Computer Graphics 2018

5. Geometry and Transform

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1024

程序员节日快乐！

Today outline

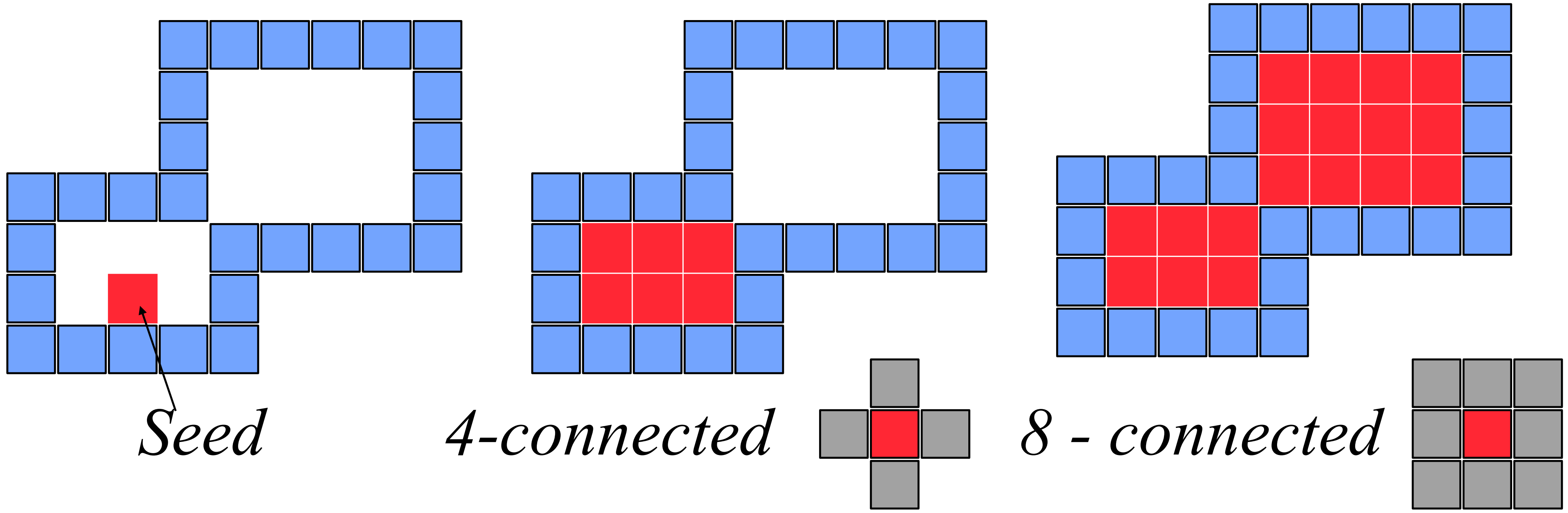
- Triangle rasterization
    - Basic vector algebra ~ geometry
- Antialiasing revisit
- Clipping
- Transforms (I)

Previous lesson

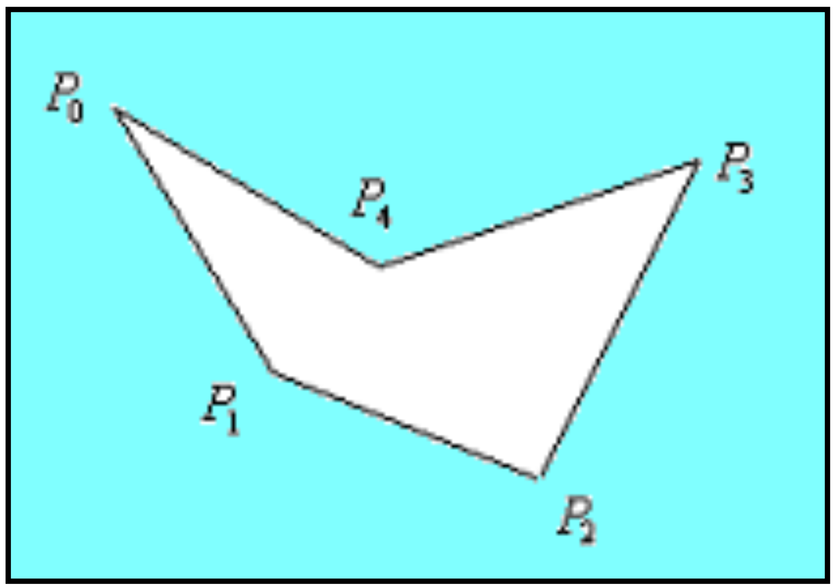
- Primitive attributes
- Rasterization and scan line algorithm
  - line,
  - general polygon

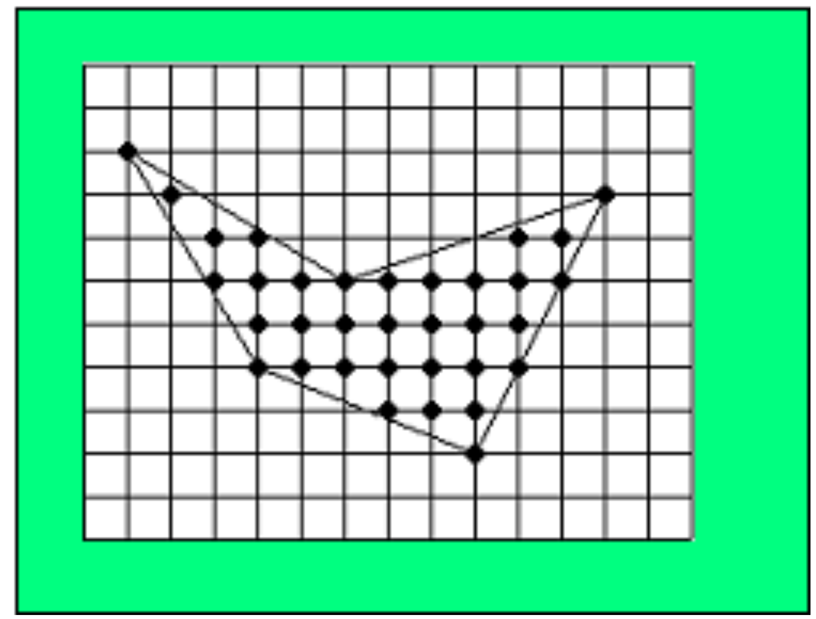
Seed Fill Algorithms

- Assumes that at least one pixel interior to the polygon is known
- It is a recursive algorithm
- Useful in interactive paint packages



Polygon filling

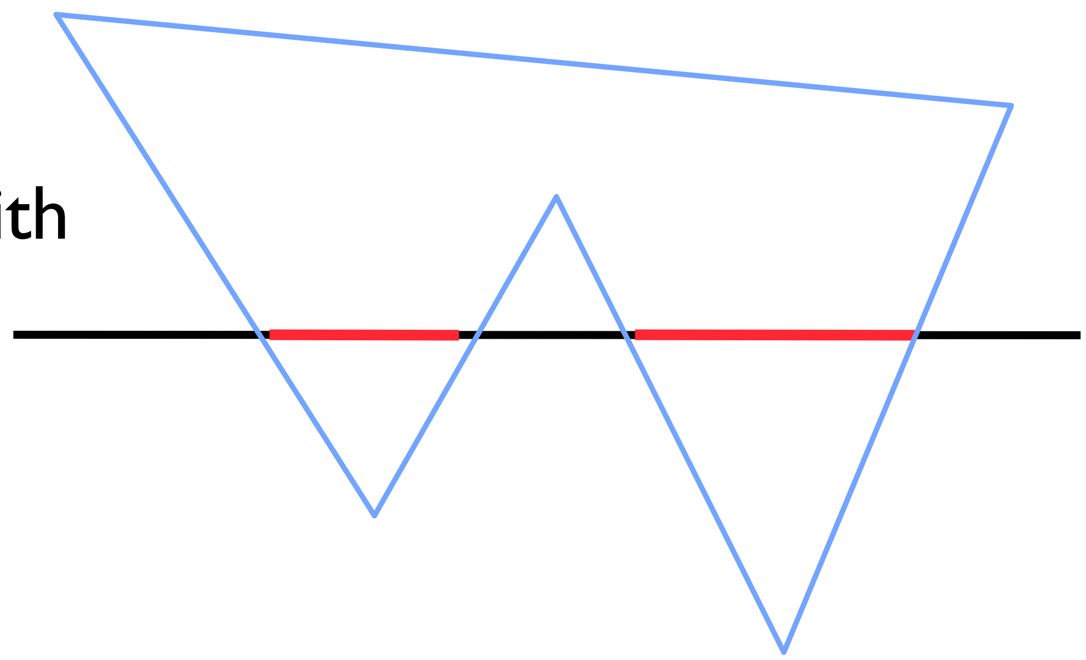
- Polygon representation
- 


- By vertex By lattice
- Polygon filling:
- vertex representation vs lattice representation

Scan Line Method

- Proceeding from left to right the intersections are paired and intervening pixels are set to the specified intensity
- Algorithm
  - Find the intersections of the scan line with all the edges in the polygon
  - Sort the intersections by increasing X-coordinates
  - Fill the pixels between pair of intersections

From top to down



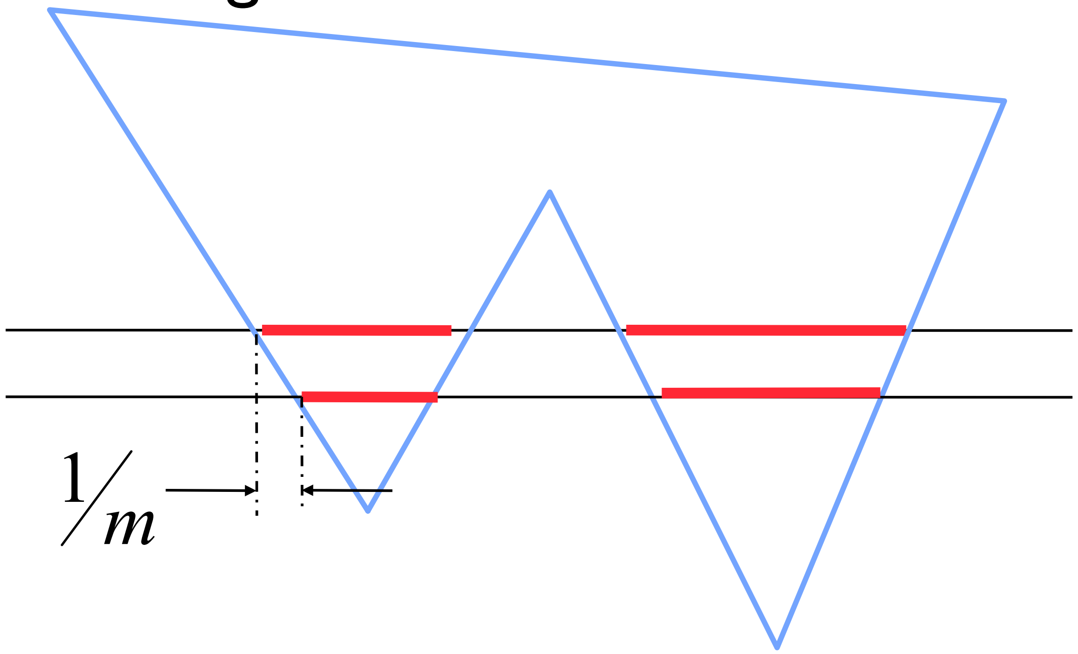
http://www.cecs.csulb.edu/~pnguyen/cecs449/lectures/fillalgorithm.pdf

Efficiency Issues in Scan Line Method

- Intersections could be found using edge coherence

  the X-intersection value x_{i+1} of the lower scan line can be computed from the X-intersection value x_i of the proceeding scan line as

$$x_{i+1} = x_i + 1/m$$



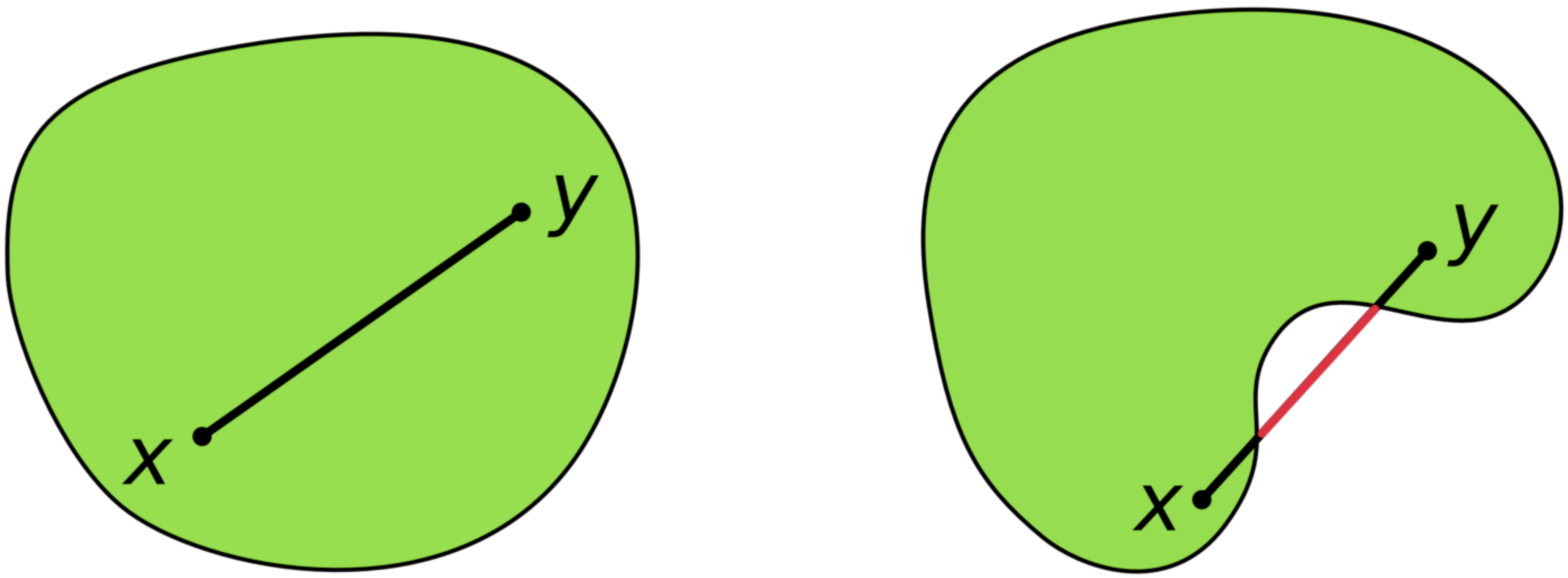
- List of active edges could be maintained to increase efficiency

Advantages of Scan Line method

- The algorithm is efficient
- Each pixel is visited only once
- Shading algorithms could be easily integrated with this method to obtain shaded area

- Efficiency could be further improved if polygons are **convex**,
- much better if they are **only triangles**

Convex?

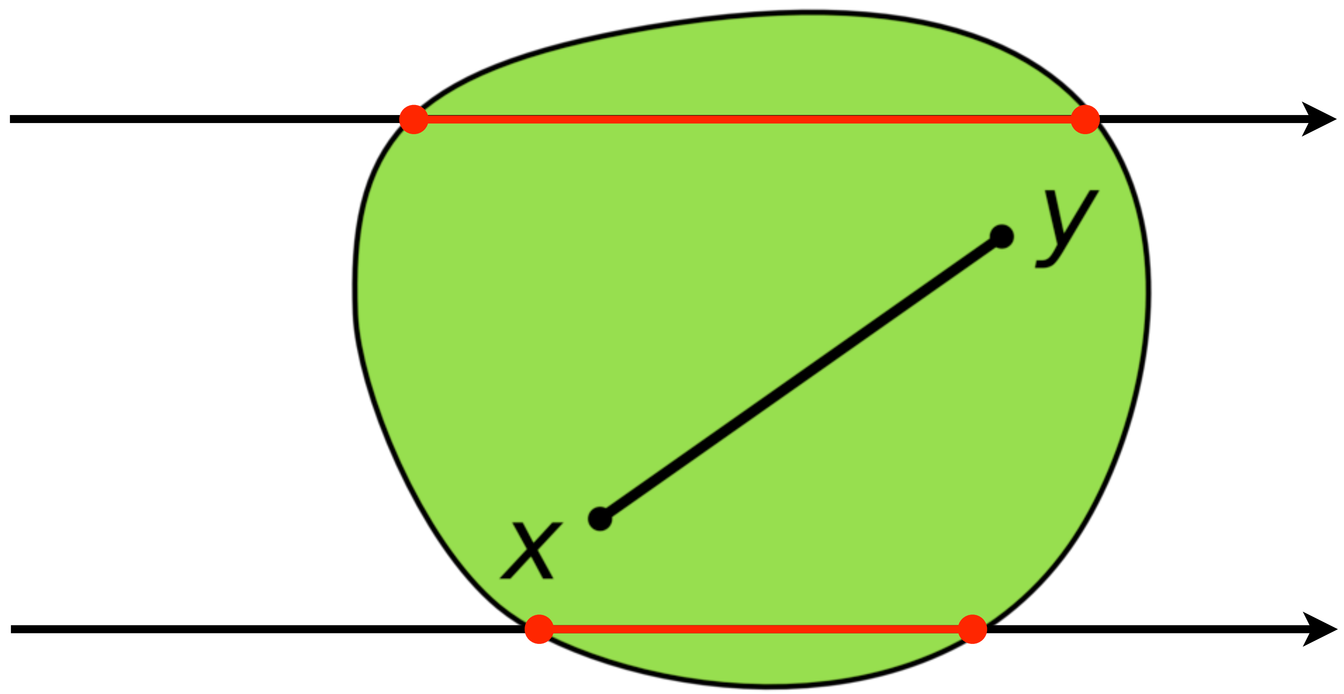


A set C in S is said to be **convex** if, for all x and y in C and all *t* in the interval [0,1], the point

$$(1 - t)\,x + t\,y$$

is in C.

Convex polygon rasterization

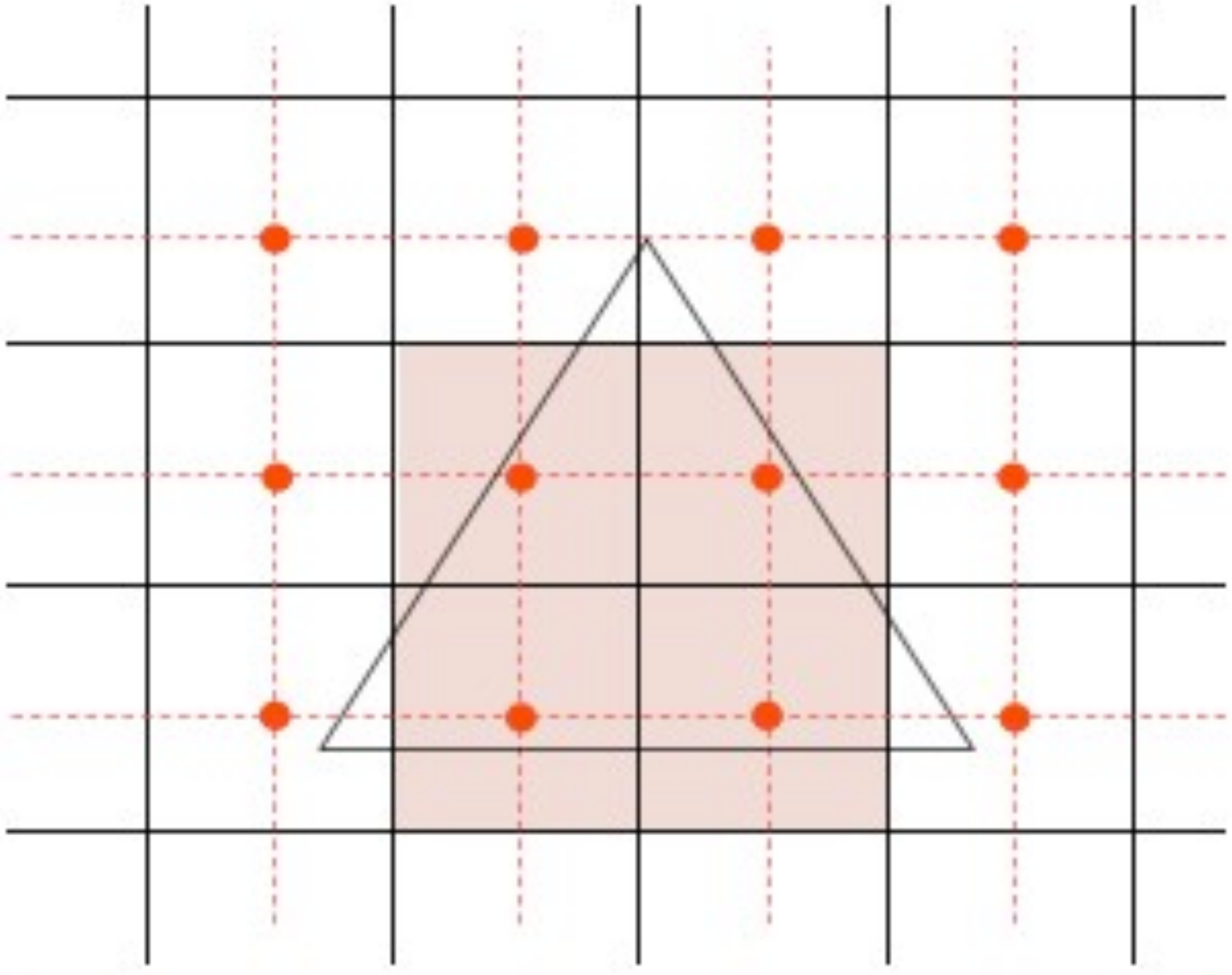


One in and one out

Triangle Rasterization

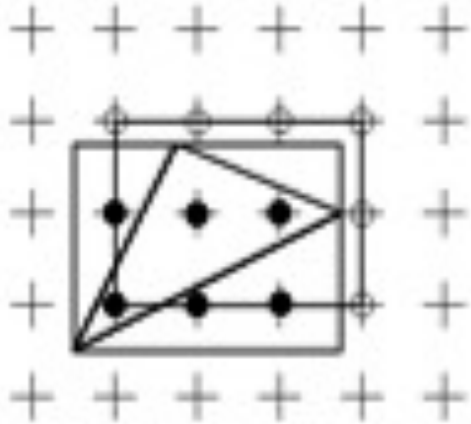
Triangle Rasterization?

Output fragment if pixel center is inside the triangle



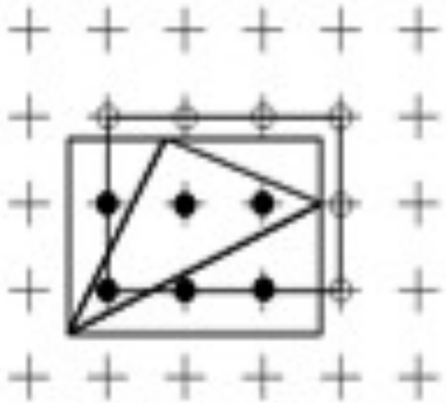
Triangle Rasterization

```
rasterize( vert v[3] )
{
  bbox b; bound3(v,b);
  for( int y=b.ymin; y<b.ymax, y++ )
    for( int x=b.xmin; x<b.xmax, x++ )
      if( inside3(v,x,y) )
        fragment(x,y);
}
```



GPUs contain triangle rasterization hardware
Can output billions of fragments per second

Compute Bounding Box

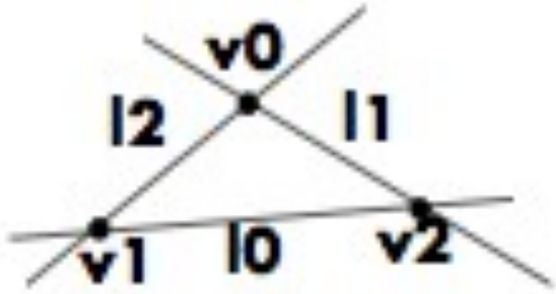


```
bound3( vert v[3], bbox& b )
{
  b.xmin = ceil(min(v[0].x, v[1].x, v[2].x));
  b.xmax = ceil(max(v[0].x, v[1].x, v[2].x));
  b.ymin = ceil(min(v[0].y, v[1].y, v[2].y));
  b.ymax = ceil(max(v[0].y, v[1].y, v[2].y));
}
```

Calculate tight bound around the triangle
Round coordinates upward (ceil) to the nearest integer

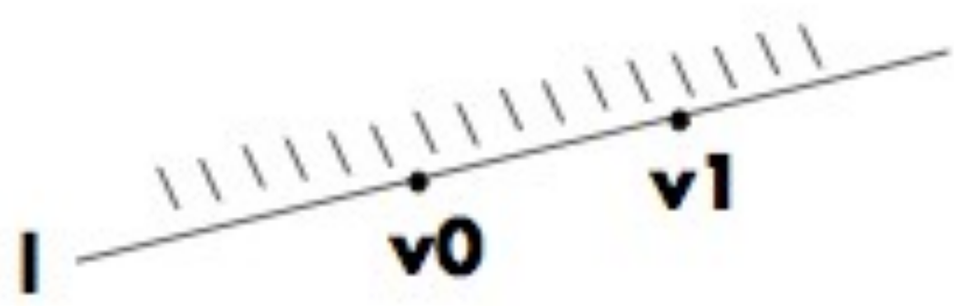
Point Inside Triangle Test

```
rasterize( vert v[3] )
{
  bbox b; bound3(v, b);
  line l0, l1, l2;

  makeline(&v[0],&v[1],&l2);
  makeline(&v[1],&v[2],&l0);
  makeline(&v[2],&v[0],&l1);

  for( y=b.ymin; y<b.ymax, y++ ) {
    for( x=b.xmin; x<b.xmax, x++ ) {
      e0 = l0.A * x + l0.B * y + l0.C;
      e1 = l1.A * x + l1.B * y + l1.C;
      e2 = l2.A * x + l2.B * y + l2.C;
      if( e0<=0 && e1<=0 && e2<=0 )
        fragment(x,y);
    }
  }
}
```



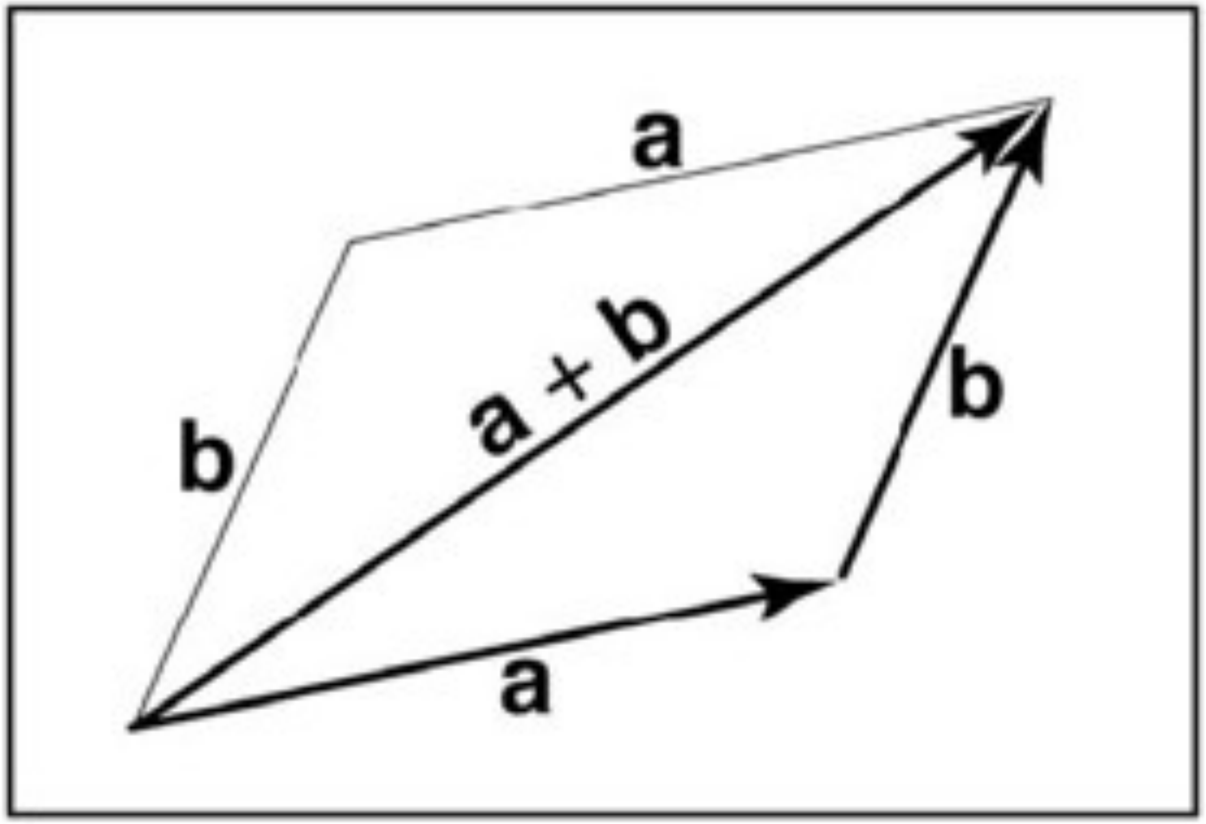
Line equation

Inside on the left for CCW polygons



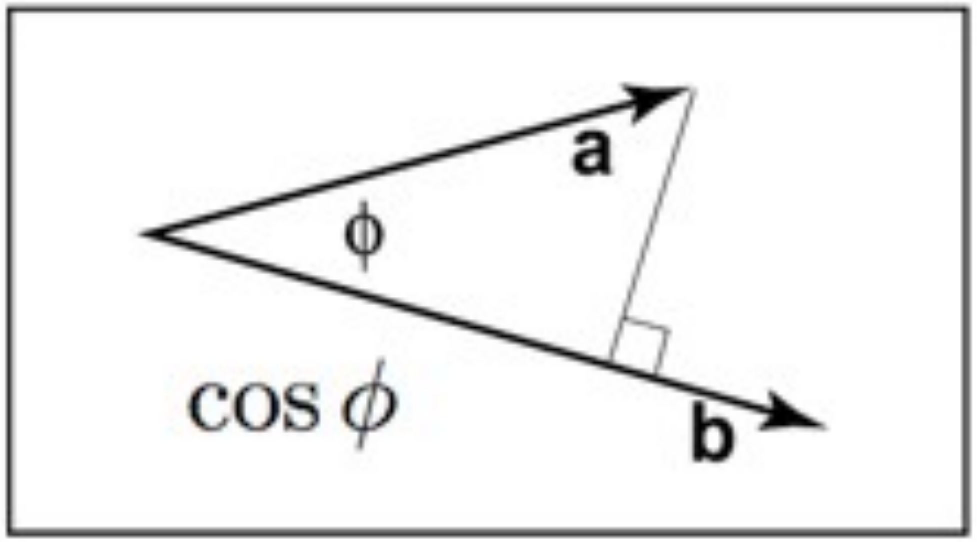

```
makeline( vert& v0, vert& v1, line& l )
{
    l.a = v1.y - v0.y;
    l.b = v0.x - v1.x;
    l.c = -(l.a * v0.x + l.b * v0.y);
}
```

The Parallelogram Rule



Vector addition define for any number of dimensions

Dot Product

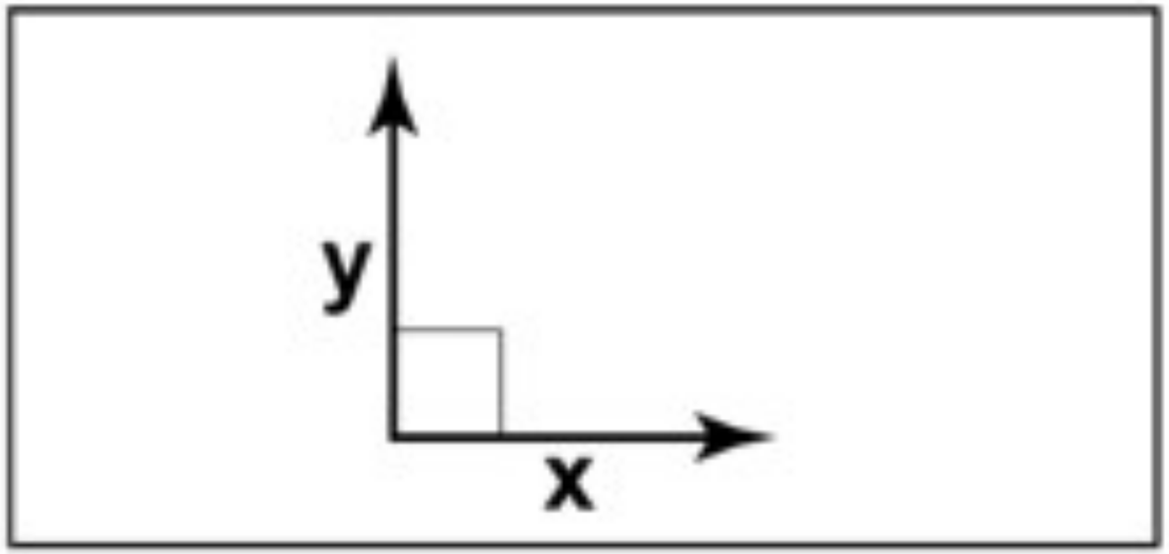


$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \phi$$

The projection of a onto b

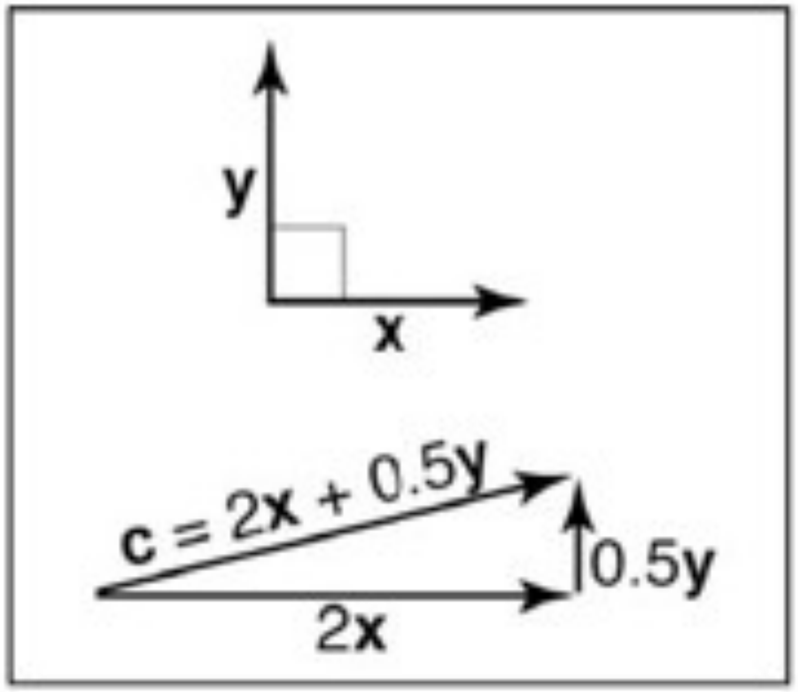
N. B. the projection is 0 if a is perpendicular to b

Orthonormal Vectors



Perpendicular	$\mathbf{x} \cdot \mathbf{y}$	$=$	0
Unit length	$\mathbf{x} \cdot \mathbf{x}$	$=$	1
	$\mathbf{y} \cdot \mathbf{y}$	$=$	1

Coordinates and Vectors



$$\mathbf{c} = \alpha \mathbf{x} + \beta \mathbf{y}$$

$$\begin{aligned} \alpha &= \mathbf{x} \cdot \mathbf{c} = \alpha\, \mathbf{x} \cdot \mathbf{x} + \beta\, \mathbf{x} \cdot \mathbf{y} \\ \beta &= \mathbf{y} \cdot \mathbf{c} = \alpha\, \mathbf{y} \cdot \mathbf{x} + \beta\, \mathbf{y} \cdot \mathbf{y} \end{aligned}$$

Dot product between two vectors

$$\mathbf{a} = x_a \mathbf{x} + y_a \mathbf{y}$$

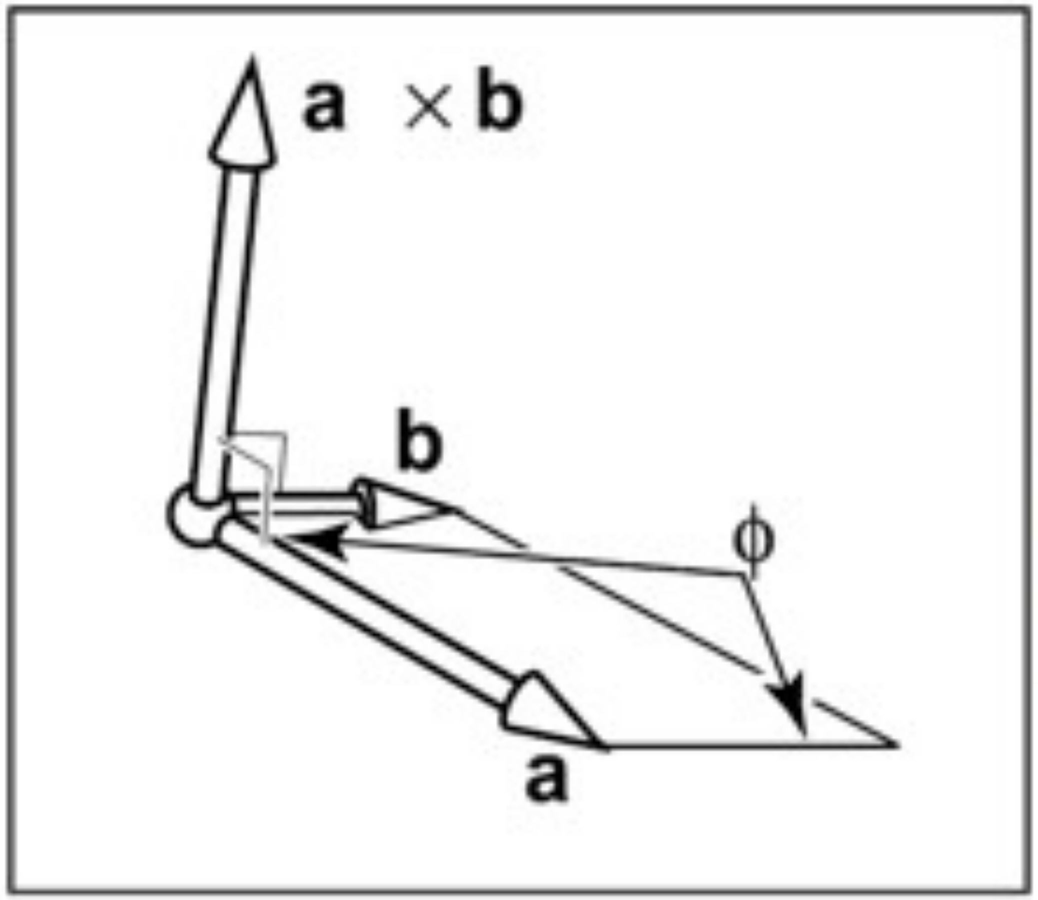
$$\mathbf{b} = x_b \mathbf{x} + y_b \mathbf{y}$$

$$\mathbf{a} \cdot \mathbf{b} = x_a x_b + y_a y_b$$

$$\mathbf{a} \cdot \mathbf{a} = x_a^2 + y_a^2 = |\mathbf{a}|^2$$

$$|\mathbf{a}| = \sqrt{x_a^2 + y_a^2} = \sqrt{\mathbf{a} \cdot \mathbf{a}}$$

Cross Product

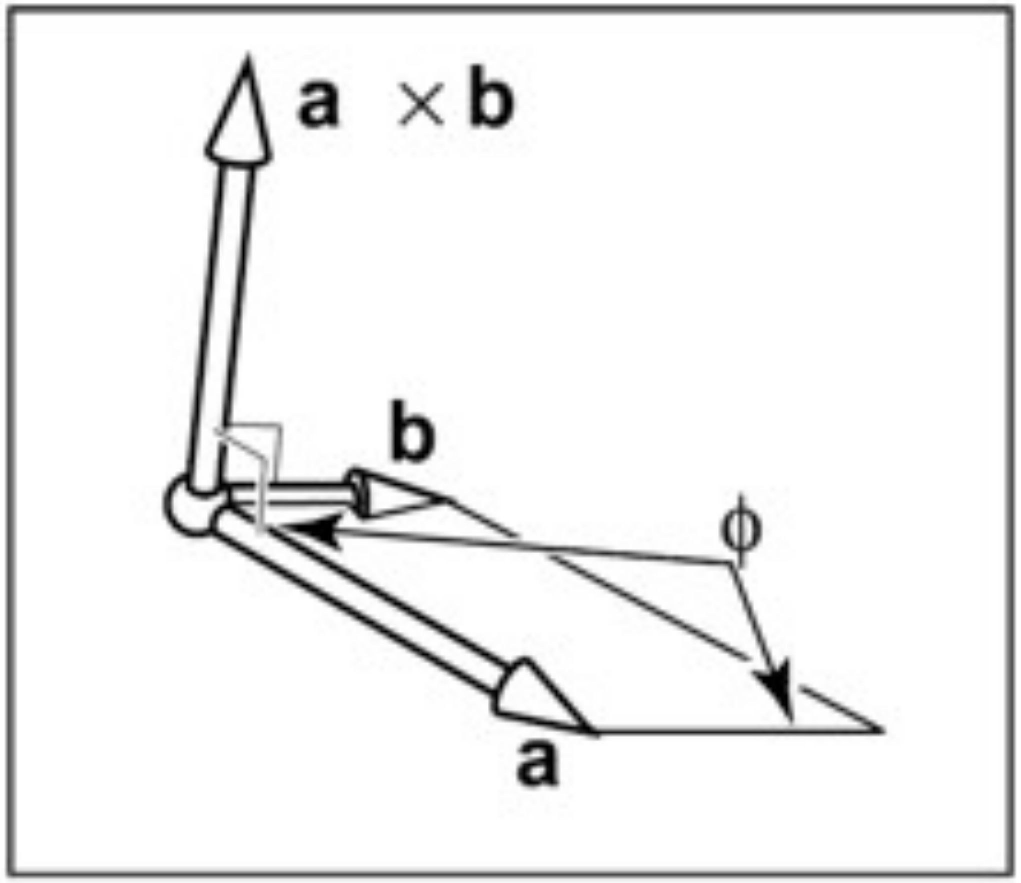


$$\mathbf{c} = \mathbf{a} \times \mathbf{b}$$

$$\begin{aligned} x_c &= y_a z_b - z_a y_b \\ y_c &= z_a x_b - x_a z_b \\ z_c &= x_a y_b - z_a x_b \end{aligned}$$

c perpendicular to both a and b
|c| is equal to the area of quadrilateral a b

Cross Product



Right-Hand Rule

$$\begin{aligned}\mathbf{x} \times \mathbf{y} &= \mathbf{z} \\ \mathbf{y} \times \mathbf{z} &= \mathbf{x} \\ \mathbf{z} \times \mathbf{x} &= \mathbf{y}\end{aligned}$$

$$\begin{aligned}\mathbf{x} \times \mathbf{x} &= 0 \\ \mathbf{y} \times \mathbf{y} &= 0 \\ \mathbf{z} \times \mathbf{z} &= 0\end{aligned}$$

2~3D

```
typedef float float2[2];
typedef float float3[3];

float2 p2;
float3 p3;

glVertex2fv( p2 );
glVertex3fv( p3 );
```

Vector operations

Vectors: $\mathbf{u}, \mathbf{v}, \mathbf{w}$

```
<Vector> = <Scalar> * <Vector>
```

$$\mathbf{v} = \alpha \mathbf{w}$$

```
<Vector> = <Vector> + <Vector>
```

$$\mathbf{u} = \mathbf{v} + \mathbf{w}$$

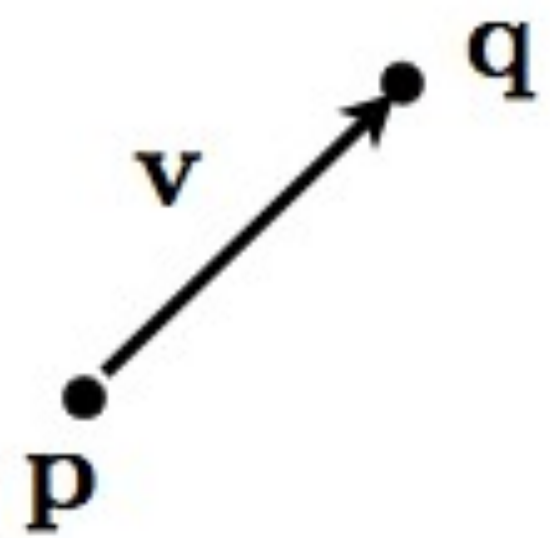
Implementation of parallelogram rule

Point operations

Points: $\mathbf{p}, \mathbf{q}, \mathbf{r}$

<Point> = <Point> + <Vector>

$$\mathbf{q} = \mathbf{p} + \mathbf{v}$$



<Vector> = <Point> - <Point>

$$\mathbf{v} = \mathbf{q} - \mathbf{p}$$

A point is an origin and a vector displacement

illegal operations

```
<Point> = <Scalar> * <Point>
```

$$\mathbf{p} = \alpha \mathbf{q}$$

```
<Point> = <Point> + <Point>
```

$$\mathbf{p} = \mathbf{q} + \mathbf{r}$$

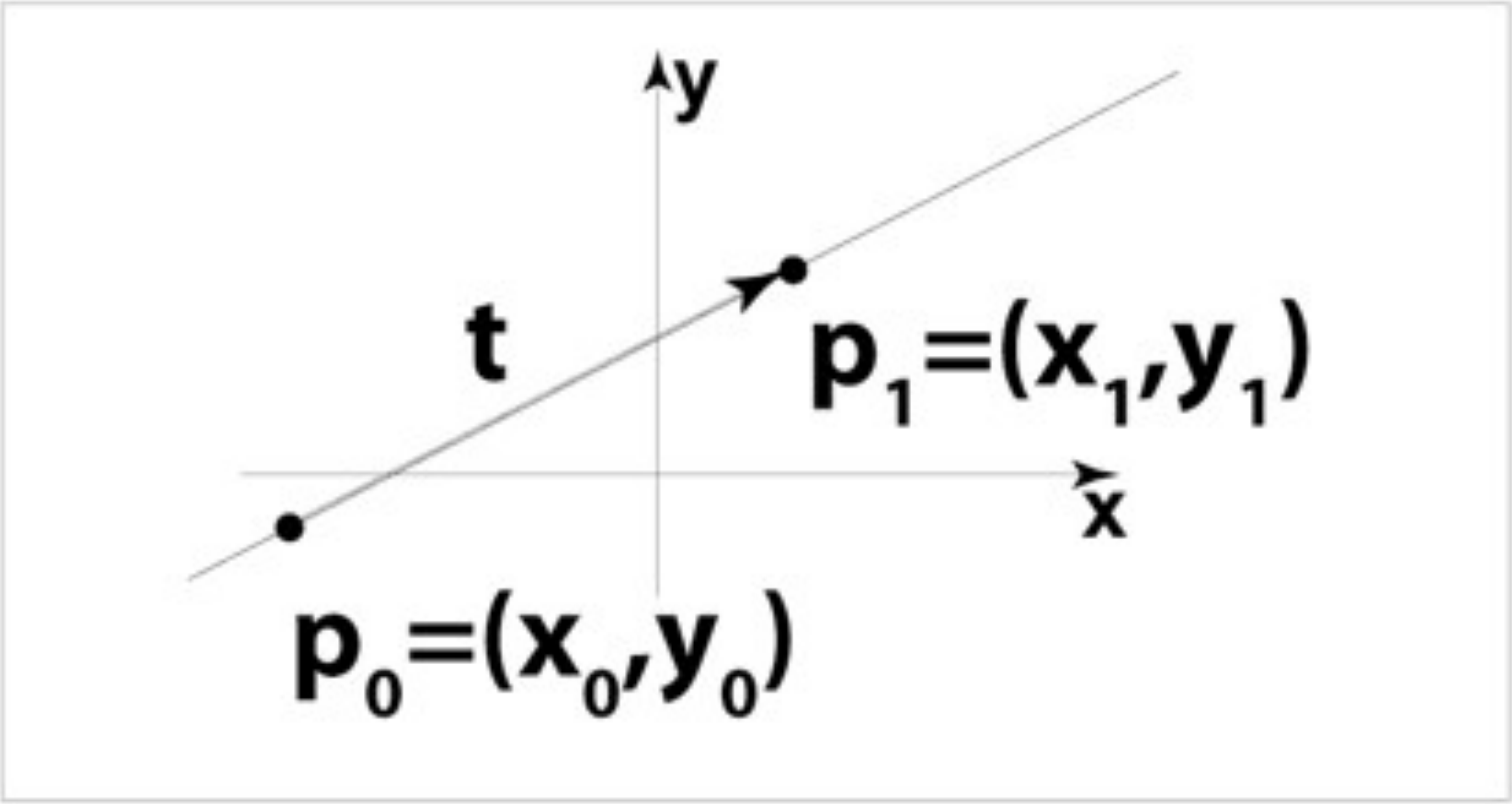
```
<Vector> = <Point> + <Vector>
```

$$\mathbf{v} = \mathbf{p} + \mathbf{w}$$

```
<Point> = <Point> - <Point>
```

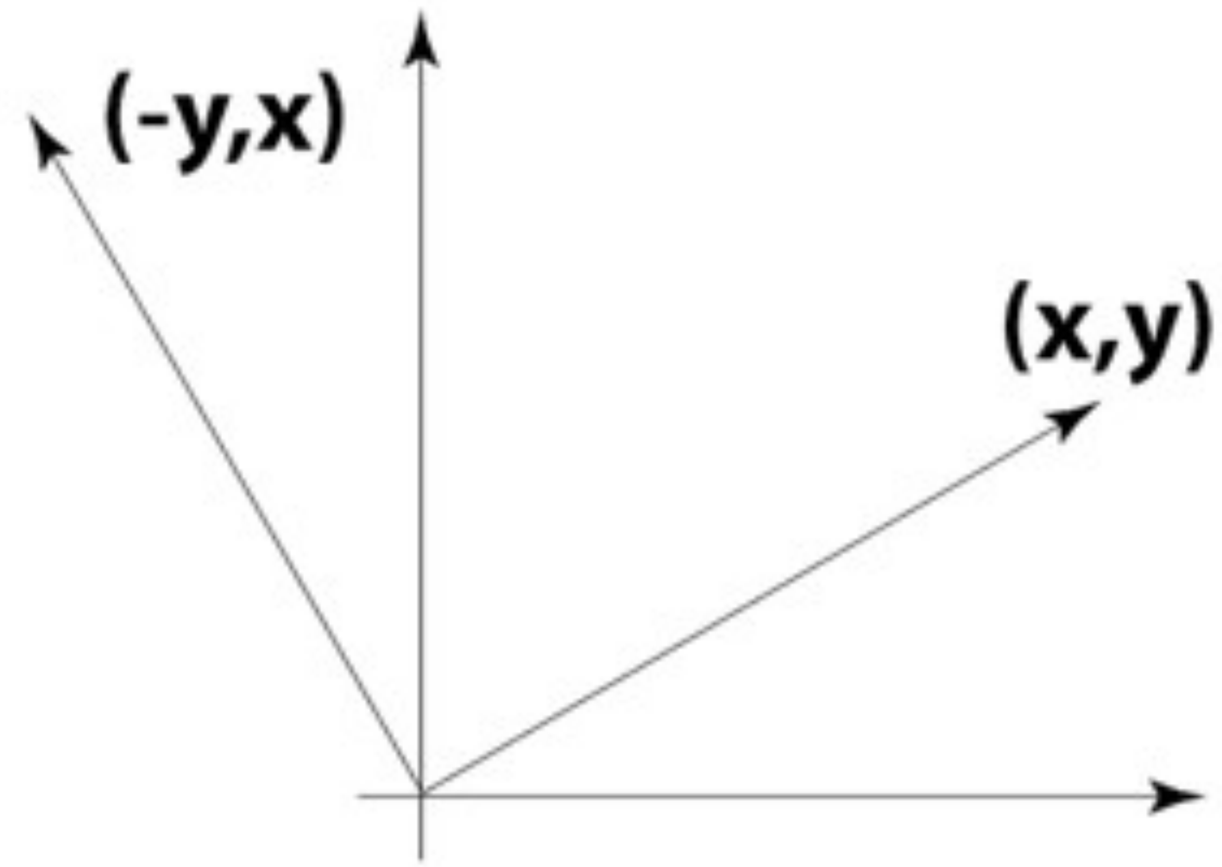
$$\mathbf{p} = \mathbf{q} - \mathbf{r}$$

Directed line



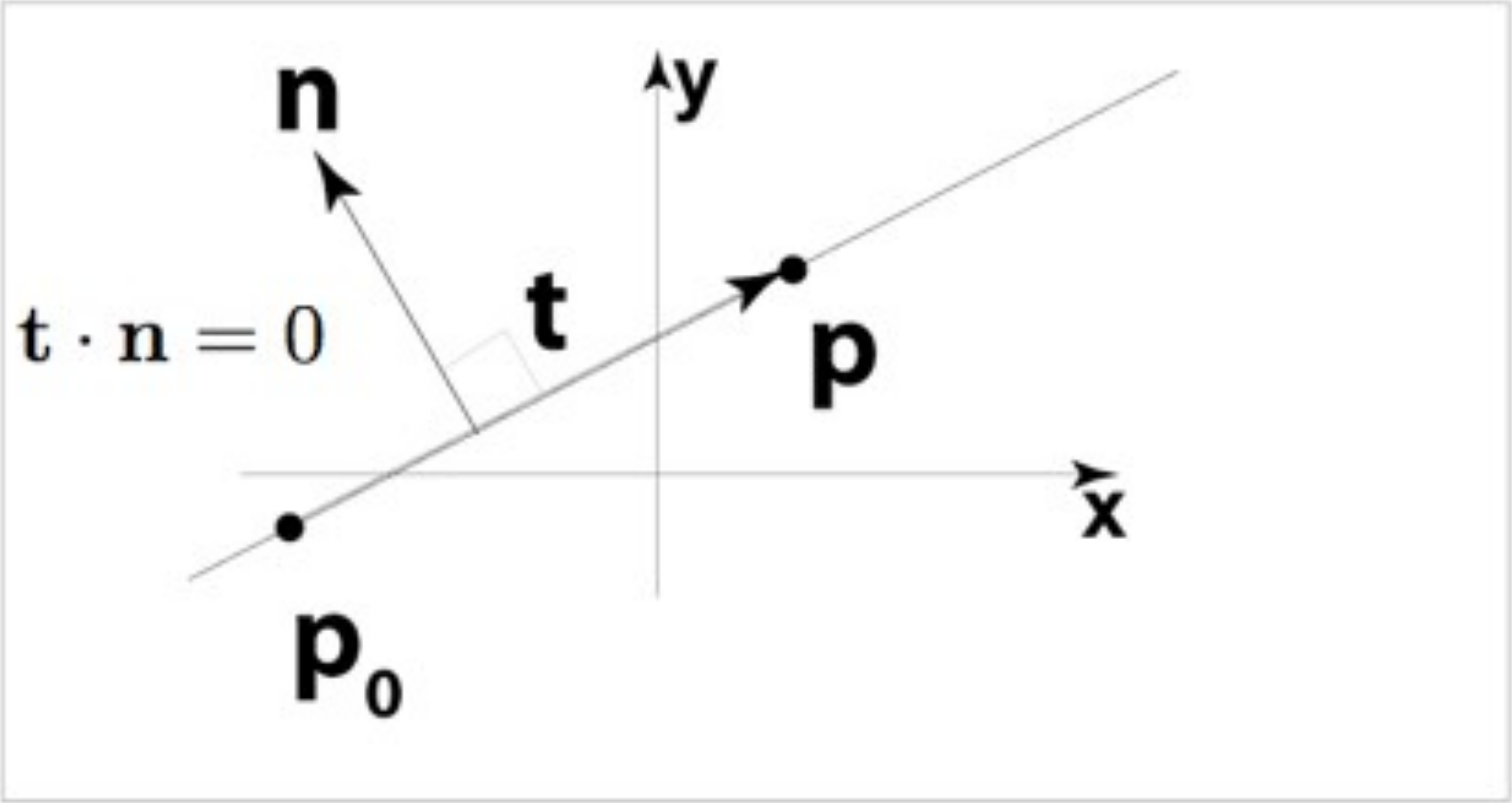
$$\mathbf{t} = \mathbf{p_1} - \mathbf{p_0} = (x_1 - x_0, y_1 - y_0)$$

Perpendicular vector in 2D



Perp((x,y))=(-y,x)

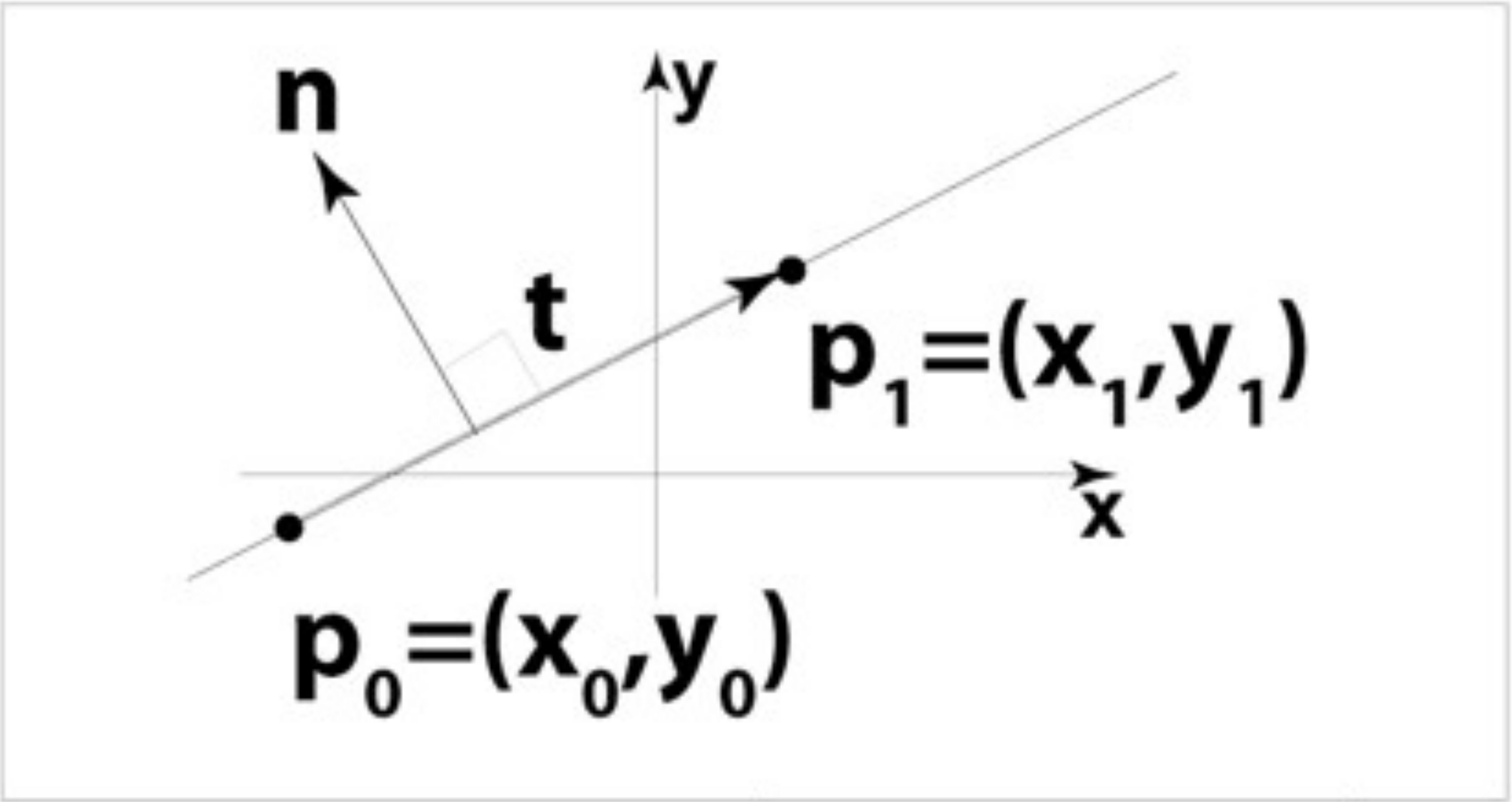
Line equation



$$(\mathbf{p} - \mathbf{p}_0) \cdot \mathbf{n} = 0$$

This equation must be true for all point p on the line

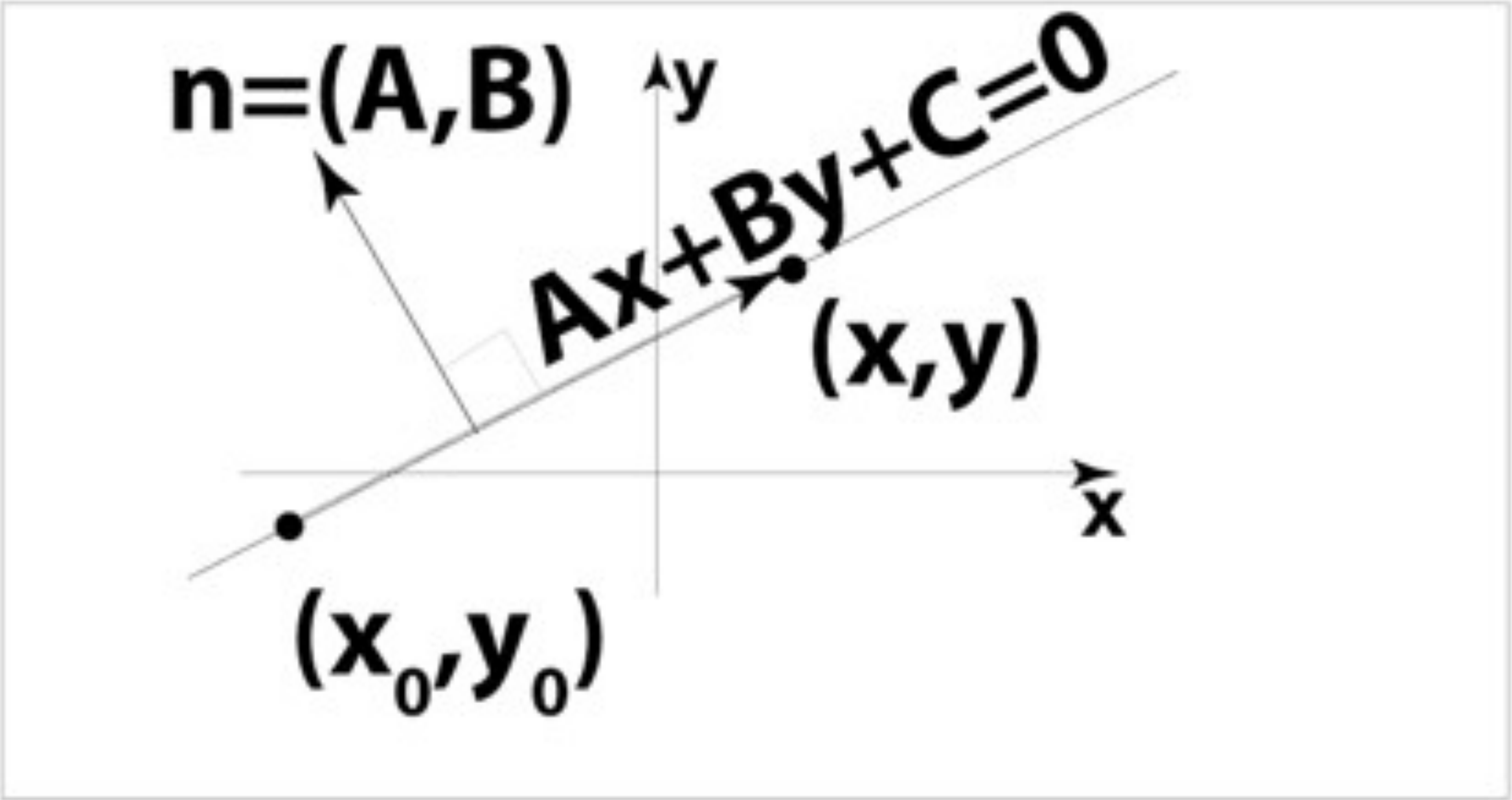
Normal to the line



$$\mathbf{t} = \mathbf{p}_1 - \mathbf{p}_0 = (x_1 - x_0, y_1 - y_0)$$

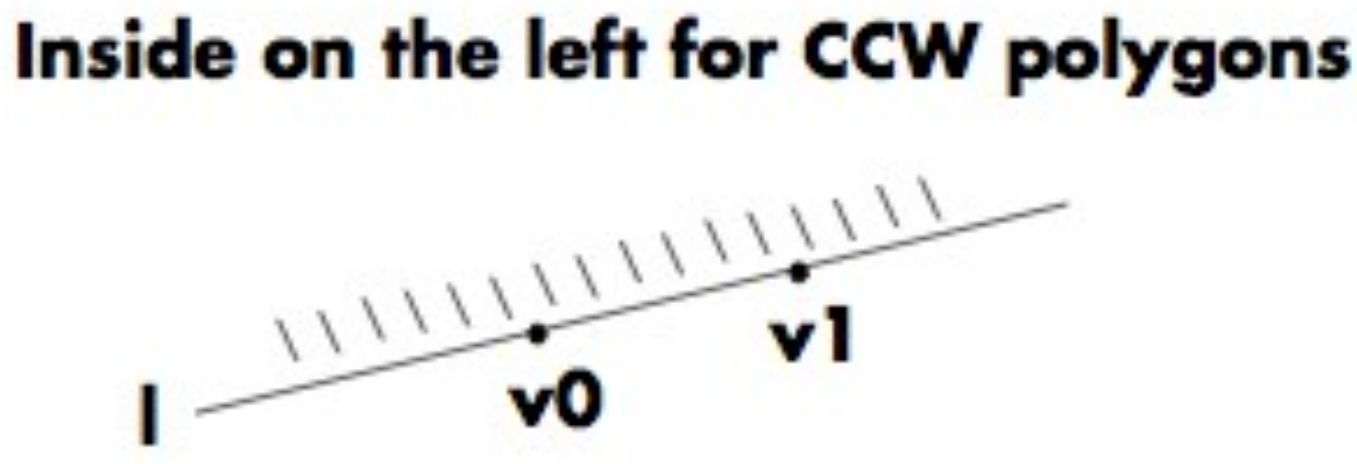
$$\mathbf{n} = \mathrm{Perp}(\mathbf{t}) = (y_0 - y_1, x_1 - x_0)$$

Line equation



$$A = y_1 - y_0$$
$$B = x_0 - x_1$$
$$C = x_0 y_1 - y_0 x_1$$

Line equation



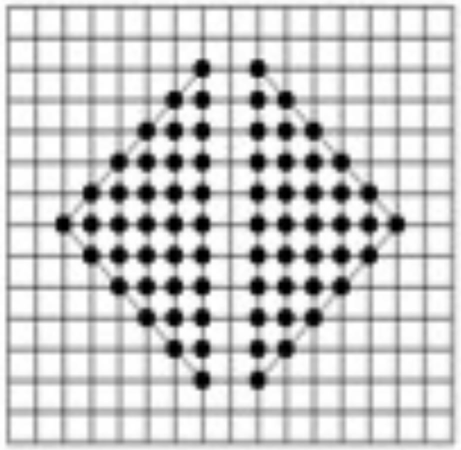

```
makeline( vert& v0, vert& v1, line& l )
{
    l.a = v1.y - v0.y;
    l.b = v0.x - v1.x;
    l.c = -(l.a * v0.x + l.b * v0.y);
}
```

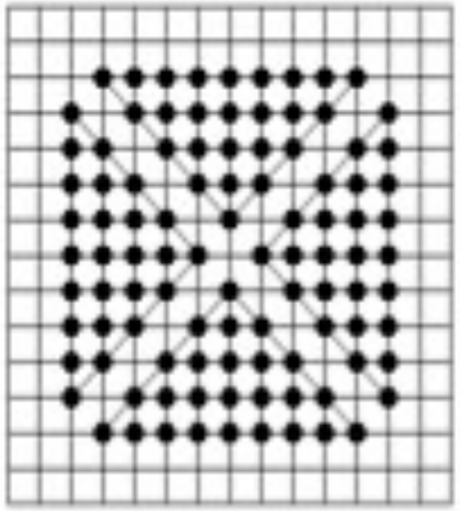
Singularities

Singularities: Edges that touch pixels (e == 0)

Causes two fragments to be generated

- Wasted effort
- Problems with transparency (later lecture)





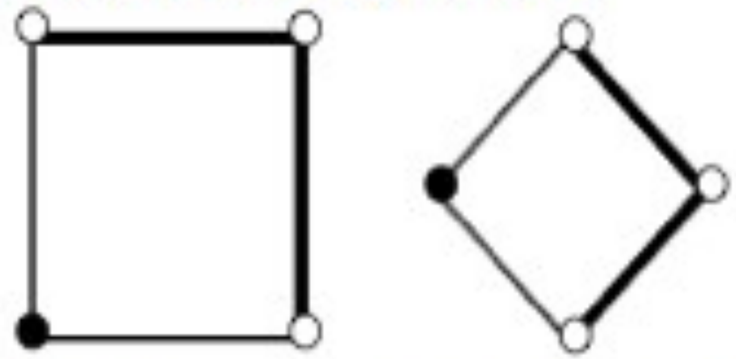
Not including singularities (e < 0) causes gaps

Handling singularity

Create shadowed edges (thick lines)

Don't draw pixels on shadowed edges

Solid drawn; hollow not drawn

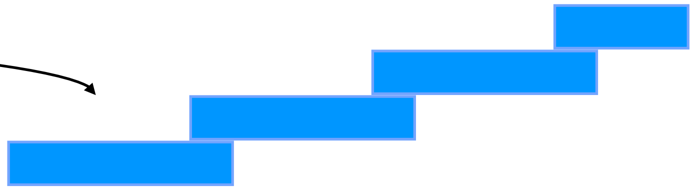


```
int shadow( value a, value b ) {
  return (a>0) || (a==0 && b > 0);
}
int inside( value e, value a, value b ) {
  return (e == 0) ? !shadow(a,b) : (e < 0);
}
```

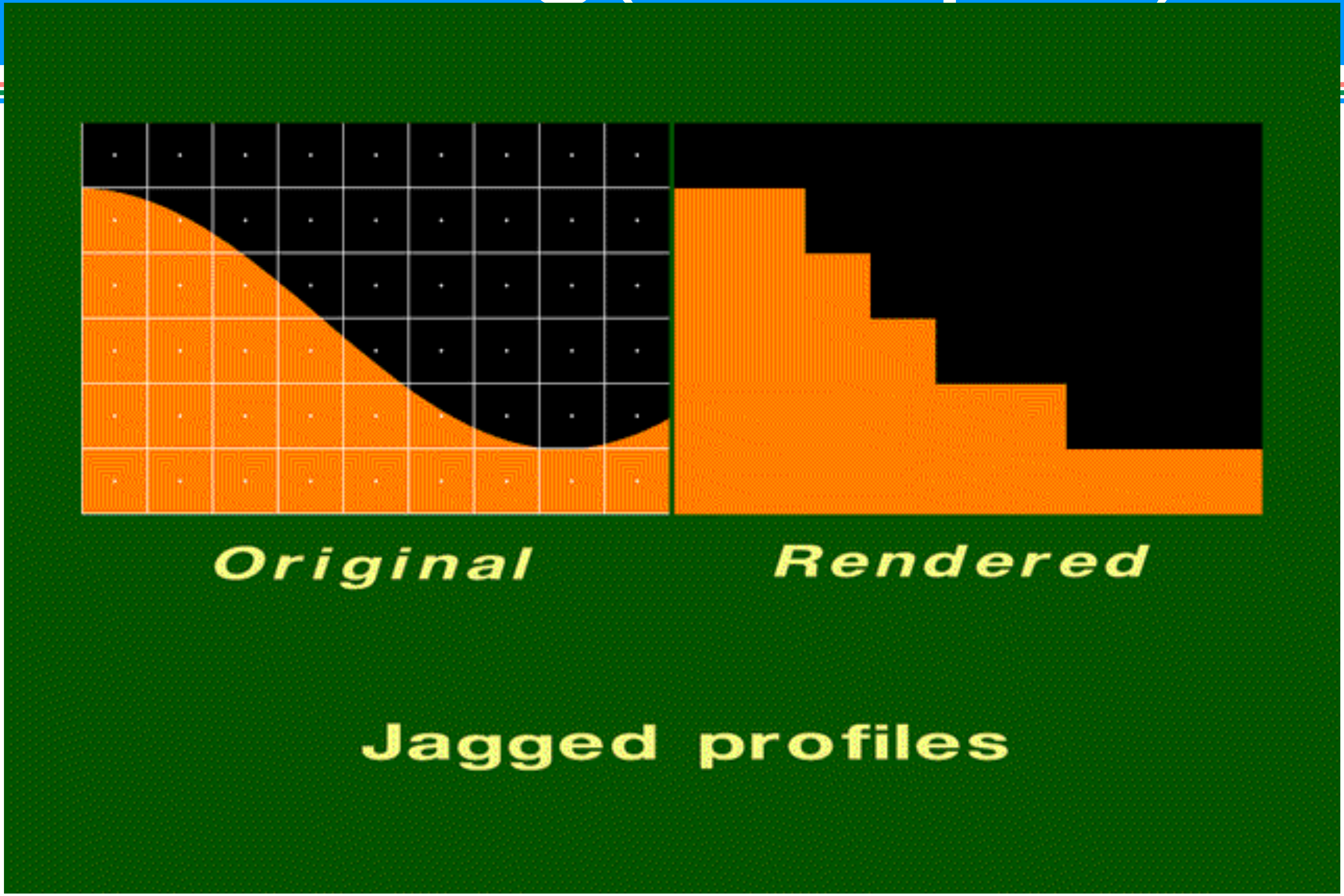
Antialiasing

Aliasing

- Aliasing is caused due to the discrete nature of the display device
- Rasterizing primitives is like sampling a continuous signal by a finite set of values (point sampling)
- Information is lost if the rate of sampling is not sufficient. This sampling error is called ***aliasing***.
- Effects of aliasing are
  - – Jagged edges
  - – Incorrectly rendered fine details
  - – Small objects might miss

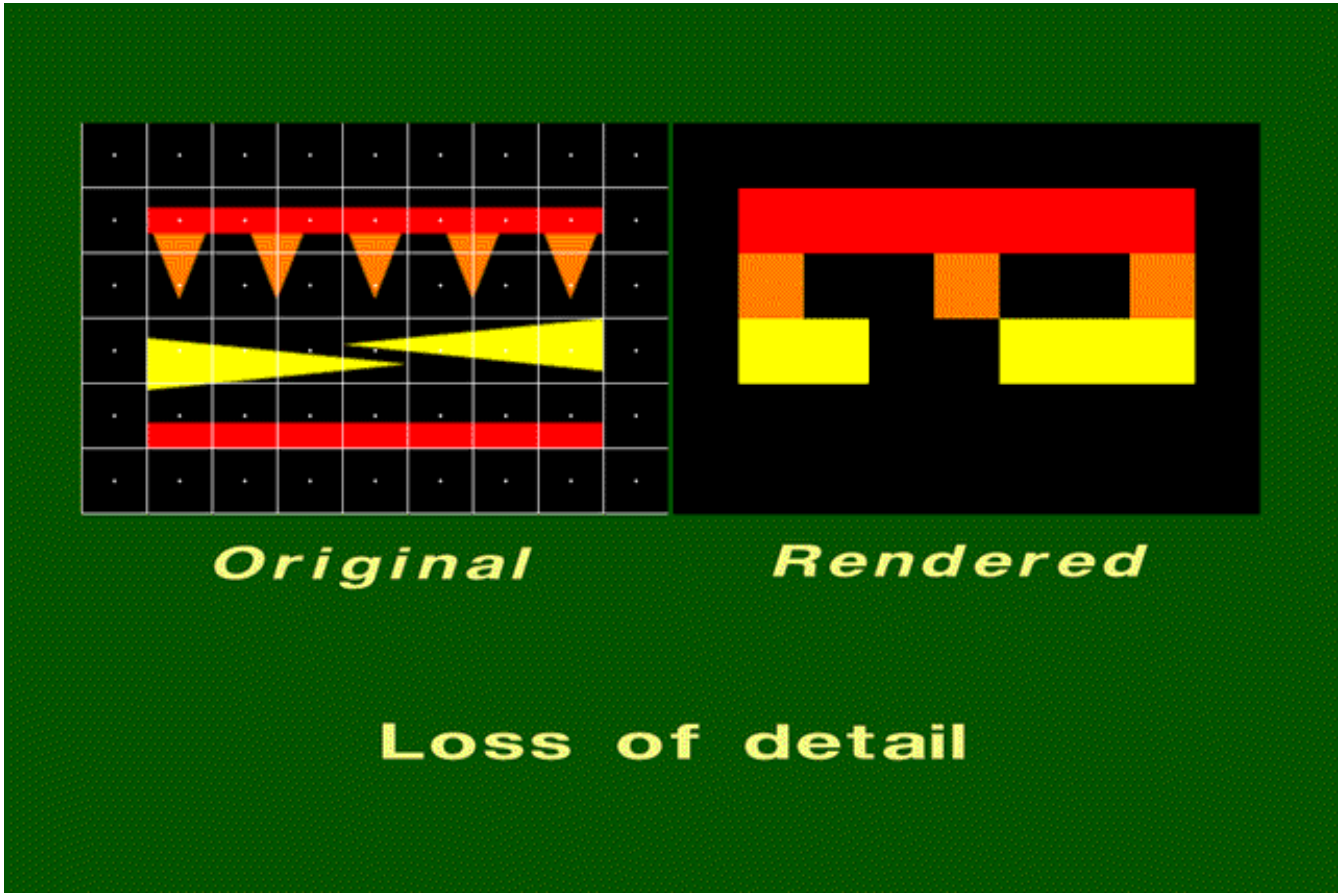


Aliasing(examples)



Aliasing(examples)

Aliasing(examples)

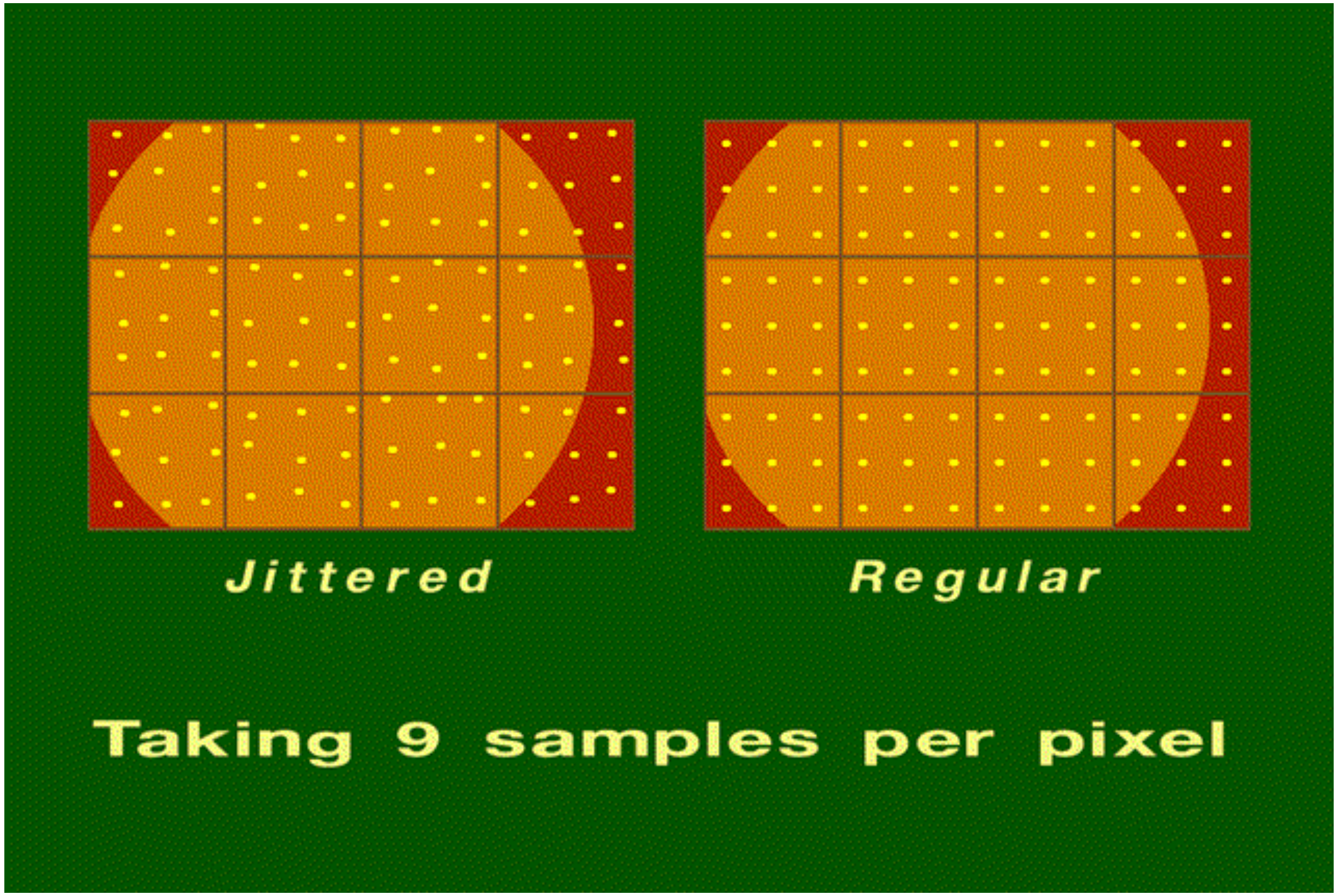


Antialiasing

- Application of techniques to reduce/eliminate aliasing artifacts
- Some of the methods are
  - increasing sampling rate by increasing the resolution. Display memory requirements increases four times if the resolution is doubled
  - averaging methods (post processing). Intensity of a pixel is set as the weighted average of its own intensity and the intensity of the surrounding pixels
  - Area sampling, more popular

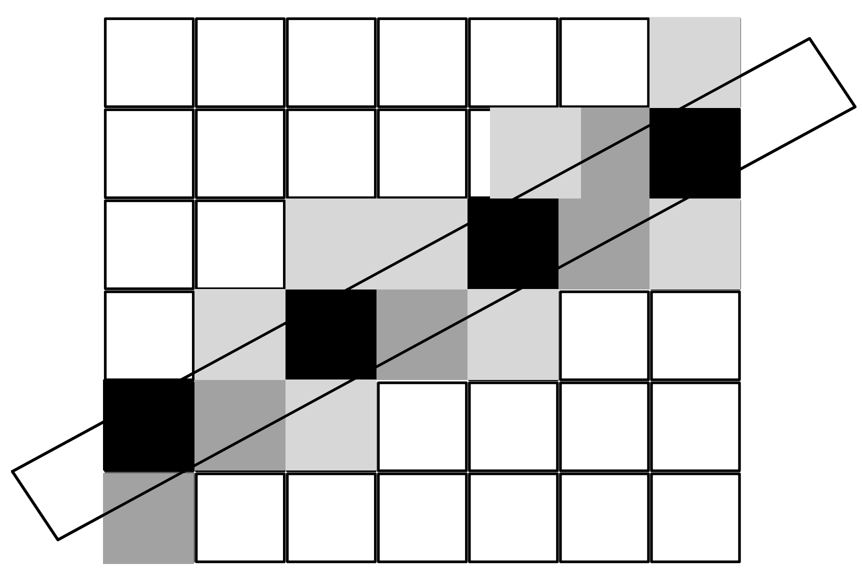
Antialiasing (postfiltering)

How should one supersample?



Area Sampling

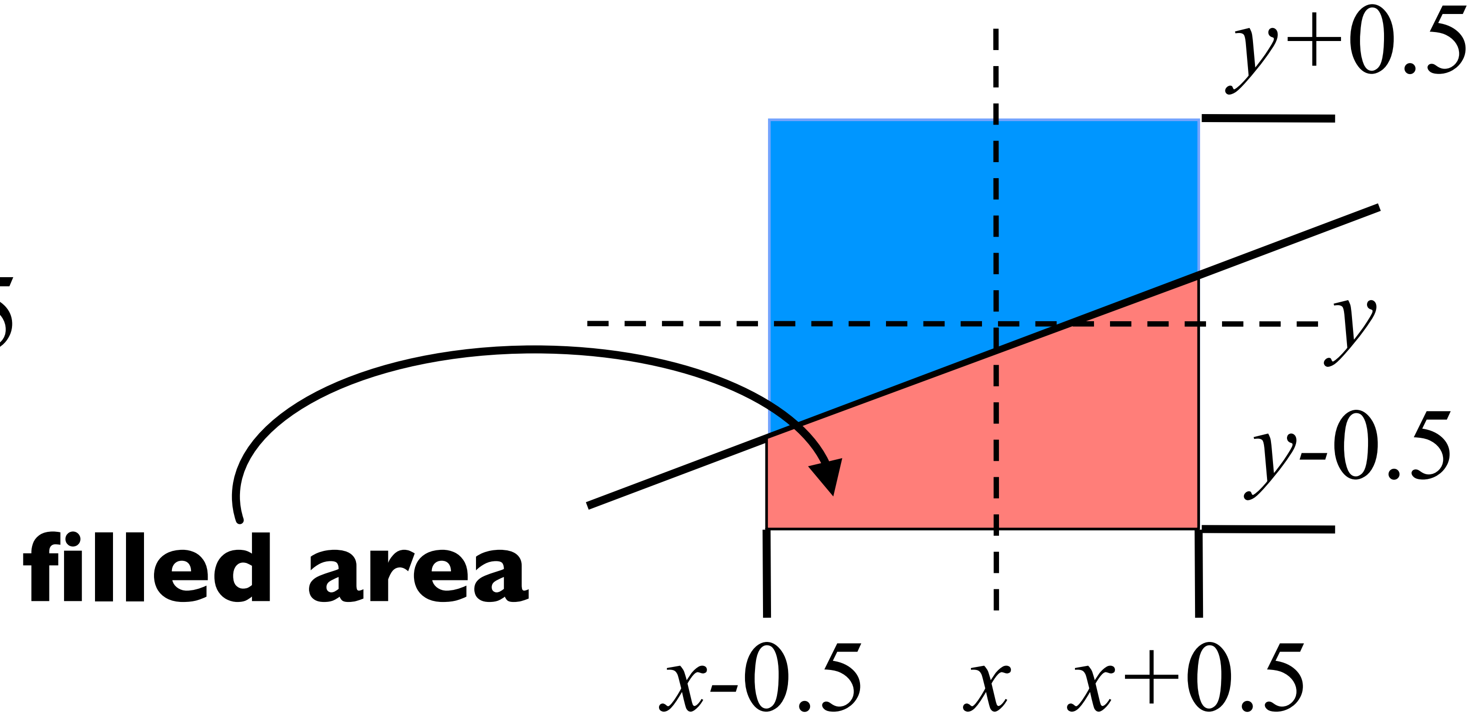
- A scan converted primitive occupies finite area on the screen
- Intensity of the boundary pixels is adjusted depending on the percent of the pixel area covered by the primitive. This is called weighted area sampling



Area Sampling

- Methods to estimate percent of pixel covered by the primitive
  - subdivide pixel into sub-pixels and determine how many sub-pixels are inside the boundary
  - Incremental line algorithm can be extended, with area calculated as

$$Area = m \times x - y + c + 0.5$$



Clipping

Clipping

- Clipping of primitives is done usually before scan converting the primitives
- Reasons being
  - —scan conversion needs to deal only with the clipped version of the primitive, which might be much smaller than its unclipped version
  - —Primitives are usually defined in the real world, and their mapping from the real to the integer domain of the display might result in the overflowing of the integer values resulting in unnecessary artifacts

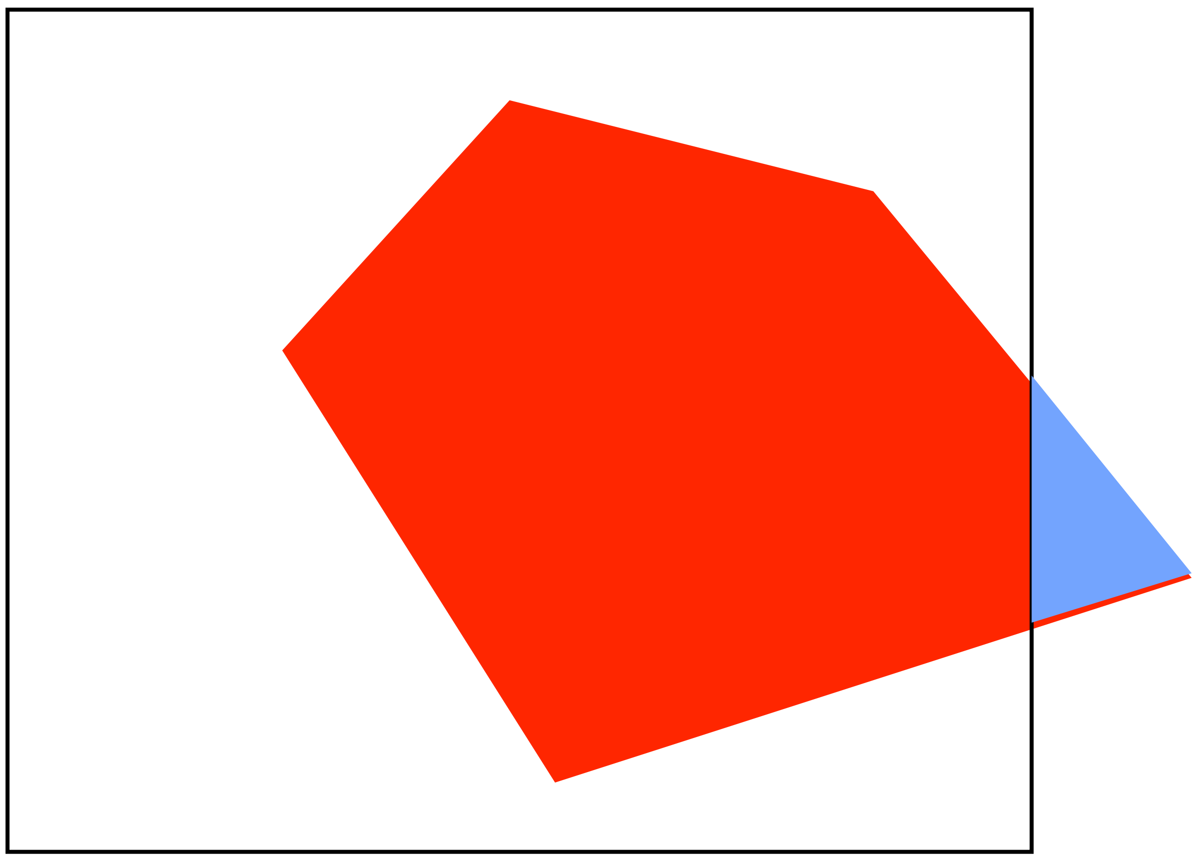
Clipping

- Why Clipping?
- How Clipping?
  - Lines
  - Polygons

- Note: Content from chapter 4.
  - Lots of stuff about rendering systems and mathematics in that chapter.

Definition

- Clipping – Removal of content that is not going to be displayed
  - —Behind camera
  - —Too close
  - —Too far
  - —Off sides of the screen

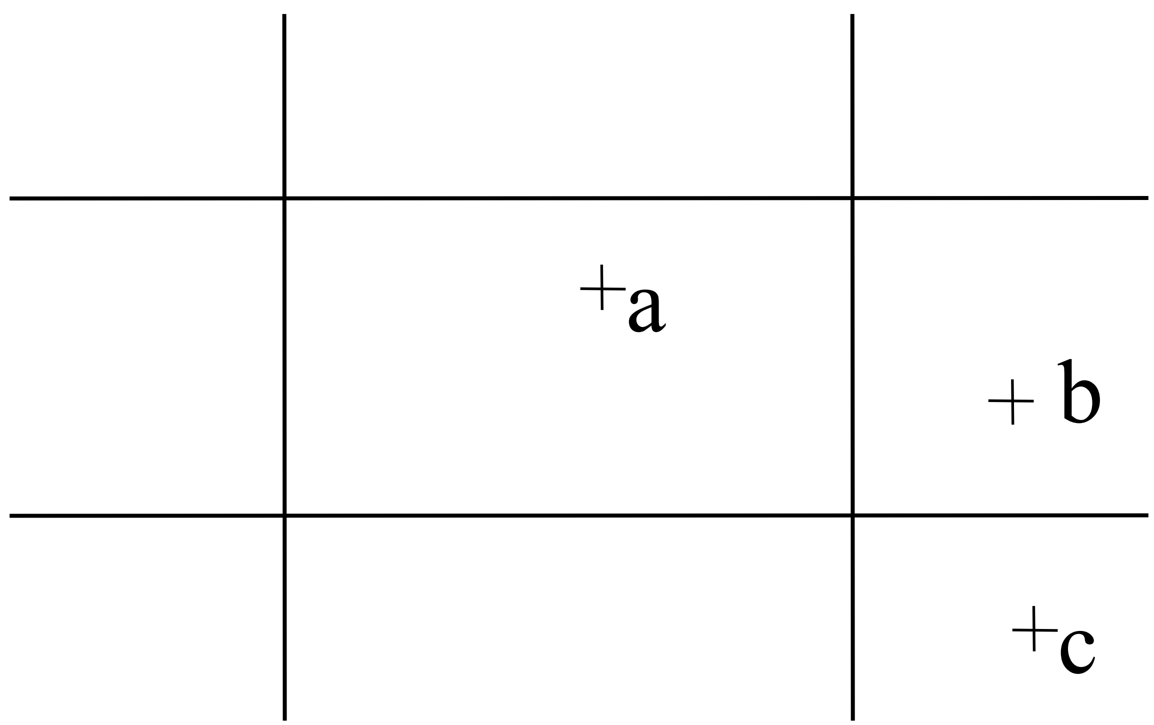


How would we clip?

- Points?
- Lines?
- Polygons?
- Other objects?

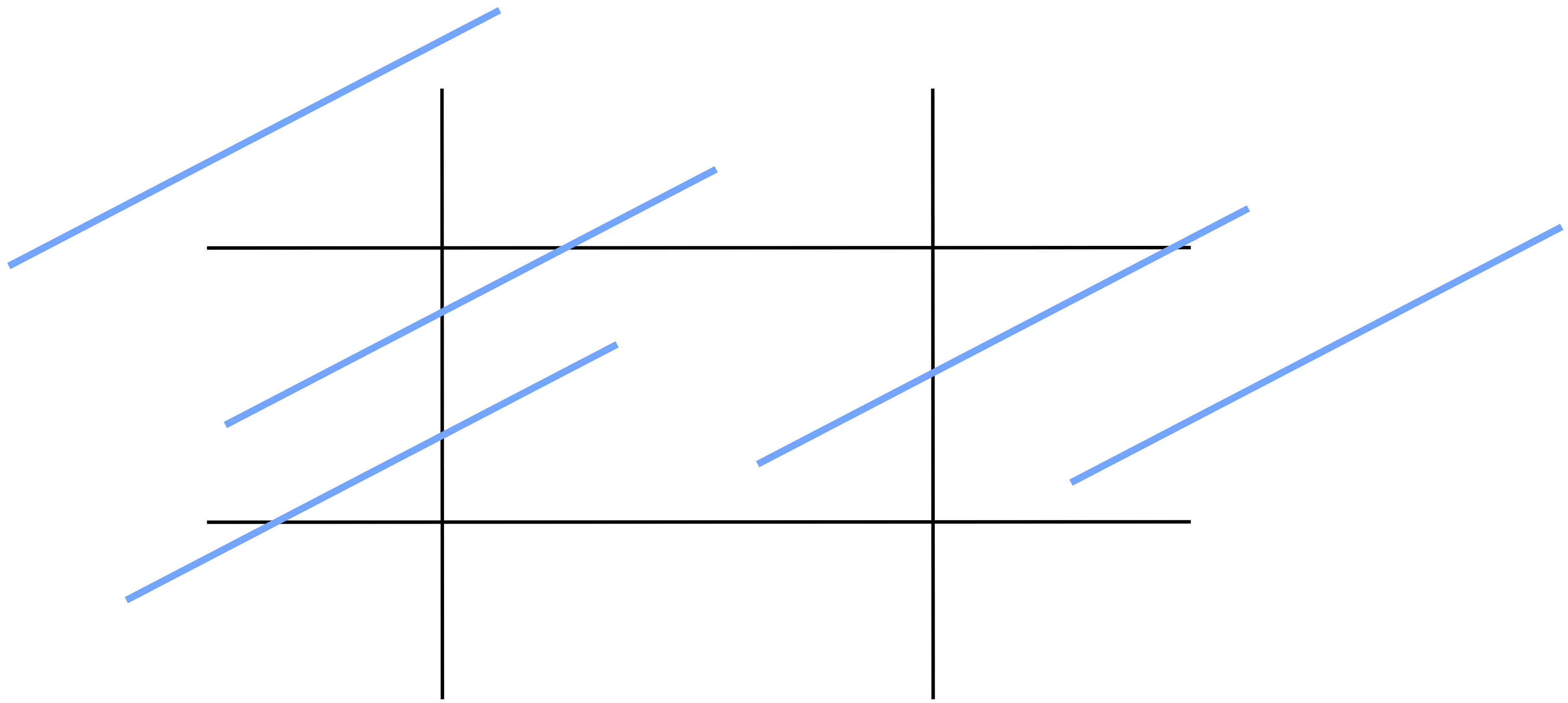
We'll start in 2D

- Assume a 2D upright rectangle we are clipping against
  - —Common in windowing systems
  - —Points are trivial
    - >= minx and <= maxx and >= miny and <= maxy



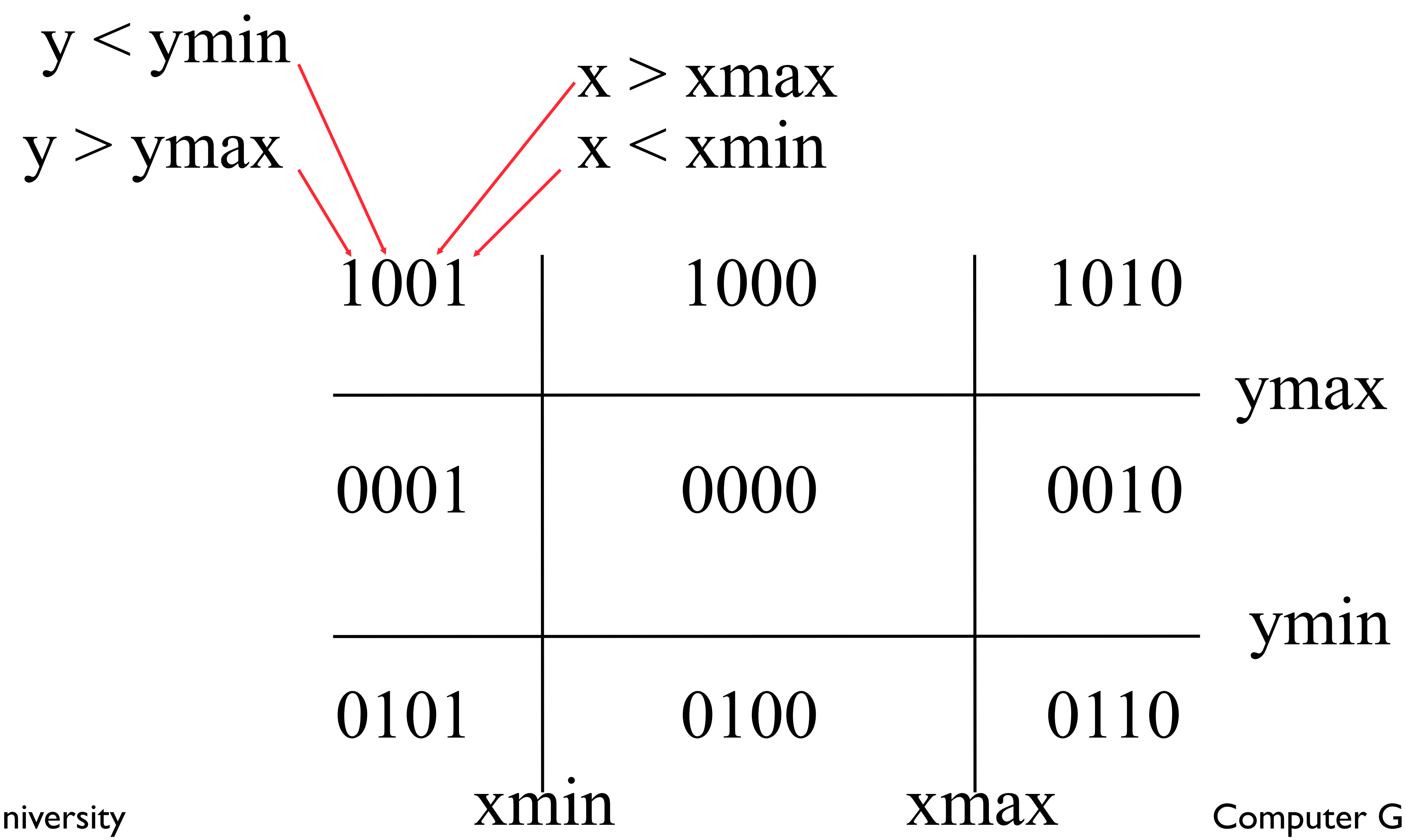
Line Segments

- What can happen when a line segment is clipped?



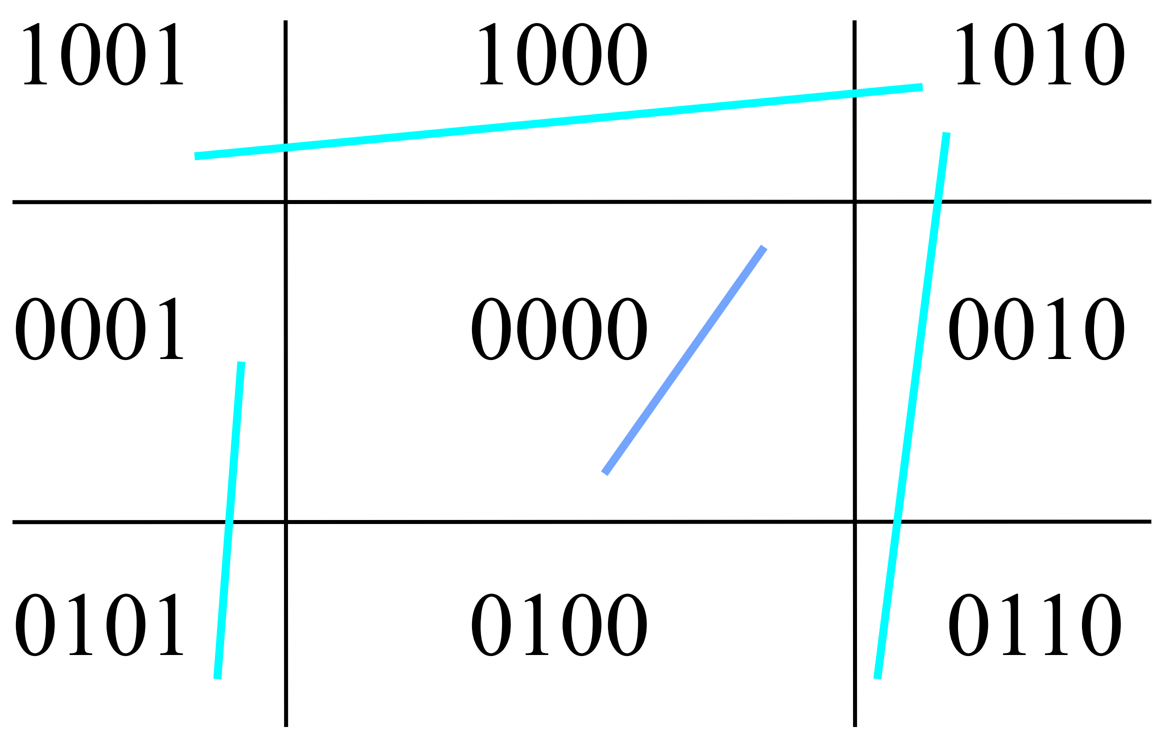
Cohen-Sutherland Line Clipping

- We'll assign the ends of a line "outcodes", 4 bit values that indicate if they are inside or outside the clip area.



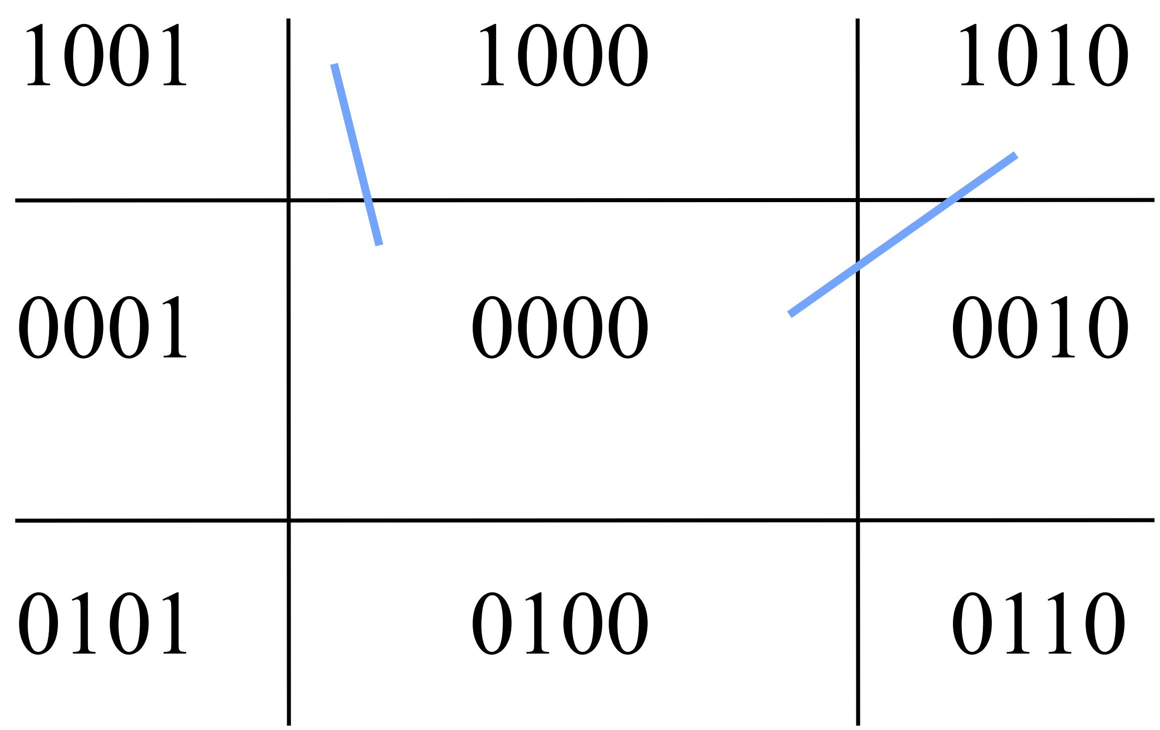
Outcode cases

- We'll call the two endpoint outcodes o_1 and o_2.
  - If $o_1 = o_2 = 0$, both endpoints are inside.
  - else if $(o_1 \,\&\, o_2)$!= 0, both ends points are on the same side, the edge is discarded.



More cases

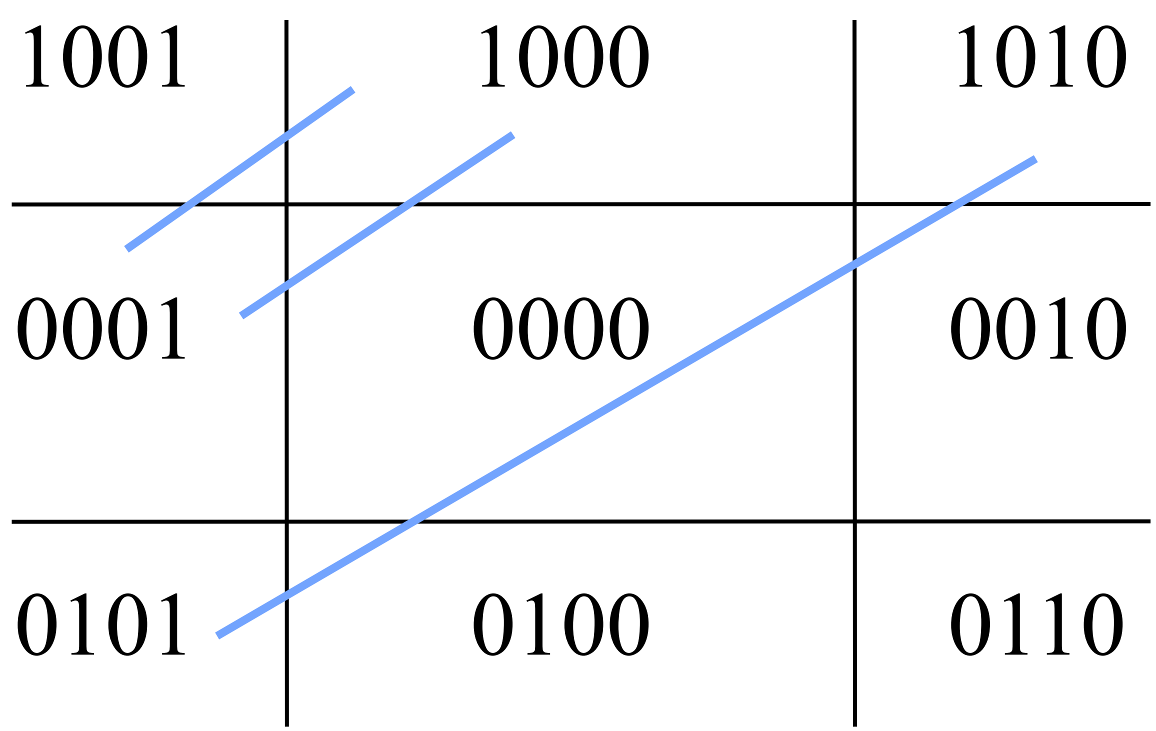
- else if (o_1 != 0) and (o_2 = 0), (or vice versa), one end is inside, other is outside.
  - —Clip and recompute *one that's outside* until inside.
  - —Clip edges with bits set…
  - —May require two clip computations



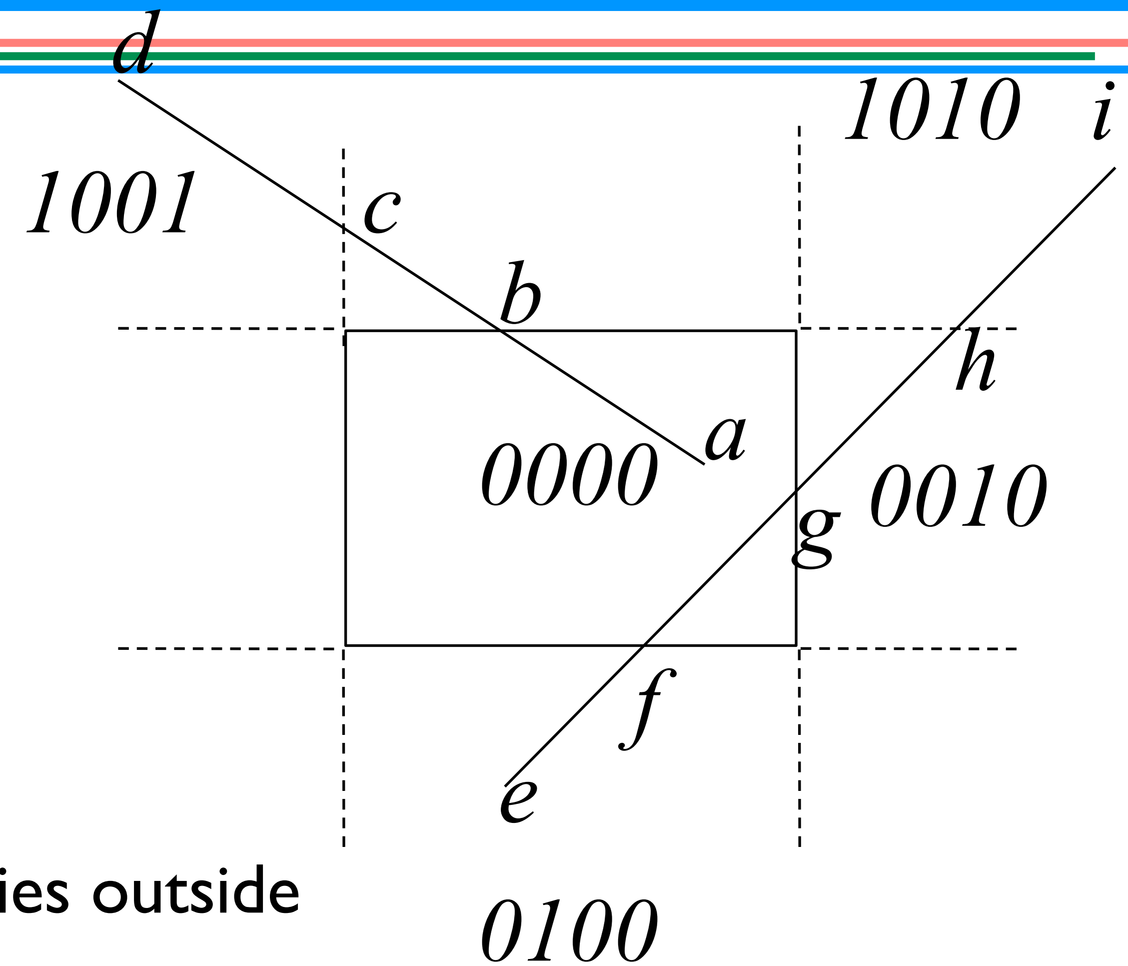
Last case…

—else if (o1 & o2) = 0, end points are on different sides.

- Clip and recompute.
- May have some inside part or may not…
- May require up to 4 clips!

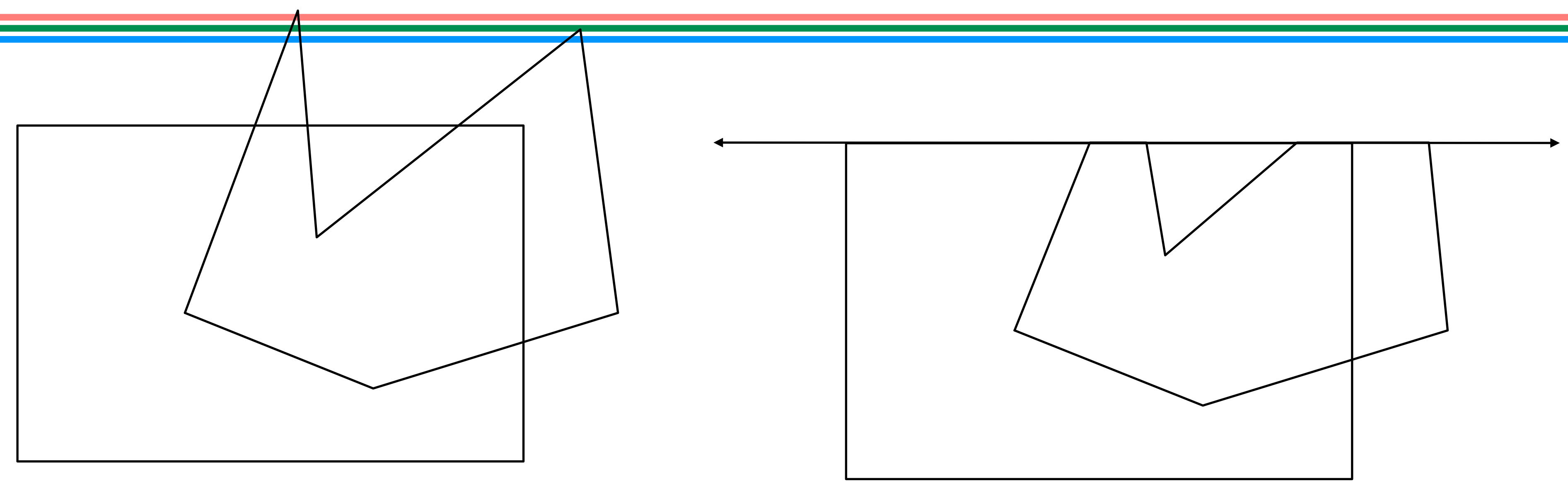


Cohen-Sutherland Line-Clipping Algorithm



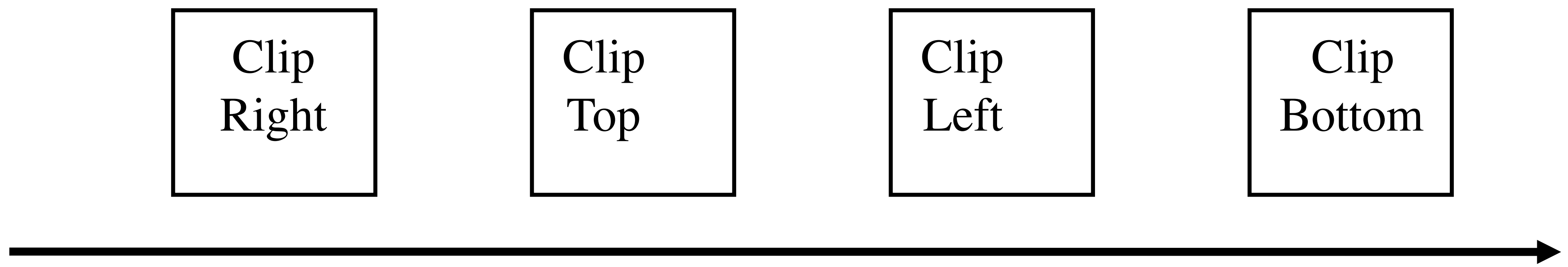
- To do the clipping find the end point that lies outside
- Test the outcode to find the edge that is crossed and determine the corresponding intersection point
- Replace the outside end-point by intersection-point
- Repeat the above steps for the new line

Sutherland-Hodgeman Polygon-Clipping Algorithm



- Polygons can be clipped against each edge of the window one edge at a time. Window/edge intersections, if any, are easy to find since the X or Y coordinates are already known.
- Vertices which are kept after clipping against one window edge are saved for clipping against the remaining edges.

Pipelined Polygon Clipping



- Because polygon clipping does not depend on any other polygons, it is possible to arrange the clipping stages in a **pipeline**. the input polygon is clipped against one edge and any points that are kept are passed on as input to the next *stage* of the pipeline.
- This way four polygons can be at different *stages* of the clipping process simultaneously. This is often implemented in hardware.

```
typedef int OutCode;

const int INSIDE = 0; // 0000
const int LEFT = 1;   // 0001
const int RIGHT = 2;  // 0010
const int BOTTOM = 4; // 0100
const int TOP = 8;    // 1000

// Compute the bit code for a point (x, y) using the clip rectangle
// bounded diagonally by (xmin, ymin), and (xmax, ymax)

// ASSUME THAT xmax, xmin, ymax and ymin are global constants.

OutCode ComputeOutCode(double x, double y)
{
    OutCode code;

    code = INSIDE;          // initialised as being inside of [[clip window]]

    if (x < xmin)           // to the left of clip window
        code |= LEFT;
    else if (x > xmax)      // to the right of clip window
        code |= RIGHT;
    if (y < ymin)           // below the clip window
        code |= BOTTOM;
    else if (y > ymax)      // above the clip window
        code |= TOP;

    return code;
}

// Cohen–Sutherland clipping algorithm clips a line from
// P0 = (x0, y0) to P1 = (x1, y1) against a rectangle with
// diagonal from (xmin, ymin) to (xmax, ymax).
void CohenSutherlandLineClipAndDraw(double x0, double y0, double x1, double y1)
{
    // compute outcodes for P0, P1, and whatever point lies outside the clip rectangle
    OutCode outcode0 = ComputeOutCode(x0, y0);
    OutCode outcode1 = ComputeOutCode(x1, y1);
    bool accept = false;

    while (true) {
        if (!(outcode0 | outcode1)) { // Bitwise OR is 0. Trivially accept and get out of loop
            accept = true;
```

```
// Cohen–Sutherland clipping algorithm clips a line from
// P0 = (x0, y0) to P1 = (x1, y1) against a rectangle with
// diagonal from (xmin, ymin) to (xmax, ymax).
void CohenSutherlandLineClipAndDraw(double x0, double y0, double x1, double y1)
{
    // compute outcodes for P0, P1, and whatever point lies outside the clip rectangle
    OutCode outcode0 = ComputeOutCode(x0, y0);
    OutCode outcode1 = ComputeOutCode(x1, y1);
    bool accept = false;

    while (true) {
        if (!(outcode0 | outcode1)) { // Bitwise OR is 0. Trivially accept and get out of loop
            accept = true;
            break;
        } else if (outcode0 & outcode1) { // Bitwise AND is not 0. Trivially reject and get out of loop
            break;
        } else {
            // failed both tests, so calculate the line segment to clip
            // from an outside point to an intersection with clip edge
            double x, y;

            // At least one endpoint is outside the clip rectangle; pick it.
            OutCode outcodeOut = outcode0 ? outcode0 : outcode1;

            // Now find the intersection point;
            // use formulas y = y0 + slope * (x - x0), x = x0 + (1 / slope) * (y - y0)
            if (outcodeOut & TOP) {           // point is above the clip rectangle
                x = x0 + (x1 - x0) * (ymax - y0) / (y1 - y0);
                y = ymax;
            } else if (outcodeOut & BOTTOM) { // point is below the clip rectangle
                x = x0 + (x1 - x0) * (ymin - y0) / (y1 - y0);
                y = ymin;
            } else if (outcodeOut & RIGHT) {  // point is to the right of clip rectangle
                y = y0 + (y1 - y0) * (xmax - x0) / (x1 - x0);
                x = xmax;
            } else if (outcodeOut & LEFT) {   // point is to the left of clip rectangle
                y = y0 + (y1 - y0) * (xmin - x0) / (x1 - x0);
                x = xmin;
            }

            // Now we move outside point to intersection point to clip
            // and get ready for next pass.
            if (outcodeOut == outcode0) {
                x0 = x;
                y0 = y;
```

```
                break;
            } else {
                // failed both tests, so calculate the line segment to clip
                // from an outside point to an intersection with clip edge
                double x, y;

                // At least one endpoint is outside the clip rectangle; pick it.
                OutCode outcodeOut = outcode0 ? outcode0 : outcode1;

                // Now find the intersection point;
                // use formulas y = y0 + slope * (x - x0), x = x0 + (1 / slope) * (y - y0)
                if (outcodeOut & TOP) {           // point is above the clip rectangle
                    x = x0 + (x1 - x0) * (ymax - y0) / (y1 - y0);
                    y = ymax;
                } else if (outcodeOut & BOTTOM) { // point is below the clip rectangle
                    x = x0 + (x1 - x0) * (ymin - y0) / (y1 - y0);
                    y = ymin;
                } else if (outcodeOut & RIGHT) {  // point is to the right of clip rectangle
                    y = y0 + (y1 - y0) * (xmax - x0) / (x1 - x0);
                    x = xmax;
                } else if (outcodeOut & LEFT) {   // point is to the left of clip rectangle
                    y = y0 + (y1 - y0) * (xmin - x0) / (x1 - x0);
                    x = xmin;
                }

                // Now we move outside point to intersection point to clip
                // and get ready for next pass.
                if (outcodeOut == outcode0) {
                    x0 = x;
                    y0 = y;
                    outcode0 = ComputeOutCode(x0, y0);
                } else {
                    x1 = x;
                    y1 = y;
                    outcode1 = ComputeOutCode(x1, y1);
                }
            }
        }
        if (accept) {
               // Following functions are left for implementation by user based on
               // their platform (OpenGL/graphics.h etc.)
               DrawRectangle(xmin, ymin, xmax, ymax);
               LineSegment(x0, y0, x1, y1);
        }
}
```

Liang–Barsky algorithm

Consider first the usual parametric form of a straight line:

$$x = x_0 + u(x_1 - x_0) = x_0 + u\Delta x$$
$$y = y_0 + u(y_1 - y_0) = y_0 + u\Delta y$$

A point is in the clip window, if

$$x_{\text{min}} \leq x_0 + u\Delta x \leq x_{\text{max}}$$

and

$$y_{\text{min}} \leq y_0 + u\Delta y \leq y_{\text{max}},$$

which can be expressed as the 4 inequalities

$$up_k \leq q_k, \quad k = 1, 2, 3, 4,$$

where

$p_1 = -\Delta x, q_1 = x_0 - x_{\text{min}}$ (left)
$p_2 = \Delta x, q_2 = x_{\text{max}} - x_0$ (right)
$p_3 = -\Delta y, q_3 = y_0 - y_{\text{min}}$ (bottom)
$p_4 = \Delta y, q_4 = y_{\text{max}} - y_0$ (top)

Liang–Barsky algorithm

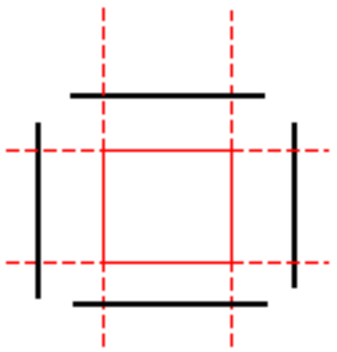
To compute the final line segment:

1. A line parallel to a clipping window edge has $p_k = 0$ for that boundary.
2. If for that k, $q_k < 0$, the line is completely outside and can be eliminated.
3. When $p_k < 0$ the line proceeds outside to inside the clip window and when $p_k > 0$, the line proceeds inside to outside.
4. For nonzero p_k, $u = \frac{q_k}{p_k}$ gives the intersection point.
5. For each line, calculate u_1 and u_2. For u_1, look at boundaries for which $p_k < 0$ (i.e. outside to inside). Take u_1 to be the largest among $\left\{0, \frac{q_k}{p_k}\right\}$. For u_2, look at boundaries for which $p_k > 0$ (i.e. inside to outside). Take u_2 to be the minimum of $\left\{1, \frac{q_k}{p_k}\right\}$. If $u_1 > u_2$, the line is outside and therefore rejected.

Liang–Barsky algorithm

1、初始化线段交点的参数：$u_1=0$，$u_2=1$；
2、计算出各个裁剪边界的p、q值；
3、根据p、q来判断：是舍弃线段还是改变交点的参数。
（1）当$p<0$时，参数r用于更新u_1；（$u_1=\max\{u_1, \ldots, r_k\}$）
（2）当$p>0$时，参数r用于更新u_2。（$u_2=\min\{u_2, \ldots, r_k\}$）
（3）如果更新了u_1或u_2后，使$u_1>u_2$，则舍弃该线段。
（4）当$p=0$且$q<0$时，因为线段平行于边界并且位于边界之外，则舍弃该线段。

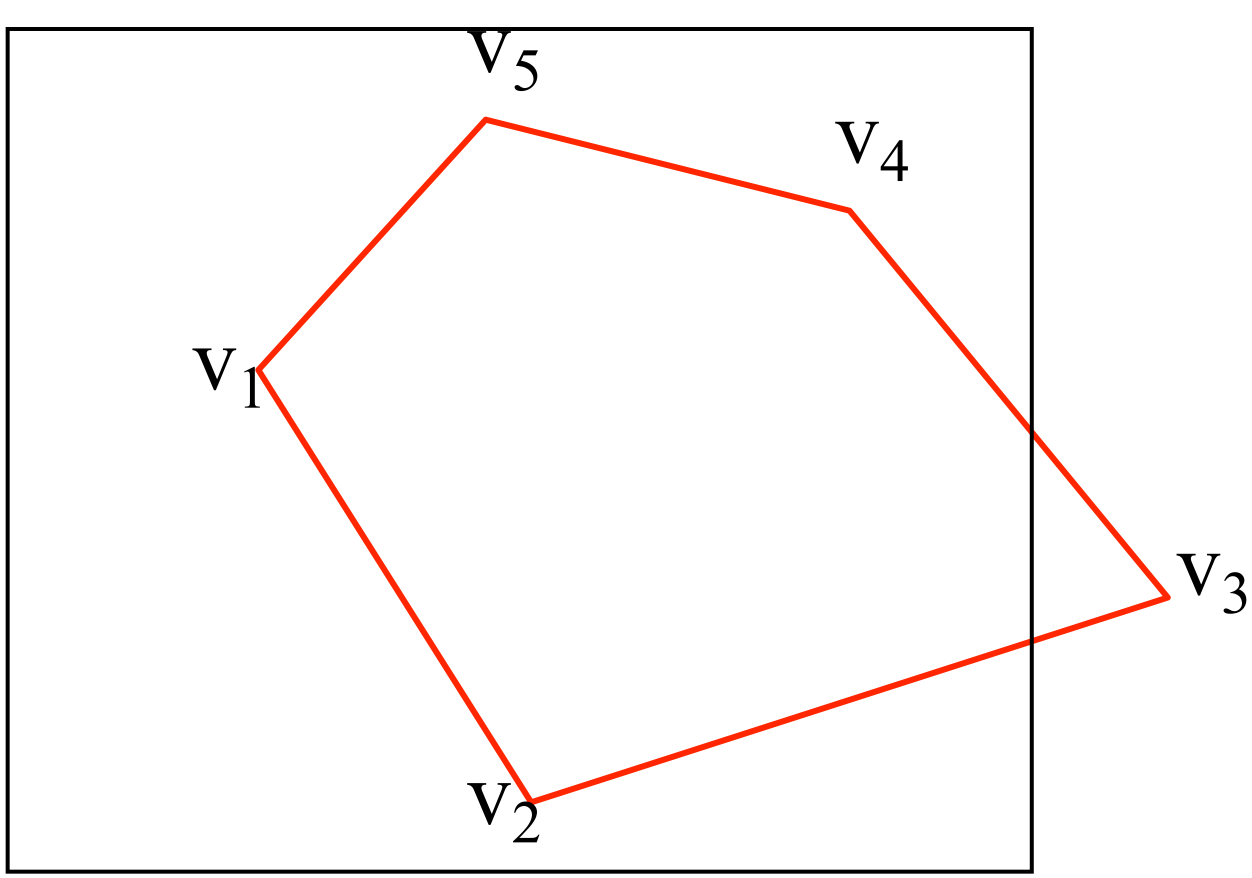
4、p、q的四个值经判断后，如果该线段未被舍弃，则裁剪线段的端点坐标由参数u_1和u_2的值决定。



p=0且q<0的情况

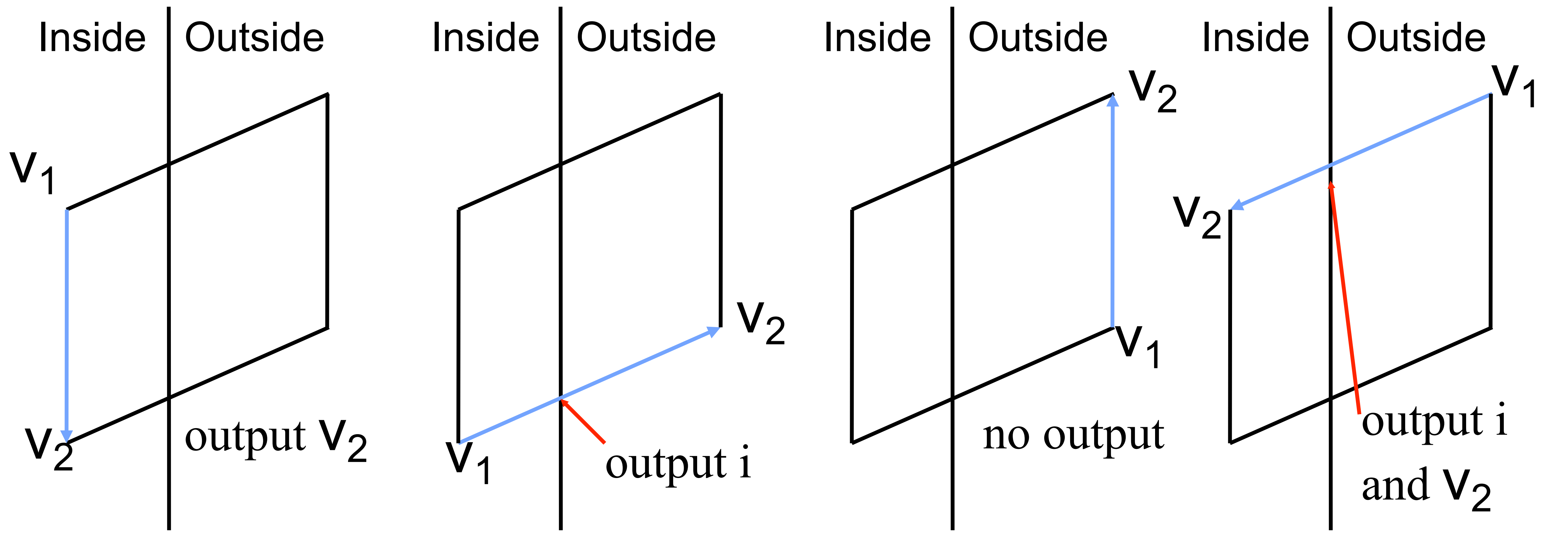
Sutherland-Hodgeman Polygon Clipping Algorithm

- Polygon clipping is similar to line clipping except we have to keep track of inside/outside relationships
  - Consider a polygon as a list of vertices
  - Note that clipping can increase the number of vertices!
  - Typically clip one edge at a time…

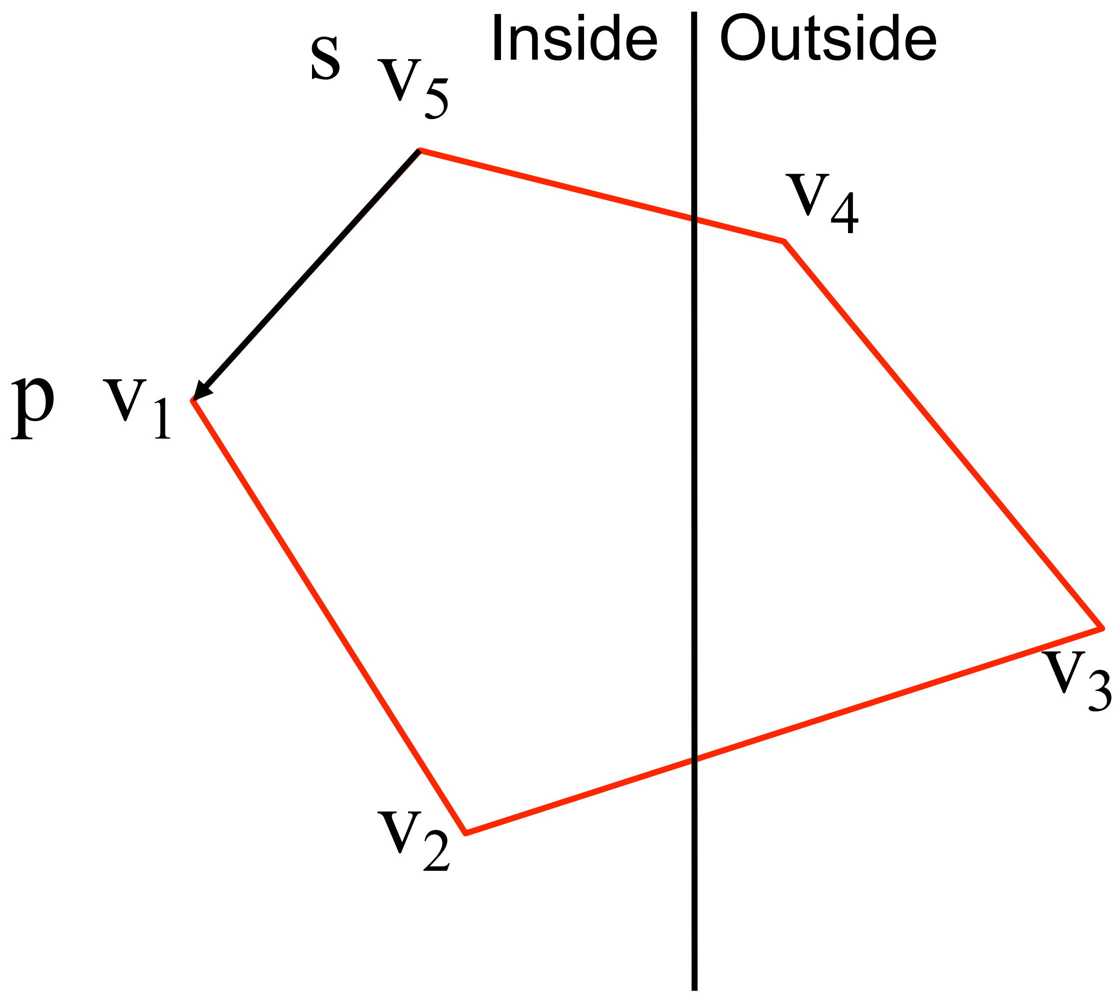


Sutherland-Hodgeman algorithm

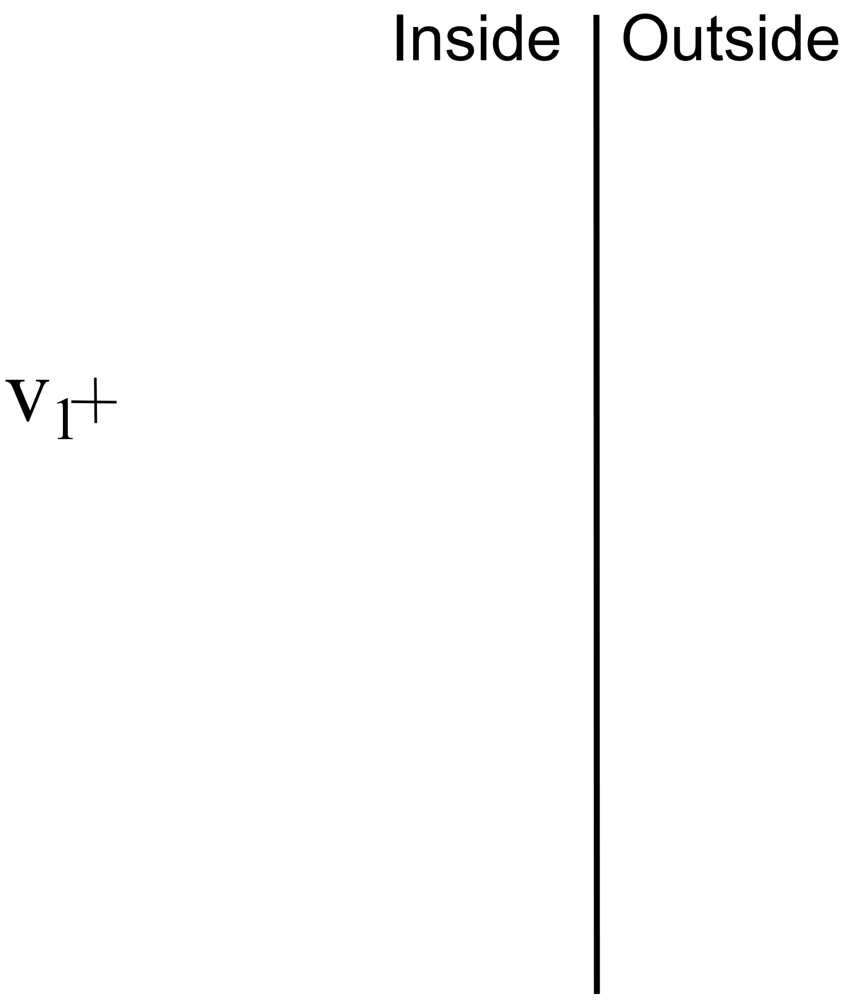
- Present the vertices in pairs
  - $(v_n,v_1), (v_1, v_2), (v_2,v_3), \ldots, (v_{n-1},v_n)$
  - For each pair, what are the possibilities?
  - Consider v_1, v_2



Example
v_5, v_1

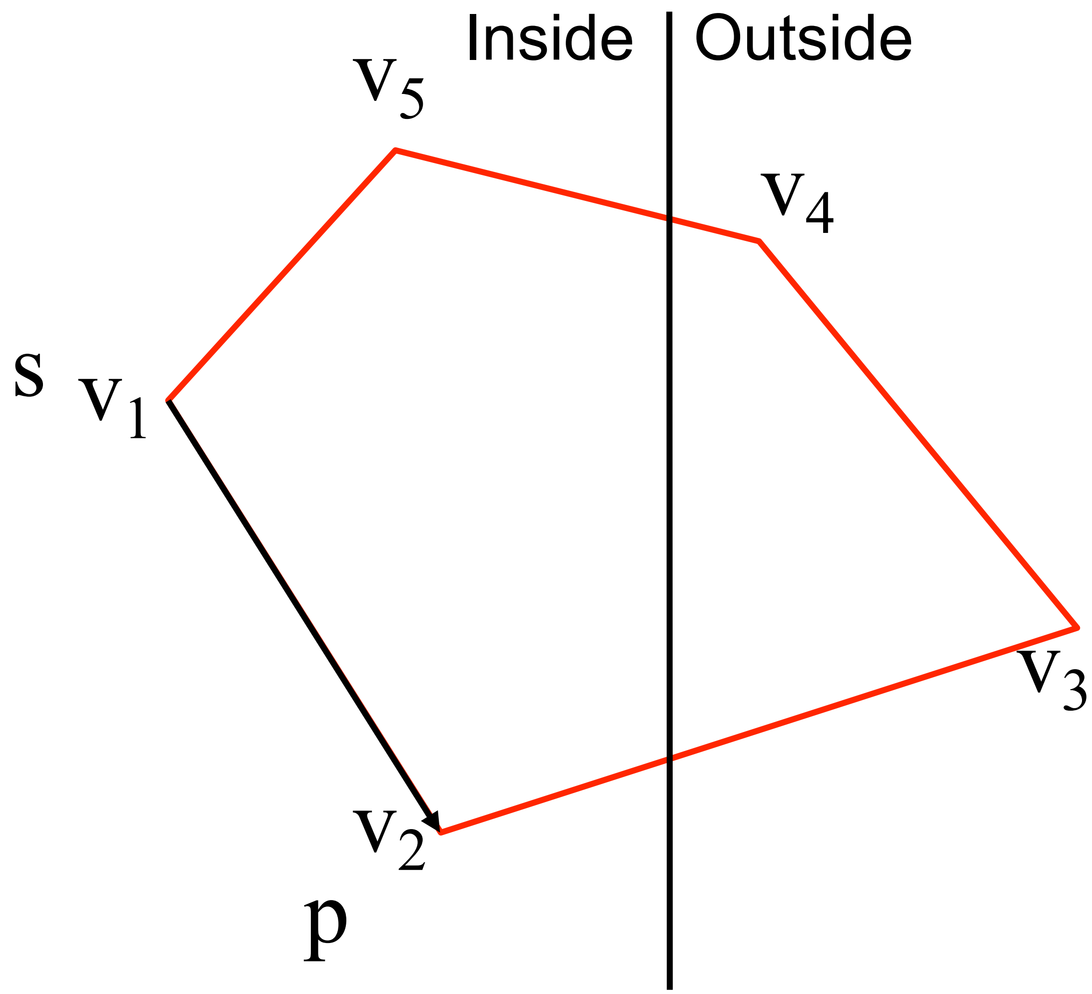


Inside, Inside
Output v_1

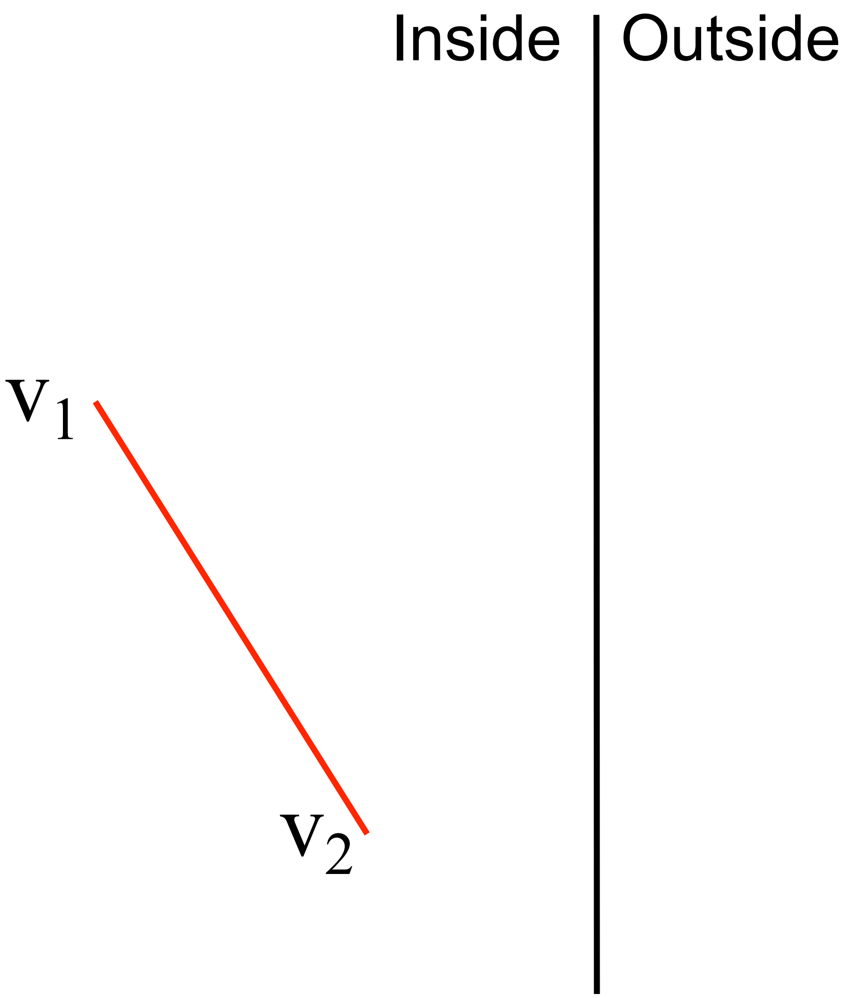


Current
Output

v_1, v_2

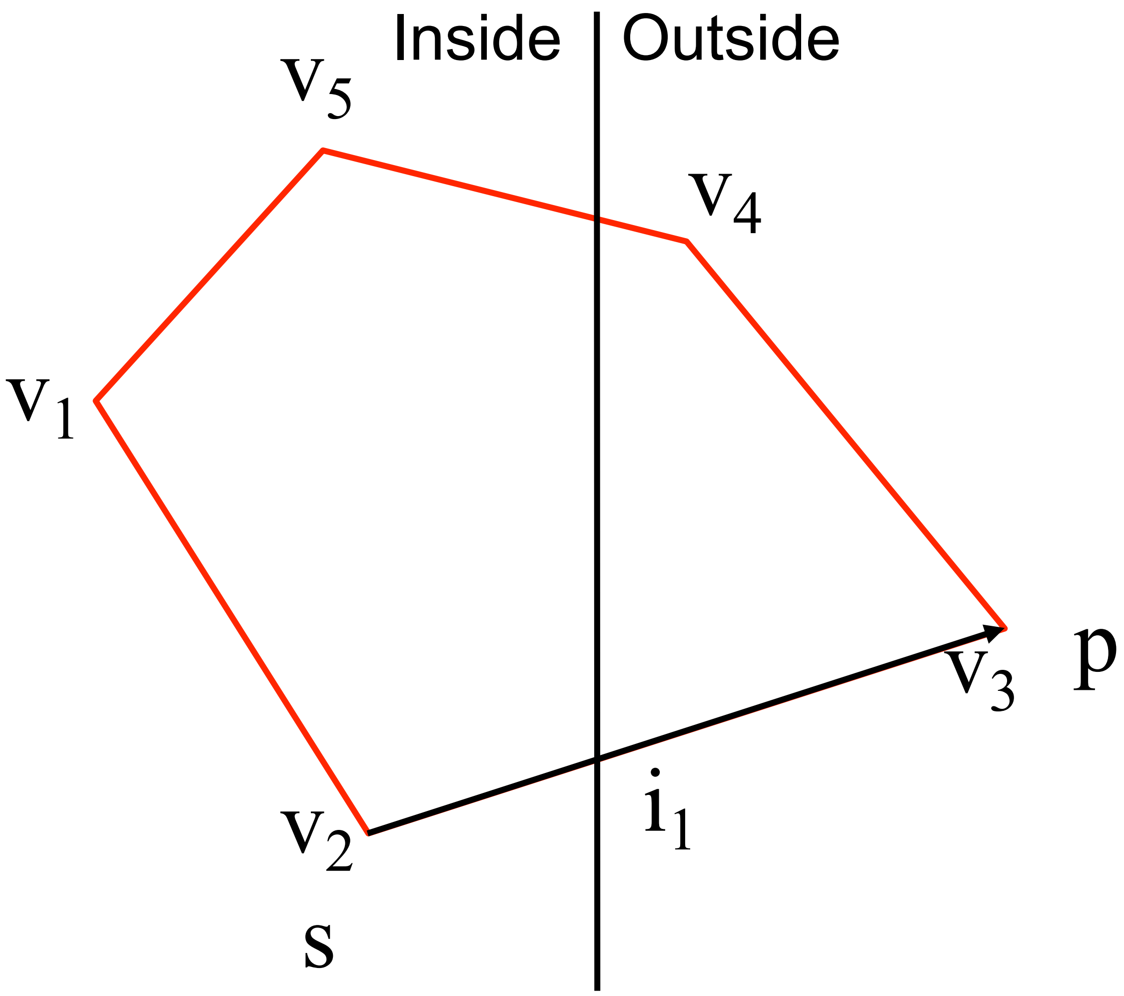


Inside, Inside
Output v_2

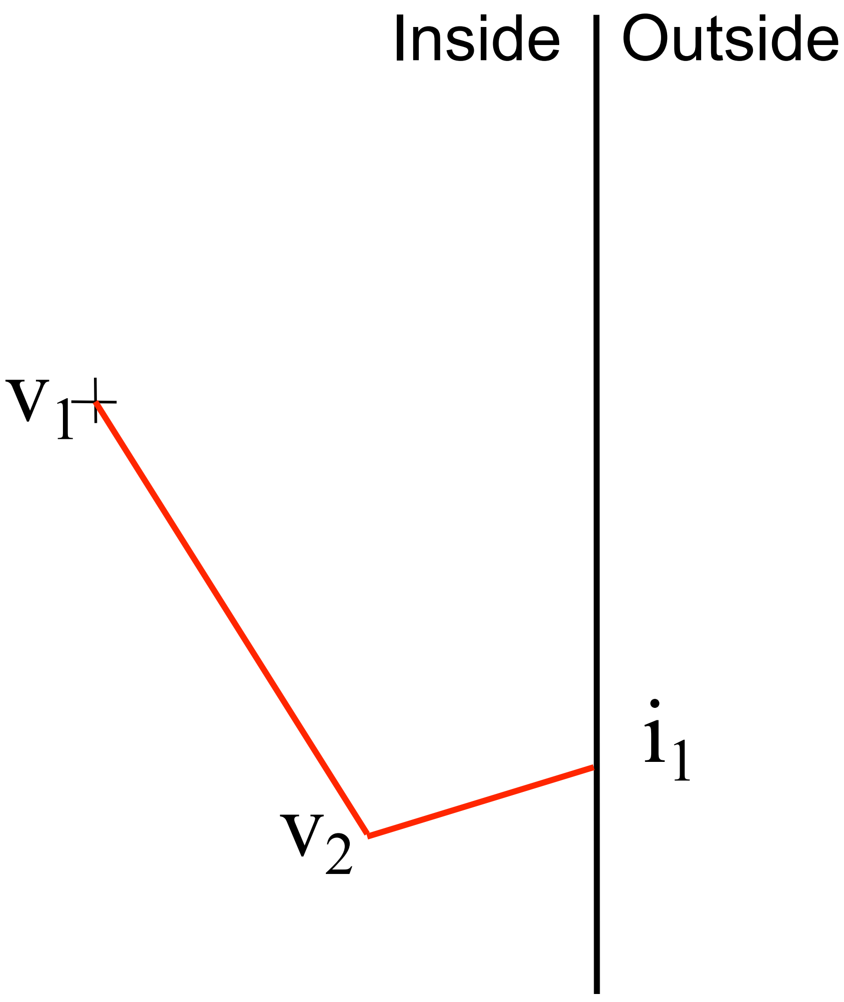


Current
Output

v_2, v_3

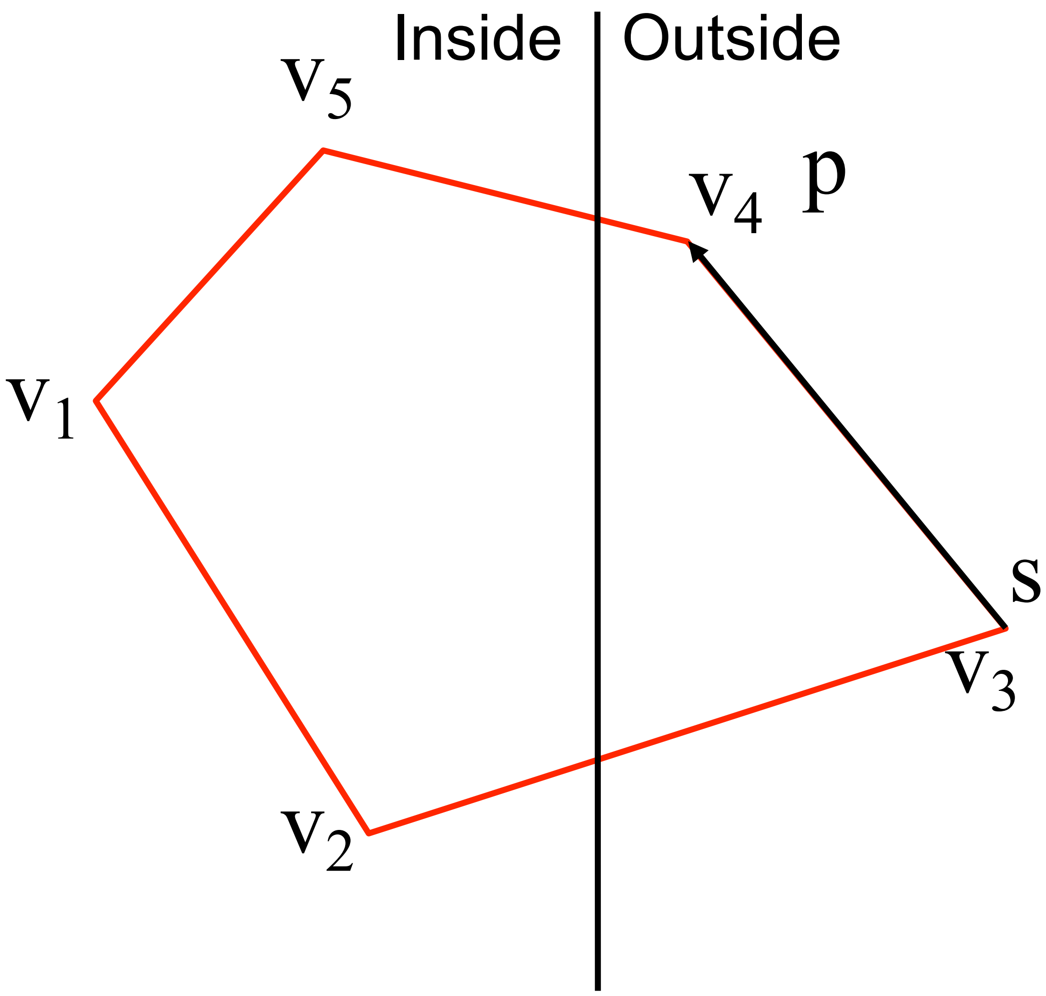


Inside, Outside
Output i_1

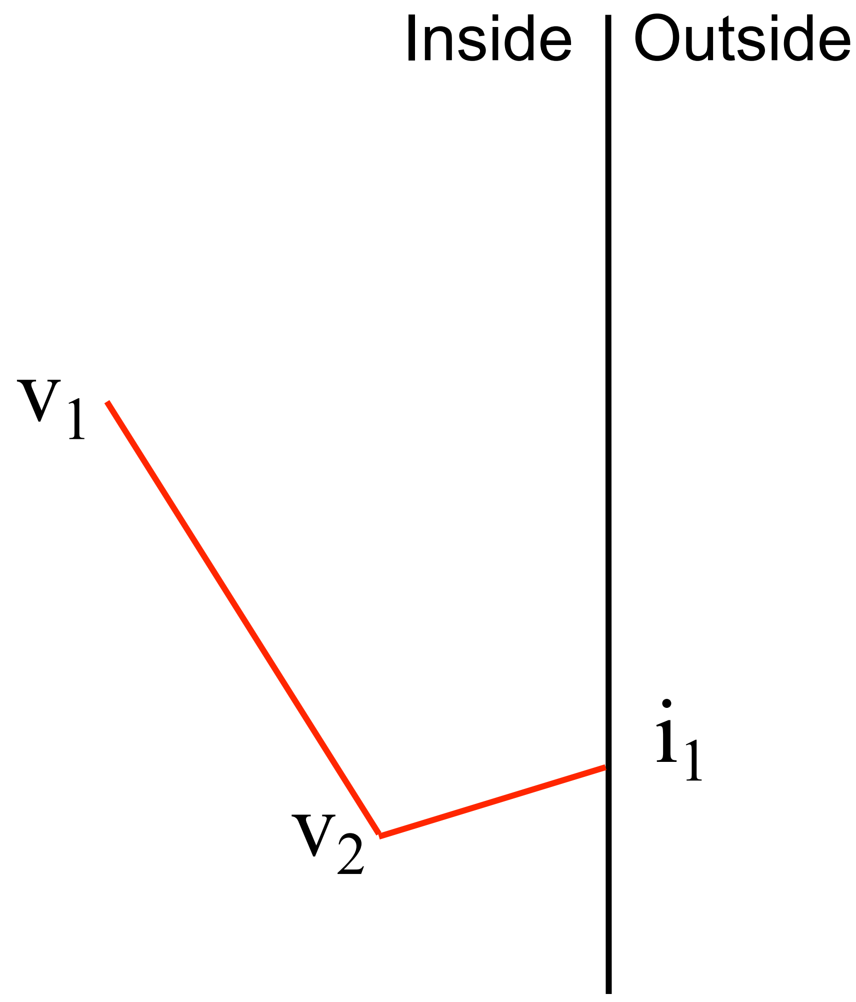


Current
Output

v_3, v_4

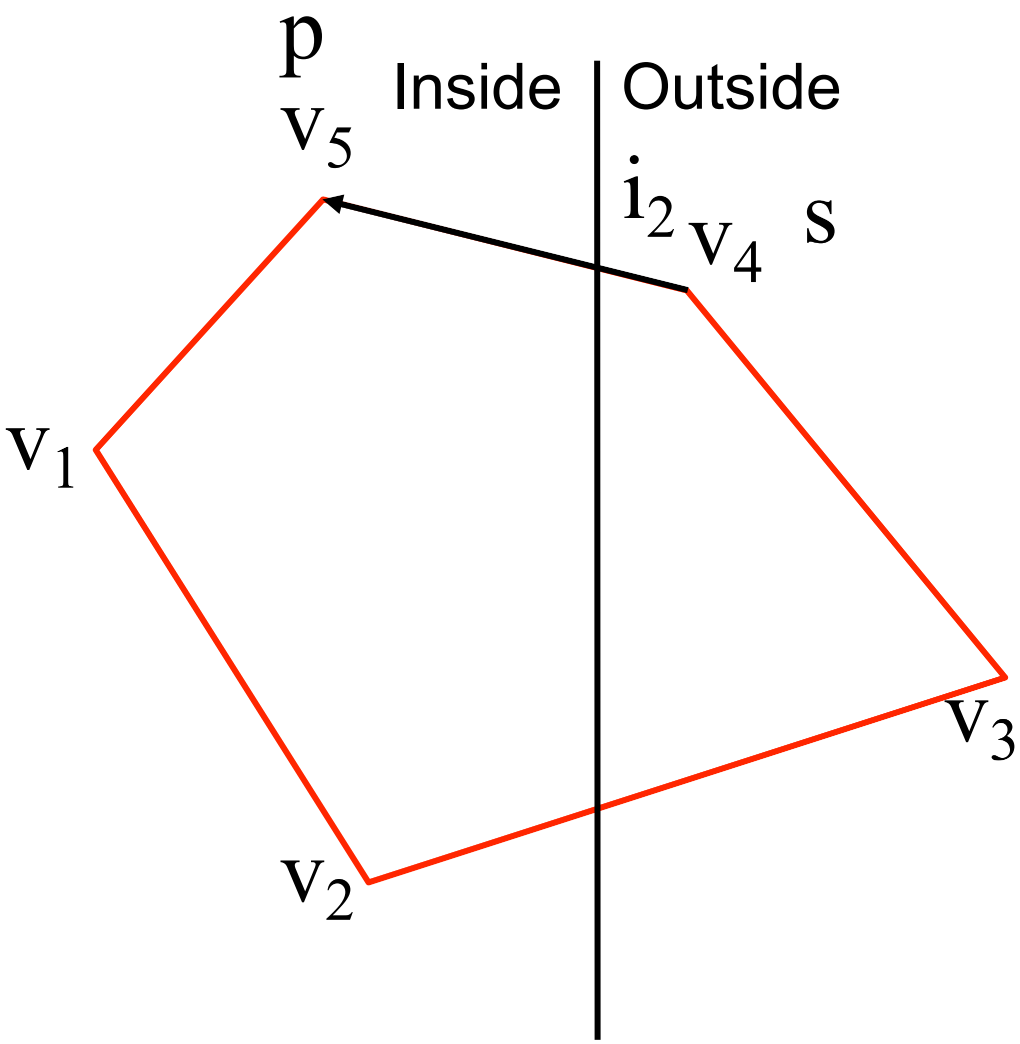


Outside, Outside
No output

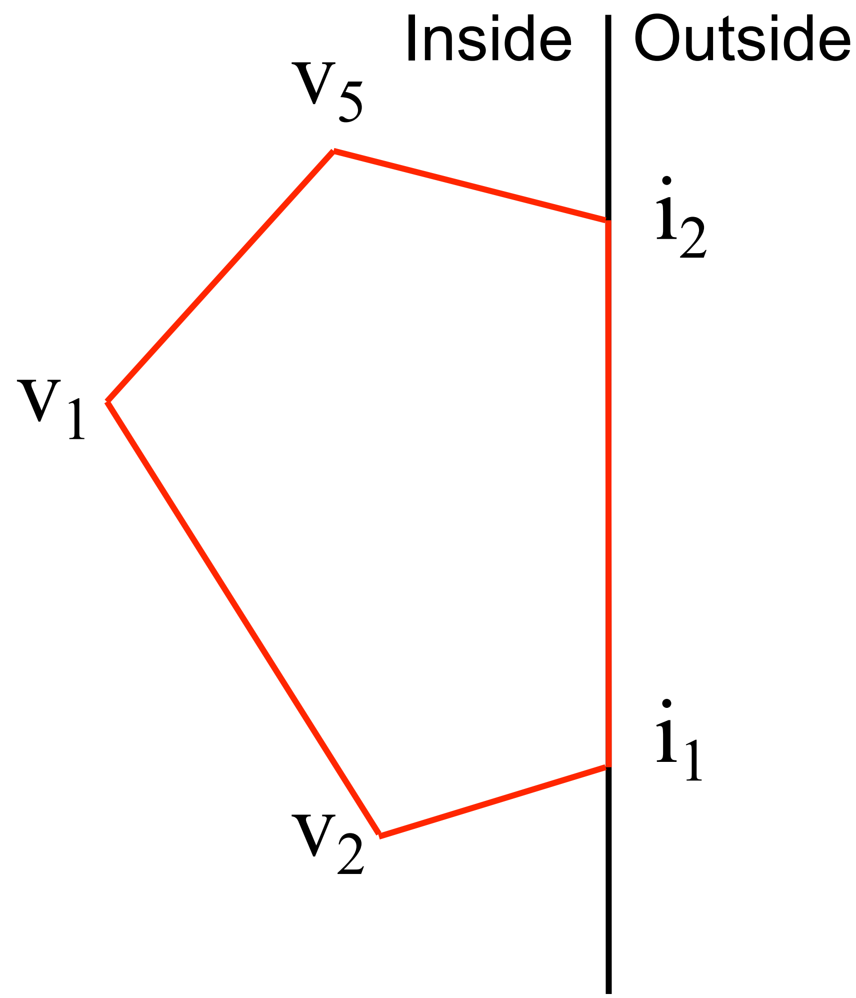


Current
Output

v_4, v_5 – last edge…



Outside, Inside
Output i_2, v_5



Current
Output

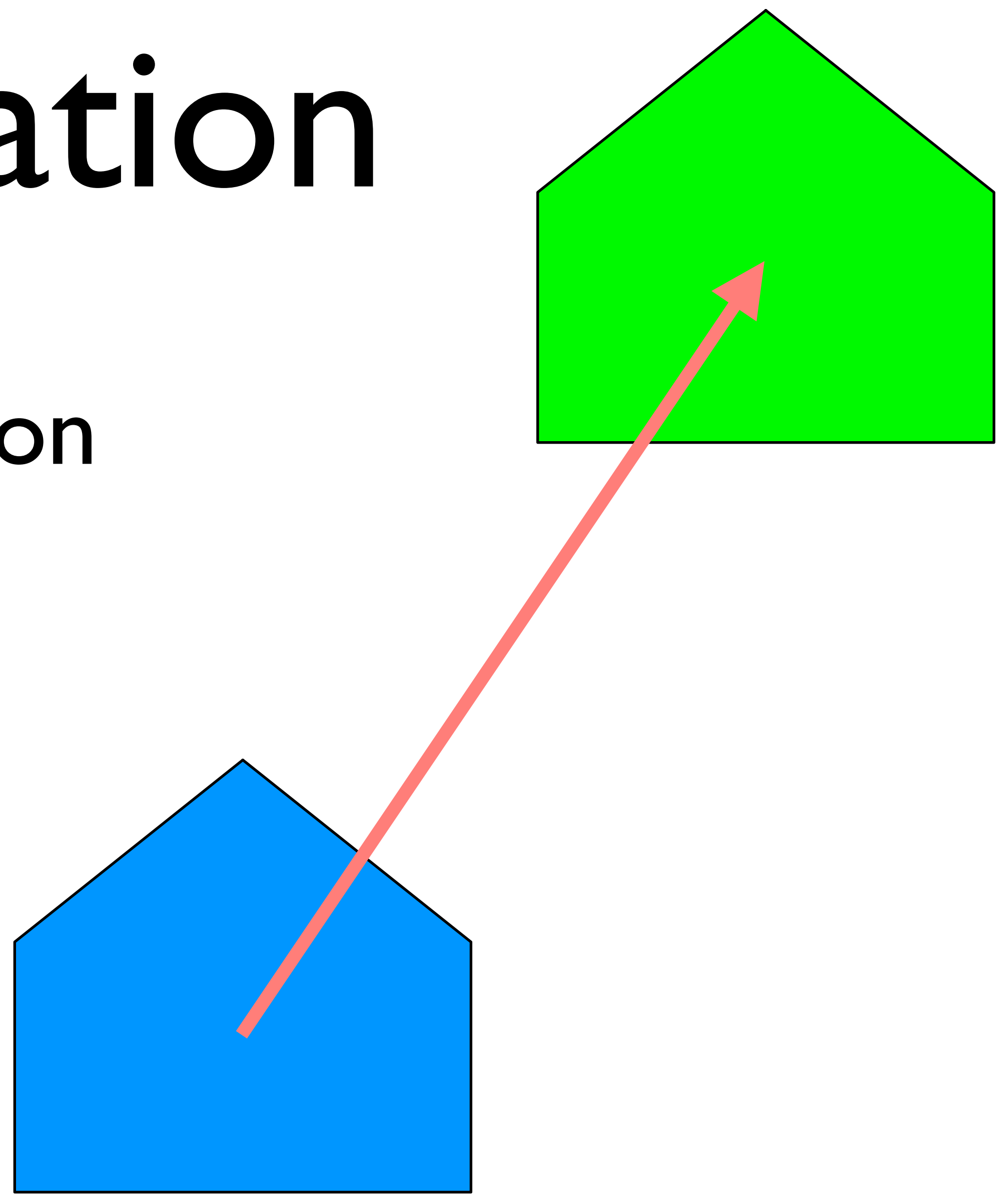
Transforms

Transformations

- Procedures to compute new positions of objects
- Used to modify objects or to transform (map) from one co-ordinate system to another co-ordinate system

As all objects are eventually represented using points, it is enough to know how to transform points.

Translation



- Is a Rigid Body Transformation

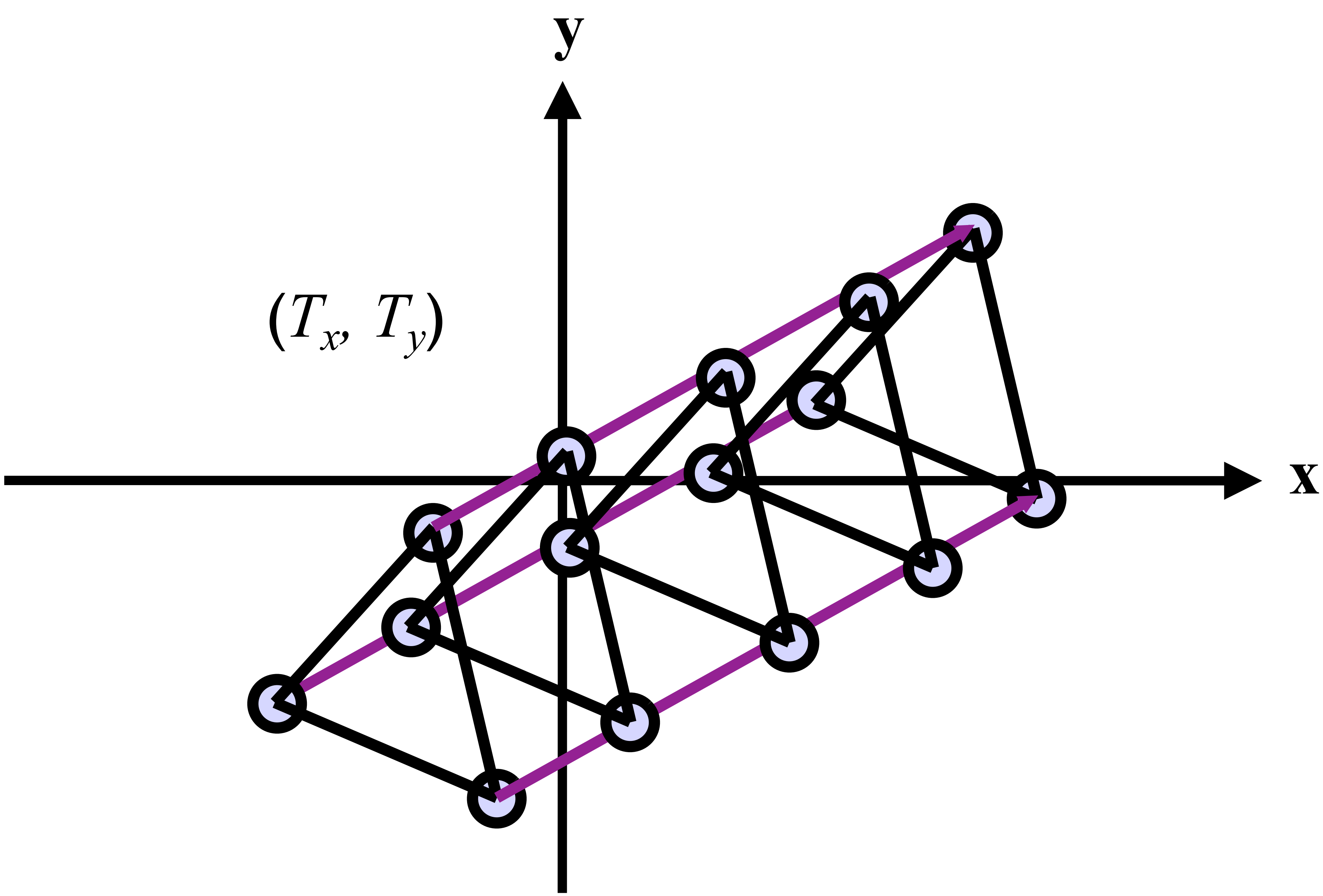
$$x \Rightarrow x + T_x$$

$$y \Rightarrow y + T_y$$

$$z \Rightarrow z + T_z$$

- Translation vector (T_x, T_y, T_z) or shift vector

Translating an Object



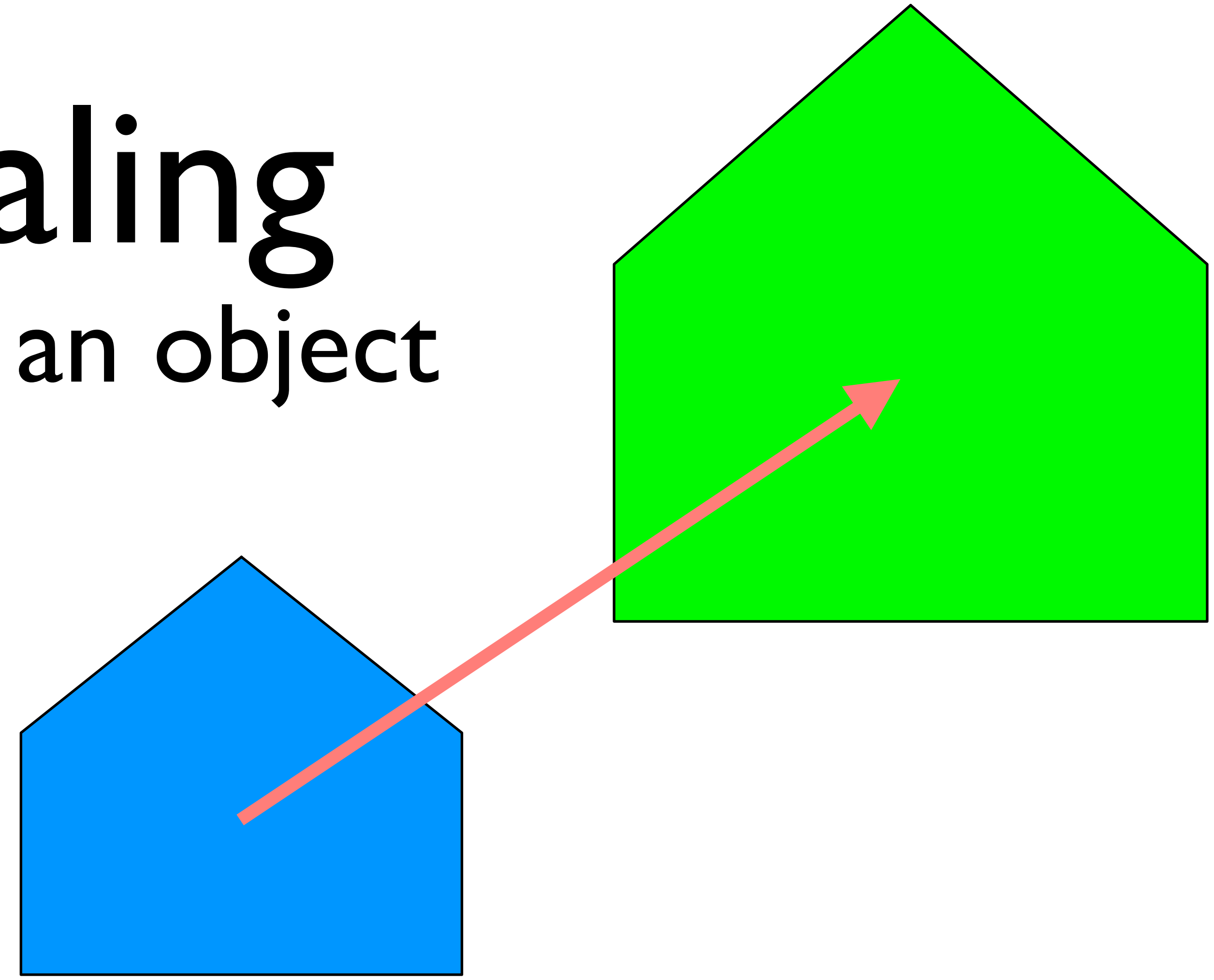
Scaling

- Changing the size of an object

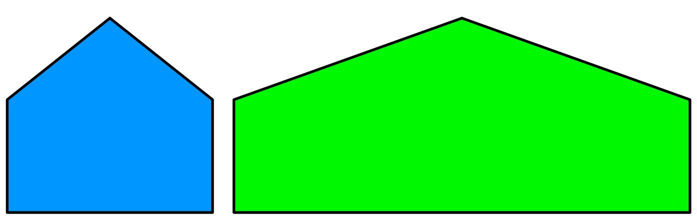
$$x \Rightarrow x * S_x$$

$$y \Rightarrow y * S_y$$

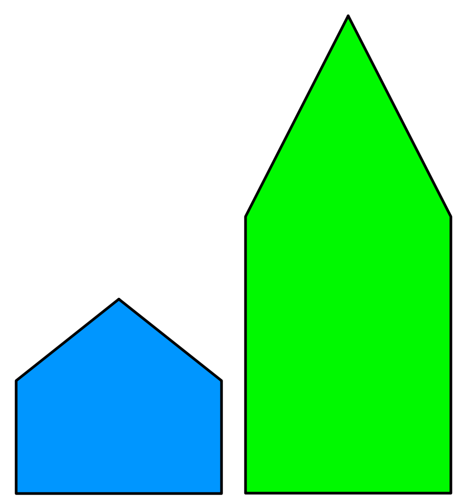
$$z \Rightarrow z * S_z$$



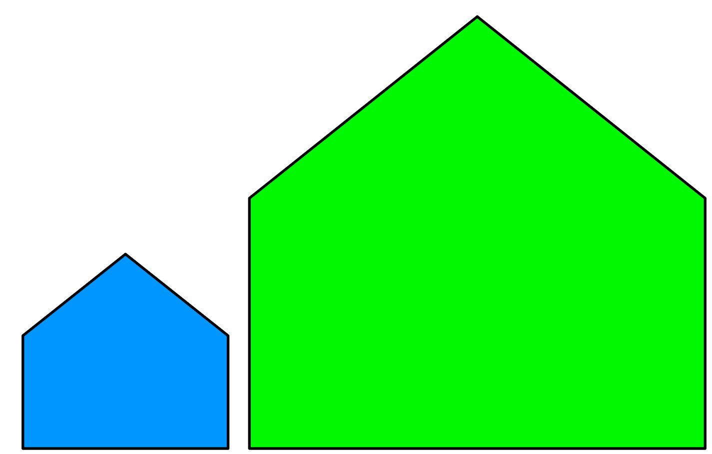
- Scale factor (S_x, S_y, S_z)



$S_y = 1$

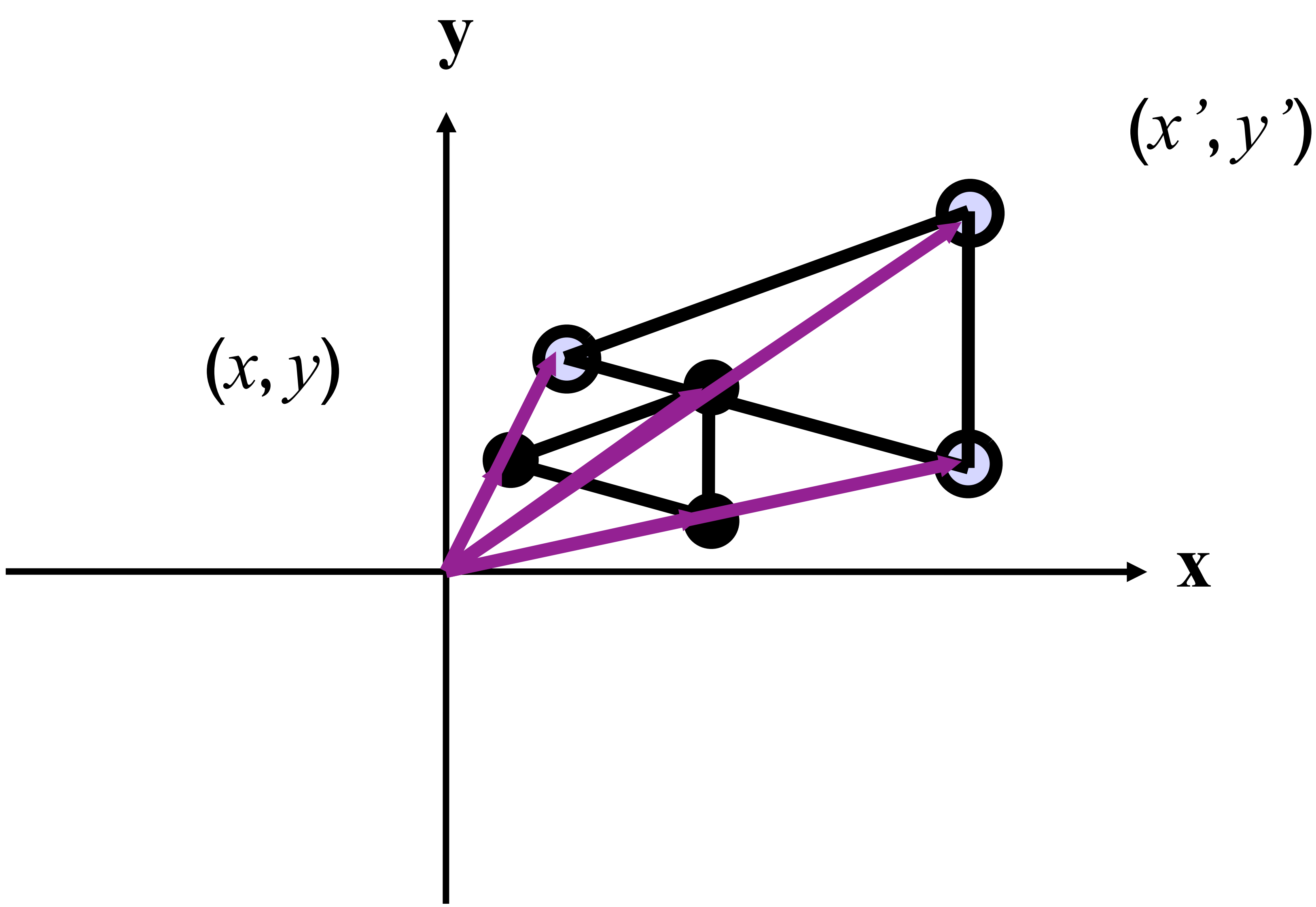


$S_x = 1$



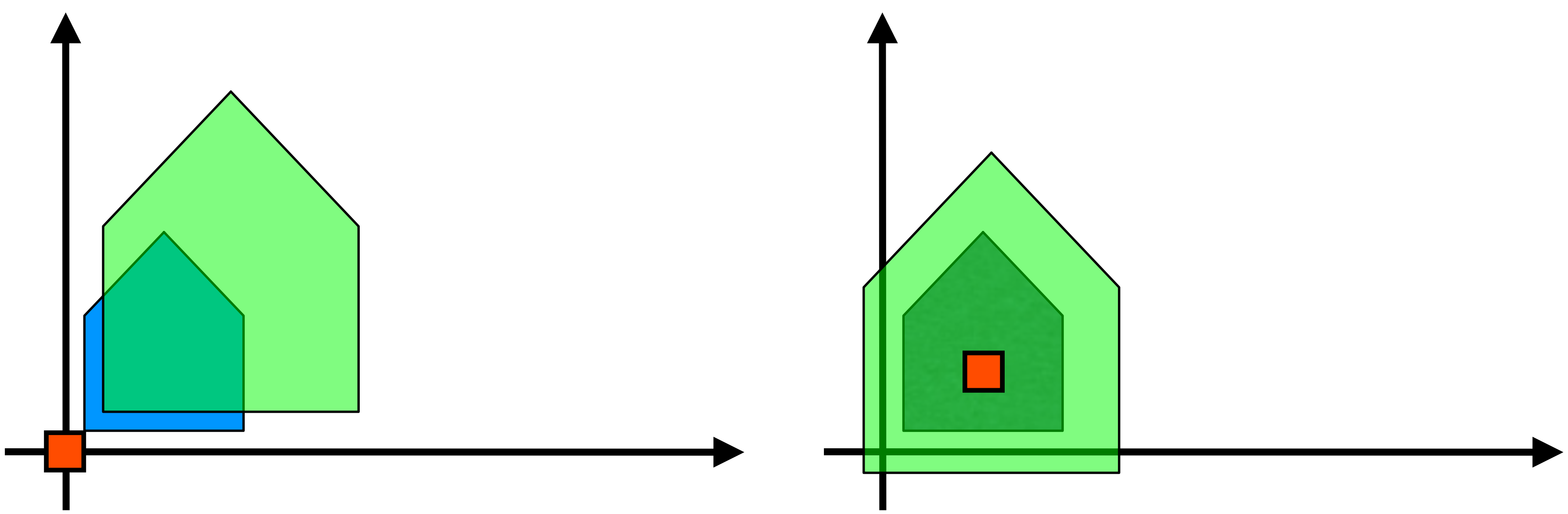
$S_x = S_y$

Scaling an Object



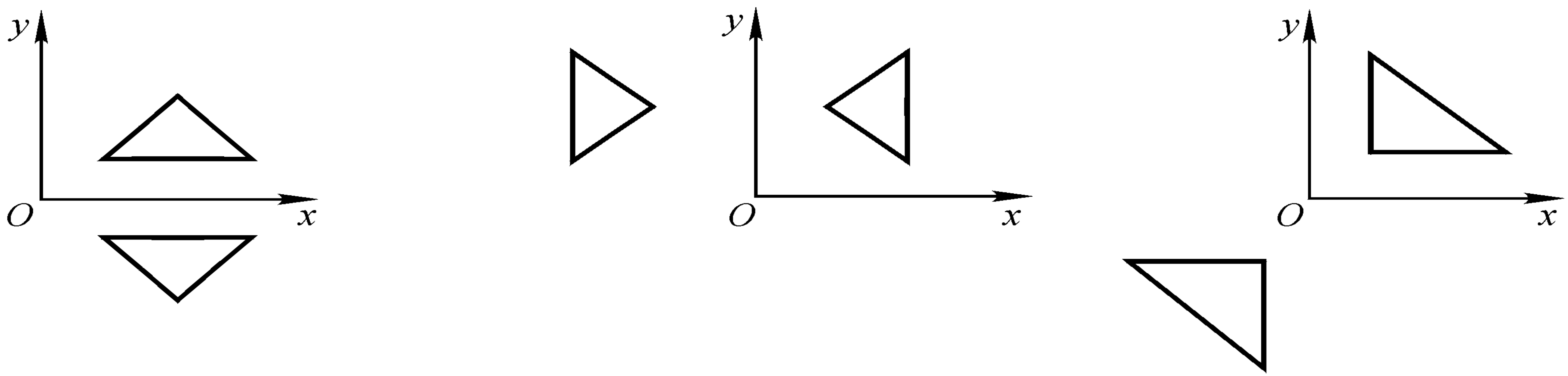
Scaling (contd.)

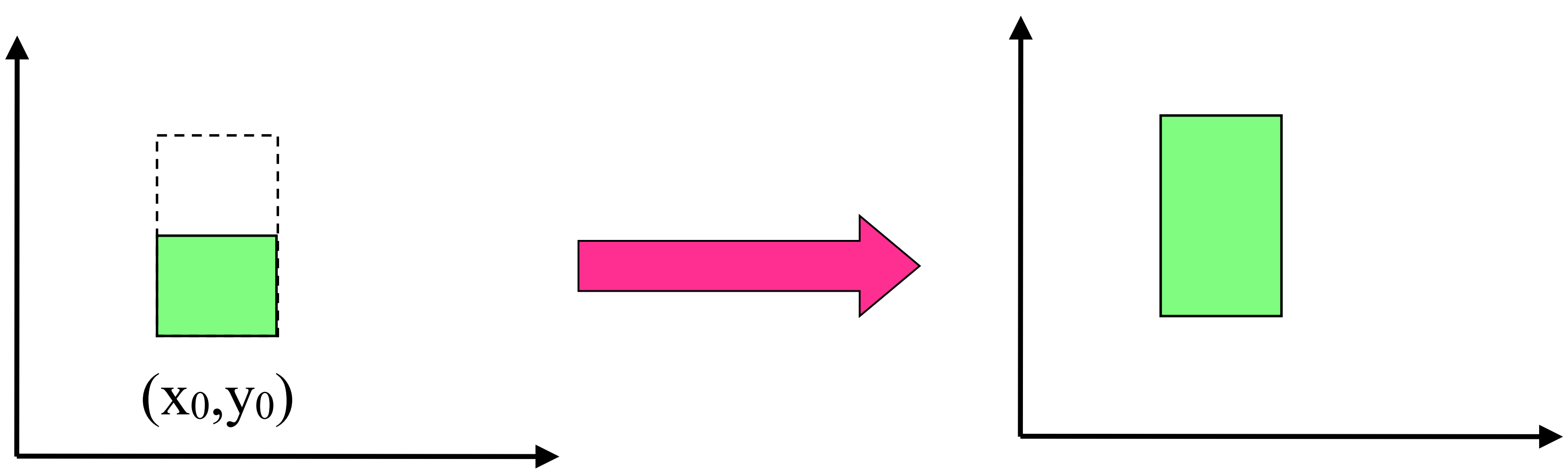
Scaling is always with respect to the origin. The origin does not move.



Scaling wrt a reference point can be achieved as a **composite transformation**

Scaling an Object





Question8 : How ?

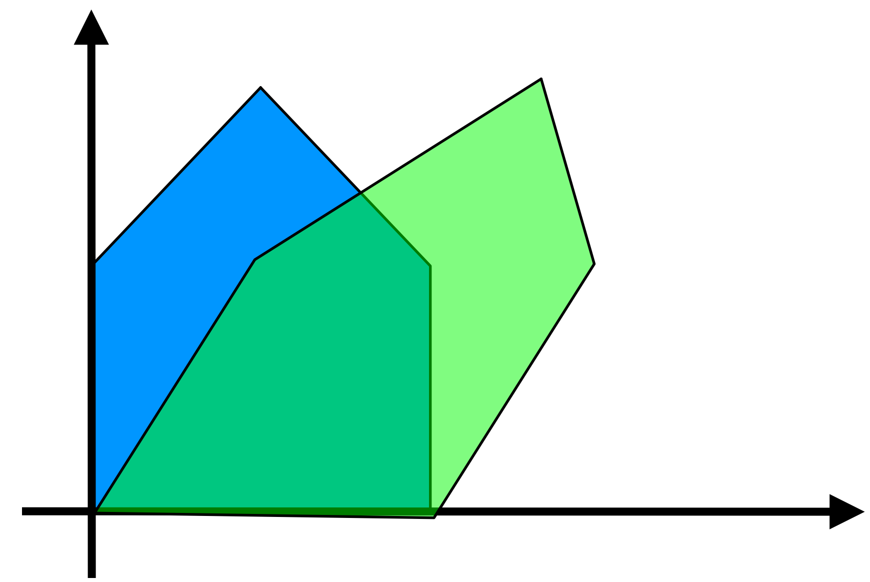
Shearing

- Produces shape distortions
- Shearing in x-direction

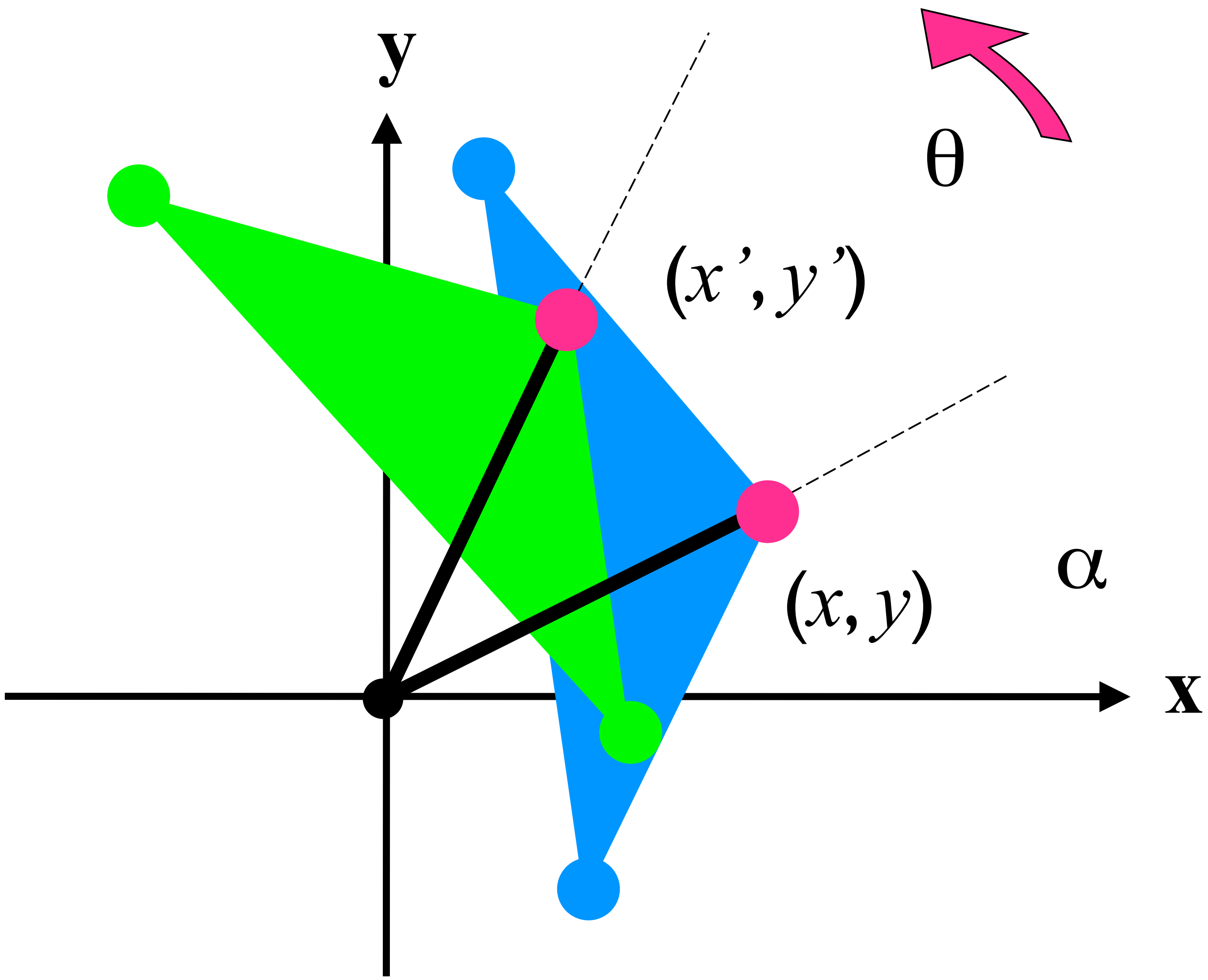
$x \Rightarrow x + a * y$

$y \Rightarrow y$

$z \Rightarrow z$



Rotation



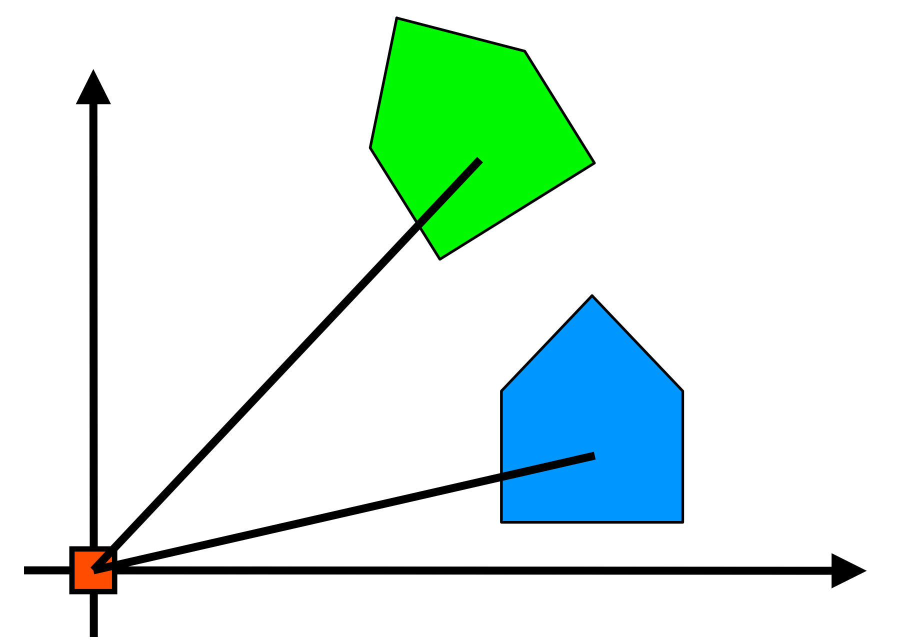
Rotation

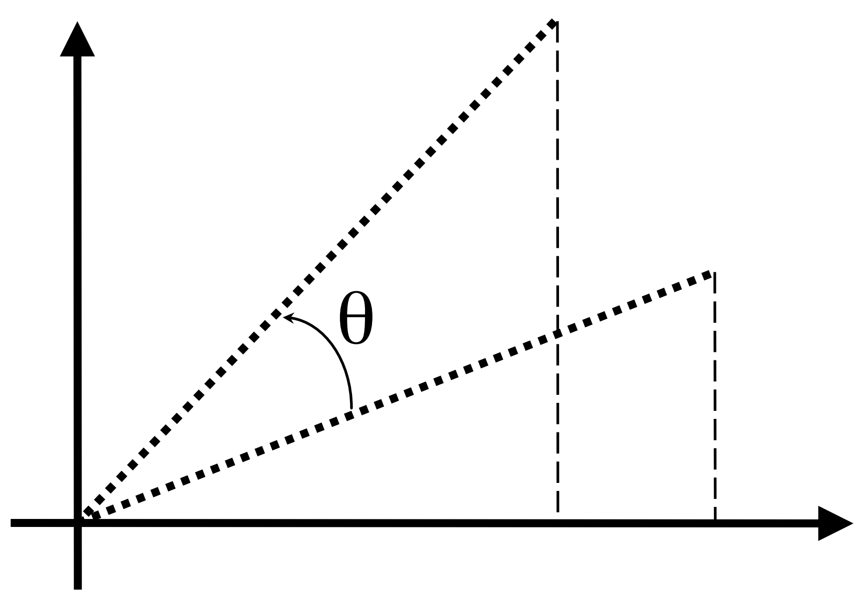
- Is a Rigid Body Transformation

$$x = x * \cos(\theta) - y * \sin(\theta)$$

$$y = x * \sin(\theta) + y * \cos(\theta)$$

$$z = z$$



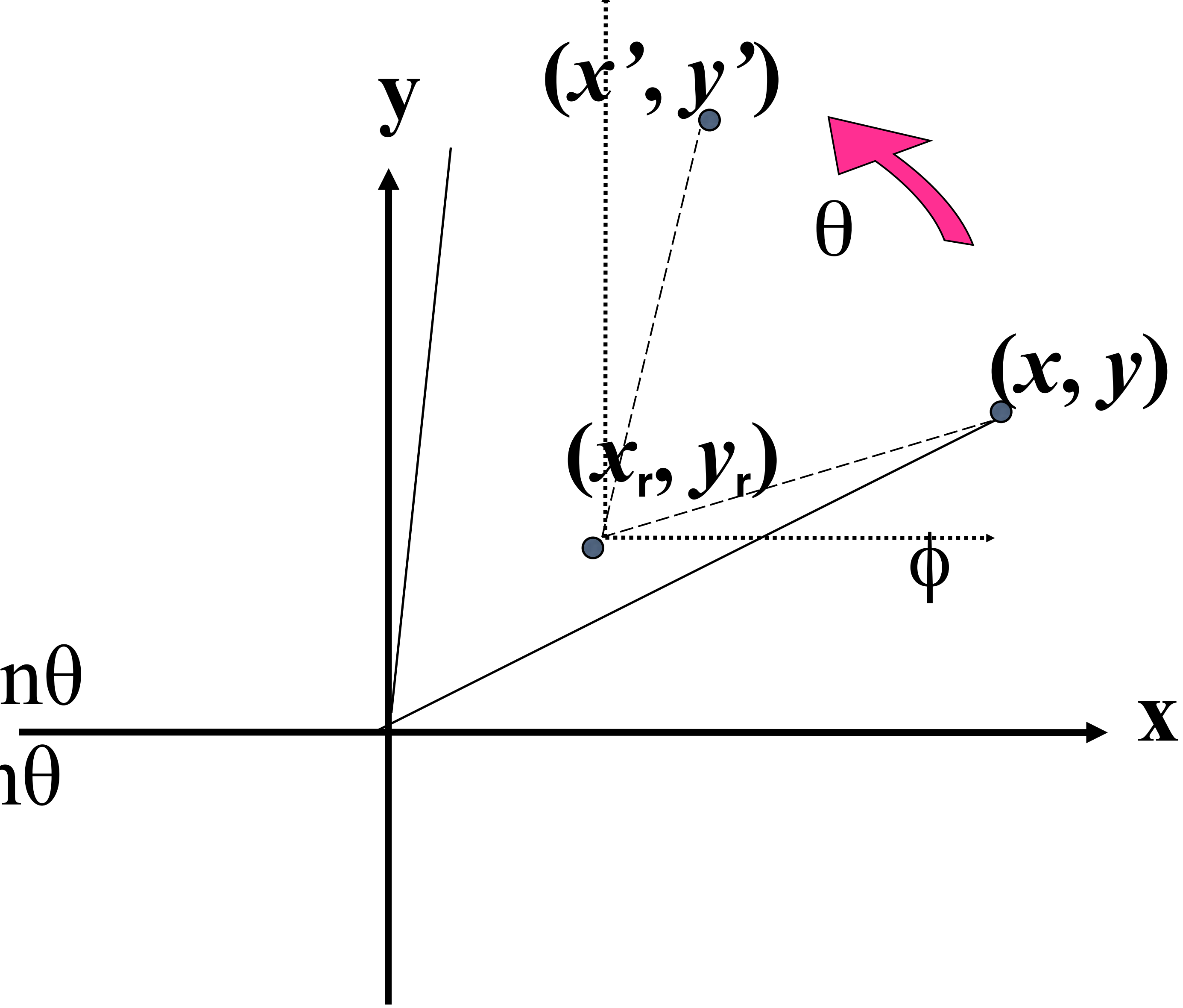


$$x' = r \cdot \cos(\alpha+\theta)$$
$$= r \cos \alpha \cos \theta - r \sin \alpha \sin \theta$$
$$= x \cos \theta - y \sin \theta$$

Rotation (contd.)

- Rotation also is wrt to a reference -
  - A Reference Line in 3D
  - A Reference Point in 2D
  - Define 2D rotation about arbitrary point

Rotate around (x_r, y_r)



$$newx = x - x_r$$
$$newy = y - y_r$$

$$newx' = newx \cos\theta - newy \sin\theta$$
$$newy' = newy \cos\theta + newx \sin\theta$$

$$x' = newx' + x_r$$
$$y' = newy' + y_r$$

$$\mapsto \quad x' = x_r + (x - x_r)\cos\theta - (y - y_r)\sin\theta$$
$$y' = y_r + (y - y_r)\cos\theta + (x - x_r)\sin\theta$$

General Linear Transformation

$$
\begin{aligned}
x &\circledR\; a*x + b*y + c*z \\
y &\circledR\; d*x + e*y + f*z \\
z &\circledR\; g*x + h*y + i*z
\end{aligned}
\quad or \quad
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix}
$$

- Which of the following can be represented in this form?
  - Translation
  - Scaling
  - Rotation