CS499 Homework 6

Intersteller

Exercise 6.1

(1)As the picture shows:

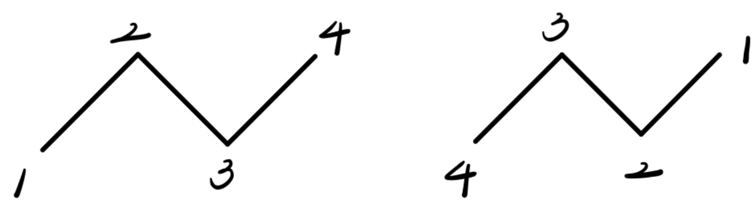

Figure 1:

The number of automorphisms is 2. (2)Suppose the original picture is

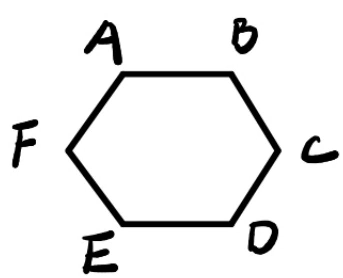

Figure 2:

Original A position now is A',so original B position now is the point connected with A' before and there are two possibilities.

If B' is determined,the original F position is the other point connected with A' apart from B'.

Therefore other positions is also determined for the same reason, as in the following figure.

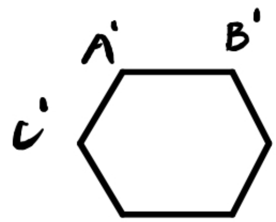

Figure 3:

So the number of automorphisms is $6 \times 2 = 12$.

(3)For the same reason of (2),if there are 3 points determined , the graph is determined.

So the number of automorphisms is $8 \times 3! = 48$.

Exercise 6.2

The complete graph on 999 vertices has C_{999}^2 edges.The $|E|$ is odd,so the $|E|$ of original graph and the $|E|$ of complement graph is not equal. So there is no self-complementary graph on 999 vertices.

Exercise 6.3

Theorem: There is a self-complementary graph on n vertices if and only if $n = 4k$ or $n = 4k + 1$. (here and in the following, $k = 1, 2, 3, \cdots$)

proof:
(1) If $n = 4k + 2$ or $n = 4k + 3$, since a complete graph has $\frac{(|V|-1)(|V|-2)}{2}$ edges, $|E|$ will be an odd number. It is obvious that if $|E|$ is an odd number, the graph can not be self-complementary.

(2) If $n = 4k$, we can show the graph is self-complementary in the following method:
We divide the vertices into 4 sequences A, B, C and D, each sequence has k verticies. Then we add $\frac{(k-1)(k-2)}{2}$ edges into sequence A and sequence B respectively.After that we add $\frac{(k)(k-1)}{2}$ edges between sequence A and sequence B, sequence A and C, sequence B and sequence D respectively, as in the following figure.

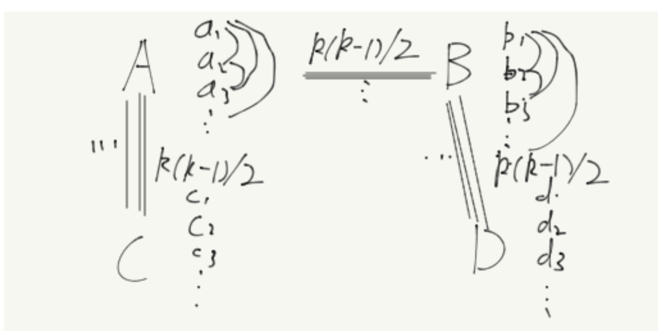

Figure 4:

In this way we get a graph G, whose complementary graph is G', which has $\frac{(k-1)(k-2)}{2}$ edges in sequence C and in sequence D respectively, and $\frac{(k)(k-1)}{2}$ edges between sequence A and sequence D, sequence B and sequence C, sequence C and sequence D respectively. Through the bijection $C \to A$ ($c_1 \to a_1, c_2 \to a_2, \cdots$ (and the same in the following)), $A \to D$, $B \to C$ and $D \to B$, we can show that G and G' are automorphism, which means G or G' is a self-complementary graph.

(3) If $n = 4k + 1$, we divide the vertices into 4 sequences same as above and 1 special dot. Then for the sequences we use the same method to add edges, except in graph G we add k edges between A and the dot, B and the dot respectively. Then we will have k edges between C and the dot, D and the dot respectively in G'. In the bijection we add $dot \rightarrow dot$. In this way, G or G' is still self-complementary graph.

Exercise 6.4

For $k = 3$ and $k = 4$, the corresponding graphs are in the following figure.

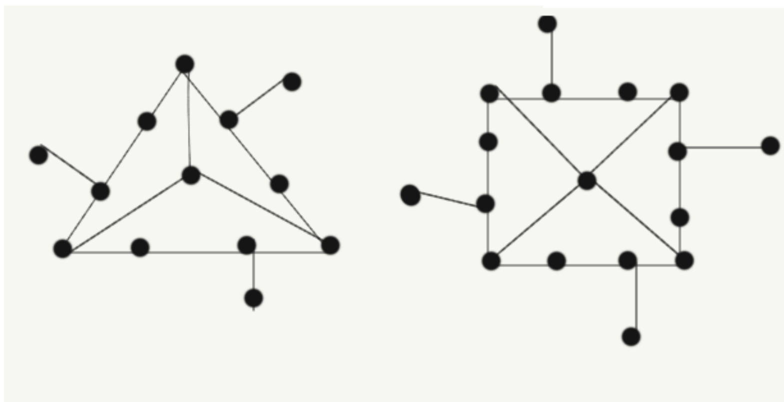

Figure 5:

One general construction method: For $k = n$, first we draw a polygon that has n edges. Then we add one vertex in the middle of the polygon and link it with n vertices respectively. After that we add 2 verticies on each edge. Finally, we add one vertex and one edge in every 3 vertices clockwise.

Exercise 6.5

For every $n \leq 6$, there is an asymmetric graph on n vertices, like Figure 1.

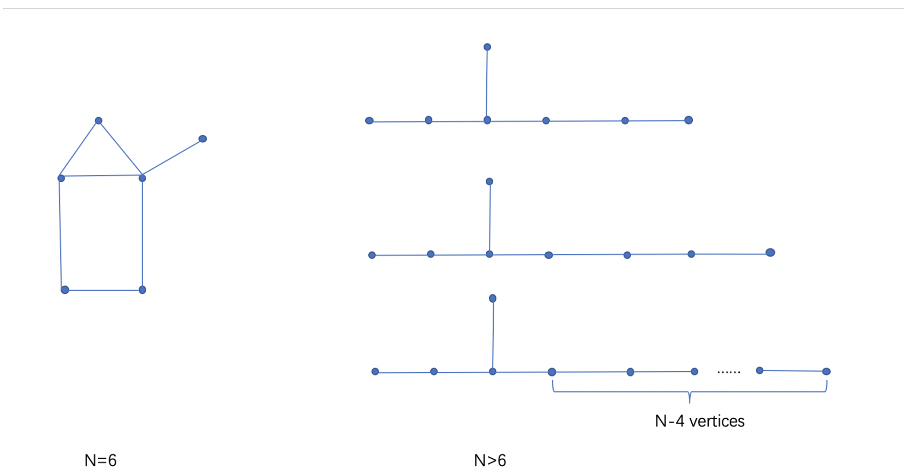

Figure 6:

When $n = 6$, the left graph is correct. When $n > 6$, we just need to construct trees like right graphs. There are only one vertex that has three degrees and $n - 4$ vertices are at the right of this vertex.

Exercise 6.6

For graphs in **Exercise 6.5**, we just need to add cherries for every vertex, like Figure 2.

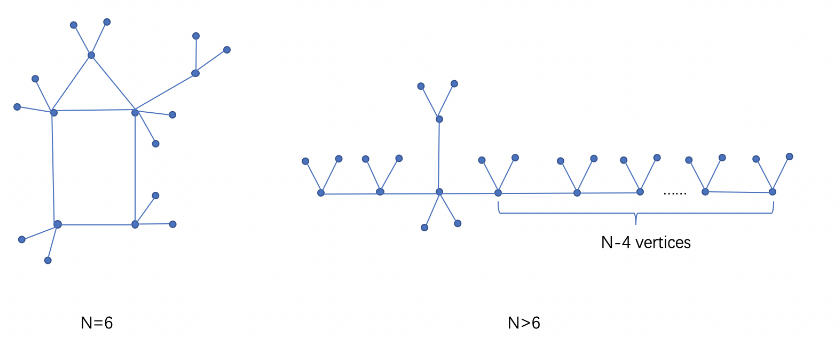

Figure 7:

The graph has $3n$ vertices with 2^n automorphisms.

Exercise 6.7

1. We can use $4p + 1$ to form such graph with the method showed in **Exercise 6.4**

2. There is no such graph with less than p vertexes.

As the conclusion we got from class goes, for a graph with p vertexes, the number of its automorphisms can divide $p!$. Any prime number bigger than p cannot divide $p!$, so There is no such graph with less than p vertexes.

Question:

In Exercise 6.7, if we do not include the constraint that P is a prime number, what conclusion can we get? First, we make a prime factorization on P, which is $P = P_1^{A_1} \cdot P_2^{A_2} \cdot \cdots \cdot P_n^{A_n}$ For every i, P_i is prime number and A_i is positive integer. We have a hypothesis that the number of vertices might be $O(P_{max} + A_{max})$. But we dont know how to prove it. Or do we have another answer?