Explicit Local Class Field Theory à la Lubin and Tate with an Application to Algebraic Topology

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Abstract

This is part of my senior thesis I am working on. The thesis has two goals. Firstly, we state and prove explicit local class field theory à la Lubin and Tate. Secondly, we prove the existence and uniqueness of Ando's norm-coherent coordinates on deformations of formal groups in the context of structured multiplicative orientations in algebraic topology. The first part is finished, while some algebraic topological backgrounds need to filled into the second part.

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1 Motivation and History of Local Class Field Theory

The motivation of class field theory is to generate all the Galois extensions of a field. In particular, local class field theory wants to generate all the Galois extensions of a local field.

Historically, local class field theory arises from the problem proposed by Emil Artin(1929) that whether one can generalize the norm residue symbol to arbitrary fields that do not contain n-th roots of unity [FLR14]. Helmut Hasse(1930) solved this problem using the global Artin reciprocity law. For an abelian extension L/K, where K, L may not be local fields, and $\alpha \in K^*$ and v a place of K, the new norm residue symbol $(\alpha, L/K)_v$ is an element in the decomposition group of any $w \mid v$ [Con] (Since L/K is abelian, all decomposition groups are the same). This led Hasse to the discovery of local class field theory. We first need a lemma to see this.

Lemma 1.1. *Suppose F/K_v is a field extension for some number field K and a finite place v of K. Then there exists a number field L/K such that $F = LK_v$, $[L : K] = [F : K_v]$ and $F = L_w$ for some place w of L extending v.*

Proof. Suppose $F = K_v(\alpha)$ and $f \in K_v[X]$ is the minimal polynomial of α over K_v. By [Hu21, Corollary 3.2.16], there is a separable and irreducible polynomial $g \in K[X]$ close enough to f with $\deg(g) = \deg(f)$ such that $K_v(\beta) = K_v(\alpha)$ for some root β of g. Then $[F : K_v] = \deg(f) = \deg(g) = [K(\beta) : K]$. Since F is a finite extension of a complete field K_v, F is itself complete. Since $F \supset L$, F is a completion of L with respect to some valuation w of L. $\square$

Here is how local class field theory shows up: Given an abelian extension F/K_v, there exists a field extension L/K such that $F = LK_v$, $[L : K] = [F : K_v]$ and $F = L_w$ for some place w of L extending v by the lemma. Thus, $\mathrm{Gal}(F/K_v) \cong \mathrm{Gal}(L_w/K_v)$. Note that there is a natural inclusion $\mathrm{Gal}(L_w/K_v) \to \mathrm{Gal}(L/K)$ by $\sigma \mapsto \sigma|_L$, mapping $\mathrm{Gal}(L_w/K_v)$ to the decomposition group of $w \mid v$. For any $\alpha \in K^*$, let $(\alpha, F/K_v)$ be the image of $(\alpha, L/K)_v$ in $\mathrm{Gal}(F/K_v)$. Therefore, we get a homomorphism

$$K^* \to \mathrm{Gal}(F/K_v) \quad \alpha \mapsto (\alpha, F/K_v)$$

The definition of $(\alpha, L/K)_v$ implies that $(\alpha, L/K)_v = Id$ when $v(\alpha)$ is large enough [Con]. Thus, the above map can be extended to $K_v^* \mapsto \mathrm{Gal}(F/K_v)$, which is now called the local Artin map.

As discussed above, local class field theory is derived from the global class field theory originally and there is no explicit description of the local Artin map. The significance of the proof by Lubin and Tate is to give an explicit description of the local Artin map.

2 Main Theorems of Local Class Field Theory and Proof by Lubin-Tate Formal Group Laws

2.1 Statements of the Main Theorems

By a local field, we mean a field K that is one of the following case:

1. $K = \mathbb{R}$ or $K = \mathbb{C}$ with the usual absolute value.
2. K is complete with respect to a discrete valuation whose valuation ring has finite residue field.

By Proposition 4.1.4 in [Hu21], the latter case is either a finite extension of $\mathbb{Q}_p$ or a finite extension of $\mathbb{F}_p((t))$. The former one is called **archimedean** while the latter case is called **non-archimedean**.

Let K be a local field, $K^{al} \supset K^{ab} \supset K^{un}$ be its separable and abelian closure respectively. Let $\mathcal{O}_K$ be the integer ring of K and $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_K$ and $k = \mathcal{O}_K/\mathfrak{m}$ is the residue field with q elements, where q is a power of a prime number p. Suppose L/K is a finite extension, $\mathrm{Nm}_{L/K}(x)$ is the norm of $x \in L$ with respect to L/K.

Let $\mathrm{Gal}(K^{ab}/K)$ be the Galois group of K^{ab}/K. We assign Krull topology to $\mathrm{Gal}(K^{ab}/K)$, i.e., $\mathrm{Gal}(K^{ab}/E)$ forms a fundamental system of neighborhoods of 1 in $\mathrm{Gal}(K^{ab}/K)$, where E runs through all finite abelian extensions of K.

The main theorems of the abelian local class field theory are the following:

Theorem 2.1 (Local Reciprocity Law)**.** *For any non-archimedean local field K, there exists a unique homomorphism*

$$\phi_K \colon K^* \to \mathit{Gal}(K^{ab}/K)$$

satisfying:

(a) For any uniformizer π of K, $\phi_K(\pi)$ is the Frobenius element of $\mathit{Gal}(K^{un}/K)$ under the restriction $\mathit{Gal}(K^{ab}/K) \to \mathit{Gal}(K^{un}/K)$.

(b) *For any finite abelian extension L of K, there is an exact sequence:*

$$1 \to Nm_{L/K}(L^*) \to K^* \to Gal(L/K) \to 1$$

where the latter map is the composition of ϕ_K and the restriction map. This induces an isomorphism

$$\phi_{L/K} \colon K^*/Nm_{L/K}(L^*) \to Gal(L/K)$$

In particular, $\big(K^* : Nm_{L/K}(L^*)\big) = [L : K]$.

The map $\phi_{L/K}$ is then called the ***local Artin map****.*

The following corollary can be deduced from Theorem 2.1.

Corollary 2.2. *Let K be a non-archimedean local field. Assume that Theorem 2.1 is true. Then*

(a) *The map $L \mapsto Nm(L^*)$ is as order-reversing bijection between abelian extensions of K and norm groups in K^*.*

(b) *$Nm\big((L \cdot L')^*\big) = Nm(L^*) \cap Nm(L'^*)$.*

(c) *$Nm\big((L \cap L')^*\big) = Nm(L^*) \cdot Nm(L'^*)$*

(d) *Every subgroup of K^* containing a norm group is a norm group itself.*

Proof. See [Mil20] Corollary 1.2. □

Theorem 2.3 (Local Existence Theorem)**.** *The norm groups in K^* are exactly the open subgroups of finite index.*

Thus, the remaining of this section is to prove Theorem 2.1 and Theorem 2.3.

The following remarks of the main theorems are essential to the proof. Recall in the finite case, if L/K is a totally ramified extension of degree n and F/K is an unramified extension of degree m, then LF/K is of degree mn (Here we do not require K, L, F to be local fields). Actually K^{ab} can also be decomposed into the composition of a maximal unramified extension and a maximal totally ramified extension as follows.

Given the isomorphisms

$$\phi_{L/K} \colon K^*/\mathrm{Nm}(L^*) \to \mathrm{Gal}(L/K) \cong \mathrm{Gal}(K^{ab}/K)/\mathrm{Gal}(K^{ab}/L)$$

for each finite abelian extension L of K, by passing to the limit we get an isomorphism:

$$\hat{\phi}_K \colon \widehat{K^*} \to \mathrm{Gal}(K^{ab}/K)$$

where $\widehat{K^*}$ is the profinite completion of K^* since $\mathrm{Nm}(L^*)$ are all open subgroups of finite index in K^* by Theorem 2.3.

Now choose an uniformizer π of K. We have

$$K^* \cong U_K \times \pi^{\mathbb{Z}} \cong U_K \times \mathbb{Z}$$

Lemma 2.4. *Under the decomposition above,* $\lim_{n\in\mathbb{N}^*, m\in\mathbb{N}^*} K^*/\big((1+\mathfrak{m}^n)\times m\mathbb{Z}\big) \cong \widehat{K^*}$.

Proof. It suffices to show that for any open subgroup of finite index H in K^*, H contains some $(1+\mathfrak{m}^n)\times m\mathbb{Z}$. Since H is open and $(1+\mathfrak{m}^n)\times\{0\}$ forms a fundamental system of neighborhoods of 1 in K^*, $H \supset (1+\mathfrak{m}^n)\times\{0\}$ for some n. Moreover, H contains a $u\pi^r$ for some integer r and $u \in U_K$. Since $U_K/(1+\mathfrak{m}^n)$ is a finite group, $u^s \in (1+\mathfrak{m}^n)$ for some integer s. Therefore, $H \supset (1+\mathfrak{m}^n)\times rs\mathbb{Z}$. $\square$

Then we have

$$\widehat{K^*} \cong U_K \times \pi^{\hat{\mathbb{Z}}} \cong U_K \times \hat{\mathbb{Z}}$$

It is well-known that profinite topological groups are equivalent to compact Hausdorff totally disconnected topological groups. Since $U_K, \hat{\mathbb{Z}}$ are profinite, they are compact. Because $\widehat{K^*}$ is Hausdorff, both $U_K, \hat{\mathbb{Z}}$ are closed subgroups in $\widehat{K^*}$. Since $\mathbb{Z}$ is dense in $\hat{\mathbb{Z}}$, $\hat{\mathbb{Z}} = \overline{\mathbb{Z}}$ in $\widehat{K^*}$. Let $K_\pi = (K^{ab})^{\hat{\phi}_K(\pi)}$ and $K^{un} = (K^{ab})^{\hat{\phi}_K(U_K)}$. Then by infinite Galois theory, $\mathrm{Gal}(K^{ab}/K_\pi) = \hat{\mathbb{Z}}$ and $\mathrm{Gal}(K^{ab}/K^{un}) = U_K$. Thus, K_π is the union of finite abelian extensions L such that $\pi \in \mathrm{Nm}(L^*)$, which are totally ramified, and K^{un} is the union of finite abelian extensions L such that $\mathrm{Nm}(L^*) \supset U_K$, which are unramified. We deduce that K^{un} is the maximal unramified extension of K in K^{ab} and $K^{un} \cap K_\pi = K$. Thus, $\mathrm{Gal}(K_\pi K^{un}/K) = \mathrm{Gal}(K_\pi/K) \times \mathrm{Gal}(K^{un}/K) = U_K \times \hat{\mathbb{Z}}$. Hence, $K^{ab} = K_\pi K^{un}$.

Under such view of point, we can show the uniqueness of ϕ_K.

Lemma 2.5. *Assume that Theorem 2.3 is true. Then there exists at most one homomorphism* $\phi \colon K^* \to$ *Gal*(K^{ab}/K) *satisfying the conditions in Theorem 2.1.*

Proof. We know that $K^{ab} = K^{un}K_\pi$. If there is a ϕ satisfies the conditions in Theorem 2.1, then $\phi(\pi)|_{K^{un}}$ is the Frobenius element for any uniformizer π of K. Since K_π is fixed by $\phi(\pi)$ from above discussion, the value of $\phi(\pi)$ is determined for all uniformizer π. Because K^* is generated by uniformizers π of $\mathcal{O}_K$, the value of ϕ is uniquely determined. □

Since we know the restriction of the local Artin map on K^{un} is the Frobenius element, we may prove the existence by constructing the fields K^{un}, K_π and the restriction of local Artin map $U_K \to \mathrm{Gal}(K_\pi/K)$. Then we need to show that the composition $K_\pi K^{un}$ and the associated map ϕ_π are independent of the choice of π. Next, we show that $K_\pi K^{un} = K^{ab}$. Finally, we have to show that ϕ_π satisfies the condition (b) of Theorem 2.1.

Example 2.6. *Suppose $K = \mathbb{Q}_p$ for some prime number p and pick the uniformizer $\pi = p$. By Kummer-Dedekind Theorem, for each positive integer n, $\mathbb{Q}_p(\mu_n)/\mathbb{Q}_p$ is unramified if $(n,p) = 1$ and is totally ramified if $n = p^i$ for some positive integer i. Moreover, the Galois group Gal$\big(\mathbb{Q}_p(\mu_n)/\mathbb{Q}_p\big)$ is $(\mathbb{Z}/n\mathbb{Z})^*$. By taking the colimit, we see that the Galois groups of $\left(\bigcup_{(n,p)=1} \mathbb{Q}_p(\mu_n)\right)/\mathbb{Q}_p$ and $\left(\bigcup_{i=1}^{\infty} \mathbb{Q}_p(\mu_{p^i})\right)/\mathbb{Q}_p$ are $\hat{\mathbb{Z}}$ and $(\mathbb{Z}_p)^*$ respectively. Thus, we have*

$$(\mathbb{Q}_p)_p = \bigcup_{i=1}^{\infty} \mathbb{Q}_p(\mu_{p^i}) \qquad Q_p^{un} = \left(\bigcup_{(n,p)=1} \mathbb{Q}_p(\mu_n)\right)$$

By above discussion,

$$\mathbb{Q}_p^{ab} = \left(\bigcup_{(n,p)=1} \mathbb{Q}_p(\mu_n)\right) \cdot \left(\bigcup_{i=1}^{\infty} \mathbb{Q}_p(\mu_{p^i})\right)$$

The above method of construction Q_p^{un} applies to arbitrary local field K. Suppose $p \nmid n$, μ_n is the primitive n-th root of unity over K and $L = K(\mu_n)$. Suppose $\Phi_n(t)$ is the minimal polynomial of μ_n over K and $\overline{\Phi_n}(t)$ is the reduction of $\Phi_n(t)$ to the residue field k. Thus, $\overline{\Phi_n}(t) \mid (t^n - 1)$, so it is separable. By Hensel's Lemma, $\overline{\Phi_n}(t)$ is also irreducible. Thus, $\overline{\Phi_n}(t)$ is the minimal polynomial of $\bar{\mu_n}$ over k. Therefore,

$$[L:K] = \deg \Phi_n(t) = \deg \overline{\Phi_n}(t) = [k(\bar{\mu_n}) : k] \leqslant [l:k] \leqslant [L:K]$$

where l is the residue field of L. Hence, $[L:K] = [l:k]$ implying that L/K is unramified. By field theory, we know that $l = k(\bar{\mu_n})$ is the splitting field of $t^{q^f} - t$, where f is the smallest number such that $n \mid (q^f - 1)$. Therefore, $\left(\bigcup_{(n,p)=1} K(\mu_n)\right)/K$ is an unramified extension and has the residue field $\bar{k}$, implying that $K^{un} = \bigcup_{(n,p)=1} K(\mu_n)$.

However, we cannot simply add of roots of unity to K to construct K_π. Indeed, if $K =$

$\mathbb{F}_p((t))$, then K itself contains p^i-th roots of unity. Lubin-Tate theory generalizes this method to arbitrary local field via Lubin-Tate formal group laws. If we let $\mathbb{G}_m$ to be the multiplication formal group law on $\mathbb{Z}_p$, $\mathbb{G}_m(X,Y) = X + Y + XY$, then there exists a natural map $\mathbb{Z}_p \to \mathrm{End}(\mathbb{G}_m)$ given by the following: for any $n \in \mathbb{Z}$, $\big((1+T)^n - 1\big) \in \mathrm{End}(\mathbb{G}_m)$. This can be extended to $\mathbb{Z}_p$. For any $a \in \mathbb{Z}_p$,

$$(1+T)^a = \sum_{m \geqslant 0} \binom{a}{m} T^m \qquad \binom{a}{m} = \frac{a(a-1)\cdots(a-m+1)}{m(m-1)\cdots 1}$$

By continuity, $\binom{a}{m} \in \mathbb{Z}_p$ and $\big((1+T)^a - 1\big) \in \mathrm{End}(\mathbb{G}_m)$. Then we see that $(\mu_{p^i} - 1)$ is a p^n-torsion point. Thus, $\mathbb{Q}_p(\mu_{p^i}) = \mathbb{Q}_p(\mu_{p^i} - 1)$ can be viewed as adding p^n-torsion points in $\mathbb{Q}_p^{al}$.

2.2 Lubin-Tate Formal Group Laws

Note that for power series f, g, h, $f \circ (g + h) \neq f \circ g + f \circ h$ in general. In order to make the distribution law possible, we need to rewrite the addition. Suppose F is the new addition. Then we need $f \circ F(g,h) = F(f \circ g, f \circ h)$. We use the formal group law to capture this.

Definition 2.7 (One-Parameter Commutative Formal Group Law)**.** Let R be a commutative ring. A **one-parameter commutative formal group law** is a power series $F \in R[[X,Y]]$ such that

(a) $F(X,Y) \equiv X + Y \pmod{(X,Y)^2}$.

(b) (Associativity) $F\big(X, F(Y,Z)\big) = F\big(F(X,Y), Z\big)$.

(c) (Commutativity) $F(X,Y) = F(Y,X)$.

We can prove that with the conditions (a)(b), there exists a unique $i_F(T) \in R[[T]]$ such that $F\big(X, i_F(X)\big) = 0$.

We denote End(F) by the set of $f \in R[[X]]$ such that $f \circ F(X,Y) = F\big(f(X), f(Y)\big)$ and $f +_F g = F(f,g)$. Then we see from the beginning of this subsection that End(F) admits a ring structure with the addition $+_F$ and the multiplication $\circ$.

Definition 2.8. Let $\mathcal{F}_\pi$ be the set of $f(X) \in \mathcal{O}_K[[X]]$ such that

(a) $f \equiv \pi X \pmod{X^2}$.

(b) $f \equiv X^q \pmod{\pi}$.

Example 2.9. *Let $K = \mathbb{Q}_p$, $\pi = p$. Then $f(X) = (1+X)^p - 1$ lies in $\mathcal{F}_p$.*

Lemma 2.10. *Let $f, g \in \mathcal{F}_\pi$ and $\phi_1(X_1, \cdots, X_n) \in \mathcal{O}_K[X_1, \cdots, X_n]$ is a linear form. Then there exists a unique $\phi \in \mathcal{O}_K[[X_1, \cdots, X_n]]$ such that*

(a) $\phi \equiv \phi_1 \pmod{(X_1, \cdots, X_n)^2}$.

(b) $f\big(\phi(X_1, \cdots, X_n)\big) = \phi\big(g(X_1), \cdots, g(X_n)\big)$.

Proof. See [Mil20], Lemma 2.11.

The idea is doing induction on the degree of ϕ and taking the limit, i.e., show that there exists a unique polynomial $\phi_r(X_1, \cdots, X_n)$ of degree r such that $\phi_r \equiv \phi_1 \pmod{(X_1, \cdots, X_n)^2}$ and $f\big(\phi_r(X_1, \cdots, X_n)\big) \equiv \phi_r\big(g(X_1), \cdots, g(X_n)\big) \pmod{(X_1, \cdots, X_n)^{r+1}}$. $\square$

The following three propositions can be deduced by repeatedly applying the above lemma.

Proposition 2.11. *For every $f \in \mathcal{F}_\pi$, there is a unique formal group law $F_f \in \mathcal{O}_K[[X, Y]]$ admitting f as an endomorphism.*

Proposition 2.12. *For $f, g \in \mathcal{F}_\pi$ and $a \in \mathcal{O}_K$, let $[a]_{g,f}$ be the unique element of $\mathcal{O}_K[[T]]$ such that*

(a) $[a]_{g,f} \equiv aT \pmod{T^2}$.

(b) $g \circ [a]_{g,f} = [a]_{g,f} \circ f$.

Then $[a]_{g,f}$ is a homomorphism $F_f \to F_g$.

Proposition 2.13. *For any $a, b \in \mathcal{O}_K$, $[a+b]_{g,f} = [a]_{g,f} +_{F_g} [b]_{g,f}$ and $[ab]_{h,f} = [a]_{h,g} \circ [b]_{g,f}$.*

This proposition has two direct corollaries.

Corollary 2.14. *For any $f, g \in \mathcal{F}_\pi$, $F_f \cong F_g$.*

Proof. Given every $u \in \mathcal{O}_K^*$, $[u]_{f,g}$ and $[u^{-1}]_{g,f}$ are inverse to each other. $\square$

Corollary 2.15. *For each $a \in \mathcal{O}_K$, there is a unique endomorphism $[a]_f \colon F_f \to F_f$ such that $[a]_f \equiv aT \pmod{T^2}$ and $[a]_f$ commutes with f. The map*

$$\mathcal{O}_K \to \mathit{End}(F_f) \colon a \mapsto [a]_f$$

is an injective ring homomorphism. In particular, $[\pi]_f = f$.

The formal group law F_f associated to an uniformizer π is called the **Lubin-Tate formal group law**.

Example 2.16. *When $K = \mathbb{Q}_p$, $\pi = p$, $f(X) = (1+X)^p - 1$, $F_f = \mathbb{G}_m$ is the multiplication group law. The power series $[a]_f = (1+X)^a - 1$ is the one we defined before.*

2.3 Construction of K_π and the Local Artin Map

For any $f \in \mathcal{F}_\pi$, let $\Lambda_f = \{\alpha \in K^{al} : |\alpha| < 1\}$. Define a $\mathcal{O}_K$-module structure on Λ_f by $\alpha + \beta = \alpha +_{F_f} \beta = F_f(\alpha, \beta)$ and $a \cdot \alpha = [a]_f(\alpha)$. Let $\Lambda_{f,n}$ be the submodule of Λ_f consisting of elements killed by $[\pi]_f^n$.

Remark. *The canonical isomorphism $[1]_{g,f} \colon F_f \to F_g$ induces isomorphisms $\Lambda_f \to \Lambda_g$ and $\Lambda_{f,n} \to \Lambda_{g,n}$ for each n.*

Proposition 2.17. *For each n, $\Lambda_{f,n} = \mathcal{O}_K/(\pi^n)$ as $\mathcal{O}_K$-modules. Thus, End$(\Lambda_{f,n}) \cong \mathcal{O}_K/(\pi^n)$ and Aut$(\Lambda_{f,n}) \cong \big(\mathcal{O}_K/(\pi^n)\big)^*$.*

Proof. By the above remark, it suffices to take $f = \pi X + X^q$. Thus, $[\pi^n]_f = \pi^n X + \cdots + X^{q^n}$. From the Newton polygon of $[\pi^n]_f$, we see that all the roots of $[\pi^n]_f$ lie in $\Lambda_{f,n}$.

Since $f = \pi X + X^q$ is an Eisenstein polynomial, f is irreducible and has q distinct roots. Thus, $\Lambda_{f,1}$ has exactly q elements. By the structure theorem of modules over PID, $\Lambda_{f,1} \cong \mathcal{O}_K/(\pi)$ since $\mathcal{O}_K/(\pi^n)$ contains q^n elements.

For each $\alpha \in K^{al}$ with $|\alpha| < 1$, $f(X) - \alpha = X^q + \cdots + \pi X - \alpha$. From the Newton polygon of $f(X) - \alpha$, we see that all roots of $f(X) - \alpha$ lie in Λ_f. Therefore, $[\pi]_f$ is surjective.

Suppose $\Lambda_{f,n} \cong \mathcal{O}_K/(\pi^n)$ for some n. Since $[\pi]_f$ is surjective, we have the following exact sequence:

$$0 \to \Lambda_{f,1} \to \Lambda_{f,n+1} \overset{[\pi]_f}{\to} \Lambda_{f,n} \to 0$$

Thus, $\Lambda_{f,n+1}$ has q^{n+1} elements. Suppose $\Lambda_{f,n+1} \cong \mathcal{O}_K/(\pi^{n_1}) \oplus \cdots \mathcal{O}_K/(\pi^{n_r})$ by the structure theorem of modules over PID. Then the exact sequence implies that $\Lambda_{f,1} \cong (\pi^{n_1-1})/(\pi^{n_1}) \oplus \cdots \oplus (\pi^{n_r-1})/(\pi^{n_r})$. Therefore, $r = 1$ and $\Lambda_{f,n+1} \subset \mathcal{O}_K/(\pi^{n+1})$. □

Lemma 2.18. *Every subfield E in K^{al} containing K is closed in the topological sense.*

Proof. Let $G = \mathrm{Gal}(K^{al}/E)$. By the uniqueness of the extension of the absolute valuation, G fixes the closure of E. Thus, $\overline{E} = (K^{al})^G = E$. □

Theorem 2.19. *Let $K_{\pi,n} = K(\Lambda_{f,n})$. Then we have*

(a) $K_{\pi,n}$ is independent of the choice of f.

(b) For each n, $K_{\pi,n}/K$ is a totally ramified extension of degree $(q-1)q^{n-1}$.

(c) The action of $\mathcal{O}_K$ on Λ_n induces an isomorphism

$$(\mathcal{O}_K/\mathfrak{m}^n)^* \to Gal(K_{\pi,n}/K)$$

Thus, $K_{\pi,n}/K$ is an abelian extension.

(d) For each n, Nm$(K_{\pi,n}^*) \ni \pi$.

Proof. (a) Via the isomorphisms $[1]_{g,f} : \Lambda_{f,n} \to \Lambda_{g,n}$, we have that

$$\widehat{K(\Lambda_{g,n})} = K\big(\widehat{[1]_{g,f}(\Lambda_{f,n})}\big) \subset \widehat{K\big(\Lambda_{f,n}\big)} = K\big(\widehat{[1]_{f,g}(\Lambda_{g,n})}\big) \subset \widehat{K(\Lambda_{g,n})}$$

Thus, $\widehat{K(\Lambda_{g,n})} = \widehat{K(\Lambda_{f,n})}$. By the above lemma,

$$K(\Lambda_{g,n}) = \widehat{K(\Lambda_{g,n})} \cap K^{al} = \widehat{K(\Lambda_{f,n})} \cap K^{al} = K(\Lambda_{f,n})$$

(b)(c) Since $K_{\pi,n}$ is independent on the choice of f, we may assume again that $f = [\pi]_f = \pi X + \cdots + X^q$.

Choose a nonzero root π_1 of f and π_{s+1} of $f(X) - \pi_s$ for each $s = 1, 2, \cdots, n-1$. Then there is a sequence of field extensions:

$$K(\pi_n) \supset K(\pi_{n-1}) \supset \cdots \supset K(\pi_1) \supset K$$

Note that each extension is Eisenstein, so is totally ramified. The degree of $K(\pi_1)/K$ is $q-1$ and the degree of $K(\pi_{s+1})/K(\pi_s)$ is q for each s. Therefore, $K(\pi_n)/K$ is a totally ramified extension of degree $q^{n-1}(q-1)$. Since $[\pi^n]_f(\pi_n) = 0$, $K(\Lambda_{f,n}) \supset K(\pi_n)$.

Note that $K(\Lambda_{f,n})$ is the splitting field of $[\pi^n]_f$ over K. Thus, Gal$\big(K(\Lambda_{f,n})/K\big)$ can be identified as a subgroup of permutations on $\Lambda_{f,n}$. By passing to limit of the power series, we can prove that the action of Gal$\big(K(\Lambda_{f,n})/K\big)$ on $\Lambda_{f,n}$ is compatible with the A-module structure on $\Lambda_{f,n}$. Thus, $\mathrm{Gal}\big(K(\Lambda_{f,n})/K\big) < \mathrm{Aut}(\Lambda_{f,n}) = \big(\mathcal{O}_K/(\pi^n)\big)^*$. Therefore,

$$(q-1)q^{n-1} = |\big(\mathcal{O}_K/(\pi^n)\big)^*| \geqslant [K(\Lambda_{f,n})/K] \geqslant [K(\pi_n)/K] = (q-1)q^{n-1}$$

Hence, $K(\Lambda_{f,n}) = K(\pi_n)$ is a totally ramified extension of degree $(q-1)q^{n-1}$ over K and $\mathrm{Gal}(K_{\pi,n}/K) \cong (\mathcal{O}_K/\mathfrak{m}^n)^*$ and $u \in \mathcal{O}_K^*$ acts on $\Lambda_{f,n}$ by $[u]_f$.

(d) Since the degree of $[\pi^n]_f/X = \pi + \cdots + X^{(q-1)q^{n-1}}$ is $(q-1)q^n$, it is the minimal polynomial of π_n over K. Hence, $\mathrm{Nm}_{K_{\pi,n}/K}(\pi_n) = (-1)^{(q-1)q^{n-1}}\pi$, so $\pi \in \mathrm{Nm}(K_{\pi,n}^*)$.

□

Let $K_\pi = \cup_{n=1}^\infty K_{\pi,n}$. By passing to the limit, we have that $\tilde{\phi}_f\colon U_K \cong \mathrm{Gal}(K_\pi/K)$ given by $u \mapsto [u^{-1}]_f$. The inverse here will make the formula elegant in the future.

Let $\phi_f\colon K^* \to \mathrm{Gal}(K_\pi K^{un}/K)$ given as follows: for each $a = u\pi^m \in K^*$, $\phi_f(a)|_{K^{un}}$ is the m-th power of the Frobenius element and $\phi_f(a)(\lambda) = \tilde{\phi}_f(u)(\lambda) = [u^{-1}]_f(\lambda)$ for all $\lambda \in \bigcup_{n=1}^\infty \Lambda_{f,n}$.

Next, we want to show that $K_\pi K^{un}$ and ϕ_f are independent of the choice of π, f. Note that in the proof of the part (a) of Theorem 2.19, the essential part is the $\mathcal{O}_K$-isomorphisms $[1]_{g,f}\colon \Lambda_{f,n} \to \Lambda_{g,n}$, where $[1]_{g,f}$ is a power series with coefficients in $\mathcal{O}_K$. Now suppose π, ω are two uniformizers of $\mathcal{O}_K$ and $\omega = u\pi$ for some $u \in U_K$. Let $B, \hat{B}$ be the integer ring of $K^{un}, \hat{K}^{un}$ respectively. Suppose we have such $\mathcal{O}_K$-isomorphisms $\theta\colon \Lambda_{f,n} \to \Lambda_{g,n}$, where $f \in \mathcal{F}_\pi$, $g \in \mathcal{F}_\omega$ and θ is a power series with coefficients in $\hat{B}$ (Since we took completion in the proof of the part (a) of Theorem 2.19, the coefficients of θ to can be taken in $\hat{B}$ and the proof of part (a) of Theorem 2.19 still work). We need to explore properties θ need for proving that ϕ_f is independent of π, f.

In order to show that $\phi_f = \phi_g$, it suffices to show that they agree on every uniformizer of $\mathcal{O}_K$. Given any uniformizer π' of $\mathcal{O}_K$, $\phi_f(\pi')|_{K^{un}} = \phi_g(\pi')|_{K^{un}}$ is the Frobenius element. Suppose $\pi' = v\pi = vu^{-1}\omega$. Let θ^σ be the power series obtained by acting σ on each coefficient of θ. Then for each $\lambda \in \Lambda_{f,n}$,

$$\phi_f(\pi')\big(\theta(\lambda)\big) = \theta^\sigma\big(\phi_f(v)(\lambda)\big) = \theta^\sigma \circ [v^{-1}]_f(\lambda)$$

We want that the right-hand side is equal to $\phi_g(\pi')\big(\theta(\lambda)\big) = [uv^{-1}]_g \circ \theta(\lambda) = \theta \circ [uv^{-1}]_f(\lambda)$ since θ is a $\mathcal{O}_K$-homomorphism. Therefore, we need that $\theta^\sigma = \theta \circ [u]_f$. Note that $\theta^\sigma = \theta \circ [u]_f$

implies that θ induces isomorphisms $\Lambda_{f,n} \to \Lambda_{g,n}$ because $(\sigma\circ f)^\sigma = \theta\circ[u\pi]_f = [\omega]_g\circ\theta = g\circ\theta$.

Suppose $\theta(X) = \epsilon X + \cdots$ for some $\epsilon \in \hat{B}$. Then $\sigma(\epsilon) = \epsilon u$. We claim that $\sigma(\cdot)/\cdot\colon \hat{B} \to \hat{B}$ is surjective while it is not true that $\sigma(\cdot)/\cdot\colon B \to B$ is surjective. That is why we require the coefficients of θ to be in $\hat{B}$.

Lemma 2.20. *The homomorphism $\sigma(\cdot)/\cdot\colon \hat{B}^* \to \hat{B}^*$ is surjective with kernel $\mathcal{O}_K^*$.*

Proof. Let $\mathfrak{n}$ be the maximal ideal in B. It suffices to show that the sequence

$$1 \to (\mathcal{O}_K/\mathfrak{m}^n)^* \to (B/\mathfrak{n}^n)^* \overset{\sigma(\cdot)/\cdot}{\to} (B/\mathfrak{n}^n)^* \to 1$$

is exact for each n and then pass to the limit.

For $n = 1$, $B/\mathfrak{n} = \overline{k}$ and the result follows easily. Assume that the sequence is exact for $n-1$. Then we have the following diagram:

$$\begin{tikzcd}
 & 1 \arrow[d] & & 1 \arrow[d] & \\
 & (\mathcal{O}_K/\mathfrak{m})^* \arrow[d] & & (\mathcal{O}_K/\mathfrak{m}^{n-1})^* \arrow[d] & \\
1 \arrow[r] & (B/\mathfrak{n})^* \arrow[r] \arrow[d, "\sigma(\cdot)/\cdot"] & (B/\mathfrak{n}^n)^* \arrow[r] \arrow[d, "\sigma(\cdot)/\cdot"] & (B/\mathfrak{n}^{n-1})^* \arrow[r] \arrow[d, "\sigma(\cdot)/\cdot"] & 1 \\
1 \arrow[r] & (B/\mathfrak{n})^* \arrow[r] \arrow[d] & (B/\mathfrak{n}^n)^* \arrow[r] & (B/\mathfrak{n}^{n-1})^* \arrow[r] \arrow[d] & 1 \\
 & 1 & & 1 &
\end{tikzcd}$$

By the snake lemma, $\sigma(\cdot)/\cdot\colon (B/\mathfrak{n}^n)^* \to (B/\mathfrak{n}^n)^*$ is surjective with kernel of q^n elements. Since $(\mathcal{O}_K/\mathfrak{m}^n)^*$ contains q^n elements and is contained in the kernel, the kernel is $(\mathcal{O}_K/\mathfrak{m}^n)^*$. □

The following proposition says that there exists the required $\theta \in \hat{B}[[X]]$, so it finishes the proof that $K_\pi K^{un}$ and ϕ_f are independent on the choice of π, f.

Proposition 2.21. *Let F_f and F_g be the Lubin-Tate formal group law defined by $f \in \mathcal{F}_\pi$ and $g \in \mathcal{F}_\omega$, where $\omega = u\pi$ are two uniformizers of $\mathcal{O}_K$. Then there exists an $\epsilon \in \hat{B}^*$ such that $\sigma(\epsilon) = \epsilon u$ and a power series $\theta \in \hat{B}[[X]]$ such that*

(a) $\theta(X) \equiv \epsilon X \pmod{X^2}$.

(b) $\theta^\sigma = \theta \circ [u]_f$.

(c) $\theta\big(F_f(X,Y)\big) = F_g\big(\theta(X),\theta(Y)\big)$.

(d) $\theta \circ [a]_f = [a]_g \circ \theta$.

Proof. The proof has four steps:

1. Show that there exists a $\theta \in \hat{B}[[X]]$ satisfying (a)(b). This can be shown by induction on the degree of θ as Lemma 2.10.

2. Show that the θ in the first step can be chosen so that $g = \sigma \circ f \circ \theta^{-1}$. Let $h = \theta^\sigma \circ f \circ \theta^{-1}$. Then show that $h \in \mathcal{O}_K[[X]]$. Let $\theta' = [1]_{g,h} \circ \theta$. Then θ' satisfies (a)(b) and $(\theta')^\sigma \circ f \circ (\theta')^{-1} = [1]_{g,h} \circ h \circ [1]_{h,g} = g$.

3. Show that $\theta\Big(F_f\big(\theta^{-1}(X), \theta^{-1}(Y)\big)\Big) = F_g(X,Y)$.

4. Show that $\theta \circ [a]_f \circ \theta^{-1} = [a]_g$.

Both the third and the fourth steps can be shown by directly Lemma 2.10. For details, see [Mil20] Proposition 3.10. $\square$

2.4 Local Kronecker-Weber Theorem

The main propose of this section is to prove the following theorem:

Theorem 2.22. *(Local Kronecker-Weber Theorem)* $K_\pi K^{un} = K^{ab}$.

Lemma 2.23. *Let L be a finite abelian extension of K_π of degree m. Let K_m be the unramified extension of K_π of degree m. Then there exists a totally ramified extension L_t/K_π such that $L \subset L_t K_m = LK_m$.*

Proof. Note that Gal(LK_m/K_π) is a subgroup of Gal$(L/K_\pi)\times$Gal(K_m/K_π), so every element in Gal(LK_m/K_π) has torsion m. Pick a $\tau \in$ Gal(LK_m/K_π) such that $\tau|_{K_m}$ is the Frobenius element. Then τ has order m in Gal(LK_m/K_π). By the structure theorem of finite abelian groups, we have that Gal(LK_m/K_π) can be decomposed into $\langle\tau\rangle \times H$ for some subgroup $H <$ Gal(LK_m/K_π). Let $L_t = L^{\langle\tau\rangle}$. Then $L_t \cap K_m = K_\pi$ since Gal$(K_m/K_\pi) = \langle\tau|_{K_m}\rangle$, so L_t/K_π is totally ramified and Gal$(L_t/K_\pi) = H$. Therefore, $L_t K_m = LK_m \supset L$. $\square$

Remark. *The above proof actually works for all henselian valuation field with finite residue field K and finite abelian extension L/K.*

Lemma 2.24. *Let L be a totally ramified extension of K and $L \supset K_\pi$. Then $L = K_\pi$.*

Proof. See [Mil20] Lemma 4.9.

The idea is that $\mathrm{Gal}(L/K_\pi) = \bigcap_{n=1}^{\infty} \mathrm{Gal}(L/K_{n,\pi})$. In fact, $\mathrm{Gal}(L/K_{\pi,n})$ is some ramification group of $\mathrm{Gal}(L/K)$, so their intersection is trivial. □

Lemma 2.25. *Every finite unramified extension of K_π is contained in $K_\pi K^{un}$*

Proof. Suppose L/K_π is a finite unramified extension. Then $L = K_\pi(\alpha)$ for some $\alpha \in K^{al}$. Suppose $f \in K_\pi[X]$ is the minimal polynomial of α over K_π. Then $f \in K_{\pi,n}[X]$ for some n. Since L/K_π is unramified, f is irreducible in the residue field of K_π, which is the same with the residue field of K_π, n. Thus, $K_{\pi,n}(\alpha)/K_{\pi,n}$ is unramified. Suppose T/K is the maximal unramified extension of $K_{\pi,n}(\alpha)/K$, so the residue field of T equals the residue field of $K_{\pi,n}(\alpha)$. Then $[T : K]$ equals the inertia index of $K_{\pi,n}(\alpha)/K$. Thus, $K_{\pi,n}(\alpha) = TK_{\pi,n}$. Hence, $L = K_\pi T \subset K_\pi K^{un}$. □

Proof. (of Theorem 2.22): Suppose L/K is a finite abelian extension. Then LK_π/K_π is also a finite abelian extension. Thus, there exists a totally ramified extension L_t/K_π and an unramified extension K_m/K_π such that $LK_\pi \subset L_t K_m$. By the two lemmas above, $L_t = K_\pi$ and $K_m \subset K_\pi K^{un}$. Therefore, $L \subset LK_\pi \subset K_\pi K^{un}$. Hence, $K_\pi K^{un} = K^{ab}$. □

2.5 End of the Proof

Now we finish the proof of the main theorems of local class field theory by showing that the ϕ_K we constructed satisfies the Theorem 2.1 and that Theorem 2.3 is true.

By construction, we know that $\phi_K(\pi)|_{K^{un}}$ is the Frobenius element for each uniformizer π of K.

To prove the part (b) of the Theorem 2.1, take a finite abelian extension L/K.

Lemma 2.26. *The following diagram is commutative*

$$\begin{array}{ccc} L^* & \xrightarrow{\phi_L} & Gal(K^{ab}/L) \\ {\scriptstyle Nm}\downarrow & & \downarrow \\ K^* & \xrightarrow[\phi_K]{} & Gal(K^{ab}/K) \end{array}$$

Proof. Since L^* is generated by all uniformizers, it suffices to show that $\phi_L(\Pi) = \phi_K\big(\mathrm{Nm}(\Pi)\big)$ for all uniformizers Π of L. By taking the maximal unramified extension of K in L, it suffices to show the cases when L/K is totally ramified and unramified respectively.

For detail, see [Iwa86] Theorem 6.9. □

Thus, ϕ_K induces a homomorphism $\phi_{L/K}\colon K^*/\mathrm{Nm}(L^*) \to \mathrm{Gal}(L/K)$.

From the construction of ϕ_K, it is easy to see that

Lemma 2.27. *The homomorphism ϕ_K is injective and continuous. Moreover, $\phi_K(K^*)$ is dense in Gal(K^{ab}/K), consisting of all elements τ such that $\tau|_{K^{un}}$ is a power of the Frobenius element.*

The following proposition finishes the proof of the part (b) of Theorem 2.1.

Proposition 2.28. *As notations above, $\phi_{L/K}\colon K^*/Nm(L^*) \to Gal(L/K)$ is an isomorphism.*

Proof. If $\phi_K(x)|_L = Id$ for some $x \in K^*$, then there is $\tau = \phi_K(x)|_L \in \mathrm{Gal}(K^{ab}/L)$. Let $T = L \cap K^{un}$. Suppose $[T : K] = m$. Then $\phi_K(x)|_T = Id$ implies that $\phi_K(x)|_{K^{un}}$ is a power of σ^m by the above lemma. Note that $\mathrm{Gal}(K^{un}/T) \cong \mathrm{Gal}(LK^{un}/L) = \mathrm{Gal}(L^{un}/L)$ and σ^m corresponds to the Frobenius element of L under this isomorphism. Therefore, $\phi_K(x)|_L^{un}$ is a power of the Frobenius element of L. By the above lemma again, there is $y \in L$ such that $\phi_L(y) = \phi_K(x)$. Since $\phi_L(y) = \phi_K\big(\mathrm{Nm}(y)\big)$ and ϕ_K is injective, $x = \mathrm{Nm}(y)$. Thus, $\phi_{L/K}$ is injective.

In order to prove the surjectivity, identify $\mathrm{Gal}(L/K)$ as $\mathrm{Gal}(K^{ab}/K)/\mathrm{Gal}(K^{ab}/L)$. For each $[\tau] \in \mathrm{Gal}(L/K)$, $\tau\mathrm{Gal}(K^{ab}/L)$ is an open subset of $\mathrm{Gal}(K^{ab}/K)$. Since $\phi_K(K^*)$ is dense in $\mathrm{Gal}(K^{ab}/K)$, there is $x \in K^*$ such that $\phi_K(x) \in \tau\mathrm{Gal}(K^{ab}/L)$. Therefore, $\phi_{L/K}(x) = [\tau]$. □

Finally, we should prove Theorem 2.3.

Lemma 2.29. *Let K be a non-archimedean local field and L/K is a field extension. If $Nm(L^*)$ is of finite index in K^*, then it is open.*

Proof. Since U_L is profinite, U_L is compact. Thus, $\mathrm{Nm}(U_L)$ is compact in K^*, which is Hausdorff. Therefore, $\mathrm{Nm}(U_L)$ is closed in K^*. Since $\mathrm{Nm}(U_L) = \mathrm{Nm}(L^*) \cap U_K$, U_L is a closed subgroup with finite index in U_K, so is open in U_K. Note that U_K is open in K^*. Hence, $\mathrm{Nm}(L^*)$ contains an open subgroup of K^*, so is open. □

Proof. (of Theorem 2.3): By the part (b) of Theorem 2.1, we see that every norm group in K^* is of finite index. Thus, by the lemma above, they are open. Conversely, by the part (d) of the Corollary 2.2, it suffices to show that each open subgroup of finite index H in K^* contains a norm group. Since H is open, $H \supset (1 + \mathfrak{m}^n)$ for some n. Since H is of finite index, there is an integer s such that $H \supset (1+\mathfrak{m}^n) \times s\mathbb{Z}$ by the same proof as in Lemma 2.4. Let K_s be the unramified extension of K of degree s and $L = K_{\pi,n}K_s$. Therefore, $\phi_{L/K}\big((1+\mathfrak{m}^n)\times s\mathbb{Z}\big) = 1$. It follows that $(1+\mathfrak{m}^n)\times s\mathbb{Z} \subset \mathrm{Nm}(L^*)$. Since they have the same index in K^*, $(1+\mathfrak{m}^n)\times s\mathbb{Z} = \mathrm{Nm}(L^*)$. $\square$

3 Backgrounds in Algebraic Topology

3.1 Generalized Cohomology and Homology Theory and Spectrum

It is well-known that the singular cohomology and homology theory are characterized by several axioms on the contravariant functor, called the Eilenberg-Steenrod axioms. Actually there are other cohomology and homology theories share similar properties. We can generalize such axioms by dropping out the dimension axiom. It turns out that the resulted generalized cohomology and homology theories are very useful in stable homotopy theory.

Definition 3.1 (Generalized Cohomology and Homology Theory)**.** A **generalized cohomology theory** is a sequence of contravariant functors h^n from the homotopy category of pointed CW-complexes to abelian groups with natural isomorphisms $\partial^n \colon h^{n+1}\circ\Sigma \to h^n$ such that for any cofiber sequence $A \xrightarrow{i} X \xrightarrow{j} X/A \xrightarrow{q} \Sigma A$, there is a natural long exact sequence

$$\cdots \xrightarrow{i^*} h^{n-1}(A) \xrightarrow{\delta} h^n(X/A) \xrightarrow{j^*} h^n(X) \xrightarrow{i^*} h^n(A) \xrightarrow{\delta} \cdots$$

where δ is the composition of q^* and ∂^n.

A **generalized homology theory** is just the dual definition.

Actually such algebraic objects can be constructed from some geometric objects.

Definition 3.2 (Spectrum)**.** (a) A **prespectrum** E is a family of pointed topological spaces $\{E_n\}_{n\in\mathbb{Z}}$ and the structure maps $\Sigma E_n \to E_{n+1}$, where ΣE_n is the suspension of E_n.

(b) A **spectrum** is a prespectrum E such that the adjoint maps of the structure maps $E_n \to \Omega E_{n+1}$ (we will also call these the structure maps) are weak equivalences, where ΩE_{n+1} is the loop space of E_{n+1}.

(c) For a spectrum E, the **homotopy groups of** E is well-defined by

$$\pi_n(E) := \pi_{n+k}(E_k), n+k \geqslant 0$$

(d) Suppose E, F are two spectra. A map $f\colon E \to F$ between spectra is a sequence of maps $f_n\colon E_n \to F_n$ such that the following diagram commutes for each n

$$\begin{tikzcd} E_n \arrow[r, "f_n"] \arrow[d] & F_n \arrow[d] \\ \Omega E_{n+1} \arrow[r, "\Omega f_{n+1}"'] & \Omega F_{n+1} \end{tikzcd}$$

(e) Suppose E is a spectrum. Then $\Sigma^m E$ is the spectrum defined by $(\Sigma^m E)_n := E_{m+n}$.

Example 3.3. *Given a space X, we can define the $\Sigma^\infty X'$ by $\Sigma^\infty X'_n := \Sigma^n X$ if $n \geqslant 0$ and just a point if $n < 0$. This is surely a prespectrum. However, it is not a spectrum. The structure maps are just injective. We can a make it to a spectrum by a process called **spectrification**. If there is a spectrum E_n with injective structure maps $\omega_n\colon E_n \to \Omega E_{n+1}$, then we define $\mathbb{L}E_n := colim_k \Omega^k E_{n+k}$ and $\mathbb{L}\omega := colim_k \Omega^k \omega_{n+k}$. It can be shown that the result sequence of spaces with structure maps is a spectrum and the spectrification is left adjoint to the inclusion functor from prespectra to spectra. [1] From the construction, we see that the homotopy groups invariant after the spectrification. We define the $\Sigma^\infty X$ to be the spectrification of $\Sigma^\infty X'$. In particular, we define the **sphere spectrum** S as the suspension spectrum of S^0.*

It can be shown that Σ^∞ is left adjoint to the functor from spectra to spaces by taking the space at degree 0.[2] Therefore, maps between $\Sigma^\infty X$ and E is the same with pointed maps between X and E_0.

We can further define the smash product between spectra. However, the true definition is very tedious. (See [EKMM97] for example) We just point out here the true smash product makes the homotopy category of spectra into a monoidal category with the unit element S.

Definition 3.4. A **ring spectrum** is a spectrum with maps $\eta\colon S \to E$, called the unit map, and $m\colon E \wedge E \to E$, called the multiplication map, such that the following diagrams commute up to homotopy

$$\begin{tikzcd} E \arrow[r, "\eta\wedge Id_E"] \arrow[d, "Id_E\wedge\eta"'] \arrow[dr, "Id_E"] & E \wedge E \arrow[d, "m"] \\ E \wedge E \arrow[r, "m"'] & E \end{tikzcd}$$

[1] https://ncatlab.org/nlab/show/spectrification

[2] https://ncatlab.org/nlab/show/stabilization

$$\begin{array}{ccc} E \wedge E \wedge E & \xrightarrow{m \wedge Id_E} & E \wedge E \\ {\scriptstyle Id_E \wedge m}\downarrow & & \downarrow{\scriptstyle m} \\ E \wedge E & \xrightarrow[m]{} & E \end{array}$$

If E is a ring spectrum, we define the **coefficient ring** of E as the ring $E^{-*}(*) = \pi_*(E) = E_*(S)$. The ring structure is induced by the ring structure on E. We will simply denote it as E_*.

Definition 3.5. Let E be a spectrum. The **generalized cohomology and homology theory associated with** E, E^*, is defined by

$$E^n(X) := [\Sigma^{-n}X, E]$$
$$E_n(X) := \pi_n(X \wedge E)$$

for any spectrum X. This is a generalized cohomology theory by [Ada95, Chapter III, Proposition 6.1]

Example 3.6. *(a) Let $K(A,n)$ be the Eilenberg-Maclane space. Then $\Omega K(A,n+1) \simeq K(A,n)$. Define the Eilenberg-Maclane spectrum HA by the spectrification of $HA'_n := K(A,n)$ for $n \geqslant 0$ and a point for $n < 0$. Then $HA_n = K(A,n)$ for $n \geqslant 0$ and $HA^n(X) = H^n(X;A)$.*

(b) For the sphere spectrum S,

$$S_n(X) = \pi_n(\Sigma^\infty X \vee S) = \pi_n(\Sigma^\infty X) = \pi_n^S(X)$$

is the degree n stable homotopy group of X.

Besides the axioms given in the definition of generalized cohomology theories, the generalized cohomology theories associated with spectra have another important property, which is sometimes called the additivity axiom or the wedge axiom.

Proposition 3.7. *Suppose E is a spectrum. Then*

$$E^*(\vee_{\alpha \in I} X_\alpha) \cong \prod_{\alpha \in I} E^*(X_\alpha)$$

Proof. By definition,

$$E^n(\vee_{\alpha \in I} X_\alpha) = [\vee_{\alpha \in I} X_\alpha, E_n] \cong \prod_{\alpha \in I} [X_\alpha, E_n] = \prod_{\alpha \in I} E^*(X_\alpha)$$

□

A beautiful and fundamental result is that there is a correspondence between spectra and generalized cohomology theories with the wedge axiom.

Theorem 3.8 (Brown Representability Theorem)**.** *If h^* is a generalized cohomology theory satisfying*

$$h^*(\vee_{\alpha\in I} X_\alpha) \cong \prod_{\alpha\in I} h^*(X_\alpha)$$

then there is a spectrum E, such that $h^* = E^*$. Moreover, multiplicative cohomology theories correspond to ring spectra. If E is a ring spectrum, the associated generalized cohomology theory is called ***multiplicative****.*

Proof. For further references, see [Ada95, Chapter III, Remark 6.5]. □

3.2 Complex Orientation

In differentiable manifolds, we have the following definition of orientation of a manifold.

Definition 3.9 (Orientability of a Manifold)**.** Suppose M is an n-manifold. Pick any two charts $(U, \phi), (V, \psi)$ of M. Then M is said to be **orientable** if there is a smooth atlas such that the Jacobi matrix of each transition map $\psi \circ \phi^{-1}$ at each point has positive determinant.

Note that the Jacobi matrix of the transition map is just the differential map of the transition map. Therefore, the above definition can be rephrased in terms of the transition maps on the tangent bundle. Then we can say that the tangent bundle TM is orientable if M is orientable. More generally, we have the following definition of the orientability of a real vector bundle, which is equivalent to the condition that M is orientable when we restrict to $TM \to M$.

Definition 3.10 (Orientability of a Real Vector Bundle)**.** Suppose $p\colon E \to B$ is a real vector bundle of dimension n. Pick two $(U, \phi), (V, \psi)$ bundle charts for p. Then the transition map gives a map $g\colon U \cap V \to \mathrm{GL}_n(\mathbb{R})$ by

$$\psi \circ \phi^{-1}\colon (U \cap V) \times \mathbb{R}^n \to (U \cap V) \times \mathbb{R}^n, \ \ (x, v) \mapsto \big(x, g_x(v)\big)$$

Then p is said to be **orientable** if there is a bundle atlas such that every element in the image of g_x have positive determinant for all x.

In fact, the orientability of a bundle is encoded in the cohomology group.

Proposition 3.11. *Suppose $p\colon E \to B$ is a real vector bundle of dimension n. Let $p'\colon E' \to B$ be the subbundle where E' is E minus the zero section of p. Then p is orientable if and only if there exists a $t \in H^n(E, E'; \mathbb{Z})$ such that t restricts to a generator in $H^n(F_b, F_b'; \mathbb{Z})$ for each $b \in B$, where F_b, F_b' are fibers over b in E, E' respectively.*

Proof. See [TD08, Theorem 17.9.4]. □

We can generalize this to arbitrary generalized cohomology theories associated to some ring spectrum.

Definition 3.12 (E-Orientation)**.** Suppose E is a ring spectrum. Let $p\colon V \to B$ be a vector bundle of dimension n. Then an E**-orientation** on p is an element in $E^n\big(Th(V)\big)$ restricting to a generator in $E^n(S^n) \cong \pi_0(E)$ on each fiber, where $Th(V)$ is the Thom space.

Note that all real manifolds are $H\mathbb{Z}/2$-orientable. It inspires us to define the orientability of the generalized cohomology theory itself so that all vector bundles have a canonical choice of orientation. Here we only want to focus on the complex vector bundles.

Definition 3.13 (Complex Orientation)**.** A **complex orientation** on a ring spectrum E is a family of elements $c_V \in E^{2n}\big(Th(V)\big)$ for each $n \in \mathbb{Z}_+$ and complex vector bundle $V \to B$ of dimension n such that

(a) For any $b \in B$, c_V restricts to a generator in $E^{2n}\big(Th(V_x)\big) \cong E^{2n}(S^{2n}) \cong \pi_0(E)$.

(b) For any map $f\colon B' \to B$, $c_{f^*V} = f^*(c_V)$.

(c) For any two complex vector bundles V_1, V_2 over B, $c_{V_1 \oplus V_2} = c_{V_1} \cdot c_{V_2}$.

We know that there is a universal 1-dimensional complex vector bundle γ_1 over $\mathbb{CP}^\infty$.

Theorem 3.14. *A complex orientation is determined by the element $c_{\gamma_1} \in E^2\big(Th(\gamma_1)\big)$. There is a bijection between the elements in $E^2\big(Th(\gamma_1)\big) \cong E^2(\mathbb{CP}^\infty)$ that restricts to a generator in $E^2(S^2)$ and complex orientations of E.*

Proof. See [TD08, Theorem 19.0.1]. □

Due to [TD08, Theorem 19.1.4, Proposition 19.1.6], we have $E^*\big(\mathbb{CP}^\infty\big) = E_*[[T]]$ and $E^*\big(\mathbb{CP}^\infty \times \mathbb{CP}^\infty\big) = E_*[[X,Y]]$. Note that $\mathbb{CP}^\infty \simeq BU(1)$. Therefore, there is a symmetric multiplication map $m\colon \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$. The induced map on cohomology rings sends T to an element $f(X,Y) \in E_*[[T]]$. By the associativity and commutativity of m, we have

Proposition 3.15. *The above $f(X,Y)$ is a formal group law with coefficients in E_*.*

3.3 Complex Bordism Theory

3.4 Morava E-Theory

3.5 H_∞-Map

3.6 Power Operation

4 Proof of Ando's Theorem via Coleman Norm Operator

In this section, we suppose that $\pi = p$. Suppose Φ is a Honda formal group law over $k = F_q$, i.e., $[p]_\Phi = T^q$, where $[p]_\Phi$ is the p-th composition of Φ with itself. Then every lifting F of Φ to $\mathcal{O}_K$ is a Lubin-Tate formal group law. Let $f \in \mathcal{F}_\pi$ be the element associated to F. Then $f = [p]_f = [p]_F$.

By [And95, Theorem 4],

Definition 4.1 (Ando's criterion)**.** We say a formal group law F over $\mathcal{O}_K$ that is a lifting of a Honda formal group law over k satisfies Ando's criterion if

$$[p]_F(T) = \prod_{\lambda \in \Lambda_1} (T +_F \lambda)$$

Suppose $\mathcal{O}_K((T))$ is the ring of Laurent series with coefficients in $\mathcal{O}_K$. We assign the "compact-open" topology to $\mathcal{O}_K((T))$, i.e., a sequence $\{g_n\}$ converges to g if and only if for any closed annulus A around zero in $\mathfrak{m}$, and for each $\epsilon > 0$, there exists a positive integer $N = N(A, \epsilon)$ such that $|g_n(a) - g(a)| < \epsilon$ for all $a \in A$ and $n \geqslant N$. Then the Coleman Norm Operator is given by:

Theorem 4.2. *There exists a unique $\mathscr{N}_f\colon \mathcal{O}_K((T)) \to \mathcal{O}_K((T))$ satisfying*

$$\mathscr{N}_f(g) \circ [\pi]_f = \prod_{\lambda \in \Lambda_1} g(T +_F \lambda)$$

for every $g \in \mathcal{O}_K((T))$. Moreover, $\mathscr{N}_f$ is continuous and multiplicative.

Proof. See [Col79, Theorem 11, Corollary 12]. □

Therefore, we see that a Lubin-Tate formal group law satisfies Ando's criterion if and only if

$$[p]_F(T) = \mathscr{N}_f(T) \circ [p]_F$$

Since $[p]_F$ has a composition inverse in $K[[T]]$, we can cancel the $[p]_F$ on both sides, so that Ando's criterion is equivalent to

$$\mathscr{N}_f(T) = T$$

Let $\mathscr{M}_\infty = \{g \in \mathcal{O}_K((T))\colon \mathscr{N}_f(g) = g\}$ and $\varprojlim \Lambda_{f,n}$ be the inverse limit taken with respect to $[\pi]_f$. Suppose $v_f = (v_{f,n}) \in \varprojlim \Lambda_{f,n}$ is a generator, i.e., $v_{f,n}$ is a generator of $\Lambda_{f,n}$ as a $\mathcal{O}_K$-module for each n. Let $X_\infty = \varprojlim K_n^*$, where the inverse limit is taken with respect to norm maps $N_{n+1,n}\colon K_{n+1}^* \to K_n^*$. According to [Col79, Corollary 17], $T \in \mathscr{M}_\infty$ if and only if $v \in X_\infty$, i.e.,

$$N_{n+1,n}(v_{f,n+1}) = v_{f,n}$$

for each n.

Proposition 4.3. *If $[p]_f = \sum_{i=0}^{q} a_i T^i$ where $a_0 = \pi, a_q = 1$, then F satisfies Ando's criterion.*

Proof. By the above discussion, it suffices to show that $N_{n+1,n}(v_{f,n+1}) = v_{f,n}$ for each n. Since $v_{f,n} = [p]_f(v_{f,n+1})$ and $[p]_f$ is an Eisenstein polynomial, $[p]_f(T) - v_{f,n}$ is the minimal polynomial of $v_{f,n+1}$ over K_n. Hence, $N_{n+1,n}(v_{f,n+1}) = (-1)^q(-v_{f,n}) = v_{f,n}$. □

We say two liftings $F, \tilde{F}$ of Φ are $\star$-isomorphic if the isomorphism $\psi \in T\mathcal{O}_K[[T]]$ between $F, \tilde{F}$ satisfies $\psi \equiv T \pmod{\pi}$.

We have the following description for Ando's criterion ([And95, Theorem 4]):

Theorem 4.4 (Ando)**.** *In each $\star$-isomorphism class of lifting of Φ to a complete local ring with residue field k, there is a unique formal group law F satisfying Ando's criterion.*

The goal of this section is to prove Theorem 4.4 via Coleman Norm Operator. First we need some notations. Some of them follow the notations in [Col79].

$$I := \mathcal{O}_K[[T]]$$

$$I^0 := 1 + \pi I$$

$$J := TI = T\mathcal{O}_K[[T]]$$

$$J^0 := TI^0 = T + \pi TI$$

Proposition 4.5. *Suppose $g, h \in \mathcal{F}_\pi$. Then F_g, F_h are $\star$-isomorphic if and only if there is $u \in J^0$ such that $h = u \circ g \circ u^{-1}$.*

Proof. If F_g, F_h are $\star$-isomorphic with isomorphism u, then $u \in J^0$ and $u \circ g = h \circ u$.

Conversely, if there is a $u \in J^0$ such that $u \circ g = h \circ u$, $u = [u'(0)]_{h,g}$ by Proposition 2.12. Thus, $u \in \mathrm{Hom}(F_g, F_h)$ is an isomorphism and $u \equiv t \pmod{\pi}$. □

Fix $f \in \mathcal{F}_\pi$ and $u \in J^0$. Let $f_u := u \circ f \circ u^{-1}$. By above discussion, F_{f_u} satisfies the Ando's criterion if and only if $N_{n+1,n}(v_{f_u,n+1}) = v_{f_u,n}$ for all n. Note that u induces isomorphisms between $\Lambda_{f,n} \to \Lambda_{f_u,n}$ for all n. Let $v_{f,n} = u^{-1}(v_{f_u,n})$. Since $f_u(v_{f_u,n+1}) = v_{f_u,n}$ for each n, $f(v_{f,n+1}) = v_{f,n}$ for each n. Thus, $v_f = (v_{f,n}) \in \varprojlim \Lambda_{f,n}$ is a generator. Then by [Col79, Corollary 12(ii)],

$$N_{n+1,n}(v_{f_u,n+1}) = v_{f_u,n} \Leftrightarrow u(v_{f,n}) = N_{n+1,n}\big(u(v_{f,n+1})\big) = \mathscr{N}_f(u)(v_{f,n})$$

For any $\lambda \in \Lambda_{f,n} - \Lambda_{f,n-1}$, $\lambda = \tau(v_{f,n})$ for some $\tau \in \mathrm{Gal}(K_{\pi,n}/K)$. Since $u, \mathscr{N}_f(u) \in \mathcal{O}_K[[T]]$,

$$u(\lambda) = \tau\big(u(v_{f,n})\big) = \tau\big(\mathscr{N}_f(u)(v_{f,n})\big) = \mathscr{N}_f(\lambda)$$

Thus, $\mathscr{N}_f(u), u$ agree on $v_{f,n}$ for all n if and only if they agree on $\Lambda_{f,n} - \{0\}$ for all n. By [Col79, Uniqueness Principle], the latter condition is equivalent to say that $\mathscr{N}_f(u) = u$. Thus, we are left to prove

Proposition 4.6. *For an $f \in \mathcal{F}_\pi$, there is a unique $u \in J^0$, such that $\mathscr{N}_f(u) = u$.*

We first need the following lemma from [Col79].

Lemma 4.7. *Let $i \geqslant 1$, $g \in 1 + \pi^i I$ and $h \in I$. Then* *(a)* $\mathscr{N}_f(g) \in 1 + \pi^{i+1} I$.

(b) $\mathscr{N}_f^i(h)/\mathscr{N}_f^{i-1}(h) \in 1 + \pi^i I$.

Proof. See [Col79, Lemma 13]. □

Then we see that $\mathscr{N}_f^\infty(h) := \lim_{i\to\infty} \mathscr{N}_f^i(h)$ exists. By Lemma 4.7(a), $\mathscr{N}_f^\infty$ vanishes on I^0. Since $\mathscr{N}_f$ is continuous,

$$\mathscr{N}_f\big(\mathscr{N}_f^\infty(h)\big) = \mathscr{N}_f\big(\lim_{i\to\infty} \mathscr{N}_f^i(h)\big) = \lim_{i\to\infty} \mathscr{N}_f\big(\mathscr{N}_f^i(h)\big) = \mathscr{N}_f^\infty(h)$$

Moreover, $\mathscr{N}_f^\infty$ is multiplicative since $\mathscr{N}_f$ is.

Proof of Proposition 4.6. If $\mathscr{N}_f(u) = u$, then $\mathscr{N}_f^i(u) = u$ for each i. Thus, $\mathscr{N}_f^\infty(u) = u$ after taking the limit. Since $u \in J^0$, there is $\tilde{u} \in I^0$ such that $u = t\tilde{u}$. Then

$$u = \mathscr{N}_f^\infty(u) = \mathscr{N}_f^\infty(t)\mathscr{N}_f^\infty(\tilde{u}) = \mathscr{N}_f^\infty(t)$$

It suffices to show that $\mathscr{N}_f^\infty(t) \in J^0$. Suppose $g_i := \mathscr{N}_f^i(t)/\mathscr{N}_f^{i-1}(t) \in 1 + \pi^i I$. Then $\mathscr{N}_f^\infty(t) = t g_1 g_2 \cdots$. Calculating each coefficient we see that $\mathscr{N}_f^\infty(t) \in J^0$. □

5 Where to Go from Here

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