

# A partial description of the chromatic support of $\mathbb{E}_n$ -algebras

## Abstract

For a commutative ring spectrum, using power operations, Hahn proved that its chromatic support is an interval containing zero. A natural question is how the chromatic support of  $\mathbb{E}_n$ -algebras looks like. In this article, we give a partial description of the problem.

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# 1 Introduction

Throughout, we fix a prime number  $p$  and  $0 < m < h$  in  $\mathbb{Z}$ .

In [Hah22], Hahn proved that the chromatic support of an  $\mathbb{E}_\infty$ -algebra is an interval containing zero. In other words, we have the following theorem.

**Theorem 1.1** (Hahn). *Suppose  $R \in \text{CAlg}(\text{Sp})$ . If  $R$  is  $K(m)$ -acyclic, then  $R$  is also  $K(m+1)$ -acyclic.*

**Remark 1.2.** This theorem inspires us to define the **chromatic height** for commutative ring spectra to be the upper bound of the chromatic support, which provides a well-behaved notion for the chromatic complexity. Using the notion of chromatic height, we are able to formulate the chromatic redshift conjecture for commutative ring spectra, which was solved in the series of works [BSY22], [CMNN22], [LMMT24] and [Yua21]. ┘

A natural question is that

**Question.** *What does the chromatic support of an  $\mathbb{E}_n$ -algebra look like?*

We will give a partial answer to this problem here (Corollary 4.5).

**Theorem 1.3.** *Let  $0 < m < h$ . Suppose  $R \in \text{Alg}_{\mathbb{E}_{2n}}(\text{Sp})$  and  $R$  is  $K(m)$ -acyclic. Thus,  $u_m^k = 0$  in  $\pi_0(L_{K(h)}(R \otimes E_h)/(p, u_1, \dots, u_{m-1}))$  for some  $k \in \mathbb{Z}_{\geq 0}$ . If  $k(p^h - p^{m+1} + p - 1) \leq n$ , then  $R$  is  $K(h)$ -acyclic. In particular, if  $k(p - 1) \leq n$ , then  $R$  is  $K(m+1)$ -acyclic.*

The proof of the theorem is a simple combination of the techniques used in the two papers by Hahn ([Hah17] and [Hah22]). In particular, we generalize the method in [Hah17] of using power operations on the universal  $\mathbb{E}_n$ -torsion-algebra to odd primes via a filtration on  $BC_p$ . After that, we conclude by computing the power operation depending on the result [Hah22, Proposition 4.5].

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## 2 A filtration on $BC_p$

Suppose  $n \in \mathbb{Z}_{\geq 1}$  and  $\mu_p \in \mathbb{C}^*$  be a primitive  $p$ -th root of unity. We have a  $C_p$ -action on  $S^{2n-1} \hookrightarrow \mathbb{C}^n$  by multiplication by  $\mu_p$ . Let  $BC_p^{(2n)}$  be the homotopy cofiber of the canonical map  $S^{2n-1} \rightarrow S_{hC_p}^{2n-1}$  and  $BC_p^{(2n-1)} := S_{hC_p}^{2n-1}$ . When  $p = 2$ , we have  $BC_p^{(n)} \simeq \mathbb{RP}^n$ .

Let  $S^{2n-1} \rightarrow S^{2n+1}$  be the natural inclusion induced from  $\mathbb{C}^n \rightarrow \mathbb{C}^{n+1}$ . This map is  $C_p$ -equivariant, so it induces a map  $S_{hC_p}^{2n-1} \rightarrow S_{hC_p}^{2n+1}$  and further a diagram in  $\mathcal{S}_*$

$$\begin{array}{ccccc} S^{2n-1} & \longrightarrow & S_{hC_p}^{2n-1} & & \\ \downarrow & & \downarrow & \searrow & \\ * & \longrightarrow & BC_p^{(2n)} & \dashrightarrow & S_{hC_p}^{2n+1} \end{array} \quad (1)$$

where the dashed arrow exists because  $\pi_{2n-1}(S_{hC_p}^{2n+1}) = 0$ . Composing with the inclusion  $S_{hC_p}^{2n+1} \rightarrow S_{hC_p}^\infty \simeq BC_p$ , we get a map  $BC_p^{(n)} \rightarrow BC_p$  for each  $n \in \mathbb{Z}_{\geq 1}$ . Let  $BC_{p,n+1}$  denote the cofiber of this map.

Similarly, if  $a \in S^{2n-1}$ , then  $(a, \mu_p a, \mu_p^2 a, \dots, \mu_p^{p-1} a)$  is a point in  $\text{Conf}_p(\mathbb{R}^{2n})$  by identifying  $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ , so we get a  $C_p$ -equivariant map  $S^{2n-1} \rightarrow \text{Conf}_p(\mathbb{R}^{2n})$ , where  $C_p$  acts on  $\text{Conf}_p(\mathbb{R}^{2n})$  by translation of coordinates. Since  $\pi_{2n-1}(\text{Conf}_p(\mathbb{R}^{2n+1})) = 0$ , we get the following diagram

$$\begin{array}{ccccccc} S^{2n-1} & \longrightarrow & S_{hC_p}^{2n-1} = BC_p^{(2n-1)} & \xrightarrow{\gamma_{2n-1}} & \text{Conf}_p(\mathbb{R}^{2n})_{hC_p} & & \\ \downarrow & & \downarrow & & \downarrow & & \\ * & \longrightarrow & BC_p^{(2n)} & \dashrightarrow^{\gamma_{2n}} & \text{Conf}_p(\mathbb{R}^{2n+1})_{hC_p} & & \end{array}$$

Now suppose  $E$  is a Morava E-theory of height  $h > 0$  with an orientation  $T \in E^2[\mathbb{CP}^\infty]$ . From the fiber sequence

$$S^1 \rightarrow S_{hC_p}^{2n-1} \rightarrow \mathbb{CP}^{n-1}$$

and the Gysin sequence, we obtain that

$$E^0(S_{hC_p}^{2n-1}) \cong E_0[[T]]/([p](T), T^n) \quad \text{and} \quad E^1(S_{hC_p}^{2n-1}) \cong E_0.$$

Since  $BC_p^{(2n)}$  is the cofiber of the map  $S^{2n-1} \rightarrow S_{hC_p}^{2n-1}$ , we obtain an exact sequence

$$0 \rightarrow E^{-1}(BC_p^{(2n)}) \rightarrow E_0 \xrightarrow{p} E_0 \rightarrow E^0(BC_p^{(2n)}) \rightarrow E_0[[T]]/([p](T), T^n) \rightarrow 0.$$

Thus,  $E^0(BC_p^{(2n)})$  fits into a short exact sequence

$$0 \rightarrow E_0/p \rightarrow E^0(BC_p^{(2n)}) \rightarrow E_0[[T]]/([p](T), T^n) \rightarrow 0.$$

Since  $E^0(S_{hC_p}^{2n+1}) \cong E_0[[T]]/([p](T), T^{n+1})$  also fits into this short exact sequence by sending  $1 \in E_0/p$  to  $T^n$ , we may compare to  $S_{hC_p}^{2n+1}$  via the Diagram 1 and learn that

$$E^0(BC_p^{(2n)}) \cong E_0[[T]]/([p](T), T^{n+1}) \quad \text{and} \quad E^1(BC_p^{(2n)}) = 0.$$

Therefore,

$$E^0(BC_{p,2n+1}) \cong T^{n+1} E_0[[T]]/([p](T)) \quad \text{and} \quad E^1(BC_{p,2n+1}) = 0,$$

which are free  $E_0$ -modules. It follows from [Rez09, 3.11 and 3.17] that the **completed  $E$ -homology**

$$E_0^\wedge(BC_{p,2n+1}) := \pi_0(L_{K(h)} E \otimes (\Sigma_+^\infty BC_{p,2n+1})) \cong E^0(BC_{p,n+1})^\vee. \quad (2)$$

### 3 Power operations in universal torsion $\mathbb{E}_n$ -algebras

In this section, we recall some constructions in [Hah17, Section 2 and 5].

Let  $A$  be an  $\mathbb{E}_\infty$ -ring spectrum and  $\text{Alg}_{\mathbb{E}_n,A}(\text{Sp})$  be the  $\infty$ -category of  $\mathbb{E}_n$ - $A$ -algebras. Let  $\text{BGL}_1(A)$  be the maximal groupoid of the full subcategory of  $\text{Mod}_A$  consisting only  $A$ .

**Definition 3.1.** Suppose  $x \in \pi_0(A)$  such that  $1 + x \in \pi_0(A)^\times$ . For each  $n \geq 0$ , let

$$\widetilde{1+x}: \Omega^n S^{n+1} \rightarrow \text{BGL}_1(A) \quad (3)$$

be the  $n$ -fold loop map adjoint to the map  $1+x: S^1 \rightarrow \text{BGL}_1(A)$ . Let  $R_{n,x}^A$  be the Thom spectrum associated to this map, i.e.,  $R_{n,x}^A$  is the colimit of the map  $\widetilde{1+x}$  in  $\text{Mod}_A$ . According to [ACB19, Corollary 3.2],  $R_{n,x}^A$  promotes to an element in  $\text{Alg}_{\mathbb{E}_n,A}(\text{Sp})$ .  $\lrcorner$

**Theorem 3.2** ([ACB19, Theorem 4.10]). *Suppose  $B \in \text{Alg}_{\mathbb{E}_n,A}(\text{Sp})$ . If  $x = 0 \in \pi_0(B)$ , then we have*

$$\text{Map}_{\text{Alg}_{\mathbb{E}_n,A}(\text{Sp})}(R_{n,x}^A, B) \simeq \Omega^{\infty+1} B.$$

*Otherwise, the mapping space is contractible.*

When  $x = 0 \in \pi_0(B)$ , then  $\Omega^{\infty+1} B$  is homotopic to the space of homotopies between 0 and  $x$  in  $\text{Map}_{\text{Mod}_A}(A, B)$ . In light of this theorem, we will say  $R_{n,x}^A$  is a **universal  $\mathbb{E}_n$ - $A$ -algebra in which  $x = 0$** .

Now we have the following diagram by looping once of [Eq. \(3\)](#).

$$\begin{array}{ccccccc}
BC_p^{(n)} & \longrightarrow & BC_p & & & & \\
\downarrow \gamma_n & & \downarrow \simeq & & & & \\
\text{Conf}_p(\mathbb{R}^{n+1})_{hC_p} & \longrightarrow & \text{Conf}_p(\mathbb{R}^\infty)_{hC_p} & & & & \\
\downarrow & & \downarrow & & & & \\
\Omega^{n+1}S^{n+1} & \longrightarrow & \Omega^\infty S^\infty & \xrightarrow{\Omega^\infty(1+x)} & GL_1(A) & \xrightarrow{GL_1(\text{unit})} & GL_1(R_{n,x}^A) \\
& & & & \downarrow & & \downarrow \\
& & & & \Omega^\infty A & \xrightarrow{\Omega^\infty(\text{unit})} & \Omega^\infty R_{n,x}^A
\end{array}$$

Note that the composite map  $BC_p \rightarrow \Omega^\infty A$  is exactly  $P(1+x)$ . We get the following lemma.

**Lemma 3.3.** *The composition*

$$\Sigma_+^\infty BC_p^{(n)} \rightarrow \Sigma_+^\infty BC_p \xrightarrow{P(1+x)} A \rightarrow R_{n,x}^A$$

is  $1 \in (R_{n,x}^A)^0(BC_p^{(n)})$ .

**Corollary 3.4.** *The image of  $P(x) \in \text{Map}(BC_p^{(2n)}, R_{n,x}^A)$  is null-homotopic, so we have the following diagram.*

$$\begin{array}{ccccc}
& & A & \xrightarrow{\text{unit}} & R_{n,x}^A \\
& & \uparrow P(x) & & \uparrow \\
\Sigma_+^\infty BC_p^{(n)} & \longrightarrow & \Sigma_+^\infty BC_p & \longrightarrow & BC_{p,n+1}
\end{array}$$

*Proof.* In light of the last lemma, it suffices to show that  $P(1+x) - 1$  is homotopic to  $P(x)$  after composing with the unit map  $A \rightarrow R_{n,x}^A$ . Note that  $P(1+x)$  is computed by the composition

$$\Sigma_+^\infty BC_p \simeq \mathbb{S}_{hC_p}^{\otimes p} \xrightarrow{\Delta^{\otimes p}} (\mathbb{S} \oplus \mathbb{S})_{hC_p}^{\otimes p} \xrightarrow{(1+x)^p} A_{hC_p}^{\otimes p} \rightarrow A.$$

We have the following identification

$$(\mathbb{S} \oplus \mathbb{S})^{\otimes p} \simeq \mathbb{S} \oplus \text{Ind}(X_1) \oplus \text{Ind}(X_2) \cdots \text{Ind}(X_{p-1}) \oplus \mathbb{S},$$

where each  $X_i$  is a direct sum of  $\mathbb{S}$ . After taking the  $C_p$ -orbit, the map  $X_i \rightarrow A$  is divisible by  $x$ . Furthermore, the maps  $\Sigma_+^\infty BC_p \rightarrow A$  corresponding to the first and the last direct summands are 1 and  $P(x)$  respectively. Since  $x = 0$  in  $R_{n,x}^A$ , we find out that  $P(1+x) - 1$

is homotopic to  $P(x)$  in  $R_{n,x}^A$ . □

## 4 Chromatic support of $\mathbb{E}_n$ -algebras

In this section, let  $E$  be a Morava E-theory of height  $h$ , so that

$$\pi_*(E) \cong W(\overline{\mathbb{F}}_p)[[u_1, \dots, u_{h-1}]] [u^{\pm 1}],$$

where  $|u| = -2$ .

Recall (from, e.g., the formula at the top of [GHMR05, p. 788]) that for each  $0 \leq k < h$ ,  $[p](T)$  satisfies the equation

$$[p](T) \equiv u_k z^{p^k} \pmod{p, \dots, u_{k-1}, z^{p^k+1}}.$$

Thus, we introduce the following notion.

**Definition 4.1.** For  $0 \leq k < h$ , let  $g_k(T) \in E_0[[T]]$  be a lift of  $\bar{g}_k(T) \in E_0/(p, \dots, u_{k-1})[[T]]$  such that  $[p](z) = z^{p^k} \bar{g}_k(T)$  in  $E_0/(p, \dots, u_{k-1})[[T]]$ .

Let  $I_k := (p, \dots, u_{k-1}, u_{k+1}, \dots, u_{h-1}) \subset E_0$ . ⌋

**Corollary 4.2.** Taking reduced completed  $E$ -homology in Corollary 3.4, we have the following diagram in  $\text{Mod}_{E_0}^\heartsuit$  from the results in Section 2.

$$\begin{array}{ccc} E_0 & \xrightarrow{\quad} & \pi_0(L_{K(h)} R_{2n,x}^E) \\ \uparrow P(x) & & \uparrow \text{---} \\ (TE_0[[T]]/[p](T))^\vee & \longrightarrow & (T^{n+1}E_0[[T]]/[p](T))^\vee \end{array}$$

Furthermore, modding out  $I_m$ , we get the following diagram.

$$\begin{array}{ccc}
& K[[u_m]] & \longrightarrow \pi_0(L_{K(h)}R_{2n,x}^E/I_m) \\
& \uparrow & \uparrow \\
& E_0 & \longrightarrow \pi_0(L_{K(h)}R_{2n,x}^E) \\
& \uparrow P(x) & \uparrow \\
(T E_0[[T]]/[p](T))^\vee & \longrightarrow & (T^{n+1} E_0[[T]]/[p](T))^\vee \\
\uparrow & & \uparrow \\
(T K[[u_m, T]]/[p](T))^\vee & \longrightarrow & (T^{n+1} K[[u_m, T]]/[p](T))^\vee \\
\uparrow & & \uparrow \\
(T K[[u_m, T]]/g_m(T))^\vee & \longrightarrow & (T^{n+1} K[[u_m, T]]/g_m(T))^\vee
\end{array}$$

$\bar{P}(x)$  (curved arrow from bottom left to top left)

By Weierstrass preparation theorem, there exists a unique monic polynomial  $g(T) \in K[[u_m]][[T]]$  such that  $g_m(T)$ , modding out  $I_m$ , is a unit multiple of  $g(T)$  in  $K[[u_m]][[T]]$ . Moreover, the degree of  $g(T)$  is  $d := p^h - p^m$  and the constant term is  $u_m$  times a unit in  $K[[u_m]]$  (cf. [Hah22, Proposition 4.2]). Suppose  $g(T) = T^{p^h-p^m} - u_m u(T)$  for some unit  $u(T) \in K[[T]]$ . Now the above diagram can be written as the following.

$$\begin{array}{ccc}
K[[u_m]] & \longrightarrow & \pi_0(L_{K(h)}R_{2n,x}^E/I_m) \\
\bar{P}(x) \uparrow & & \uparrow \\
(T K[[u_m, T]]/(g(T)))^\vee & \longrightarrow & (T^{n+1} K[[u_m, T]]/(g(T)))^\vee
\end{array}$$

We pick a basis  $\delta_T, \delta_{T^2}, \dots, \delta_{T^d}$  for  $\left(\frac{TK[[u_m, T]]}{(g(T))}\right)^\vee$  over  $K[[u_m]]$  and a basis  $\delta_{T^{n+1}}, \dots, \delta_{T^{n+d}}$  for  $\left(\frac{T^{n+1}K[[u_m, T]]}{(g(T))}\right)^\vee$ , where  $\delta_{T^i}$  denotes the dual basis of  $T^i$ . Thus, there is a matrix  $Q \in M_d(K[[u_m]])$  such that  $(\delta_T, \dots, \delta_{T^d})$  are mapped to  $(\delta_{T^{n+1}}, \dots, \delta_{T^{n+d}})Q$ . Note that for any  $q \in \mathbb{Z}_{\geq 0}$  and  $1 \leq r \leq d$ ,  $T^{qd+r} = u_m^q u^q(T) T^r$  in  $\frac{TK[[u_m, T]]}{(g(T))}$ . We can read off from this that  $v_{u_m}(Q_{qd+r,r}) = q$ ,  $v_{u_m}(Q_{qd+r,i}) \geq q$  for  $i > r$  and  $v_{u_m}(Q_{qd+r,i}) \geq q+1$  for  $i < r$ , where  $v_{u_m}$  is the  $u_m$ -valuation on  $K[[u_m]]$ .

Suppose  $P(x) = a_1 T + a_2 T^2 + \dots + a_d T^d \in \frac{TK[[u_m, T]]}{(g(T))}$ . Thus,  $\bar{P}(x)(\delta_{T^i}) = a_i$ . Suppose for each  $i$ ,  $\delta_{T^{n+i}}$  is mapped to  $b_{n+i}$  under the map

$$(T^{n+1} K[[u_m, T]]/(g(T)))^\vee \rightarrow \pi_0(L_{K(h)}R_{2n,u_m^k}^E/I_m).$$

We have

$$(a_1, \dots, a_d) = (b_{n+1}, \dots, b_{n+d})Q$$

in  $\pi_0(L_{K(h)}R_{2n, u_m^k}^E/I_m)$ . Let  $n+1 = qd + r$  for some  $q \in \mathbb{Z}_{\geq 0}$  and  $1 \leq r \leq d$ . By the above computation, we learn that  $u_m^q \mid a_i$  for all  $i$  and  $u_m^{q+1} \mid a_i$  for  $1 \leq i < r$  in  $\pi_0(L_{K(h)}R_{2n, u_m^k}^E/I_m)$ .

Now if we let  $x = u_m^k$ , then we have the following computational result for  $\overline{P}(u_m^k)$ .

**Proposition 4.3.** *We have*

$$\overline{P}(u_m^k) = T^{k(p^h - p^{m+1} + p - 1)}U(T)$$

in  $K[[u_m]][[T]]/(g(T))$  for some unit  $U(T)$ .

*Proof.* By the same proof of [Hah22, Proposition 4.5], we have that

$$\overline{P}(u_m^k) = \left( u_m \left( \frac{1}{T^{p-1}\tilde{u}(T)} \right)^{p^m-1} \right)^k = T^{k(p^h - p^{m+1} + p - 1)}\tilde{u}^{-k(p^m-1)}(T)u^{-k}(T)$$

in  $K[[u_m]][[T]]/(g(T))$  for some unit  $\tilde{u}(T)$ . □

**Theorem 4.4.** *When  $0 < m < h$  and  $k(p^h - p^{m+1} + p - 1) \leq n$ ,  $L_{K(h)}R_{2n, u_m^k}^E \simeq 0$ .*

*Proof.* Suppose  $k(p^h - p^{m+1} + p - 1) = q'd + r'$  for some integer  $q'$  and  $1 \leq r' \leq d$ . Similar to above, we have that  $v_{u_m}(a_{r'}) = q'$ . Hence, there is a unit  $u'_{r'} \in K[[u_m]]^\times$ , such that  $a_{r'} = u_m^{q'}u'_{r'}$ . Therefore, if  $k(p^h - p^{m+1} + p - 1) + 1 \leq n + 1$ , then  $u_m^{q'+1} \mid u_m^{q'}u'_{r'}$  in  $\pi_0(L_{K(h)}R_{2n, u_m^k}^E/I_m)$ , which implies that  $u_m^{q'} = 0$  in  $\pi_0(L_{K(h)}R_{2n, u_m^k}^E/I_m)$  since  $u_m$  is in the maximal ideal. When  $m > 0$ ,  $p^h - p^{m+1} + p - 1 < d$ , so  $q' < k$ . By infinite descent, we learn that  $L_{K(h)}R_{2n, u_m^k}^E/I_m \simeq 0$ . Since  $L_{K(h)}R_{2n, u_m^k}^E$  is  $K(h)$ -local,  $L_{K(h)}R_{2n, u_m^k}^E \simeq 0$ . □

**Corollary 4.5.** *Let  $0 < m < h$ . Suppose  $R \in \text{Alg}_{\mathbb{E}_{2n}}(\text{Sp})$  and  $R$  is  $K(m)$ -acyclic. Thus,  $u_m^k = 0$  in  $\pi_0(L_{K(h)}(R \otimes E)/(p, u_1, \dots, u_{m-1}))$  for some  $k \in \mathbb{Z}_{\geq 0}$ . If  $k(p^h - p^{m+1} + p - 1) \leq n$ , then  $R$  is  $K(h)$ -acyclic. In particular, if  $k(p - 1) \leq n$ , then  $R$  is  $K(m + 1)$ -acyclic.*

*Proof.* Since  $K(m) \otimes L_{K(h)}(R \otimes E)$  is a module over  $K(m) \otimes R$ ,  $L_{K(h)}(R \otimes E)$  is also  $K(m)$ -acyclic. Therefore,  $L_{K(h)}(R \otimes E)$  is  $T(m)$ -acyclic for some telescope  $T(m)$  by [LMMT24, Lemma 2.3], which implies that  $u_m^k = 0$  in  $\pi_0(L_{K(h)}(R \otimes E)/(p, u_1, \dots, u_{m-1}))$ . Therefore, there is an  $\mathbb{E}_{2n}$ -algebra map  $L_{K(h)}R_{2n, u_m^k}^E \rightarrow L_{K(h)}(R \otimes E)$  classifying the nullhomotopy of  $u_m^k$  by Theorem 3.2. We now conclude by the previous theorem. □

**Remark 4.6.** When  $h$  tends to infinity, the lower bound  $k(p^h - p^{m+1} + p - 1)$  also tends to infinity. However, the Hopkins–Mahowald theorem says that  $R_{2,p} \simeq \mathbb{F}_p$ , which is  $K(h)$ -acyclic for all  $h \geq 0$ . Thus, we suspect the above lower bound is far from being sharp.  $\lrcorner$

**Remark 4.7.** The above proof does not work when  $m = 0$ . However,  $\overline{P}(p^k)$  is more computable as in [Hah17, Lemma 6.2]. Thus, one direction of future works is trying to give an answer to [Hah17, Question 4] depending on computations of  $\overline{P}(p^k)$ .  $\lrcorner$

**Remark 4.8.** Another future direction is determining the exponent of  $u_h^{k'}$  such that  $u_h^{k'}$  vanishes in  $R$ . Therefore, we can give a better description of the chromatic support of  $\mathbb{E}_n$ -algebra. In particular, if  $u_{m+1}^k$  also vanishes in  $R$ , then the chromatic support of  $\mathbb{E}_n$ -algebras are also an interval containing 0 using the latter part of Corollary 4.5, which is same as the  $\mathbb{E}_\infty$ -case and what expected by the Hopkins–Mahowald theorem.  $\lrcorner$

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