

A partial description of the chromatic support of \mathbb{E}_n -algebras

Abstract

For a commutative ring spectrum, using power operations, Hahn proved that its chromatic support is an interval containing zero. A natural question is how the chromatic support of \mathbb{E}_n -algebras looks like. In this article, we give a partial description of the problem.

Contents

1	Introduction	2
2	A filtration on BC_p	2
3	Power operations in universal torsion \mathbb{E}_n-algebras	4
4	Chromatic support of \mathbb{E}_n-algebras	6
References		10

1 Introduction

Throughout, we fix a prime number p and $0 < m < h$ in \mathbb{Z} .

In [Hah22], Hahn proved that the chromatic support of an \mathbb{E}_∞ -algebra is an interval containing zero. In other words, we have the following theorem.

Theorem 1.1 (Hahn). *Suppose $R \in \text{CAlg}(\text{Sp})$. If R is $K(m)$ -acyclic, then R is also $K(m+1)$ -acyclic.*

Remark 1.2. This theorem inspires us to define the **chromatic height** for commutative ring spectra to be the upper bound of the chromatic support, which provides a well-behaved notion for the chromatic complexity. Using the notion of chromatic height, we are able to formulate the chromatic redshift conjecture for commutative ring spectra, which was solved in the series of works [BSY22], [CMNN22], [LMMT24] and [Yua21]. \square

A natural question is that

Question. *What does the chromatic support of an \mathbb{E}_n -algebra look like?*

We will give a partial answer to this problem here ([Corollary 4.5](#)).

Theorem 1.3. *Let $0 < m < h$. Suppose $R \in \text{Alg}_{\mathbb{E}_{2n}}(\text{Sp})$ and R is $K(m)$ -acyclic. Thus, $u_m^k = 0$ in $\pi_0(L_{K(h)}(R \otimes E_h)/(p, u_1, \dots, u_{m-1}))$ for some $k \in \mathbb{Z}_{\geq 0}$. If $k(p^h - p^{m+1} + p - 1) \leq n$, then R is $K(h)$ -acyclic. In particular, if $k(p - 1) \leq n$, then R is $K(m+1)$ -acyclic.*

The proof of the theorem is a simple combination of the techniques used in the two papers by Hahn ([Hah17] and [Hah22]). In particular, we generalize the method in [Hah17] of using power operations on the universal \mathbb{E}_n -torsion-algebra to odd primes via a filtration on BC_p . After that, we conclude by computing the power operation depending on the result [Hah22, Proposition 4.5].

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2 A filtration on BC_p

Suppose $n \in \mathbb{Z}_{\geq 1}$ and $\mu_p \in \mathbb{C}^*$ be a primitive p -th root of unity. We have a C_p -action on $S^{2n-1} \hookrightarrow \mathbb{C}^n$ by multiplication by μ_p . Let $BC_p^{(2n)}$ be the homotopy cofiber of the canonical map $S^{2n-1} \rightarrow S_{hC_p}^{2n-1}$ and $BC_p^{(2n-1)} := S_{hC_p}^{2n-1}$. When $p = 2$, we have $BC_p^{(n)} \simeq \mathbb{RP}^n$.

Let $S^{2n-1} \rightarrow S^{2n+1}$ be the natural inclusion induced from $\mathbb{C}^n \rightarrow \mathbb{C}^{n+1}$. This map is C_p -equivariant, so it induces a map $S_{hC_p}^{2n-1} \rightarrow S_{hC_p}^{2n+1}$ and further a diagram in \mathcal{S}_*

$$\begin{array}{ccccc} S^{2n-1} & \longrightarrow & S_{hC_p}^{2n-1} & & \\ \downarrow & & \downarrow & & \\ * & \xrightarrow{\Gamma} & BC_p^{(2n)} & \dashrightarrow & S_{hC_p}^{2n+1} \end{array} \quad (1)$$

where the dashed arrow exists because $\pi_{2n-1}(S_{hC_p}^{2n+1}) = 0$. Composing with the inclusion $S_{hC_p}^{2n+1} \rightarrow S_{hC_p}^\infty \simeq BC_p$, we get a map $BC_p^{(n)} \rightarrow BC_p$ for each $n \in \mathbb{Z}_{\geq 1}$. Let $BC_{p,n+1}$ denote the cofiber of this map.

Similarly, if $a \in S^{2n-1}$, then $(a, \mu_p a, \mu_p^2 a, \dots, \mu_p^{p-1} a)$ is a point in $\text{Conf}_p(\mathbb{R}^{2n})$ by identifying $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, so we get a C_p -equivariant map $S^{2n-1} \rightarrow \text{Conf}_p(\mathbb{R}^{2n})$, where C_p acts on $\text{Conf}_p(\mathbb{R}^{2n})$ by translation of coordinates. Since $\pi_{2n-1}(\text{Conf}_p(\mathbb{R}^{2n+1})) = 0$, we get the following diagram

$$\begin{array}{ccccccc} S^{2n-1} & \longrightarrow & S_{hC_p}^{2n-1} = BC_p^{(2n-1)} & \xrightarrow{\gamma_{2n-1}} & \text{Conf}_p(\mathbb{R}^{2n})_{hC_p} & & \\ \downarrow & & \downarrow & & \downarrow & & \\ * & \xrightarrow{\Gamma} & BC_p^{(2n)} & \dashrightarrow & \text{Conf}_p(\mathbb{R}^{2n+1})_{hC_p} & & \end{array}$$

Now suppose E is a Morava E-theory of height $h > 0$ with an orientation $T \in E^2[\mathbb{CP}^\infty]$. From the fiber sequence

$$S^1 \rightarrow S_{hC_p}^{2n-1} \rightarrow \mathbb{CP}^{n-1}$$

and the Gysin sequence, we obtain that

$$E^0(S_{hC_p}^{2n-1}) \cong E_0[\![T]\!]/([p](T), T^n) \quad \text{and} \quad E^1(S_{hC_p}^{2n-1}) \cong E_0.$$

Since $BC_p^{(2n)}$ is the cofiber of the map $S^{2n-1} \rightarrow S_{hC_p}^{2n-1}$, we obtain an exact sequence

$$0 \rightarrow E^{-1}(BC_p^{(2n)}) \rightarrow E_0 \xrightarrow{p} E_0 \rightarrow E^0(BC_p^{(2n)}) \rightarrow E_0[\![T]\!]/([p](T), T^n) \rightarrow 0.$$

Thus, $E^0(BC_p^{(2n)})$ fits into a short exact sequence

$$0 \rightarrow E_0/p \rightarrow E^0(BC_p^{(2n)}) \rightarrow E_0[\![T]\!]/([p](T), T^n) \rightarrow 0.$$

Since $E^0(S_{hC_p}^{2n+1}) \cong E_0[\![T]\!]/([p](T), T^{n+1})$ also fits into this short exact sequence by sending $1 \in E_0/p$ to T^n , we may compare to $S_{hC_p}^{2n+1}$ via the Diagram 1 and learn that

$$E^0(BC_p^{(2n)}) \cong E_0[\![T]\!]/([p](T), T^{n+1}) \quad \text{and} \quad E^1(BC_p^{(2n)}) = 0.$$

Therefore,

$$E^0(BC_{p,2n+1}) \cong T^{n+1}E_0[\![T]\!]/([p](T)) \quad \text{and} \quad E^1(BC_{p,2n+1}) = 0,$$

which are free E_0 -modules. It follows from [Rez09, 3.11 and 3.17] that the ***completed E -homology***

$$E_0^\wedge(BC_{p,2n+1}) := \pi_0(L_{K(h)}E \otimes (\Sigma_+^\infty BC_{p,2n+1})) \cong E^0(BC_{p,n+1})^\vee. \quad (2)$$

3 Power operations in universal torsion \mathbb{E}_n -algebras

In this section, we recall some constructions in [Hah17, Section 2 and 5].

Let A be an \mathbb{E}_∞ -ring spectrum and $\mathrm{Alg}_{\mathbb{E}_n,A}(\mathrm{Sp})$ be the ∞ -category of \mathbb{E}_n - A -algebras. Let $BGL_1(A)$ be the maximal groupoid of the full subcategory of Mod_A consisting only A .

Definition 3.1. Suppose $x \in \pi_0(A)$ such that $1 + x \in \pi_0(A)^\times$. For each $n \geq 0$, let

$$\widetilde{1+x}: \Omega^n S^{n+1} \rightarrow BGL_1(A) \quad (3)$$

be the n -fold loop map adjoint to the map $1+x: S^1 \rightarrow BGL_1(A)$. Let $R_{n,x}^A$ be the Thom spectrum associated to this map, i.e., $R_{n,x}^A$ is the colimit of the map $\widetilde{1+x}$ in Mod_A . According to [ACB19, Corollary 3.2], $R_{n,x}^A$ promotes to an element in $\mathrm{Alg}_{\mathbb{E}_n,A}(\mathrm{Sp})$. \dashv

Theorem 3.2 ([ACB19, Theorem 4.10]). *Suppose $B \in \mathrm{Alg}_{\mathbb{E}_n,A}(\mathrm{Sp})$. If $x = 0 \in \pi_0(B)$, then we have*

$$\mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_n,A}(\mathrm{Sp})}(R_{n,x}^A, B) \simeq \Omega^{\infty+1}B.$$

Otherwise, the mapping space is contractible.

When $x = 0 \in \pi_0(B)$, then $\Omega^{\infty+1}B$ is homotopic to the space of homotopies between 0 and x in $\mathrm{Map}_{\mathrm{Mod}_A}(A, B)$. In light of this theorem, we will say $R_{n,x}^A$ is a ***universal \mathbb{E}_n - A -algebra in which $x = 0$*** .

Now we have the following diagram by looping once of Eq. (3).

$$\begin{array}{ccc}
BC_p^{(n)} & \longrightarrow & BC_p \\
\downarrow \gamma_n & & \downarrow \simeq \\
\text{Conf}_p(\mathbb{R}^{n+1})_{hC_p} & \longrightarrow & \text{Conf}_p(\mathbb{R}^\infty)_{hC_p} \\
\downarrow & & \downarrow \\
\Omega^{n+1}S^{n+1} & \longrightarrow & \Omega^\infty S^\infty \xrightarrow{\Omega^\infty(1+x)} \text{GL}_1(A) \xrightarrow{\text{GL}_1(\text{unit})} \text{GL}_1(R_{n,x}^A) \\
& & \downarrow \\
& & \Omega^\infty A \xrightarrow{\Omega^\infty(\text{unit})} \Omega^\infty R_{n,x}^A
\end{array}$$

Note that the composite map $BC_p \rightarrow \Omega^\infty A$ is exactly $P(1+x)$. We get the following lemma.

Lemma 3.3. *The composition*

$$\Sigma_+^\infty BC_p^{(n)} \rightarrow \Sigma_+^\infty BC_p \xrightarrow{P(1+x)} A \rightarrow R_{n,x}^A$$

is $1 \in (R_{n,x}^A)^0(BC_p^{(n)})$.

Corollary 3.4. *The image of $P(x) \in \text{Map}(BC_p^{(2n)}, R_{n,x}^A)$ is null-homotopic, so we have the following diagram.*

$$\begin{array}{ccc}
A & \xrightarrow{\text{unit}} & R_{n,x}^A \\
\uparrow P(x) & & \uparrow \\
\Sigma_+^\infty BC_p^{(n)} & \longrightarrow & \Sigma_+^\infty BC_p \longrightarrow BC_{p,n+1}
\end{array}$$

Proof. In light of the last lemma, it suffices to show that $P(1+x) - 1$ is homotopic to $P(x)$ after composing with the unit map $A \rightarrow R_{n,x}^A$. Note that $P(1+x)$ is computed by the composition

$$\Sigma_+^\infty BC_p \simeq \mathbb{S}_{hC_p}^{\otimes p} \xrightarrow{\Delta^{\otimes p}} (\mathbb{S} \oplus \mathbb{S})_{hC_p}^{\otimes p} \xrightarrow{(1+x)^p} A_{hC_p}^{\otimes p} \rightarrow A.$$

We have the following identification

$$(\mathbb{S} \oplus \mathbb{S})^{\otimes p} \simeq \mathbb{S} \oplus \text{Ind}(X_1) \oplus \text{Ind}(X_2) \cdots \text{Ind}(X_{p-1}) \oplus \mathbb{S},$$

where each X_i is a direct sum of \mathbb{S} . After taking the C_p -orbit, the map $X_i \rightarrow A$ is divisible by x . Furthermore, the maps $\Sigma_+^\infty BC_p \rightarrow A$ corresponding to the first and the last direct summands are 1 and $P(x)$ respectively. Since $x = 0$ in $R_{n,x}^A$, we find out that $P(1+x) - 1$

is homotopic to $P(x)$ in $R_{n,x}^A$. □

4 Chromatic support of \mathbb{E}_n -algebras

In this section, let E be a Morava E-theory of height h , so that

$$\pi_*(E) \cong W(\overline{\mathbb{F}}_p)[[u_1, \dots, u_{h-1}][u^{\pm 1}],$$

where $|u| = -2$.

Recall (from, e.g., the formula at the top of [GHMR05, p. 788]) that for each $0 \leq k < h$, $[p](T)$ satisfies the equation

$$[p](T) \equiv u_k z^{p^k} \pmod{p, \dots, u_{k-1}, z^{p^k+1}}.$$

Thus, we introduce the following notion.

Definition 4.1. For $0 \leq k < h$, let $g_k(T) \in E_0[[T]]$ be a lift of $\bar{g}_k(T) \in E_0/(p, \dots, u_{k-1})[[T]]$ such that $[p](z) = z^{p^k} \bar{g}_k(T)$ in $E_0/(p, \dots, u_{k-1})[[T]]$.

Let $I_k := (p, \dots, u_{k-1}, u_{k+1}, \dots, u_{h-1}) \subset E_0$. ⊣

Corollary 4.2. Taking reduced completed E -homology in Corollary 3.4, we have the following diagram in $\text{Mod}_{E_0}^\heartsuit$ from the results in Section 2.

$$\begin{array}{ccc} E_0 & \longrightarrow & \pi_0(L_{K(h)}R_{2n,x}^E) \\ \uparrow P(x) & & \uparrow \\ (TE_0[[T]]/[p](T))^\vee & \longrightarrow & (T^{n+1}E_0[[T]]/[p](T))^\vee \end{array}$$

Furthermore, modding out I_m , we get the following diagram.

$$\begin{array}{ccc}
K[\![u_m]\!] & \longrightarrow & \pi_0(L_{K(h)}R_{2n,x}^E/I_m) \\
\uparrow & & \uparrow \\
E_0 & \longrightarrow & \pi_0(L_{K(h)}R_{2n,x}^E) \\
\uparrow P(x) & & \uparrow \\
(T E_0[\![T]\!]/[p](T))^\vee & \longrightarrow & (T^{n+1} E_0[\![T]\!]/[p](T))^\vee \\
\uparrow & & \uparrow \\
(T K[\![u_m, T]\!]/[p](T))^\vee & \longrightarrow & (T^{n+1} K[\![u_m, T]\!]/[p](T))^\vee \\
\uparrow & & \uparrow \\
(T K[\![u_m, T]\!]/g_m(T))^\vee & \longrightarrow & (T^{n+1} K[\![u_m, T]\!]/g_m(T))^\vee
\end{array}$$

(
 $\overline{P}(x)$
)

By Weierstrass preparation theorem, there exists a unique monic polynomial $g(T) \in K[\![u_m]\!][T]$ such that $g_m(T)$, modding out I_m , is a unit multiple of $g(T)$ in $K[\![u_m]\!][T]$. Moreover, the degree of $g(T)$ is $d := p^h - p^m$ and the constant term is u_m times a unit in $K[\![u_m]\!]$ (cf. [Hah22, Proposition 4.2]). Suppose $g(T) = T^{p^h-p^m} - u_m u(T)$ for some unit $u(T) \in K[\![T]\!]$. Now the above diagram can be written as the following.

$$\begin{array}{ccc}
K[\![u_m]\!] & \longrightarrow & \pi_0(L_{K(h)}R_{2n,x}^E/I_m) \\
\uparrow \overline{P}(x) & & \uparrow \\
(T K[\![u_m, T]\!]/(g(T)))^\vee & \longrightarrow & (T^{n+1} K[\![u_m, T]\!]/(g(T)))^\vee
\end{array}$$

We pick a basis $\delta_T, \delta_{T^2}, \dots, \delta_{T^d}$ for $\left(\frac{TK[\![u_m, T]\!]}{(g(T))}\right)^\vee$ over $K[\![u_m]\!]$ and a basis $\delta_{T^{n+1}}, \dots, \delta_{T^{n+d}}$ for $\left(\frac{T^{n+1}K[\![u_m, T]\!]}{(g(T))}\right)^\vee$, where δ_{T^i} denotes the dual basis of T^i . Thus, there is a matrix $Q \in M_d(K[\![u_m]\!])$ such that $(\delta_T, \dots, \delta_{T^d})$ are mapped to $(\delta_{T^{n+1}}, \dots, \delta_{T^{n+d}})Q$. Note that for any $q \in \mathbb{Z}_{\geq 0}$ and $1 \leq r \leq d$, $T^{qd+r} = u_m^q u^q(T)T^r$ in $\frac{TK[\![u_m, T]\!]}{(g(T))}$. We can read off from this that $v_{u_m}(Q_{qd+r,r}) = q$, $v_{u_m}(Q_{qd+r,i}) \geq q$ for $i > r$ and $v_{u_m}(Q_{qd+r,i}) \geq q+1$ for $i < r$, where v_{u_m} is the u_m -valuation on $K[\![u_m]\!]$.

Suppose $P(x) = a_1 T + a_2 T^2 + \dots + a_d T^d \in \frac{TK[\![u_m, T]\!]}{(g(T))}$. Thus, $\overline{P}(x)(\delta_{T^i}) = a_i$. Suppose for each i , $\delta_{T^{n+i}}$ is mapped to b_{n+i} under the map

$$(T^{n+1} K[\![u_m, T]\!]/(g(T)))^\vee \rightarrow \pi_0(L_{K(h)}R_{2n,u_m^k}^E/I_m).$$

We have

$$(a_1, \dots, a_d) = (b_{n+1}, \dots, b_{n+d})Q$$

in $\pi_0(L_{K(h)}R_{2n, u_m^k}^E / I_m)$. Let $n + 1 = qd + r$ for some $q \in \mathbb{Z}_{\geq 0}$ and $1 \leq r \leq d$. By the above computation, we learn that $u_m^q \mid a_i$ for all i and $u_m^{q+1} \mid a_i$ for $1 \leq i < r$ in $\pi_0(L_{K(h)}R_{2n, u_m^k}^E / I_m)$.

Now if we let $x = u_m^k$, then we have the following computational result for $\overline{P}(u_m^k)$.

Proposition 4.3. *We have*

$$\overline{P}(u_m^k) = T^{k(p^h - p^{m+1} + p - 1)} U(T)$$

in $K[\![u_m]\!][\![T]\!]/(g(T))$ for some unit $U(T)$.

Proof. By the same proof of [Hah22, Proposition 4.5], we have that

$$\overline{P}(u_m^k) = \left(u_m \left(\frac{1}{T^{p-1} \tilde{u}(T)} \right)^{p^m-1} \right)^k = T^{k(p^h - p^{m+1} + p - 1)} \tilde{u}^{-k(p^m-1)}(T) u^{-k}(T)$$

in $K[\![u_m]\!][\![T]\!]/(g(T))$ for some unit $\tilde{u}(T)$. □

Theorem 4.4. *When $0 < m < h$ and $k(p^h - p^{m+1} + p - 1) \leq n$, $L_{K(h)}R_{2n, u_m^k}^E \simeq 0$.*

Proof. Suppose $k(p^h - p^{m+1} + p - 1) = q'd + r'$ for some integer q' and $1 \leq r' \leq d$. Similar to above, we have that $v_{u_m}(a_{r'}) = q'$. Hence, there is a unit $u'_{r'} \in K[\![u_m]\!]^\times$, such that $a_{r'} = u_m^{q'} u'_{r'}$. Therefore, if $k(p^h - p^{m+1} + p - 1) + 1 \leq n + 1$, then $u_m^{q'+1} \mid u_m^{q'} u'_{r'}$ in $\pi_0(L_{K(h)}R_{2n, u_m^k}^E / I_m)$, which implies that $u_m^{q'} = 0$ in $\pi_0(L_{K(h)}R_{2n, u_m^k}^E / I_m)$ since u_m is in the maximal ideal. When $m > 0$, $p^h - p^{m+1} + p - 1 < d$, so $q' < k$. By infinite descent, we learn that $L_{K(h)}R_{2n, u_m^k}^E / I_m \simeq 0$. Since $L_{K(h)}R_{2n, u_m^k}^E$ is $K(h)$ -local, $L_{K(h)}R_{2n, u_m^k}^E \simeq 0$. □

Corollary 4.5. *Let $0 < m < h$. Suppose $R \in \text{Alg}_{\mathbb{E}_{2n}}(\text{Sp})$ and R is $K(m)$ -acyclic. Thus, $u_m^k = 0$ in $\pi_0(L_{K(h)}(R \otimes E) / (p, u_1, \dots, u_{m-1}))$ for some $k \in \mathbb{Z}_{\geq 0}$. If $k(p^h - p^{m+1} + p - 1) \leq n$, then R is $K(h)$ -acyclic. In particular, if $k(p - 1) \leq n$, then R is $K(m + 1)$ -acyclic.*

Proof. Since $K(m) \otimes L_{K(h)}(R \otimes E)$ is a module over $K(m) \otimes R$, $L_{K(h)}(R \otimes E)$ is also $K(m)$ -acyclic. Therefore, $L_{K(h)}(R \otimes E)$ is $T(m)$ -acyclic for some telescope $T(m)$ by [LMMT24, Lemma 2.3], which implies that $u_m^k = 0$ in $\pi_0(L_{K(h)}(R \otimes E) / (p, u_1, \dots, u_{m-1}))$. Therefore, there is an \mathbb{E}_{2n} -algebra map $L_{K(h)}R_{2n, u_m^k}^E \rightarrow L_{K(h)}(R \otimes E)$ classifying the nullhomotopy of u_m^k by Theorem 3.2. We now conclude by the previous theorem. □

Remark 4.6. When h tends to infinity, the lower bound $k(p^h - p^{m+1} + p - 1)$ also tends to infinity. However, the Hopkins–Mahowald theorem says that $R_{2,p} \simeq \mathbb{F}_p$, which is $K(h)$ -acyclic for all $h \geq 0$. Thus, we suspect the above lower bound is far from being sharp. \square

Remark 4.7. The above proof does not work when $m = 0$. However, $\overline{P}(p^k)$ is more computable as in [Hah17, Lemma 6.2]. Thus, one direction of future works is trying to give an answer to [Hah17, Question 4] depending on computations of $\overline{P}(p^k)$. \square

Remark 4.8. Another future direction is determining the exponent of $u_h^{k'}$ such that $u_h^{k'}$ vanishes in R . Therefore, we can give a better description of the chromatic support of \mathbb{E}_n -algebra. In particular, if u_{m+1}^k also vanishes in R , then the chromatic support of \mathbb{E}_n -algebras are also an interval containing 0 using the latter part of Corollary 4.5, which is same as the \mathbb{E}_∞ -case and what expected by the Hopkins–Mahowald theorem. \square

References

- [ACB19] Omar Antolín-Camarena and Tobias Barthel, *A simple universal property of Thom ring spectra*, J. Topol. **12** (2019), no. 1, 56–78. MR 3875978 [3.1](#), [3.2](#)
- [BSY22] Robert Burklund, Tomer M. Schlank, and Allen Yuan, *The chromatic Nullstellensatz*, <https://arxiv.org/abs/2207.09929>, 2022. [1.2](#)
- [CMNN22] Dustin Clausen, Akhil Mathew, Niko Naumann, and Justin Noel, *Descent and vanishing in chromatic algebraic K-theory via group actions*, <https://arxiv.org/abs/2011.08233>, 2022. [1.2](#)
- [GHMR05] P. Goerss, H.-W. Henn, M. Mahowald, and C. Rezk, *A resolution of the $K(2)$ -local sphere at the prime 3*, Ann. of Math. (2) **162** (2005), no. 2, 777–822. MR 2183282 [4](#)
- [Hah17] Jeremy Hahn, *Nilpotence in \mathbb{E}_n -algebras*, 2017. [1](#), [3](#), [4.7](#)
- [Hah22] ———, *On the bousfield classes of H_∞ -ring spectra*, 2022. [1](#), [1](#), [4](#), [4.3](#)
- [LMMT24] Markus Land, Akhil Mathew, Lennart Meier, and Georg Tamme, *Purity in chromatically localized algebraic K-theory*, J. Amer. Math. Soc. **37** (2024), no. 4, 1011–1040. MR 4777639 [1.2](#), [4.5](#)
- [Rez09] Charles Rezk, *The congruence criterion for power operations in Morava E-theory*, Homology Homotopy Appl. **11** (2009), no. 2, 327–379. MR 2591924 [2](#)
- [Yua21] Allen Yuan, *Examples of chromatic redshift in algebraic K-theory*, <https://arxiv.org/abs/2111.10837>, 2021. [1.2](#)