modulus** or absolute value, written as $|z|$, and defined as $|z| = \sqrt{x^2 + y^2}$.

One useful result is that $|z|^2 = zz^*$, which is often used to divide complex numbers; for example,

$$\frac{3 + 4i}{3 - 4i} = \frac{(3 - 4i)^2}{(3 - 4i)(3 + 4i)} = \frac{9 - 24i - 16}{25} = \frac{-7 - 24i}{25}. \tag{3.1}$$

Aside from writing $z = x + iy$, it is also common to write complex numbers in **polar form**,

$$z = r(\cos\theta + i\sin\theta), \tag{3.2}$$

where $x = r\cos\theta$ and $y = r\sin\theta$, and $r, \theta \in \mathbb{R}$. In fact, it is often helpful to visualize the complex plane, with the real component on the x-axis and the imaginary component on the y-axis, so that complex numbers $z = x + iy$ can be visualized as points (x, y), or in polar coordinates in terms of (r, θ).

You can check that $r = |z|$. We call the value of θ the **argument** of the complex number, written as $\theta = \arg(z)$. You can see, however, that given θ as the argument of z, $\theta + 2n\pi$ for any integer n gives you the same complex number as well. We therefore define the **principal value of the argument**

Arg(z), which is the unique value of θ such that $-\pi < \theta \leq \pi$. Note that Arg(-1) = π.

Euler's formula, which we will study in more detail later, is given by

$$e^{i\theta} = \cos\theta + i\sin\theta\,, \tag{3.3}$$

and so we can also write $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$, which we call the **exponential form**. Switching between forms can help make algebra easier. In particular, for $z = x + iy$,

$$z^* = x - iy = r(\cos\theta - i\sin\theta) = re^{-i\theta}\,. \tag{3.4}$$

Euler's formula also directly leads to **de Moivre's formula**, which reads

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta\,, \tag{3.5}$$

which you can check by writing $(e^{i\theta})^n = e^{in\theta}$.

One particularly important operation is to find the nth roots of any complex number $z_0 = r_0 e^{i\theta_0}$, which by definition are complex numbers $z = re^{i\theta}$ such that $z^n = z_0$. This means that we require $r^n = r_0$, and $n\theta = \theta_0 + 2k\pi$ for some $k \in \mathbb{Z}$, since adding an additional 2π to the argument doesn't change the complex number at all. We can see here that there are n unique values of θ, namely

$$\theta = \frac{\theta_0 + 2k\pi}{n}\,, \quad k = 0, \cdots, n-1\,. \tag{3.6}$$

Therefore the nth roots of z_0 are

$$z = r_0^{1/n}\exp\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right]\,, \quad k = 0, \cdots, n-1\,. \tag{3.7}$$

(**End of Lecture**: Wednesday Oct 16 2024)

3.2 Complex Functions and Derivatives

We next move on to studying functions of one complex variable $f(z)$, which in general maps a complex number to another complex number on some subset of $\mathbb{C}$. In a rigorous study of complex analysis, we would make sure to define things like continuity and taking limits rigorously, but as this is a physics class, we will just rely on the intuitive definitions of continuity and limit; you can read more about robust definitions in CB15–19. At this point, some functions that you can easily define are things like polynomials, e.g. $f(z) = z^2$, as well as familiar functions like exponential and trigonometric functions, which all can be generalized onto the complex plane (more on that later).

3.2.1 Complex derivative

We will however define the **complex derivative** carefully. Let f be a function defined over some small neighborhood around a point z_0. Then the derivative of f and z_0 is the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}, \tag{3.8}$$

and the function f is said to be **differentiable** at z_0 when $f'(z_0)$ exists. This definition should come as no surprise if you're familiar with the derivative over $\mathbb{R}$. However, some thought should now convince you that this definition is quite a bit *stronger* than that in $\mathbb{R}$; the limit that you obtain has to be the same *regardless of which direction you approach z_0 from*, as otherwise the limit cannot be defined.

Take for example the function $f(z) = |z|^2$, which intuitively looks like a nice function over $\mathbb{R}^2$; on the complex plane, we find that $f(z)$ is in fact not differentiable anywhere except at $z = 0$! You can see this by picking a point x_0 on the real axis, and trying to find the limit when 1) you approach along the real axis, where the result of the limit is $2x_0$, and 2) you approach along the purely imaginary direction, where the result of the limit is 0. *Complex differentiability is a much more stringent requirement.*

Assuming that the derivative does exist, all of our intuition from calculus on $\mathbb{R}$ carries over. So, for example, we have:

1. The usual differentiation formula for a polynomial,

$$\frac{d}{dz} z^n = n z^{n-1}. \tag{3.9}$$

2. Complex differentiation is additive, i.e.

$$\frac{d}{dz}[f(z) + g(z)] = f'(z) + g'(z). \tag{3.10}$$

3. We have the product rule, i.e.

$$\frac{d}{dz}[f(z)g(z)] = f(z)g'(z) + f'(z)g(z). \tag{3.11}$$

4. Finally, we have the chain rule: suppose that f has a derivative at z_0, and g has a derivative at the point $f(z_0)$. Then the function $F(z) = g[f(z)]$ has a derivative at z_0, and

$$F'(z_0) = g'[f(z_0)]f'(z_0). \tag{3.12}$$

Thus, everything we know and love about real analysis carries over neatly here!

3.2.2 Holomorphic functions and the Cauchy-Riemann equations

At this point, you might still be really uncomfortable with taking complex derivatives, because you've got to be real careful about whether a given function is differentiable or not. In this part, we're going to develop some nice necessary and sufficient conditions for a complex function to be differentiable. Let me just remind you of what "necessary" and "sufficient" mean:

- The statement "A is necessary for B" means that A has to be true in order for B to be true, or in other words, if B is true, A must also be true. This is sometimes written as $B \implies A$. So, for example, "passing your ACE exams is necessary for obtaining your PhD"; if you successfully obtained your PhD, you must have passed your ACE exams, because it was a necessary condition.

- The statement "A is sufficient for B" means that if A is true, B is true as well. This is sometimes written as $A \implies B$. So, for example, "getting a 70 on this course is sufficient to pass", and so if you do get a 70, that's sufficient for you to pass; but you might also have passed with a 60.

Let's examine more closely how derivatives work on the complex plane, and derive a *necessary* condition for a derivative to exist. Any complex function $f(z)$ can be written as a pair of component functions u and v, such that for any $z = x + iy$,

$$f(z) \equiv u(x,y) + iv(x,y), \tag{3.13}$$

where u, v take in a pair of real numbers, and return a real number. Suppose the derivative $f'(z_0)$ exists at the point $z_0 = x_0 + iy_0$. Then

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}. \tag{3.14}$$

This should apply as I approach z_0 from any direction on the complex plane, so let's start by approaching z_0 along a direction parallel to the real axis. We can write $(x, y) = (x_0 + \Delta x, y_0)$, so that going along the real direction, we find

$$f'(z_0) = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0) + iv(x_0 + \Delta x, y_0) - iv(x_0, y_0)}{x_0 + \Delta x - x_0}$$

$$= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + \lim_{\Delta x \to 0} i\frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}$$

$$= \partial_x u + i\partial_x v, \tag{3.15}$$

where I have just used the usual definition of the partial derivative of a function of two real numbers. On the other hand, I can also approach z_0 parallel to the imaginary axis. In this case, we should write $(x, y) = (x_0, y_0 + \Delta y)$, so that going along the imaginary direction, we find

$$f'(z_0) = \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0) + v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i(y_0 + \Delta y) - iy_0}$$

$$= \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + \lim_{\Delta y \to 0} i\frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i\Delta y}$$

$$= -i\partial_y u + \partial_y v \tag{3.16}$$

Comparing the two expressions we just obtained, we find

$$\partial_x u = \partial_y v, \qquad \partial_y u = -\partial_x v, \tag{3.17}$$

where all functions are to be evaluated at (x_0, y_0). These are known as the **Cauchy-Riemann equations**. Once again, these are *necessary* conditions for the existence of a derivative: if they are not satisfied, then the derivative does not exist, but if they are satisfied, it does *not* imply that the derivative must therefore exist. Furthermore, one can see that

$$f'(z_0) = \partial_x u + i\partial_x v, \tag{3.18}$$

when u and v are evaluated at (x_0, y_0).

What about *sufficient* conditions? In addition to satisfying the Cauchy-Riemann equations, for $f'(z_0)$ to exist, *1)* the first order partial derivatives must exist everywhere in some neighborhood of z_0, and *2)* these partial derivatives

are continuous at (x_0, y_0). I won't prove this statement, since it is not very essential in physics, but you can read about it in CB22.

Let's try a simple example. Consider the function $f(z) = z^*$. We can write $f(z) = u(x, y) + iv(x, y)$, where $u(x, y) = x$, and $v(x, y) = -y$. From this, we see that $\partial_x u = 1$, but $\partial_y v = -1$, and therefore $\partial_x u \neq \partial_y v$. We can thus conclude that $f(z) = z^*$ is not differentiable anywhere.

Now for some terms that can thrown around a lot: a function f is called **holomorphic** in some region of $\mathbb{C}$ if it has a complex derivative everywhere in the region. Sometimes people use the word **analytic** instead; technically, analytic means something else that we will discuss later on the course, but fortunately *all holomorphic functions are analytic and vice-versa in complex analysis, so to all intents and purposes, they are synonyms for complex functions*, and we won't worry about this distinction any further. A function that is holomorphic on $\mathbb{C}$ is an **entire** function; for example, any polynomial is an entire function.

Frequently, a function f is not holomorphic at a point z_0, but is holomorphic at points arbitrarily close to z_0; for example, $f(z) = 1/z$ is holomorphic everywhere except at $z = 0$. Such points z_0 are called **singularities**. Lots more to come about singularities ahead.

3.2.3 Harmonic functions

A **harmonic function** $H(x, y)$ is a real-valued function of two real variables x and y which satisfies **Laplace's equation**,

$$(\partial_x^2 + \partial_y^2)H = 0\,. \tag{3.19}$$

Laplace's equation comes up frequently in physics; for example, the electrostatic and gravitational potentials in vacuum must satisfy Laplace's equation. Complex analysis turns out to be very useful for solving Laplace's equation, due to the following relation:

> If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in some domain D, then u and v are harmonic in D.

Let's see why this is true. Since f is analytic in D, the Cauchy-Riemann equations must be satisfied, i.e.

$$\partial_x u = \partial_y v\,, \qquad \partial_y u = -\partial_x v\,. \tag{3.20}$$

Let's take the x-derivative of the first equation, and the y-derivative of the second equation. This gives

$$\partial_x^2 u = \partial_x \partial_y v\,, \qquad \partial_y^2 u = -\partial_y \partial_x v\,. \tag{3.21}$$

However, we can commute the partial derivatives,[30] from which we find

$$(\partial_x^2 + \partial_y^2)u = 0\,, \tag{3.22}$$

i.e. u is harmonic. By taking the y-derivative of the first Cauchy-Riemann equation, and taking the x-derivative of the second one, we can also see that $(\partial_x^2 + \partial_y^2)v = 0$, thus proving the relation above.

Suppose we have two functions u and v that are both solutions to Laplace's equation. We say that v is the **harmonic conjugate** of u if they also satisfy the Cauchy-Riemann equations, i.e. $\partial_x u = \partial_y v$ and $\partial_y u = -\partial_x v$. Confusingly, v being the harmonic conjugate of u doesn't mean that u is the harmonic conjugate of v, so be careful! This gives us powerful way of checking if a function is analytic:

[30] This is not just us being sloppy as physicists: since f is analytic, we must have the partial derivatives of u and v being continuous everywhere in D, which implies mathematically that the partial derivatives do indeed commute.

> A function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D if and only if v is a harmonic conjugate of u, i.e. v being the harmonic conjugate of u is a necessary *and* sufficient condition for $f(z)$ to be analytic.

Let's see why this is so. In one direction, this is easy: f being analytic implies that u and v satisfy the Cauchy-Riemann equations, and are also harmonic. On the other hand, u and v being harmonic conjugates implies that they satisfy the Cauchy-Riemann equations, and also that u and v have well-defined partial derivatives in D that are continuous everywhere. This therefore satisfies the sufficient condition for f to be analytic that we discussed in Sec. 3.2.2.

What is the big-picture takeaway here? For a complex function $f(z) = u(x, y) + iv(x, y)$ to be analytic, the component functions u and v face very strong constraints: they satisfy the Cauchy-Riemann equations, and are harmonic. The result of this is that analytic complex functions are extremely well-behaved, and that is going to lead to all kinds of very pretty results as we go forward.

3.2.4 Elementary functions

Let's take a look at some functions that you're already familiar with from real space, and see how they work on the complex plane. Most of the properties of these functions that you're used to in real space carries over to the complex plane; I'll only highlight things that work differently between $\mathbb{R}$ and $\mathbb{C}$.

The **exponential function** e^z for $z = x + iy$ is simply given by

$$e^z = e^x e^{iy} = e^x (\cos y + i \sin y). \tag{3.23}$$

One important property of e^z that we should note is that $e^z \neq 0$ for any $z \in \mathbb{C}$, since $|e^z| = e^x > 0$ for $z = x + iy$. We say that e^z has no **zeros**, where a zero z_0 of a function $f(z)$ is defined as $f(z_0) = 0$. In addition, unlike the exponential function on the real line, e^z is actually *periodic*, since

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z. \tag{3.24}$$

Furthermore, even though $e^z \neq 0$, we still can have $e^z < 0$, e.g.

$$e^{i\pi} = \cos \pi + i \sin \pi = -1. \tag{3.25}$$

Finally, e^z is an *entire function*, with

$$\frac{d}{dz} e^z = e^z \tag{3.26}$$

everywhere on the complex plane. Otherwise, all of our intuition on real space about the exponential carries over to $\mathbb{C}$.

Next, we move on to the **trigonometric functions**. From Euler's formula, we have $e^{i\theta} = \cos \theta + i \sin \theta$, from which we can see that for a real number θ, $\cos \theta = (e^{i\theta} + e^{-i\theta})/2$, and $\sin \theta = (e^{i\theta} - e^{-i\theta})/(2i)$. We can therefore generalize this and define the following complex trigonometric functions:

$$\sin z \equiv \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z \equiv \frac{e^{iz} + e^{-iz}}{2}. \tag{3.27}$$

Once again, all of the usual properties in trigonometry and calculus generalize to the complex plane, which you can check from the definition. One thing to note from the definition of $\sin z$ and $\cos z$ is that for $y \in \mathbb{R}$,

$$\sin(iy) = i \sinh y, \qquad \cos(iy) = \cosh y. \tag{3.28}$$

From these definitions, we can go on to define

$$\tan z \equiv \frac{\sin z}{\cos z}, \qquad \cot z \equiv \frac{\cos z}{\sin z},$$
$$\sec z \equiv \frac{1}{\cos z}, \qquad \csc z \equiv \frac{1}{\sin z}. \qquad (3.29)$$

Similarly, we have the **hyperbolic functions**, defined as

$$\sinh z = \frac{e^z - e^{-z}}{2}, \qquad \cosh z = \frac{e^z + e^{-z}}{2}. \qquad (3.30)$$

Nothing really surprising happens in the generalization here.

We next turn our attention to the **logarithmic function**, which requires more careful treatment. Let's write a general complex number z in polar form, i.e. $z = re^{i\theta}$. Then we define

$$\log z \equiv \ln r + i(\theta + 2n\pi), n \in \mathbb{Z}, \qquad (3.31)$$

where we use $\ln$ to denote the real-valued natural logarithm, to distinguish it from the complex logarithm $\log$. Note that $\log$ defined in this way is *multivalued*. Importantly, since for any $z = x + iy$, $e^z = e^x e^{iy}$, we have

$$\log e^z = x + i(y + 2n\pi) = z + 2in\pi, n \in \mathbb{Z}. \qquad (3.32)$$

With this definition, we can also define a **complex exponent**, with the function z^c for $c \in \mathbb{C}$ being defined as

$$z^c \equiv e^{c \log z}. \qquad (3.33)$$

Note that this means that the complex exponent is a multi-valued function as well in general, so for example

$$i^{-2i} = e^{-2i \log i} = e^{-2i \cdot i(\pi/2 + 2n\pi)} = e^{(4n+1)\pi}, n \in \mathbb{Z}. \qquad (3.34)$$

Therefore, i^{-2i} is multivalued, and all of these values are actually real numbers!

3.2.5   Branches

Let's consider the multi-valued logarithm function again, which for $z = re^{i\theta}$ is $\log z = \ln r + i\theta$, where $\theta = \Theta + 2n\pi$, with Θ being the principal value of the argument. To make this single-valued, we can choose any real number α and restrict the value of θ so that $\alpha < \theta \leq \alpha + 2\pi$. In this restricted domain, $\log(z)$ becomes *single-valued*, and in fact is analytic for $r > 0$, $\alpha < \theta < \alpha + 2\pi$.[31] Fig. 12 shows an illustration of what we've done, which is to define what's known as a **branch** of the $\log$ function, with the $\theta = \alpha$ line on the complex plane being called a **branch cut**. The origin in this case is known as the **branch point**. The function *can* be defined on the branch cut; however, the domain over which the function is continuous and even analytic must include it.

[31] Note that we cannot include $\theta = \alpha$ here, as the function has a discontinuity at that point; a little bit off the $\theta = \alpha$ line, we have complex numbers with arguments close to α, while on the other side, we have numbers with arguments close to $\alpha + 2\pi$.
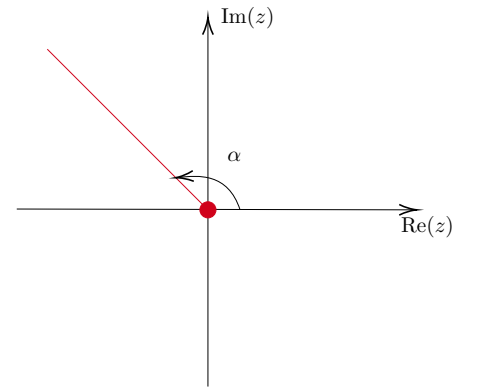

Figure 12: A branch cut on the complex plane for $\log$, restricting the values of θ to $\alpha < \theta < \alpha + 2\pi$.

The special case where we choose $\alpha = -\pi$, so that the branch cut is along the negative real axis, and $-\pi < \theta \leq \pi$ corresponds to the *principal value of the argument*, defines **the principal value of the logarithm**,

$$\mathrm{Log}(z) = \ln r + i\Theta \,, \qquad (3.35)$$

written with a capital 'L'; it is single-valued, and well-defined whenever $z \neq 0$. This is also known as the **principal branch**. Thus for example

$$\begin{aligned} \log(-1) &= i(\pi + 2n\pi) = (2n+1)\pi i \,, n \in \mathbb{Z}\,, \\ \mathrm{Log}(-1) &= i\pi\,. \end{aligned} \qquad (3.36)$$

Note that while the usual rules of how logarithms work carry over for $\log$, many of these rules don't work for Log. For example, if $z_1 = z_2 = -1$, then

$$\mathrm{Log}(z_1 z_2) = 0 \,, \qquad (3.37)$$

but

$$\mathrm{Log}(z_1) + \mathrm{Log}(z_2) = i\pi + i\pi = 2\pi i \,, \qquad (3.38)$$

i.e. $\mathrm{Log}(z_1 z_2) \neq \mathrm{Log}(z_1) + \mathrm{Log}(z_2)$.

(**End of Lecture**: Monday Oct 21 2024)

3.3 Integration Part I

For the rest of the course, the main focus will be on integration on the complex plane, which is a fascinating and beautiful area of mathematics, and also extremely powerful, mostly due to the restricted structure of differentiable functions on the complex plane.

Let's begin with integration along the real line.

3.3.1 Integration along the real line

Suppose we have some function $w(t)$ which is a complex valued function, but takes as an argument a real variable t; you can imagine that this is just a general complex function restricted to the real axis. Then we can write $w(t) = u(t) + iv(t)$ for two real-valued functions u and v. The definite integral of $w(t)$ can then be straightforwardly defined as

$$\int_a^b dt\, w(t) = \int_a^b dt\, u(t) + i\int_a^b dt\, v(t) \,, \qquad (3.39)$$

with the integral behaving in the usual ways that an integral would behave, e.g.

$$\int_a^b dt\, w(t) = \int_a^c dt\, w(t) + \int_c^b dt\, w(t) \,. \qquad (3.40)$$

You also get the fundamental theorem of calculus here, which says that

$$\int_a^b dt\, w(t) = W(b) - W(a)\,, \qquad W'(t) = w(t)\,. \qquad (3.41)$$

So, for example, we have

$$\int_0^{\pi/2} dt\, e^{it} = -i\, e^{it}\big|_0^{\pi/2} = 1 - (-i) = 1 + i\,. \qquad (3.42)$$

3.3.2 Contour integrals

We're looking at functions living on the complex plane though, so we should really be able to define integrals on the complex plane. We can generalize integrating along the real axis to an integral over any **contour** C in the complex plane, written as the **contour integral**

$$\int_C dz\, f(z)\,. \tag{3.43}$$

How would we go about defining this integral? Well, this is really the same as how you would do a line integral on a flat surface, which you already have some experience with. We try to find a parametrization of the complex numbers $z(t)$ along C, so that as t varies in some range $a \leq t \leq b$, $z(t)$ takes us along the contour. Then, following the usual change of variables formula,

$$\int_C dz\, f(z) = \int_a^b dt\, \frac{dz}{dt} f[z(t)]\,, \tag{3.44}$$

converting the contour integral to just an integral along one real variable, which we now know how to do. Note that the contour is *oriented*, i.e. there is a direction associated with it. If you choose to perform the integral along the contour $-C$, i.e. C in the opposite sense (starting at $z(b)$ and heading to $z(a)$ in the previous parametrization), then you can see quite easily that

$$\int_{-C} dz\, f(z) = -\int_C dz\, f(z)\,. \tag{3.45}$$

You are also free to split a contour up into parts, and sum the separate contributions. This is all very similar to line integrals or even integrals on the real line so far. Now let's try a few examples!

1. Let's evaluate the integral

$$I = \int_C dz\, z^*\,. \tag{3.46}$$

   where C is the contour

$$z = 2e^{i\theta}\,, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\,, \tag{3.47}$$

   shown in Fig. 13. Along the contour, we have $z^* = 2e^{-i\theta}$. We can transform the integral over the contour to just an integral over θ, which is a real variable, and so

$$\begin{aligned}
\int_C dz\, z^* &= \int_{-\pi/2}^{\pi/2} d\theta\, z^*(\theta) \frac{dz}{d\theta} \\
&= \int_{-\pi/2}^{\pi/2} d\theta\, (2e^{-i\theta}) \cdot 2ie^{i\theta} \\
&= 4i\left(\frac{\pi}{2} + \frac{\pi}{2}\right) \\
&= 4\pi i\,. \tag{3.48}
\end{aligned}$$

2. Consider an arbitrary smooth contour C, which can be parametrized as $z = z(t)$, where $a \leq t \leq b$, from a fixed point $z_1 = z(t = a)$ to a fixed point $z_2 = z(t = b)$. Now let's consider the integral
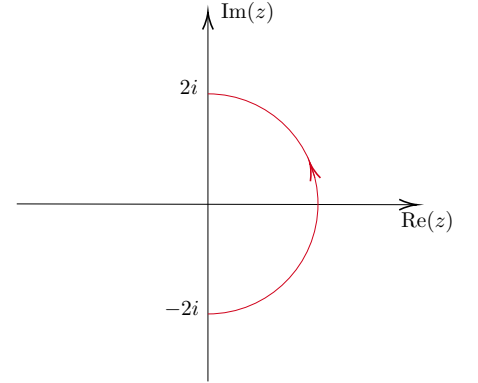
$$I = \int_C dz\, z \tag{3.49}$$

Figure 13: The contour corresponding to $z = 2e^{i\theta}$.

for this arbitrary contour. I can always write

$$
\begin{aligned}
\int_C dz\, z &= \int_a^b dt\, z(t) z'(t) \\
&= \frac{1}{2} \int_a^b dt\, \frac{d}{dt}(z^2(t)) \\
&= \frac{1}{2} z^2(b) - \frac{1}{2} z^2(a) \\
&= \frac{1}{2} \left(z_2^2 - z_1^2 \right).
\end{aligned}
\tag{3.50}
$$

Notice that this integral is independent of the actual contour, and only depends on the endpoints; and is very reminiscent of line integrals of conservative forces. In particular, if you perform the integral over a closed contour, then the result is simply zero.

3. Consider the semicircular path

$$
z = 3e^{i\theta},\, 0 \le \theta \le \pi
\tag{3.51}
$$

from the point $z = 3$ to $z = -3$. Along this path, we want to integrate the following branch of $z^{1/2}$, defined as

$$
f(z) = z^{1/2} = \exp\left(\frac{1}{2} \log z \right),\, 0 \le \arg(z) < 2\pi.
\tag{3.52}
$$

This corresponds to a branch cut taken along the positive real axis; the contour together with this branch cut is shown in Fig. 14. There's no issue with the integral starting on the branch cut. First, we note that

$$
\begin{aligned}
f(3e^{i\theta}) &= \exp\left(\frac{1}{2} \log(3e^{i\theta}) \right) \\
&= \exp\left[\frac{1}{2}(\ln 3 + i\theta) \right] \\
&= \sqrt{3} e^{i\theta/2}.
\end{aligned}
\tag{3.53}
$$

Therefore, the integral we want to compute is

$$
\begin{aligned}
\int_C dz\, z^{1/2} &= \int_0^\pi d\theta\, \sqrt{3} e^{i\theta/2} \cdot 3i e^{i\theta} \\
&= 3\sqrt{3} i \int_0^\pi d\theta\, e^{i3\theta/2} \\
&= 3\sqrt{3} i \cdot \left. \frac{e^{i3\theta/2}}{3i/2} \right|_0^\pi \\
&= 2\sqrt{3} \left(e^{i3\pi/2} - 1 \right) \\
&= -2\sqrt{3}(1 + i).
\end{aligned}
\tag{3.54}
$$

(**End of Material for Midterm 2**)

3.3.3 The modulus inequality

At this point, it seems like we know how to evaluate contour integrals, so we should just go out and do a bunch of them, right? Well, *it gets even better than that:* analytic functions on the complex plane are so well-behaved that there are
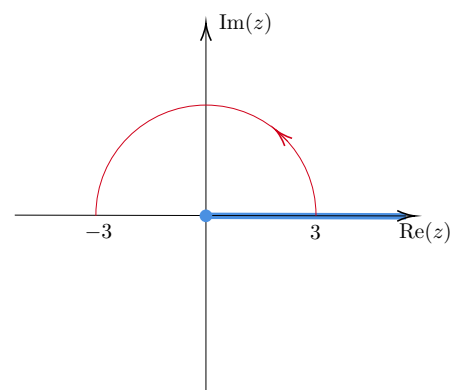
Figure 14: Contour $z = 3e^{i\theta}$ for the integral over $\sqrt{z}$, choosing the branch where $0 < \arg(z) < 2\pi$.

58

some surprisingly powerful theorems ahead of us. For now though, we'll take a little detour and prove some important results related to integrals that will come up repeatedly.

The first result relates the modulus of an integral to the integral of a modulus, and will be particularly useful when we solve improper integrals on the real line later on. In particular,

> If $w(t)$ is a complex-valued function defined on an interval $a \leq t \leq b$, then
>
> $$\left| \int_a^b dt\, w(t) \right| \leq \int_a^b dt\, |w(t)| . \tag{3.55}$$

The proof is rather straightforward, so we'll go through it: let the result of the integral of $w(t)$ over the interval be

$$\int_a^b dt\, w(t) = r_0 e^{i\theta_0} , \tag{3.56}$$

where the integral results in some arbitrary complex number determined by r_0 and θ_0. With this definition,

$$\left| \int_a^b dt\, w(t) \right| = r_0 . \tag{3.57}$$

On the other hand, we also have

$$r_0 = \int_a^b dt\, e^{-i\theta_0} w(t) \tag{3.58}$$

Since r_0 is just a real number, however, we can write the following:

$$r_0 = \mathrm{Re} \int_a^b dt\, e^{-i\theta_0} w(t) = \int_a^b dt\, \mathrm{Re}[e^{-i\theta_0} w(t)] . \tag{3.59}$$

But we also know that

$$\mathrm{Re}[e^{-i\theta_0} w(t)] \leq \left| e^{-i\theta_0} w(t) \right| = \left| e^{-i\theta_0} \right| |w(t)| = |w(t)| . \tag{3.60}$$

Putting everything together proves the statement above.

This leads immediately to another important result:

> Let C denote a contour of length L, and suppose that a function $f(z)$ is piecewise continuous on C. Then there exists M a nonnegative constant such that
>
> $$|f(z)| \leq M \tag{3.61}$$
>
> for all points z on C, and therefore by the previous result,
>
> $$\left| \int_C dz\, f(z) \right| \leq ML . \tag{3.62}$$

This is a really powerful result for taking contours to infinity. Here is an example: consider the semicircular path C given by $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$, and $z^{1/2}$ denotes the branch

$$z^{1/2} = \sqrt{r}\, e^{i\theta/2} , \, r > 0 , \, -\frac{\pi}{2} < \theta < \frac{3\pi}{2} . \tag{3.63}$$

Here, the branch cut is placed along the negative imaginary axis. Let's consider the integral

$$\lim_{R \to \infty} \int_C dz \, \frac{z^{1/2}}{z^2 + 1} \, . \tag{3.64}$$

First, we note that

$$\left| \frac{z^{1/2}}{z^2 + 1} \right| = \frac{|z^{1/2}|}{|z^2 + 1|}$$

$$= \frac{\sqrt{R}}{|R^2 e^{i2\theta} + 1|} \tag{3.65}$$

But

$$|R^2 e^{i2\theta} + 1| = \sqrt{(1 + R^2 \cos 2\theta)^2 + (R^2 \sin 2\theta)^2}$$

$$= \sqrt{1 + 2R^2 \cos 2\theta + R^4}$$

$$\geq R^2 - 1 \, , \tag{3.66}$$

where we've taken $R > 1$ since we are going to take $R \to \infty$. This means that

$$\left| \frac{z^{1/2}}{z^2 + 1} \right| \leq \frac{\sqrt{R}}{R^2 - 1} \, , \tag{3.67}$$

and that therefore

$$\left| \int_C dz \, \frac{z^{1/2}}{z^2 + 1} \right| \leq \pi R \cdot \frac{\sqrt{R}}{R^2 - 1} \tag{3.68}$$

This shows that

$$\lim_{R \to \infty} \left| \int_C dz \, \frac{z^{1/2}}{z^2 + 1} \right| = 0 \, , \tag{3.69}$$

and therefore the integral itself also becomes zero in the limit.

3.3.4   Antiderivatives

Thinking back to the example of integrating a function over an arbitrary contour in Sec. 3.3.2, we should already be able to see that something like the fundamental theorem of calculus in real analysis should carry over into the complex plane. Here is the theorem:[32]

> Suppose that a function $f(z)$ is continuous on a domain D. Then the following statements are all equivalent:
>
> 1. $f(z)$ has an antiderivative $F(z)$ throughout D, i.e. $F'(z) = f(z)$;
>
> 2. The integrals of $f(z)$ along contours lying entirely in D and extending from any fixed point z_1 to any fixed point z_2 all have the same value, namely
>
> $$\int_{z_1}^{z_2} dz \, f(z) = F(z_2) - F(z_1) \, , \tag{3.70}$$
>
> where $F'(z) = f(z)$;
>
> 3. The integrals of $f(z)$ around closed contours lying entirely in D all have value zero.

[32] Whenever the integral only depends on the endpoints and not on the particular contour, we sometimes will write the integral as having complex numbers as the upper and lower limits.

60

The fact that Statement 1) implies Statement 2) is easy to see, and is very similar to the earlier example: along any contour, we can parametrize the contour between z_1 and z_2 by $a \leq t \leq b$, and see that

$$
\begin{aligned}
\int_C dz\, f(z) &= \int_a^b dt\, f[z(t)]z'(t) \\
&= \int dt\, \frac{dF}{dz}\frac{dz}{dt} \\
&= F(z(b)) - F(z(a)) \\
&= F(z_2) - F(z_1)\,.
\end{aligned}
\tag{3.71}
$$

Statement 2) implying Statement 3) is easy too: since contour integration only depends on the end points, any closed contour is a sum of two contour integrals in opposite directions, and hence has to be zero. The hard part of the proof is that Statement 3) implies Statement 1) as well; I won't show you the proof here, but you can consult CB45.

Let's do an example where try to perform the integral

$$
I = \int_C \frac{dz}{z}\,,
\tag{3.72}
$$

where C is a circle of radius 2. Let's evaluate this integral using our knowledge of antiderivatives. We know that $d(\log z)/dz = 1/z$ with an appropriate choice of a branch, so that along the contour of interest this relation holds. First, let's start by considering C_1, the semicircle going from $-\pi/2 \leq \theta \leq \pi/2$ in the counterclockwise direction. For this, we choose the principal branch of the $\log$, i.e.

$$
\mathrm{Log}\, z = \ln r + i\Theta\,, \quad r > 0\,, \quad -\pi < \Theta < \pi\,.
\tag{3.73}
$$

Then, since we know the antiderivative, we can immediately obtain

$$
\begin{aligned}
\int_{C_1} \frac{dz}{z} = \int_{-2i}^{2i} \frac{dz}{z} &= \mathrm{Log}\,(2i) - \mathrm{Log}\,(-2i) \\
&= \ln 2 + \frac{i\pi}{2} - \ln 2 + \frac{i\pi}{2} \\
&= i\pi\,.
\end{aligned}
\tag{3.74}
$$

Next, we consider C_2, the semicircle going from $\pi/2 \leq \theta \leq 3\pi/2$ in the counterclockwise direction. This time, we'll choose another branch of the $\log$,

$$
\log z = \ln r + i\theta\,, \quad r > 0\,, \quad 0 < \Theta < 2\pi\,,
\tag{3.75}
$$

so that the branch cut lies on the positive real axis. Doing the integral, we find

$$
\begin{aligned}
\int_{C_2} \frac{dz}{z} = \int_{2i}^{-2i} \frac{dz}{z} &= \log(-2i) - \log(2i) \\
&= \ln 2 + \frac{3\pi i}{2} - \ln 2 - \frac{i\pi}{2} \\
&= i\pi\,.
\end{aligned}
\tag{3.76}
$$

Thus,

$$
\int_C \frac{dz}{z} = \int_{C_1} \frac{dz}{z} + \int_{C_2} \frac{dz}{z} = 2\pi i\,.
\tag{3.77}
$$

(**End of Lecture**: Wednesday Oct 23 2024)

3.3.5 Cauchy-Goursat theorem

We now come to one of the most remarkable result in complex analysis, known as the **Cauchy-Goursat Theorem**:

61

> If a function f is analytic at all points interior to and on a closed contour C, then
>
> $$\int_C dz \, f(z) = 0 \,. \qquad (3.78)$$

This is the result of the strong constraints on analytic functions that we have discussed so far, in particular the requirement that the Cauchy-Riemann equations are satisfied. For a full proof of the theorem, I will refer you to CB47; however, it's worth motivating where this result comes from. As we have seen, the contour integral can be written as

$$\int_C dz \, f(z) = \int_a^b dt \, f[z(t)]z'(t) \,, \qquad (3.79)$$

where we will now once again write $z = x + iy$, $f(z) = u(x,y) + iv(x,y)$, and $z'(t) = x'(t) + iy'(t)$. With this notation, we can write

$$\int_C dz \, f(z) = \int_a^b dt \, (ux' - vy') + i \int_a^b dt \, (vx' + uy') \,. \qquad (3.80)$$

On the complex plane, however, we can view the contour integral as a line integral, with line element given by

$$d\vec{\ell} = (x'\hat{x} + y'\hat{y})dt \,, \qquad (3.81)$$

where $\hat{x}, \hat{y}$ are the unit vectors in the real and imaginary directions respectively. Therefore, we can define $\vec{A} = u\hat{x} - v\hat{y}$ and $\vec{B} = v\hat{x} + u\hat{y}$, and rewrite

$$\int_C dz \, f(z) = \oint_C d\vec{\ell} \cdot \vec{A} + i \oint_C d\vec{\ell} \cdot \vec{B} \,, \qquad (3.82)$$

At this point, we can think about these integrals as simply being a line integral *in three dimensional real space*. Stokes' theorem now tells us

$$\oint_C d\vec{\ell} \cdot A = \int_S d\vec{S} \cdot (\nabla \times \vec{A}) \,, \qquad (3.83)$$

where we are now integrating over the area S enclosed by the contour C, and the vector $d\vec{S}$ is the normal vector given by the right-hand rule applied to the contour C. On the other hand,

$$\nabla \times \vec{A} = -\partial_x v - \partial_y u = 0 \,, \qquad (3.84)$$

where the last equality sign follows from the Cauchy-Riemann equations! Similarly,

$$\nabla \times \vec{B} = \partial_x u - \partial_y v = 0 \qquad (3.85)$$

as well. This does *not* constitute a full proof, since we have not been careful about whether the partial derivatives of u and v are continuous, but the details are not pertinent to us here.

The Cauchy-Goursat theorem makes many integrals basically trivial. Consider for example the contour C that is the unit circle $|z| = 1$; we have

$$\int_C dz \, \frac{z^2}{z-3} = \int_C dz \, ze^{-z} = \int_C dz \, \tan z = \int_C dz \, \mathrm{Log}(z+2) = 0 \,. \quad (3.86)$$

The only thing you have to be careful about in the above expressions is to locate where the integrated functions are *not* analytic, and check that they

lie outside of the contour. A point z_0 is a **singularity** of a function f if f is not analytic at z_0, but there are points in any neighborhood of z_0 where f is analytic. For example, $z^2/(z-3)$ is not analytic at $z = 3$, but arbitrarily close to $z = 3$, you're fine. This singularity lies outside of the contour C that we were considering above, and so the Cauchy-Goursat theorem applies. Similarly, $\text{Log}(z+2)$ has a branch cut starting at $z = -2$, and extending to the left of that point to $-\infty$, but that's all outside the contour once again.

3.3.6 Cauchy integral formula

With the Cauchy-Goursat theorem, we can now prove another powerful result, known as the **Cauchy integral formula**:

> Let f be analytic everywhere inside and on a closed contour C, oriented in the positive direction (i.e. in the counterclockwise orientation). If z_0 is any point interior to C, then
>
> $$f(z_0) = \frac{1}{2\pi i} \int_C \frac{dz\, f(z)}{z - z_0}. \tag{3.87}$$

It'll be quite instructive to see where this result comes from. Let's recall from the last problem set that we showed that given a contour C_ρ given by $z = z_0 + \rho e^{i\theta}$, $0 \leq \theta < 2\pi$ (which is simply a circle of radius ρ centered on the point z_0, going in the counterclockwise direction),

$$\int_{C_\rho} \frac{dz}{z - z_0} = 2\pi i. \tag{3.88}$$

Now let's look at the integration contours that we want to consider on the complex plane, shown in Fig. 15. We want to evaluate the integral shown Eq. (3.87) around the contour C; but first, let's take a look at the full contour that's shown in Fig. 15. It consists of *1)* the contour C, *2)* a straight path to the circle of radius ρ, *3)* a *clockwise* path around the circle (corresponding to the contour $-C_\rho$), and finally *4)* the same straight path back to the C, in the opposite direction as *2)*. Since $f(z)/(z - z_0)$ is analytic everywhere in the region enclosed by this full contour, we have that

$$\int_C \frac{dz\, f(z)}{z - z_0} = \int_{C_\rho} \frac{dz\, f(z)}{z - z_0}. \tag{3.89}$$

This illustrates a very general principle:

Figure 15: Cauchy integral formula contours. The outer contour is C, while the inner contour, a circular path of radius ρ centered around z_0 in the clockwise direction is $-C_\rho$. Two extra legs connect the C with C_ρ.

> When integrating over some function $f(z)$, you are allowed to deform the contour through any region where $f(z)$ is analytic.

Now, given Eq. (3.89), we can write

$$\int_C \frac{dz\, f(z)}{z - z_0} - f(z_0) \int_{C_\rho} \frac{dz}{z - z_0} = \int_{C_\rho} dz\, \frac{f(z) - f(z_0)}{z - z_0}. \tag{3.90}$$

Taking $\rho \to 0$, we see that the integrand on the right-hand side approaches $f'(z_0)$, and so we are left with an integral of a constant over a circle, which just goes to zero, proving (in a way that can be made rigorous) the integral formula.

Let's do a simple example here. Suppose we want to evaluate the integral

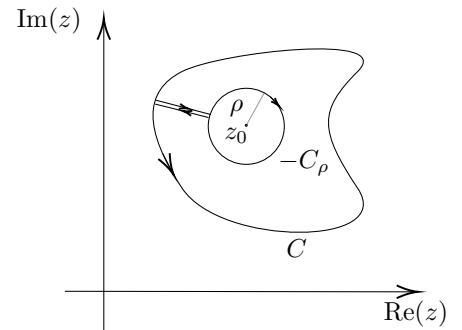$$\int_C dz\, \frac{\cos z}{z(z^2 + 8)}, \tag{3.91}$$

where C is the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. We can write this integral as

$$\int_C dz \, \frac{\cos z}{z(z^2 + 8)} = \int_C dz \, \frac{\cos z}{z(z + 2\sqrt{2}i)(z - 2\sqrt{2}i)} \,. \tag{3.92}$$

One can easily see that two of the singularities of the function are located at $\pm 2\sqrt{2}i$, which both lie outside C. Hence, defining $g(z) \equiv \cos z/((z+2\sqrt{2}i)(z-2\sqrt{2}i))$, we have

$$\int_C dz \, \frac{\cos z}{z(z^2 + 8)} = \int_C dz \, \frac{g(z)}{z - 0} = 2\pi i g(0) = \frac{i\pi}{4} \,, \tag{3.93}$$

where we have used the Cauchy integral formula in the second last equality.

Another intriguing consequence of the Cauchy integral formula can be obtained by taking the derivative of the expression; we find

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{ds \, f(s)}{(s - z)^2} \,. \tag{3.94}$$

We can go on taking derivatives, to find that

---

for a closed contour C around z_0 with f analytic inside,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C dz \, \frac{f(z)}{(z - z_0)^{n+1}} \,. \tag{3.95}$$

In fact, given that we can construct an expression for any arbitrary derivative of an analytic function f at z_0, we arrive at the following remarkable theorem:

If a function f is differentiable at a point z_0, it is infinitely differentiable; i.e. if f is analytic at a point z_0, all derivatives of f at z_0 exist.

Let's do another example: let's evaluate

$$\int_C dz \, \frac{\cosh z}{z^4} \,, \tag{3.96}$$

around the contour C which is the positively oriented boundary of the square with sides lying on lines $x = \pm 2$ and $y = \pm 2$. Using the formula above in Eq. (3.95), we have

$$f'''(0) = \frac{3!}{2\pi i} \int_C dz \, \frac{f(z)}{(z - 0)^4} \,. \tag{3.97}$$

Thus,

$$\int_C dz \, \frac{\cosh z}{z^4} = \sinh(0) \cdot \frac{2\pi i}{3!} = 0 \,. \tag{3.98}$$

3.4 Series Representations of Complex Functions

So far, the Cauchy-Goursat theorem gives us a powerful tool for integrating functions which are analytic within the contour of integration. But we can do even better! To do this, we need to take a small but important diversion to discuss how to represent complex functions as a series. The basic idea of this is already familiar to you, from your years of experience with **Taylor series**:

Let f be analytic throughout a disk $|z - z_0| < R_0$, centered at z_0 and with radius R_0. Then $f(z)$ has the power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \,, \tag{3.99}$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!} \,, n = 0, 1, 2 \cdots \,. \tag{3.100}$$

This series converges to $f(z)$ for any z on the open disk. The largest possible R_0 over which f is analytic is called the **radius of convergence**.

This is just the usual Taylor series, generalized to complex numbers, with

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots \,, \tag{3.101}$$

when we expand about the complex number z_0, in some disk about z_0 where f is analytic. In particular, if the function is entire, i.e. analytic on the entire complex plane, then this series representation converges everywhere on the complex plane.

Thus, for example, we have

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \,, \tag{3.102}$$

which can be taken to be the definition of e^z. Furthermore, we can take the product and sum of series in the natural way, with e.g.

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$
$$= \frac{1}{2i} \left(\sum_{n=0}^{\infty} \frac{(iz)^n}{n!} - \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!} \right) \,. \tag{3.103}$$

We note here that the terms with even n cancel, while the odd terms add, and so we can rewrite this as

$$\sin z = \frac{1}{2i} \left(2 \sum_{n=0}^{\infty} \frac{(iz)^{2n+1}}{(2n+1)!} \right) = \frac{1}{i} \sum_{n=0}^{\infty} \frac{i(-1)^n z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \,, \tag{3.104}$$

which you can check is the usual Taylor series for $\sin z$ for the real line, generalized to the complex plane. We can even take the *derivative* of the series, so that for example

$$\frac{d}{dz} \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \,, \tag{3.105}$$

which you can readily check is the Taylor series expansion for $\cos z$. Every function also has a *unique* series expansion in each domain, i.e. you can identify a function by its series expansion and vice-versa.

Another example is the series representation of $f(z) = 1/(1 - z)$ about $z = 0$. First, we note that

$$f^{(n)}(z) = \frac{n!}{(1 - z)^{n+1}} \,, \tag{3.106}$$

and therefore the Taylor series at $z = 0$ (sometimes also known as the **Maclaurin series**) is

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = \sum_{n=0}^{\infty} z^n. \tag{3.107}$$

Note that this series expansion only applies for $|z| < 1$, since f is clearly not analytic at $z = 0$.

Even when a function is not analytic at z_0, it turns out we can *still* obtain a series expansion about the point.

If $f(z)$ is analytic in some annulus $R_1 < |z - z_0| < R_2$ centered at z_0, we can expand f as a **Laurent series**:

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}, \, (R_1 < |z - z_0| < R_2). \tag{3.108}$$

Given a positively oriented contour C lying in the annulus, we can evaluate

$$a_n = \frac{1}{2\pi i} \int_C \frac{dz \, f(z)}{(z - z_0)^{n+1}}$$
$$b_n = \frac{1}{2\pi i} \int_C \frac{dz \, f(z)}{(z - z_0)^{-n+1}}. \tag{3.109}$$

As was the case for the Taylor series, you can add, multiply or differentiate Laurent series, and the Laurent series in any particular domain is unique for each function. I won't prove the expressions for a_n and b_n, but if you look back to Eq. (3.95), you can see that a_n has the right structure for a Taylor series, and b_n is simply a generalization of that. If you're interested in the proof, I refer you to CB61.

We can frequently obtain the Laurent series of a function by starting from its Taylor series, but being careful about the domain of applicability of the series, since the Laurent series is defined in some annulus around a singularity. So for example,

$$z^2 \sin\left(\frac{1}{z^2}\right) = z^2 \sum_{n=0}^{\infty} \frac{(-1)^n (1/z^2)^{2n+1}}{(2n + 1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!} \frac{1}{z^{4n}}, \, (0 < |z| < \infty), \tag{3.110}$$

where the domain of validity can be deduced from the location of the singularity at $z = 0$. In this case, the inner radius of the annulus can be taken to zero, since the function is analytic arbitrarily close to the singularity.

Another instructive example is the function

$$f(z) = \frac{1}{(z - 1)(z - 2)} = \frac{1}{z - 2} - \frac{1}{z - 1}, \tag{3.111}$$

which has singularities at $z = 1$ and $z = 2$. We want to obtain three different Laurent series centered about $z_0 = 0$, in the annuli given by *1)* $D_1 = |z| < 1$, *2)* $D_2 = 1 < |z| < 2$, and *3)* $D_3 = 2 < |z| < \infty$ (see Fig. 16). For the first annulus, we can use the usual Taylor series expression that we saw above,
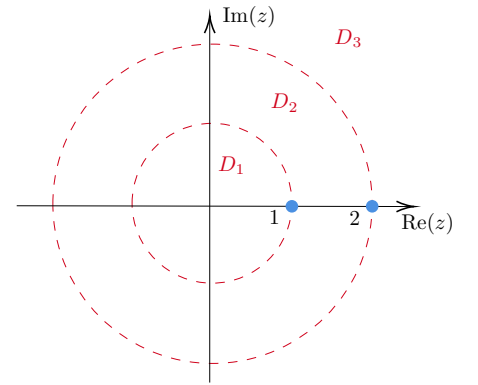

Figure 16: The annuli for computing the Laurent series of $1/((z - 1)(z - 2))$.

$1/(1-z) = \sum_{n=0}^{\infty} z^n$, to find

$$f(z) = -\left(-\sum_{n=0}^{\infty} z^n\right) + \left(-\frac{1}{2}\sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n\right)$$

$$= \sum_{n=0}^{\infty} \left(1 - 2^{-n-1}\right) z^n, \ (|z| < 1). \tag{3.112}$$

This is just a Taylor series, since there is no singularity in D_1.

Moving on, in D_2, we know that the Taylor series expression $1/(1-z) = \sum_{n=0}^{\infty} z^n$ fails, and that we must have a true Laurent series in this annulus, with nonzero b_n in Eq. (3.108). However, we know that in D_2, $|1/z| < 1$, and so we can write

$$\frac{1}{z-1} = \frac{1}{z}\left(\frac{1}{1-z^{-1}}\right) = \frac{1}{z}\sum_{n=0}^{\infty} \frac{1}{z^n}. \tag{3.113}$$

Therefore,

$$f(z) = -\frac{1}{z}\sum_{n=0}^{\infty} \frac{1}{z^n} + \left(-\frac{1}{2}\sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n\right)$$

$$= -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} - \sum_{n=1}^{\infty} \frac{1}{z^n}, \ (1 < |z| < 2). \tag{3.114}$$

Finally, in D_3, we need to do the same trick with $1/(z-2)$,

$$\frac{1}{z-2} = \frac{1}{z}\left(\frac{1}{1-2/z}\right) = \frac{1}{z}\sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n, \tag{3.115}$$

so that

$$f(z) = -\sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=1}^{\infty} \frac{2^{n-1}}{z^n}$$

$$= \sum_{n=1}^{\infty} \frac{2^{n-1}-1}{z^n}, \ (|z| > 2). \tag{3.116}$$

3.5 Integration Part II

We are now ready to continue our discussion of complex integration, armed with our ability to write down a series expansion for analytic functions.

3.5.1 Residues and Poles

First, some terminology. We've already defined a *singularity* as a point z_0 where a function f fails to be analytic, but is otherwise analytic at some point in any neighborhood around z_0. If, in addition, we can cut the point out, and find a disk $0 < |z - z_0| < \varepsilon$ where f is analytic, we call z_0 an **isolated singularity**.

Now, we can write down a Laurent series for the function on this disk, expanding about the singularity. In general, this will look like

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}, \ (0 < |z-z_0| < \varepsilon). \tag{3.117}$$

There are three possibilities for the Laurent series here:

1. $b_n = 0$ for all n. This is called a **removable singularity**. For example, consider the function $f(z) = (1 - \cos z)/z^2$. This function has a singularity at $z = 0$, but if we look at its Laurent series expanded about 0, we have

$$
\begin{aligned}
f(z) &= \frac{1}{z^2} \left(-\frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \right) \\
&= \frac{1}{2!} + \frac{z^2}{4!} + \cdots.
\end{aligned}
\tag{3.118}
$$

   We see here that if we simply redefine $f(0) = 1/2$, we get an *entire function*.

2. $b_m \neq 0$ for some m, but $b_{m+1} = b_{m+2} = \cdots = 0$. This is called a **pole of order** m. A pole of order $m = 1$ is usually referred to as a **simple pole**. For example,

$$
\frac{z^2 - 2z + 3}{z - 2} = (z - 2) + 2 + \frac{3}{z - 2}, \; (0 < |z - 2| < \infty),
\tag{3.119}
$$

   and so we have a simple pole at $z = 2$.

3. An infinite number of b_n is nonzero. These are called **essential singularities**.

Finally, given the Laurent series, we call b_1 the **residue** of f at the singularity. We often denote the residue of f at the singularity z_0 as

$$
b_1 = \operatorname*{Res}_{z=z_0} f(z).
\tag{3.120}
$$

(**End of Lecture**: Monday Nov 4 2024)

3.5.2 Cauchy's residue theorem

Let's revisit the expression for the Laurent series in Eq. (3.108). Suppose f has an isolated singularity at at z_0. We have

$$
b_1 = \operatorname*{Res}_{z=z_0} f(z) = \frac{1}{2\pi i} \int_C dz \, f(z),
\tag{3.121}
$$

if C is a closed contour around z_0, lying within a disk where f is analytic everywhere except z_0. This generalizes immediately to the immensely powerful **Cauchy residue theorem**:

> Let C be a closed contour oriented positively. If a function f is analytic inside the contour except for a finite number of singular points z_k, $k = 1, \cdots, n$, inside C, then
>
> $$
> \int_C dz \, f(z) = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).
> \tag{3.122}
> $$

The proof of this theorem is pretty straightforward. Given a contour C such as the one shown in Fig. 17, we first consider the deformed path as shown. Along the deformed path, we get the contour integral given by

$$
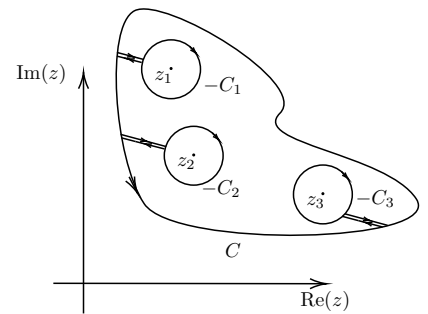\int_C dz \, f(z) - \sum_k \int_{C_k} dz \, f(z) = 0,
\tag{3.123}
$$

Figure 17: Proving Cauchy's residue theorem.

where we get zero because the deformed path encloses a region where f is analytic. But we can immediately evaluate each term using the Laurent series expression for b_1, thus proving the residue theorem.

Before we dive into numerous examples demonstrating the power of the residue theorem, let's discuss the concept of the **residue at infinity**. Pick a contour C_0 centered at the origin that is large enough that it encompasses all of the singularities of the function, *oriented clockwise*. Then we define the residue at infinity as

$$\int_{C_0} dz\, f(z) = 2\pi i \operatorname*{Res}_{z=\infty} f(z).\tag{3.124}$$

Since we can deform contours through regions that are analytic, for any contour C oriented positively encompassing all singularities of a function f,

$$\int_C dz\, f(z) = -2\pi i \operatorname*{Res}_{z=\infty} f(z).\tag{3.125}$$

How do we evaluate this residue? Well, using Eq. (3.108), if we write the Laurent series expanded about the origin, we have

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \frac{b_n}{z^n} = \sum_{n=-\infty}^{\infty} c_n z^n,\tag{3.126}$$

with $c_n = a_n$ for $n \geq 0$, and $c_{-n} = b_n$ for $n < 0$. From Eq. (3.108) that

$$c_n = \frac{1}{2\pi i} \int_{-C_0} dz\, \frac{f(z)}{z^{n+1}},\tag{3.127}$$

since C_0 is big enough that it lies away from all of the isolated singularities, and therefore lies on an annulus where f is analytic. In particular,

$$c_{-1} = \frac{1}{2\pi i} \int_{-C_0} dz\, f(z) = -\operatorname*{Res}_{z=\infty} f(z).\tag{3.128}$$

So if we have a way of evaluating c_{-1} efficiently, we would have the residue at infinity, and also the contour integrals of interest. Let's note however that

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2} \sum_{n=-\infty}^{\infty} \frac{c_n}{z^n} = \sum_{n=-\infty}^{\infty} \frac{c_{n-2}}{z^n}.\tag{3.129}$$

Therefore,

$$\operatorname*{Res}_{z=\infty} f(z) = -c_{-1} = -\operatorname{Res}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right].\tag{3.130}$$

This gives the following result:

> If a function f is analytic everywhere in the complex plane except for a finite number of singular points interior to a positively oriented closed contour C, then
>
> $$\int_C dz\, f(z) = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right].\tag{3.131}$$

On to a bunch of examples!

Example 1. Let's evaluate the integral

$$\int_C dz\, \frac{\cosh \pi z}{z(z^2+1)},\tag{3.132}$$

where C is the circle $|z| = 2$, oriented positively. There are three poles in this expression: $z = 0$, and $z = \pm i$. Instead of actually finding the Laurent series about each of these poles, let's use the following fact that makes life a lot easier:

An isolated singular point z_0 of a function f is a pole of order m if and only if $f(z)$ can be written in the form

$$f(z) = \frac{\phi(z)}{(z - z_0)^m} \,, \tag{3.133}$$

where $\phi(z)$ is analytic and nonzero at z_0. Moreover,

$$\operatorname*{Res}_{z=z_0} f(z) = \phi(z_0) \,, \ (m = 1) \,, \tag{3.134}$$

and

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \,, \ (m \geq 2) \,, \tag{3.135}$$

where $\phi^{(m-1)}(z_0)$ is the $(m-1)$th derivative, evaluated at z_0.

Let's see why the formula for the residue works. Since $\phi(z)$ is analytic at z_0, we can Taylor expand $\phi(z)$ as

$$\phi(z) = \phi(z_0) + \phi'(z_0)(z - z_0) + \frac{1}{2!}\phi''(z_0)(z - z_0)^2 + \cdots \,, \tag{3.136}$$

and therefore

$$f(z) = \frac{\phi(z_0)}{(z - z_0)^m} + \frac{\phi'(z_0)}{(z - z_0)^{m-1}} + \frac{1}{2!}\frac{\phi''(z_0)}{(z - z_0)^{m-2}} + \cdots \,. \tag{3.137}$$

We can see from this that the residue, which corresponds to the term with just $(z - z_0)$ in the denominator, is exactly what is given above.

Returning to our example, we have for $f(z) = \cosh(\pi z)/(z(z^2 + 1))$,

$$\operatorname*{Res}_{z=0} f(z) = 1 \,,$$

$$\operatorname*{Res}_{z=i} f(z) = \frac{\cosh i\pi}{i \cdot (i + i)} = \frac{1}{2} \,,$$

$$\operatorname*{Res}_{z=-i} f(z) = \frac{\cosh(-i\pi)}{-i \cdot (-i - i)} = \frac{1}{2} \,. \tag{3.138}$$

Putting everything together, we find

$$\int_C dz \, \frac{\cosh \pi z}{z(z^2 + 1)} = 2\pi i \left(1 + \frac{1}{2} + \frac{1}{2} \right) = 4\pi i \,. \tag{3.139}$$

Example 2. Let's evaluate the integral of $f(z)$ around the positively oriented circle $|z| = 3$, with

$$f(z) = \frac{z^3 e^{1/z}}{1 + z^3} \,. \tag{3.140}$$

The singularities are located at $z = 0$, as well as at $z = e^{i\pi/3}$, $z = -1$ and $z = e^{i5\pi/3}$. But these singularities are all located inside the contour, and so we can use the residue at infinity to evaluate the integral. We find

$$\frac{1}{z^2} f\left(\frac{1}{z} \right) = \frac{1}{z^2} \frac{e^z}{z^3(1 + z^{-3})} = \frac{1}{z^2} \frac{e^z}{1 + z^3} \,. \tag{3.141}$$

We can expand this as a Laurent series about the origin as

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2} \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty} (z^3)^n\right)$$

$$= \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2} + \cdots , \qquad (3.142)$$

and so the residue at $z = 0$ is simply 1. We deduce that

$$\int_C dz\, f(z) = 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right) = 2\pi i . \qquad (3.143)$$

Example 3. Let's integrate $f(z) = \tan z$ along the contour C, the positively oriented circle $|z| = 2$. Since $\tan z = \sin z / \cos z$, there are singularities within the contour at $z = \pm\pi/2$. About either of these singularities, we can expand

$$\sin z = \sin z_0 + \cdots$$
$$\cos z = \cos(z_0) - \sin z_0 \cdot (z - z_0) + \cdots$$
$$= -\sin z_0 \cdot (z - z_0) + \cdots , \qquad (3.144)$$

since $\cos(z_0) = 0$. $\cdots$ represents terms that are of order $z - z_0$ or higher. Hence,

$$\tan z = \frac{\sin z_0 + \cdots}{-\sin z_0 \cdot (z - z_0) + \cdots}$$

$$= -\frac{1}{z - z_0} \frac{1 + \cdots}{1 - \cdots}$$

$$= -\frac{1}{z - z_0} (1 + \cdots) . \qquad (3.145)$$

Clearly then, the residue at both singularities is -1, and

$$\int_C dz\, \tan z = -4\pi i . \qquad (3.146)$$

3.6 Improper integrals

By far the most powerful application of the residue theorem is in evaluating improper integrals. These are integrals of the form

$$\int_0^{\infty} dx\, f(x) = \lim_{R \to \infty} \int_0^R dx\, f(x) , \qquad (3.147)$$

and also

$$\int_{-\infty}^{\infty} dx\, f(x) = \lim_{R_1 \to \infty} \int_{-R_1}^{\infty} dx\, f(x) + \lim_{R_2 \to \infty} \int_0^{R_2} dx\, f(x) , \qquad (3.148)$$

if these limits are well-defined.

In addition, sometimes we are interested in the **Cauchy principal value** of an integral, which is

$$\text{P.V.} \int_{-\infty}^{\infty} dx\, f(x) = \lim_{R \to \infty} \int_{-R}^R dx\, f(x) . \qquad (3.149)$$

So, for example,

$$\text{P.V.} \int_{-\infty}^{\infty} dx\, x = \lim_{R \to \infty} \left[\frac{x^2}{2}\right]_{-R}^R = 0 , \qquad (3.150)$$

71

even though the limit in Eq. (3.148) doesn't exist.

For even functions, i.e. $f(x) = f(-x)$, we also have the relation that

$$\int_0^\infty dx\, f(x) = \frac{1}{2}\int_{-\infty}^\infty dx\, f(x)\,. \tag{3.151}$$

The most basic strategy to evaluate many such integrals is to choose a contour that runs from $-R$ to R along the real axis, and then a semicircular contour connecting these two ends. Finally, we take $R \to \infty$, and check that the semicircular part of the contour goes to zero in that limit. The contour integral can then be evaluated using the residue theorem.

The first example we will examine is

$$\int_0^\infty dx\, f(x) = \int_0^\infty dx\, \frac{x^2}{x^6+1} = \frac{1}{2}\int_{-\infty}^\infty dx\, \frac{x^2}{x^6+1}\,. \tag{3.152}$$

To evaluate this, we consider the contour C described above, which we write as

$$\int_C dz\, f(z) = \int_{-R}^R dx\, \frac{x^2}{x^6+1} + \int_{C_R} dz\, f(z)\,, \tag{3.153}$$

where C_R is the semicircular contour of radius R, centered on the origin. The poles of $f(z)$ are to be found at

$$z = e^{i\pi/6},\, i,\, e^{5\pi i/6},\, e^{7\pi i/6},\, -i,\, e^{11\pi i/6}\,. \tag{3.154}$$

Of these poles, three of them are in the upper half of the complex plane, and will be inside C when we take $R \to \infty$.

Let's first evaluate the residues. Defining

$$\phi_{z_0}(z) = \frac{z^2}{z^6+1}(z - z_0)\,, \tag{3.155}$$

we know from the theorem shown in Eq. (3.134) that

$$\operatorname*{Res}_{z=e^{i\pi/6}} f(z) = \phi_{e^{i\pi/6}}(e^{i\pi/6}) = -\frac{i}{6}\,,$$

$$\operatorname*{Res}_{z=i} f(z) = \phi_i(i) = \frac{i}{6}\,,$$

$$\operatorname*{Res}_{z=e^{5i\pi/6}} f(z) = \phi_{5i\pi/6}(e^{5i\pi/6}) = -\frac{i}{6}\,, \tag{3.156}$$

with the aid of Mathematica.

Next, let's turn our attention to the integral over the semicircle. The integral can be parametrized by θ where $z = Re^{i\theta}$,

$$\int_{C_R} dz\, f(z) = \int_0^\pi d\theta\, \frac{dz}{d\theta}\frac{R^2 e^{2i\theta}}{R^6 e^{6i\theta}+1}$$

$$= iR^3 \int_0^\pi \frac{e^{3i\theta}}{R^6 e^{6i\theta}+1}\,. \tag{3.157}$$

We now want to use the modulus inequality, given in Eq. (3.62), to show that this integral goes to zero as $R \to \infty$. We have

$$\left|\frac{e^{3i\theta}}{R^6 e^{6i\theta}+1}\right| = \frac{1}{|R^6 e^{6i\theta}+1|}\,. \tag{3.158}$$

A useful general result is that

$$
\begin{aligned}
|R^n e^{in\theta} \pm 1| &= \sqrt{(R^n \cos(n\theta) \pm 1)^2 + (R^n \sin(n\theta))^2} \\
&= \sqrt{R^{2n} \pm 2R^n \cos(n\theta) + 1} \\
&\geq R^n - 1
\end{aligned}
\tag{3.159}
$$

Thus,

$$
\left| \frac{e^{3i\theta}}{R^6 e^{6i\theta} + 1} \right| \leq \frac{1}{R^6 - 1} \,,
\tag{3.160}
$$

From this, we deduce from the modulus inequality in Eq. (3.62) that

$$
\lim_{R \to \infty} \left| \int_{C_R} dz \, f(z) \right| \leq \lim_{R \to \infty} \frac{\pi R \cdot R^3}{R^6 - 1} = 0 \,,
\tag{3.161}
$$

i.e. the integral itself must go to zero.

Thus, using the residue theorem, we find

$$
\lim_{R \to \infty} \int_C dz \, f(z) = \int_{-\infty}^{\infty} dx \, \frac{x^2}{x^6 + 1} = 2\pi i \left(-\frac{i}{6} + \frac{i}{6} - \frac{i}{6} \right) = \frac{\pi}{3} \,,
\tag{3.162}
$$

so that

$$
\int_0^{\infty} dx \, \frac{x^2}{x^6 + 1} = \frac{\pi}{6} \,.
\tag{3.163}
$$

Let's do another example, this time evaluating the Cauchy principal value of

$$
\int_{-\infty}^{\infty} dx \, f(x) = \int_{-\infty}^{\infty} dx \, \frac{x \sin x}{x^2 + 2x + 2}
\tag{3.164}
$$

Let's begin by locating the poles; we can solve $x^2 + 2x + 2 = 0$ to find that the poles are at

$$
z_0 = -1 \pm \frac{1}{2}\sqrt{4 - 8} = -1 \pm i \,.
\tag{3.165}
$$

To evaluate the integral, we want to do the following:

$$
\int_C dz \, f(z) = \text{Im} \int_C dz \, \frac{z e^{iz}}{z^2 + 2z + 2} \,,
\tag{3.166}
$$

where if we choose the contour to go from $-R$ to R on the real axis, and then take $R \to \infty$, the imaginary part will pick out $\sin x$ on the real axis, giving us the integral that we want. But how do we close the contour? Well, we will have to close it *upward* in a semicircle, so that when you plug in z with a large positive imaginary component into e^{iz}, you'll get something that is exponentially *suppressed*. So using the same contour as before, let's examine the integral along the semicircle given by $|z| = R$, where $z = Re^{i\theta}$:

$$
\int_{C_R} dz \, \frac{z e^{iz}}{z^2 + 2z + 2} = \int_0^{\pi} d\theta \, \frac{iR^2 e^{2i\theta} e^{iRe^{i\theta}}}{(Re^{i\theta})^2 + 2Re^{i\theta} + 2} \,.
\tag{3.167}
$$

Notice however that I can write

$$
e^{iRe^{i\theta}} = e^{iR\cos\theta} e^{-R\sin\theta} \,.
\tag{3.168}
$$

73

The exponential suppression here, which is not compensated for by any other terms, guarantees that the integrand goes to zero as $R \to \infty$. Thus

$$\lim_{R \to \infty} \int_{C_R} dz \, \frac{ze^{iz}}{z^2 + 2z + 2} = 0 \,. \tag{3.169}$$

Let's turn our attention back to the poles again. Only one pole is in the upper half plane, and that is $-1 + i$. The residue at this point is

$$\operatorname*{Res}_{z=-1+i} g(z) = \frac{(-1+i)e^{i(-1+i)}}{(-1+i-(-1-i))} = \frac{-1+i}{2i} e^{-1-i} \tag{3.170}$$

Hence,

$$\begin{aligned}
\int_{-\infty}^{\infty} dx \, \frac{x \sin x}{x^2 + 2x + 2} &= \operatorname{Im} \left(2\pi i \cdot \frac{-1+i}{2i} e^{-1-i} \right) \\
&= \operatorname{Im} \left(\pi(-1+i)e^{-1}(\cos 1 - i \sin 1) \right) \\
&= \frac{\pi}{e} \left(\cos 1 + \sin 1 \right) \,. \tag{3.171}
\end{aligned}$$

(**End of Lecture**: Wednesday Nov 6 2024)

A   Midterm 1 Review

A.1   Disk on an Inclined Plane

Consider a disk of mass m and radius a rolling without slipping down a plane inclined at an angle θ with respect to the horizontal. Denote the distance of the center of mass of the disk from the top of the plane as x, and denote the angle of rotation of the disk as ϕ. Obtain the equations of motion. Is the first integral conserved?

SOLUTION:

Rolling without slipping implies a constraint given by $x - a\phi = 0$. Therefore, the Lagrangian for this constrained system can be written as

$$L(x, \dot{x}, \phi, \dot{\phi}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\phi}^2 + mgx\sin\theta - \lambda(x - a\phi), \quad \text{(A.1)}$$

where $I = ma^2/2$ is the moment of inertia of the disk, and we have inserted the Lagrange multiplier λ. Note that for the purposes of classical mechanics, *only so-called holonomic constraints* of the form $f(x, \phi) = 0$ can be treated by adding the constraint to the Lagrangian through a Lagrange multiplier. Other types of constraints require more sophisticated treatments.

Let's write down the equations of motion for this system. The Euler-Lagrange equation with respect to x is

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \implies \ddot{x} = g\sin\theta - \frac{\lambda}{m} \quad \text{(A.2)}$$

$$\frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0 \implies I\ddot{\phi} = \lambda a, \quad \text{(A.3)}$$

with $x = a\phi$ imposed by the constraint. We thus find

$$\ddot{\phi} = \frac{g\sin\theta - \lambda/m}{a} \implies I\frac{g\sin\theta - \lambda/m}{a} = \lambda a \implies \lambda = \frac{mg\sin\theta}{1 + ma^2/I}. \quad \text{(A.4)}$$

Note that since $\ddot{x} = g\sin\theta - \lambda/m$, we can see that λ is the force of friction acting on the disk as it rolls down without slipping, which is the force responsible for enforcing the constraint $x - a\phi = 0$.

What are the conserved quantities? Since the Lagrangian depends on both x and ϕ coordinates directly, the conjugate momenta to these coordinates are not conserved. However, L does not explicitly depend on time, and therefore the first integral is conserved. This is

$$\frac{\partial L}{\partial \dot{x}}\dot{x} + \frac{\partial L}{\partial \dot{\phi}}\dot{\phi} - L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\phi}^2 - mgx\sin\theta + \lambda(x - a\phi) \quad \text{(A.5)}$$

$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\phi}^2 - mgx\sin\theta, \quad \text{(A.6)}$$

where the last term drops out since the constraint is enforced on-shell. This is the translational kinetic energy plus the rotational kinetic energy plus the total gravitational potential energy, and is conserved.

A.2 (SG 10.13) Fluid Equations

Write out Euler's equation for fluid motion

$$\dot{\vec{v}} + (\vec{v} \cdot \nabla)\vec{v} = -\nabla h \tag{A.7}$$

in Cartesian tensor notation. Transform it into

$$\dot{\vec{v}} - \vec{v} \times \vec{\omega} = -\nabla \left(\frac{1}{2}\vec{v}^2 + h \right) \tag{A.8}$$

where $\vec{\omega} = \nabla \times \vec{v}$ is the vorticity. Deduce Bernoulli's theorem, that for steady $\dot{\vec{v}} = 0$ flow the quantity $\vec{v}^2/2 + h$ is constant along streamlines.

SOLUTION:

In index notation, Euler's equation reads

$$\partial_t v_i + v_j \partial_j v_i = -\partial_i h \,. \tag{A.9}$$

In order to perform the transformation, first let's observe that

$$
\begin{aligned}
(\vec{v} \times \vec{\omega})_i &= \epsilon_{ijk} v_j \omega_k \\
&= \epsilon_{ijk} v_j (\epsilon_{kmn} \partial_m v_n) \\
&= \epsilon_{kij} \epsilon_{kmn} v_j \partial_m v_n \\
&= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) v_j \partial_m v_n \\
&= v_j \partial_i v_j - v_j \partial_j v_i \,.
\end{aligned}
\tag{A.10}
$$

Therefore, we can write Euler's equation as

$$\partial_t v_i + v_j \partial_i v_j - \epsilon_{ijk} v_j \omega_k = -\partial_i h \,. \tag{A.11}$$

Note however that $v_j \partial_i v_j = \partial_i(v_j v_j)/2 = \partial_i \vec{v}^2/2$, and so grouping terms appropriately,

$$\partial_t v_i - \epsilon_{ijk} v_j \omega_k = -\partial_i \left(\frac{1}{2}\vec{v}^2 + h \right) \implies \dot{\vec{v}} - \vec{v} \times \vec{\omega} = -\nabla \left(\frac{1}{2}\vec{v}^2 + h \right) \tag{A.12}$$

as required.

To deduce Bernoulli's theorem, we set $\partial_t v_i = 0$, and contract both sides with v_i. This gives

$$-\epsilon_{ijk} v_i v_j \omega_k = -v_i \partial_i \left(\frac{1}{2}\vec{v}^2 + h \right) \,. \tag{A.13}$$

But contracting the Levi-Civita symbol with the same vector twice is zero. Therefore,

$$v_i \partial_i \left(\frac{1}{2}\vec{v}^2 + h \right) \,, \tag{A.14}$$

which is the statement that the quantity $\vec{v}^2/2 + h$ is constant along streamlines.

A.3 Maxwell's Equations from the Lagrangian

The Lagrangian density for the electromagnetic field is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \,, \tag{A.15}$$

in Minkowski space, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor, and A^μ is the four-potential. The metric that we use is

$$\eta_{\mu\nu} \equiv \text{diag}(+1, -1, -1, -1)\,. \tag{A.16}$$

Obtain the equations of motion.

From the properties of the electromagnetic tensor, Prove the Bianchi identity

$$\partial^\mu F^{\alpha\beta} + \partial^\alpha F^{\beta\mu} + \partial^\beta F^{\mu\alpha} = 0\,. \tag{A.17}$$

Find the canonical energy-momentum tensor, and show explicitly that it is conserved.

SOLUTION:

Let's write out the Lagrangian density carefully as

$$\mathcal{L} = -\frac{1}{4}\eta^{\mu\alpha}\eta^{\nu\beta}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right)\,. \tag{A.18}$$

Now let's take the derivative. We have

$$\frac{\partial \mathcal{L}}{\partial A_\sigma} = 0 \tag{A.19}$$

and

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial(\partial_\rho A_\sigma)} &= -\frac{1}{4}\eta^{\mu\alpha}\eta^{\nu\beta}\left(\delta^\rho_\mu\delta^\sigma_\nu - \delta^\rho_\nu\delta^\sigma_\mu\right)F_{\alpha\beta} - \frac{1}{4}\eta^{\mu\alpha}\eta^{\nu\beta}F_{\mu\nu}\left(\delta^\rho_\alpha\delta^\sigma_\beta - \delta^\rho_\beta\delta^\sigma_\alpha\right) \\
&= -\frac{1}{4}(F^{\rho\sigma} - F^{\sigma\rho}) - \frac{1}{4}(F^{\rho\sigma} - F^{\sigma\rho}) \\
&= -F^{\rho\sigma}\,, \tag{A.20}
\end{aligned}$$

where in the last line I used the fact that $F^{\rho\sigma}$ is antisymmetric.

Thus the equation of motion is

$$\partial_\rho F^{\rho\sigma} = 0\,. \tag{A.21}$$

For the Bianchi identity, we note that

$$\begin{aligned}
\partial^\mu F^{\alpha\beta} &= \partial^\mu(\partial^\alpha A^\beta - \partial^\beta A^\alpha) \\
&= \partial^\alpha\partial^\mu A^\beta - \partial^\alpha\partial^\beta A^\mu + \partial^\alpha\partial^\beta A^\mu - \partial^\beta\partial^\mu A^\alpha \\
&= \partial^\alpha F^{\mu\beta} + \partial^\beta F^{\alpha\mu} \\
&= -\partial^\alpha F^{\beta\mu} - \partial^\beta F^{\mu\alpha}\,, \tag{A.22}
\end{aligned}$$

and so $\partial^\mu F^{\alpha\beta} + \partial^\alpha F^{\beta\mu} + \partial^\beta F^{\mu\alpha} = 0$ as required.

The canonical energy-momentum tensor is

$$\begin{aligned}
T^\mu{}_\nu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\sigma)}\partial_\nu A_\sigma - \delta^\mu_\nu\mathcal{L} \\
&= -F^{\mu\sigma}\partial_\nu A_\sigma + \frac{1}{4}\delta^\mu_\nu F_{\alpha\beta}F^{\alpha\beta}
\end{aligned} \tag{A.23}$$

Taking the derivative, we find

$$\partial_\mu T^\mu{}_\nu = -\partial_\mu F^{\mu\sigma}\partial_\nu A_\sigma - F^{\mu\sigma}\partial_\mu\partial_\nu A_\sigma + \frac{1}{4}\partial_\nu(F_{\alpha\beta}F^{\alpha\beta})\,. \tag{A.24}$$

The first term vanishes by the equation of motion, and we can rewrite the last term as $(1/2)\partial_\nu F_{\mu\sigma} F^{\mu\sigma}$. However, by relabeling indices, commuting derivatives, and exploiting the antisymmetry of $F^{\mu\sigma}$,

$$
\begin{aligned}
\frac{1}{2}\partial_\nu F_{\mu\sigma} F^{\mu\sigma} &= \frac{1}{2}\left(\partial_\nu\partial_\mu A_\sigma F^{\mu\sigma} - \partial_\nu\partial_\sigma A_\mu F^{\mu\sigma}\right) \\
&= \frac{1}{2}\left(\partial_\mu\partial_\nu A_\sigma F^{\mu\sigma} - \partial_\nu\partial_\mu A_\sigma F^{\sigma\mu}\right) \\
&= \partial_\mu\partial_\nu A_\sigma F^{\mu\sigma}\,,
\end{aligned}
\tag{A.25}
$$

where in the last line I used $F^{\sigma\mu} = -F^{\mu\sigma}$. Therefore,

$$
\partial_\mu T^\mu{}_\nu = -F^{\mu\sigma}\partial_\mu\partial_\nu A_\sigma + \partial_\mu\partial_\nu A_\sigma F^{\mu\sigma} = 0\,,
\tag{A.26}
$$

i.e. the canonical energy-momentum tensor is conserved.

B Midterm 2 Review

B.1 Matrix Derivatives

Consider an $n \times n$ matrix X, with each entry X_{ij} being an independent variable that can be varied. In many machine learning applications, it is useful to compute the derivative of various quantities involving the matrix X with respect to its entries. A giant compendium of such results can be found here. We now want to use index notation to demystify several of these results.

Show the following results for an invertible $n \times n$ matrix X, which you can think of representing some $(1,1)$-tensor in $\mathbb{R}^n$:

1.

$$\frac{\partial X_{kl}^{-1}}{\partial X_{ij}} = -X_{ki}^{-1} X_{jl}^{-1} . \tag{B.1}$$

> Let me review the golden rules of index notation:
>
> (a) Free indices (indices that are not summed over) must agree on both the left-hand side and the right-hand side of any equation.
>
> (b) Indices that are not free should be contracted in pairs, with one upper and one lower index (unless in $\mathbb{R}^n$, where the position doesn't matter).
>
> (c) Contracted indices can always be relabeled, since they are dummy indices.
>
> (d) There should never be more than two of the same index appearing in a term formed by the product of a bunch of tensors, vectors etc. If this happens to you, it is likely because sum of the indices are supposed to be contracted in a sum. Relabel your contracted indices!
>
> First, let's note that the multiplication of two matrices AB is written in index notation as $A_{ij}B_{jk}$, and we don't have to worry about index position here. We know that $X_{ij}^{-1}X_{jk} = \delta_{ik} = I$, where I is the $n \times n$ identity matrix, and therefore
>
> $$\frac{\partial}{\partial X_{ij}}(X_{kl}^{-1}X_{lm}) = \frac{\partial}{\partial X_{ij}}\delta_{km} = 0 . \tag{B.2}$$
>
> On the other hand, using Leibniz's rule gives
>
> $$\frac{\partial}{\partial X_{ij}}(X_{kl}^{-1}X_{lm}) = \frac{\partial X_{kl}^{-1}}{\partial X_{ij}}X_{lm} + X_{kl}^{-1}\frac{\partial X_{lm}}{\partial X_{ij}}$$
>
> $$= \frac{\partial X_{kl}^{-1}}{\partial X_{ij}}X_{lm} + X_{kl}^{-1}\delta_{il}\delta_{jm} \tag{B.3}$$

From here, we see that

$$\frac{\partial X_{kl}^{-1}}{\partial X_{ij}} X_{lm} = -X_{kl}^{-1}\delta_{il}\delta_{jm}$$

$$\frac{\partial X_{kl}^{-1}}{\partial X_{ij}} X_{lm}X_{mp}^{-1} = -X_{kl}^{-1}\delta_{il}\delta_{jm}X_{mp}^{-1}$$

$$\frac{\partial X_{kl}^{-1}}{\partial X_{ij}} \delta_{lp} = -X_{ki}^{-1}X_{jp}^{-1}$$

$$\frac{\partial X_{kp}^{-1}}{\partial X_{ij}} = -X_{ki}^{-1}X_{jp}^{-1}\,, \tag{B.4}$$

which you can check is equivalent to the required expression by just a relabeling of the indices.

2. For a constant matrix A, show explicitly that $\mathrm{Tr}(XA)$ is a scalar if these matrices are taken to represent tensors, and that

$$\frac{\partial}{\partial X_{ij}}\mathrm{Tr}(XA) = (A^T)_{ij} = A_{ji}\,, \tag{B.5}$$

where ^{T} denotes the transpose.

First, let's write the trace properly in index notation, paying some attention to the placement of indices: $\mathrm{Tr}(XA) = X^{ij}A_{ji}$. Under a coordinate transformation,

$$X^{ij} \mapsto X'^{ij} = a^i{}_k a^j{}_l X^{kl}$$
$$A_{ji} \mapsto A'_{ji} = (a^{-1})^k{}_j (a^{-1})^l{}_i A_{kl} \tag{B.6}$$

Therefore,

$$X^{ij}A_{ji} \mapsto (a^{-1})^p{}_i a^i{}_k (a^{-1})^m{}_j a^j{}_l X^{kl}A_{mp}$$
$$= \delta^p_k \delta^m_l X^{kl}A_{mp}$$
$$= X^{kl}A_{lk}\,, \tag{B.7}$$

i.e. $X^{ij}A_{ji}$ does not transform under a coordinate transformation and is a scalar. Notice how I relabeled the contracted indices on A' to avoid having more than two of the same index.
At this point, we can see immediately that

$$\frac{\partial}{\partial X_{ij}} X^{kl}A_{lk} = \delta^k_i \delta^l_j A_{lk} = A_{ji}\,, \tag{B.8}$$

proving the relation. Notice again that I was careful to label XA using indices k and l, which again avoids having more than two of the same index.

3. The following relation is true for any $n \times n$ matrix X:

$$X_{jk}\frac{\partial}{\partial X_{ik}}\det(X) = \delta_{ij}\det(X)\,. \tag{B.9}$$

Show that it is true for the case where $n = 3$.

The determinant in the 3×3 case can be written as

$$\det(X)\epsilon_{lmn} = \epsilon^{ijk}X_{il}X_{jm}X_{kn}. \tag{B.10}$$

Taking the derivative, we find

$$\frac{\partial}{\partial X_{pq}}\det(X)\epsilon_{lmn} = \frac{\partial}{\partial X_{pq}}\epsilon^{ijk}X_{il}X_{jm}X_{kn}$$

$$= \epsilon^{ijk}\left(\delta_{ip}\delta_{lq}X_{jm}X_{kn} + X_{il}\delta_{jp}\delta_{mq}X_{kn}\right.$$
$$\left. + X_{il}X_{jm}\delta_{kp}\delta_{nq}\right)$$

$$X_{rq}\frac{\partial}{\partial X_{pq}}\det(X)\epsilon_{lmn} = \epsilon^{ijk}\left(\delta_{ip}\delta_{lq}X_{jm}X_{kn} + X_{il}\delta_{jp}\delta_{mq}X_{kn}\right.$$

$$\left. + X_{il}X_{jm}\delta_{kp}\delta_{nq}\right)X_{rq} \tag{B.11}$$

Multiplying both sides by ϵ^{lmn}, and noting that $\epsilon_{lmn}\epsilon^{lmn} = 3!$, we find

$$3!X_{rq}\frac{\partial}{\partial X_{pq}}\det(X) = \epsilon^{ijk}\epsilon^{lmn}\left(\delta_{ip}\delta_{lq}X_{jm}X_{kn} + X_{il}\delta_{jp}\delta_{mq}X_{kn}\right.$$

$$\left. + X_{il}X_{jm}\delta_{kp}\delta_{nq}\right)X_{rq}$$
$$= \epsilon^{pjk}\epsilon^{lmn}X_{rl}X_{jm}X_{kn} + \epsilon^{ipk}\epsilon^{lmn}X_{il}X_{rm}X_{kn}$$
$$+ \epsilon^{ijp}\epsilon^{lmn}X_{il}X_{jm}X_{rn}. \tag{B.12}$$

You can show that for example $\epsilon^{lmn}X_{rl}X_{jm}X_{kn} = \det(X)\epsilon_{rjk}$ from Eq. (B.10), by noting that $\det(A) = \det(A^T)$. Thus, we find

$$3!X_{rq}\frac{\partial}{\partial X_{pq}}\det(X) = (\epsilon^{pjk}\epsilon_{rjk} + \epsilon^{ipk}\epsilon_{irk} + \epsilon^{ijp}\epsilon_{ijr})\det(X)$$

$$= 3\epsilon^{pjk}\epsilon_{rjk}\det(X)$$
$$= 3 \times 2\delta_{rp}\det(X), \tag{B.13}$$

giving finally

$$X_{rq}\frac{\partial}{\partial X_{pq}}\det(X) = \delta_{pr}\det(X), \tag{B.14}$$

as required.

B.2   2+1D Electrodynamics

Consider electrodynamics with 1 time-like dimension, and 2 space-like dimensions. The electromagnetic field strength two form is still defined as

$$F = -\frac{1}{2}F_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu, \tag{B.15}$$

with coordinates $(x^0, x^1, x^2) \equiv (t, x, y)$, and metric $g = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2$, adopting the mostly minus convention. In terms of the components of the usual electric and magnetic fields,

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y \\ -E_x & 0 & -B \\ -E_y & B & 0 \end{pmatrix}. \tag{B.16}$$

Furthermore, let the one-form current $J = \rho\,\mathrm{d}t - j_x\,\mathrm{d}x - j_y\,\mathrm{d}y$, where $J^\mu = (\rho, \vec{j})$ are the usual charge density and current density vector respectively. Note that

we can write

$$F = -E_x \, \mathrm{d}t \wedge \mathrm{d}x - E_y \, \mathrm{d}t \wedge \mathrm{d}y + B \, \mathrm{d}x \wedge \mathrm{d}y \,. \qquad (\text{B.17})$$

1. Write down Maxwell's equations, $\mathrm{d}F = 0$ and $\mathrm{d} \star F = \star J$, in component form for 2+1D electrodynamics.

First, we see that

$$\mathrm{d}F = (-\partial_y E_x + \partial_x E_y + \partial_t B)\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y = 0 \,, \qquad (\text{B.18})$$

which in index notation reads

$$\epsilon^{ij} \partial_i E_j = -\partial_t B \,, \qquad (\text{B.19})$$

where ϵ^{ij} is the two-dimensional Levi-Civita tensor. This is the equivalent of Faraday's law. Next, we find

$$\star F = E_x \, \mathrm{d}y - E_y \, \mathrm{d}x + B \, \mathrm{d}t \,, \qquad (\text{B.20})$$

and therefore

$$\begin{aligned}
\mathrm{d} \star F =& \partial_x E_x \, \mathrm{d}x \wedge \mathrm{d}y + \partial_t E_x \, \mathrm{d}t \wedge \mathrm{d}y - \partial_t E_y \, \mathrm{d}t \wedge \mathrm{d}x \\
& - \partial_y E_y \, \mathrm{d}y \wedge \mathrm{d}x + \partial_x B \, \mathrm{d}x \wedge \mathrm{d}t + \partial_y B \, \mathrm{d}y \wedge \mathrm{d}t \\
=& - (\partial_t E_y + \partial_x B)\mathrm{d}t \wedge \mathrm{d}x + (\partial_t E_x - \partial_y B)\mathrm{d}t \wedge \mathrm{d}y \\
& + (\partial_x E_x + \partial_y E_y)\mathrm{d}x \wedge \mathrm{d}y
\end{aligned} \qquad (\text{B.21})$$

On the other hand,

$$\star J = \rho \, \mathrm{d}x \wedge \mathrm{d}y - j_x \mathrm{d}t \wedge \mathrm{d}y + j_y \, \mathrm{d}t \wedge \mathrm{d}x \,. \qquad (\text{B.22})$$

Comparing both expressions, we find the equivalent of Gauss's law,

$$\partial_x E_x + \partial_y E_y = \rho \,, \qquad (\text{B.23})$$

and the equivalent of Ampere's law,

$$\begin{aligned}
\partial_x B &= -j_y - \partial_t E_y \,, \\
\partial_y B &= j_x + \partial_t E_x \,.
\end{aligned} \qquad (\text{B.24})$$

---

2. Derive the continuity equation.

The continuity equation can be derived from the fact that $\mathrm{d} \star J = \mathrm{d}^2 \star F = 0$. But

$$\mathrm{d} \star J = (\partial_t \rho + \partial_x j_x + \partial_y j_y)\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y = 0 \,, \qquad (\text{B.25})$$

and we obtain the continuity equation

$$\partial_t \rho + \partial_x j_x + \partial_y j_y = 0 \,. \qquad (\text{B.26})$$

B.3 Complex Numbers, Cauchy-Riemann Equations, Harmonic Functions

1. Show that for $z = x + iy$,

$$\mathrm{Re}[\log(z - 1)] = \frac{1}{2} \ln[(x - 1)^2 + y^2] \,. \qquad (\text{B.27})$$

Why must this function satisfy Laplace's equation when $z \neq 1$?

First,

$$\log(z-1) = \log((x-1) + iy)$$
$$= \ln R + i(\Theta + 2k\pi), \, k \in \mathbb{Z}, \qquad \text{(B.28)}$$

where $R = \sqrt{(x-1)^2 + y^2}$, and $\Theta = \tan^{-1}((y-1)/x)$. Therefore,

$$\text{Re}[\log(z-1)] = \ln \sqrt{(x-1)^2 + y^2} = \frac{1}{2} \ln[(x-1)^2 + y^2], \qquad \text{(B.29)}$$

as required. It must be satisfy Laplace's equation since it is the real part of the function $\log(z-1)$, which is analytic except at $z = 1$.

2. Find the values of $(1+i)^i$, as well as the principal value.

$$(1+i)^i = \exp\left(i \log(1+i)\right)$$
$$= \exp\left[i \log\left(\sqrt{2} e^{i\pi/4}\right)\right]$$
$$= \exp\left[i\left(\ln\sqrt{2} + i\left(\frac{\pi}{4} + 2k\pi\right)\right)\right], \, k \in \mathbb{Z}$$
$$= \exp\left(-\frac{\pi}{4} - 2k\pi\right) e^{i \ln\sqrt{2}}, \, k \in \mathbb{Z}. \qquad \text{(B.30)}$$

The principal value is obtained by using Log instead of log, giving

$$\text{P.V.}(1+i)^i = e^{-\pi/4} e^{i \ln\sqrt{2}}. \qquad \text{(B.31)}$$

3. Let $u(x,y) = \ln(x^2 + y^2)$, $x^2 + y^2 > 0$. Show that u is harmonic, and find its harmonic conjugate.

$$\partial_x u = \frac{2x}{x^2 + y^2} \implies \partial_x^2 u = \frac{2}{x^2 + y^2} - \frac{4x^2}{(x^2 + y^2)^2}, \qquad \text{(B.32)}$$

and similarly

$$\partial_y^2 u = \frac{2}{x^2 + y^2} - \frac{4y^2}{(x^2 + y^2)^2} \qquad \text{(B.33)}$$

by symmetry. We now see that

$$(\partial_x^2 + \partial_y^2)u = \frac{4}{x^2 + y^2} - \frac{4(x^2 + y^2)}{(x^2 + y^2)^2} = 0, \qquad \text{(B.34)}$$

and so $(\partial_x^2 + \partial_y^2)u = 0$, and u is harmonic. To find the harmonic conjugate, we need to find $v(x,y)$ such that $\partial_x u = \partial_y v$, and $\partial_y u = -\partial_x v$. We have

$$\partial_x v = -\frac{2y}{x^2 + y^2} \implies v = -2\tan^{-1}\left(\frac{x}{y}\right) + f(y), \qquad \text{(B.35)}$$

where $f(y)$ is an arbitrary function of y (including possible constants). Similarly,

$$\partial_y v = \frac{2x}{x^2 + y^2} \implies v = 2 \tan^{-1}\left(\frac{y}{x}\right) + g(x) \qquad (B.36)$$

with $g(x)$ being another arbitrary function, this time of x (including possible constants). Recall from trigonometry that

$$\tan^{-1}(1/x) + \tan^{-1}(x) = \frac{\pi}{2}, \qquad (B.37)$$

and so we see that the harmonic conjugate to $u(x, y)$ is

$$v(x, y) = 2 \tan^{-1}\left(\frac{y}{x}\right) + C \qquad (B.38)$$

for some arbitrary constant C.

B.4 Complex Integration I

Let C be the contour, consisting of the following four pieces:

1. From -2 to -1 along a straight line.

2. From -1 to $+1$ along the semicircular arc (radius 1, centered at 0) lying above the real axis.

3. From 1 to 2 along a straight line.

4. From 2 to -2 along the semicircular arc (radius 2, centered at 0) lying above the real axis.

Compute

$$\int_C dz \frac{z}{z^*}. \qquad (B.39)$$

Along leg 1, we have

$$\int_{-2}^{-1} dx \frac{x}{x} = 1. \qquad (B.40)$$

Along leg 2, we have $z = e^{i\theta}$, and

$$\int_\pi^0 d\theta \frac{dz}{d\theta} \frac{e^{i\theta}}{e^{-i\theta}} = i \int_\pi^0 d\theta\, e^{i3\theta} = \frac{1}{3}(1 - e^{i3\pi}) = \frac{2}{3}. \qquad (B.41)$$

Along leg 3, we have once again

$$\int_1^2 dx \frac{x}{x} = 1. \qquad (B.42)$$

Finally, for leg 4, we have $z = 2e^{i\theta}$, and

$$\int_0^\pi d\theta \frac{dz}{d\theta} \frac{2e^{i\theta}}{2e^{-i\theta}} = 2i \int_0^\pi e^{i3\theta} = \frac{2}{3}(e^{i3\pi} - 1) = -\frac{4}{3}, \qquad (B.43)$$

with the result being very similar to above. Therefore,

$$\int_C dz \frac{z}{z^*} = \frac{4}{3}. \qquad (B.44)$$

B.5 Complex Integration II

Let's evaluate

$$\int_C dz\, z^{a-1} \tag{B.45}$$

for some nonzero real number a, where C is the positively oriented circle $z = Re^{i\theta}$, with $-\pi \leq \theta \leq \pi$. We choose the principal branch with $-\pi < \arg(z) < \pi$.

The integral is simply

$$
\begin{aligned}
\int_{-\pi}^{\pi} d\theta\, \frac{dz}{d\theta}\, z^{a-1} &= \int_{-\pi}^{\pi} d\theta\, iRe^{i\theta} \cdot R^{a-1} e^{i(a-1)\theta} \\
&= iR^a \int_{-\pi}^{\pi} d\theta\, e^{ia\theta} \\
&= iR^a \left[\frac{e^{ia\theta}}{ia} \right]_{-\pi}^{\pi} \\
&= \frac{R^a}{a} \left(e^{ia\pi} - e^{-ia\pi} \right) \\
&= \frac{R^a}{a} \left(2i \sin(a\pi) \right) \\
&= \frac{2i}{a} R^a \sin(a\pi)\,.
\end{aligned}
\tag{B.46}
$$

References

[1] Philippe G. Ciarlet and Cristinel Mardare. "On the Brachistochrone Problem". In: *Communications in Mathematical Analysis and Applications* 1.1 (2022), pp. 213–240. ISSN: 2790-1939. DOI: https://doi.org/10.4208/cmaa.2021-0005. URL: http://global-sci.org/intro/article_detail/cmaa/20161.html.

[2] Máximo Bañados and Ignacio A. Reyes. "A short review on Noether's theorems, gauge symmetries and boundary terms". In: *Int. J. Mod. Phys. D* 25.10 (2016), p. 1630021. DOI: 10.1142/S0218271816300214. arXiv: 1601.03616 [hep-th].

[3] M. R. Flannery. "The enigma of nonholonomic constraints". In: *American Journal of Physics* 73.3 (Mar. 2005), pp. 265–272. ISSN: 0002-9505. DOI: 10.1119/1.1830501. eprint: https://pubs.aip.org/aapt/ajp/article-pdf/73/3/265/7531446/265_1_online.pdf. URL: https://doi.org/10.1119/1.1830501.

[4] July 2016. URL: https://mathematica.stackexchange.com/questions/77855/finding-basis-of-isotropic-tensors-of-rank-n.

[5] L.H. Ryder. *Quantum Field Theory*. Cambridge University Press, 1996. ISBN: 9780521478144. URL: https://books.google.com/books?id=nnuW_kVJ500C.