Chromatic Homotopy Theory

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1 Notations

For a topological space X, let $\pi_*^S(X) :=$ be its stable homotopy groups.

Let S be the sphere spectrum.

Let SG be the Moore spectrum of an abelian group G.

For spaces X, Y, let $[X, Y]_*^S := \operatorname{colim}[\Sigma^{i+*}X, \Sigma^i Y]$ be the stable homotopy groups of maps.

For a homology theory E_*, let $\overline{E}_*$ be the associated homology theory. By abuse of notation, we also denote the coefficient ring by $E_* := E_*(pt)$.

2 Rough Idea of Chromatic Homotopy Theory

In number theory, we have the Sullivan fracture square:

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \prod_{p\colon prime} \mathbb{Z}_p \\ \downarrow & & \downarrow \\ \mathbb{Q} & \longrightarrow & \mathbb{Q} \otimes \left(\prod_{p\colon prime} \mathbb{Z}_p\right) \end{array}$$

There is a similar fracture square in homotopy theory. First, we have the similar definition of localization:

Definition 2.1 (E-acyclic, E-local, E-localization)**.** Let E_* be a generalized homology theory. A spectrum X is E**-acyclic** if $E_*(X) = 0$. A space Y is E**-local** if $[X, Y]_* = 0$ for any E-acyclic X.

An E**-localization** of a spectrum X is a map $\eta\colon X \to L_E X$ such that $E_*(\eta)$ is an isomorphism.

Theorem 2.2 (Bousfield)**.** *Such $L_E X$ always exists and is functorial in X.*

Theorem 2.3. *For any spectrum X,*

$$\begin{array}{ccc} X & \longrightarrow & \prod_{p\colon prime} L_p X \\ \downarrow & & \downarrow \\ L_{\mathbb{Q}} X & \longrightarrow & L_{\mathbb{Q}}\left(\prod_{p\colon prime} L_p X\right) \end{array}$$

where $L_p X := L_{S\mathbb{F}_p} X$, $L_{\mathbb{Q}} X := L_{S\mathbb{Q}} X$ (which is the rationalization of X when X is a CW-complex).

That is, we can get the global information of the spectrum X through p-completion, rationalization and how they are glued together. Also, similar to algebra, the p-completion of the space X can be constructed from the completion of the p-localization of X, i.e., $L_{S\mathbb{Z}_{(p)}}X$. This inspires us to investigate the p-localization of a spectrum X, which turns out to have a nice structure and be computable.

Theorem 2.4 (Chromatic Convergence Theorem)**.** *Suppose that X is a p-local spectrum, i.e., it is the p-localization of some spectrum. Then we have $X \cong holim L_n X$, where $L_n X \cong L_{E_n} X$ and E_n is the Morava E-theory (also called Lubin-Tate theory).*

Theorem 2.5 (Smash Product Theorem)**.** *For any spectrum X, $L_n X \cong X \wedge L_n S$.*

Therefore, we can recover the information of X from $L_n X \cong X \wedge L_n S$. Then we are reduced to compute $L_n S$, which can be decomposed further:

Proposition 2.6. *Let $K(n)$ be the Morava K-theory and X be an arbitrary spectrum.*

$$\begin{tikzcd}
L_n X \arrow[r] \arrow[d] & L_{K(n)} X \arrow[d] \\
L_{n-1} X \arrow[r] & L_{n-1} L_{K(n)} X
\end{tikzcd}$$

Therefore, we are reduced to calculate $L_{K(n)}X$, which is somehow related to the equivariant stuff:

Theorem 2.7 (Devinatz-Hopkins)**.** *$L_{K(n)}S \cong E_n^{h\mathbb{G}_n}$, where $\mathbb{G}_n$ is called the Morava stabilizer group.*

3 Periodicity Theorem

Due to the discussion in Section 2, from now on we will focus on the case of p-local spectra. There is a sequence of useful homology theories in investigating p-local spectra called the Morava K-theory. They will give a filtration of the category of p-local spectra. The construction is tedious and artificial, so we only display some properties of $K(n)$ here:

Proposition 3.1. *For each prime p there is a sequence of homology theories $K(n)_*$ for $n \geqslant 0$ with the following properties.*

(i) $K(0)_*(X) = H_*(X;\mathbb{Q})$ *and* $\overline{K(0)}_*(X) = 0$ *when* $\overline{H}_*(X)$ *is all torsion.*

(ii) $K(1)_*(X)$ *is one of* $p-1$ *isomorphic summands of mod* p *complex K-theory.*

(iii) $K(0)_* = \mathbb{Q}$ *and for* $n > 0$, $K(n)_* = \mathbb{F}_p[v_n^{\pm}]$ *where the dimension of* v_n *is* $2p^n - 2$*. This ring is a graded field in the sense that every graded module over it is free. For each* $n \geqslant 0$, $K(n)_*(X)$ *is a module over* $K(n)_*$*.*

(iv) $K(n)_*(X \times Y) \cong K(n)_*(X) \otimes_{K(n)_*} K(n)_*(Y)$.

(v) Let X *be a* p*-local finite CW-complex. If* $\overline{K(n)}_*(X) = 0$*, then* $\overline{K(n-1)}_*(X) = 0$.

(vi) Let X *be a* p*-local finite CW-complex.*

$$\overline{K(n)}_*(X) = K(n)_* \otimes_{\mathbb{F}_p} \overline{H}_*(X; \mathbb{F}_p)$$

for n *sufficiently large. In particular, it is nontrivial if* X *is simply connected and not contractible.*

Definition 3.2 (Type)**.** A p-local finite complex X has **type** n if n is the smallest integer such that $\overline{K(n)}_*(X)$ is nontrivial. If X is contractible, it has **type** ∞.

Besides the types, Morava K-theories are useful in detecting periodic self-maps of a spectrum, which will give finer structures of the homotopy groups.

Theorem 3.3 (Periodicity theorem)**.** *Let* X, Y *be* p*-local finite CW-complexes of type* n *for finite* n*.*

(i) There is a map $f\colon \Sigma^{d+i}X \to \Sigma^i X$ *for some* $i \geqslant 0$ *such that* $K(n)_*(f)$ *is an isomorphism and* $K(m)_*(f) = 0$ *for* $m \neq n$*. (We will refer to such a map as a* v_n***-map****) When* $n = 0$ *then* $d = 0$*, and when* $n > 0$ *then* d *is a multiple of* $2p^n - 2$*.*

(ii) Suppose $h\colon X \to Y$ *is a continuous map. Assume that both have been suspended enough times to be the target of a* v_n*-map. Let* $g\colon \Sigma^e Y \to Y$ *be a self-map as in (i). Then there are positive integers* i *and* j *with* $di = ej$ *such that the following diagram commutes up to homotopy.*

$$\begin{tikzcd}
\Sigma^{di}X \arrow[r, "\Sigma^{di}h"] \arrow[d, "f^i"'] & \Sigma^{di}Y \arrow[d, "g^j"] \\
X \arrow[r, "h"'] & Y
\end{tikzcd}$$

If we take $h = Id_X$, then the second part of the periodicity theorem says that the v_n-map is unique up to powers.

4 Geometric Chromatic Filtration and Telescope Conjecture

Theorem 2.4 tells us that there is a filtration of the homotopy group:

$$\mathscr{C}_0^a(X) := \pi_*(X)$$
$$\mathscr{C}_n^a(X) := \ker\big(\pi_*(X) \to \pi_*(L_n X)\big)$$
$$\mathscr{C}_0^a(X) \supset \mathscr{C}_1^a(X) \supset \mathscr{C}_2^a(X) \supset \cdots$$

This is called the algebraic chromatic filtration, but $K(n)$ and E_n are so manufactured. In this section, we are aiming to give a geometric model for this filtration.

Lemma 4.1. *Suppose X has type n. Then the cofiber W of the map $f\colon \Sigma^{d+i}X \to \Sigma^i X$ given by Theorem 3.3 has type $n+1$.*

Proof. For each m, we have a long exact sequence:

$$\cdots \to \overline{K(m)}_t(\Sigma^{d+i}X) \xrightarrow{f_*} \overline{K(m)}_t(\Sigma^i X) \to \overline{K(m)}_t(W) \to \overline{K(m)}_{t-1}(\Sigma^{d+i}X) \xrightarrow{f_*} \cdots$$

When $m < n$, $\overline{K(m)}_*(\Sigma^{d+i}X) = \overline{K(m)}_*(\Sigma^d X) = 0$, so $\overline{K(m)}_*(W) = 0$. When $m = n$, f_* are isomorphisms, so $\overline{K(m)}_*(W) = 0$ again. When $m = n+1$, $f_* = 0$, so $\overline{K(m)}_*(W) = \overline{K(m)}_{*-1}(\Sigma^{d+i}X)$ is nontrivial by Proposition 3.1. $\square$

Proposition 4.2. *Let X be a CW-complex and $X_{(p)} := L_{S\mathbb{Z}_{(p)}}X$. Then $\overline{E}_*(X_{(p)}) \cong \overline{E}_*(X) \otimes \mathbb{Z}_{(p)}$. If X is finite, $X_{(p)}$ is also finite.*

Suppose X is a p-local complex. Then each element $x \in \pi_k^S(X)$ has infinite order or order p^i for some i. If y has infinite order, then it has a nontrivial image in $\pi_k^S(X) \otimes \mathbb{Q}$, which is left for rational homotopy theory.

On the other hand, if x has order p^i for some i, then the composite (Here we omit the suspension for simplicity)

$$S^k \xrightarrow{p^i} S^k \xrightarrow{x} X$$

is null-homotopic. Technically, we localize at p here. Then x factors through the cofiber $W(1)$ of $p^i\colon S^k_{(p)} \to S^k_{(p)}$. Note that the sphere spectrum has type 0 and the map p^i is a v_0-map. Thus, $W(1)$ has type 1. Therefore, it admits a v_1-map $f_1\colon \Sigma^{d_1}W(1) \to W(1)$. Hence, we have the following diagram:

$$\begin{tikzcd}
S^k_{(p)} \arrow[r, "p^i"] & S^k_{(p)} \arrow[r, "y"] \arrow[d] & X \\
\Sigma^{d_1} W(1) \arrow[r, "f_1"] & W(1) \arrow[ur, dashed, "g_1"'] &
\end{tikzcd}$$

If the composite of g_1 and all powers of f_1 are not null-homotopic, then g_1 has a nontrivial image in $v_1^{-1}[W(1), Y]^S_*$, which is the colimit

$$[W(1), X]^S_* \xrightarrow{f_1^*} [\Sigma^{d_1} W(1), X]^S_* \xrightarrow{f_1^*} [\Sigma^{2d_1} W(1), X]^S_* \xrightarrow{f_1^*} \cdots$$

On the other hand, if $g_1 f_1^{i_1}$ is null-homotopic for some i_1. Let $W(2)$ be the cofiber of the map $f_1^{i_1} \colon \Sigma^{d_1 i_1} W(1) \to W(1)$. Iterating this process we get a diagram:

$$\begin{tikzcd}
S^k_{(p)} \arrow[r, "p^i"] & S^k_{(p)} \arrow[r, "x"] \arrow[d] & X \\
\Sigma^{d_1 i_1} W(1) \arrow[r, "f_1^{i_1}"] & W(1) \arrow[ur, dashed, "g_1"] \arrow[d] & \\
\Sigma^{d_2 i_2} W(2) \arrow[r, "f_2^{i_2}"] & W(2) \arrow[uur, dashed, "g_2"'] \arrow[d] & \\
 & \vdots &
\end{tikzcd}$$

Definition 4.3 (Geometric chromatic filtration)**.** If an element $x \in \pi_*^S(X)$ extends to a p-local complex $W(n)$ of type n, then x is **v_{n-1}-torsion**. If in addition x does not extend to a p-local complex of type $n+1$, it is **v_n-periodic**. The **geometric chromatic filtration** of $\pi_*^S(X)$ is the decreasing family of subgroups consisting of the v_n-torsion elements for various $n \geqslant 0$.

Conjecture 4.4 (Telescope Conjecture)**.** *The algebraic chromatic filtration is the same with the geometric chromatic filtration.*

Finally, we want to talk about why this conjecture is called ”telescope” and interpret the geometric filtration in the viewpoint of Bousfield localization.

Definition 4.5 (Telescope of a self-map)**.** Let $f \colon \Sigma^d X \to X$ be a self-map. Then the **telescope of** f is the homotopy colimit

$$\hat{X} := f^{-1} X := \text{hocolim}\big(X \xrightarrow{\Sigma^{-d} f} \Sigma^{-d} X \xrightarrow{\Sigma^{-2d} f} \Sigma^{-2d} X \xrightarrow{\Sigma^{-3d} f} \cdots\big)$$

By Theorem 3.3(ii), $\hat{X}$ is independent of the choice of f, since the v_n-maps are unique up to powers.

In analogy with algebra, this likes

$$M[f^{-1}] = \mathrm{colim}(M \xrightarrow{f} M \xrightarrow{f} M \to \cdots)$$

where M is an R-module and $0 \neq f \in R$.

Definition 4.6 (Telescope Localization)**.** Let $Tel(n) := f_n^{-1}W(n)$, where $W(n)$ is defined as above. Define the **telescope localization** by

$$L_n^f X := L_{Tel(0)\vee\cdots\vee Tel(n)} X$$

We will prove that this definition does not rely on the choice of $W(n)$ when we talk more about Bousfield localization. Actually, we can take arbitrary p-local finite CW-complex of type n. That is why we take the p-localization at the beginning of the construction.

Example 4.7. If X is of type n with v_n-self map f, then $L_n^f X \cong \hat{X}$. See [Lur10, Lecture 28, Proposition 1]. That is why this is called the "telescope" localization.

Now suppose $x \in \pi_k(X)$ is v_0-torsion, i.e., it can factor through $W(1)$ defined above. Due to [Rav92, Proposition 7.2.6], $\hat{S}^k_{(p)} \wedge W(1)$ is contractible, which is similar to $p^{-1}S \otimes S/p = 0$ in algebra. Therefore, $W(1)$ is $\hat{S}^k_{(p)}$-acyclic. Since $L_0^f X = L_{\hat{S}^k_{(p)}} X$ is $\hat{S}^k_{(p)}$-local, $[W(1), L_{\hat{S}^k_{(p)}} X] = 0$. Hence, x has trivial image in $\pi_*(X) \to \pi_*(L_n^f X)$. Conversely, $Tel(0) = p^{-1}S_{(p)}$, so $H_*(p^{-1}S_{(p)}) = p^{-1}\mathbb{Z}_{(p)} = \mathbb{Q}$. Therefore, $Tel(0) = S\mathbb{Q} = H\mathbb{Q} = K(0)$. If x has trivial image in $\pi_*(X) \to \pi_*(L_0^f X)$, then it factors through the fiber of $X \to L_0^f X$. Since the fiber is $Tel(0)$-acyclic, it has type $\geqslant 1$, so x is v_0-torsion.

This is true for the general case with more knowledge about Bousfield localization. Therefore, under the viewpoint of localization, the geometric chromatic filtration becomes

$$\mathscr{C}_0^g(X) := \pi_*(X)$$
$$\mathscr{C}_n^g(X) := \ker\big(\pi_*(X) \to \pi_*(L_n^f X)\big)$$
$$\mathscr{C}_0^g(X) \supset \mathscr{C}_1^g(X) \supset \mathscr{C}_2^g(X) \supset \cdots$$

And the telescope conjecture says that $\mathscr{C}_n^g = \mathscr{C}_n^a$ or $L_n^f = L_n$ in other word. By above discussion, this is true when $n = 0$. When $n = 1$, the case of $p > 2$ is proved by Miller and the case of $p = 2$ is proved by Mahowald [Bea19, Part III].

The geometric side is more natural and conceptual while the algebraic side is more manufactured and computable. For example, we do not have a chromatic convergence theorem for $L_n^f X$ and the Adams-Novikov spectral sequence may not converge for $\pi_*(L_n^f X)$, so $\pi_*(L_n^f X)$ is hard to compute.

Now suppose that $x\colon S^k \to X$ is v_n-periodic and that it extends to $g_n\colon W(n) \to X$. Suppose $e\colon S^K \to W(n)$ is the bottom cell in $W(n)$. Then for each i, we have a composition

$$S^{K+d_n i} \xrightarrow{\Sigma^{d_n i} e} \Sigma^{d_n i} W(n) \xrightarrow{f_n^i} W(n) \xrightarrow{g_n} X$$

We can play the same game as above to get a nontrivial element in $\pi_*^S(X)$.

Definition 4.8 (v_n-periodic family)**.** Given a v_n-periodic element $x \in \pi_*^S(X)$, the element described above for various $i > 0$ constitute the v_n-periodic family associated with x.

5 Thick Category Theorem

5.1 The category CΓ

Let $L \cong \mathbb{Z}[x_1, x_2, \cdots]$ be the Lazard ring and $G(x, y)$ be the universal formal group law over L.

Definition 5.1. Let Γ be the group of power series over $\mathbb{Z}$ having the form $\gamma = x + b_1 x + b_2 x + \cdots$ where $b_1, b_2, \cdots \in \mathbb{Z}$. Then Γ acts on L by the following. Note that $\gamma^{-1}\Big(G\big(\gamma(x), \gamma(y)\big)\Big) \in FGL(L)$. It is determined by a homomorphism $L \to L$. Since γ is invertible, this endomorphism is an automorphism, which is the desired action.

Let MU be the complex cobordism theory. Then Γ also acts naturally on $MU_*(X)$ compatibly with the action on MU_*.

Remark. *According to [Rav92, Section 3.3], this action is an analogy to the action of the group of multiplicative cohomology operations. For example, in the mod 2 case, the action by the total Steenrod square $\sum_{i\geqslant 0} Sq^i$ is determined by its effect on the generator of $H^1(\mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z})$.*

Definition 5.2. Let $\mathbf{C\Gamma}$ be the category of finitely presented graded L-modules equipped with an action of Γ compatible with its action on L. Let $\mathbf{FH}$ be the category of finite CW-complexes and homotopy classes of maps between them.

Therefore, MU_* is a functor from $\mathbf{FH}$ to $\mathbf{C\Gamma}$, which is more accessible and is the main object in this subsection.

Let $v_n \in L$ denote the coefficient of x^{p^n} in the p-series for the universal formal group law. It can be shown that v_n can serve as a polynomial generator in dimension $2p^n - 2$ [Lur10, Lecture 13, Proposition 1]. Let $I_{p,n} \subset L$ denote the prime ideal $(p, v_1, \cdots, v_{n-1})$.

Theorem 5.3 (Invariant Prime Ideal Theorem)**.** *The only prime ideals in L which are invariant under the action of Γ are the $I_{p,n}$ defined above, where p is a prime integer and $n \in \mathbb{N}$, possibly ∞. By convention, $I_{p,0} = 0$.*

Moreover, $(L/I_{p,n})^\Gamma = \mathbb{F}_p[v_n]$ for $n > 0$ and $L^\Gamma = \mathbb{Z}$.

Proof. For references, see [Rav92, Theorem 3.3.6]. □

Theorem 5.4 (Landweber Filtration Theorem)**.** *Every module M in $\mathbf{C\Gamma}$ admits a finite filtration by submodules in $\mathbf{C\Gamma}$ as above in which each subquotient is isomorphic to a suspension (recall these modules are graded) of $L/I_{p,n}$ for some prime p and finite n.*

Proof. For references, see [Rav92, Theorem 3.3.7]. □

Remark. *A finitely generated module M over a Noetherian ring R has a finite filtration with each subquotient equals to R/I for some prime ideal I. Note that L is not Noetherian, but it is a limit of Noetherian rings, so finitely presented modules over it admits similar filtrations. That is why we define $\mathbf{C\Gamma}$ to be the category of such modules.*

Corollary 5.5. *Suppose M is a p-local module in $\mathbf{C\Gamma}$ and $x \in M$.*

(a) If x is annihilated by some power of v_n, then it is annihilated by some power of v_{n-1}, so if $v_n^{-1}M = 0$, then $v_{n-1}^{-1}M = 0$.

(b) If x is nonzero, then there is an n so that $v_n^k x \neq 0$ for all k, so if M is nontrivial, then so is $v_n^{-1}M$ for all sufficiently large n.

(c) If $v_{n-1}^{-1}M = 0$, then there is a positive integer k such that multiplication by v_n^k in M commutes with the action of Γ.

(d) *Conversely, if $v_{n-1}^{-1}M$ is nontrivial, then there is no positive integer k such that multiplication by v_n^k in M commutes with the action of Γ on x.*

Proof. See [Rav92, Corollary 3.3.9]. □

The first two statements are similar to the one of Morava K-theory. In fact, for a finite p-local CW-complex X, $v_n^{-1}\overline{MU}_*(X)_{(p)} = 0$ if and only if $\overline{K(n)}_*(X) = 0$. One can replace $K(n)_*$ by $v_n^{-1}MU_{(p)}$ in the statement of the periodicity theorem. The third statement is an analogy of the periodicity theorem.

Definition 5.6. A p-local module M in $\mathbf{C\Gamma}$ has **type** n if n is the smallest integer with $v_n^{-1}M$ nontrivial. A homomorphism $f\colon \Sigma^d M \to M$ in $\mathbf{C\Gamma}$ is a v_n-map if it induces an isomorphism in $v_n^{-1}M$ and the trivial homomorphism in $v_m^{-1}M$ for $m \neq n$.

Corollary 5.7. *If M in $\mathbf{C\Gamma}$ is a p-local module with $v_{n-1}^{-1}M$ nontrivial, then M does admit a v_n-map.*

Proof. See [Rav92, Corollary 3.3.11]. □

5.2 Thick Subcategories

Definition 5.8 (Thick Subcategory)**.** A full subcategory $\mathbf{C}$ of $\mathbf{C\Gamma}$ is **thick** is it satisfies that if

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence, then M is in $\mathbf{C}$ if and only if M', M'' are in it.

A full subcategory $\mathbf{F}$ of $\mathbf{FH}$ is **thick** if it satisfies the following axioms:

(a) If

$$X \xrightarrow{f} Y \to C_f$$

is a cofiber sequence in which two of the three spaces are in $\mathbf{F}$, then so is the third.

(b) If $X \vee Y$ is in $\mathbf{F}$ then so are X and Y.

Using Landweber filtration theorem we can prove that

Theorem 5.9. *Let $\mathbf{C}$ be a thick subcategory of $\mathbf{C\Gamma}_{(p)}$, the subcategory of all p-local modules in $\mathbf{C\Gamma}$. Then $\mathbf{C}$ is either all of $\mathbf{C\Gamma}_{(p)}$, or consists of all p-local modules M in $\mathbf{C\Gamma}$ with $v_{n-1}^{-1}M = 0$. We denote the latter category by $\mathbf{C\Gamma}_{p,n}$.*

Proof. See [Rav92, Theorem 3.4.2]. □

There is an analogous result about thick subcategories of $\mathbf{FH}_{(p)}$.

Theorem 5.10 (Thick Category Theorem)**.** *Let* $\mathbf{F}$ *be a thick subcategory of* $\mathbf{FH}_{(p)}$*, the category of* p*-local finite CW-complexes. Then* $\mathbf{F}$ *is either all of* $\mathbf{FH}_{(p)}$*, the trivial subcategory or consists of all* p*-local finite CW-complexes* X *with* $\overline{K(n)}_*(X) = 0$*, which is equivalent to say that* $v_{n-1}^{-1}\overline{MU}_*(X) = 0$*. We denote the latter category by* $\mathbf{FH}_{p,n}$*.*

Therefore, we have two sequences of thick subcategories, where $MU_*(\cdot)$ sends one to the other.

$$\mathbf{FH}_{(p)} = \mathbf{FH}_{p,0} \supset \mathbf{FH}_{p,1} \supset \mathbf{FH}_{p,2} \supset \cdots \supset *$$
$$\mathbf{C\Gamma}_{(p)} = \mathbf{C\Gamma}_{p,0} \supset \mathbf{C\Gamma}_{p,1} \supset \mathbf{C\Gamma}_{p,2} \supset \cdots \supset 0$$

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