CS499 Homework 3

Intersteller

Exercise 3.1

1.

For $T_{10} = n$, there are $n-1$ situations for the first $n-2$ digits of the string:

$$\begin{cases} 0,0,0,\cdots,0,0 \\ 0,0,0,\cdots,0,1 \\ \qquad\qquad\qquad\vdots \\ 1,1,1,\cdots,1,1 \end{cases}$$

Based on the probability calculation formula of classical probabilities, we have

$$Pr\left[T_{10} = n\right] = \frac{n-1}{2^{n-2}} \cdot \frac{1}{4} = \frac{n-1}{2^n} \quad n \geq 2$$

$$Pr\left[T_{10} = n\right] = 0 \quad n < 2$$

2.

Based on the knowledge about A_n (A_n=$\{x \in (0,1)^n \mid x$ does not contain $11\}$),we have

$$Pr\left[T_{11} = n\right] = \frac{|A_{n-3}|}{2^{n-3}} \cdot \frac{1}{8} = \frac{F_{n-1}}{2^n} \quad n > 3$$

$$Pr[T_{11} = 3] = \frac{1}{8}$$

$$Pr[T_{11} = 2] = \frac{1}{4}$$

$$Pr[T_{11} = 1] = 0$$

3.

There are 2 states of the coin, which are 0 and 1. Assuming the expectation of the event under each state is E_1 and E_2 respectively, we have

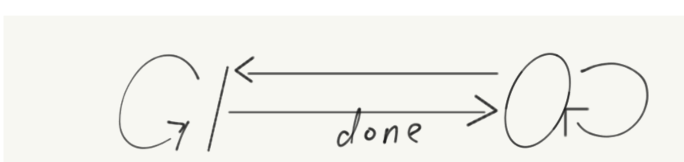

Figure 1: The state machine 3.1.3.1

$$\begin{cases} E_1 = 1 + \frac{1}{2}E_1 + \frac{1}{2} \times 0 \\ E_2 = 1 + \frac{1}{2}E_2 + \frac{1}{2}E_1 \end{cases}$$

$$\Rightarrow \begin{cases} E_1 = 2 \\ E_2 = 4 \end{cases}$$

$$\Rightarrow E[T_{10}] = 1 + \frac{1}{2}E_1 + \frac{1}{2}E_2 = 4$$

Figure 2: The state machine 3.1.3.2

$$\begin{cases} E_1 = 1 + \frac{1}{2}E_1 + \frac{1}{2}E_2 \\ E_2 = 1 + \frac{1}{2}E_1 + \frac{1}{2} \times 0 \end{cases}$$

$$\Rightarrow \begin{cases} E_1 = 6 \\ E_2 = 4 \end{cases}$$

$$\Rightarrow E[T_{11}] = 1 + \frac{1}{2}E_1 + \frac{1}{2}E_2 = 6$$

Exercise 3.2

There are 2 states of the coin, which are 0 and 1. Assuming the probability of the event under each state is P_1 and P_2 respectively, we have

$$\Rightarrow \begin{cases} P_1 = \frac{1}{2}P_1 + \frac{1}{2}P_2 \\ P_2 = \frac{1}{2} \times 0 + \frac{1}{2} \times 1 \end{cases}$$

$$\Rightarrow \begin{cases} P_1 = \frac{1}{2} \\ P_2 = \frac{1}{2} \end{cases}$$

$$\Rightarrow Pr[\varepsilon] = \frac{1}{2}(P_1 + P_2) = \frac{1}{2}$$

Exercise 3.3

1. There are 4 states of last bit pair of $\{x\}$ and $\{y\}$ sequence, which are $\{0,0\},\{1,0\},\{0,1\}and\{1,1\}$. Assuming the expectation of T under each state is E_1, E_2, E_3, E_4 respectively, we have

$$\begin{cases} E_1 = 1 + \frac{1}{4}E_1 + \frac{1}{4}E_2 + \frac{1}{4}E_3 + \frac{1}{4}E_4 \\ E_2 = 1 + \frac{1}{4}E_2 + \frac{1}{4}E_4 \\ E_3 = 1 + \frac{1}{4}E_1 + \frac{1}{4}E_2 \\ E_4 = 1 + \frac{1}{4}E_2 \end{cases}$$

$$\Rightarrow \begin{cases} E_1 = \frac{384}{121} \\ E_2 = \frac{272}{121} \\ E_3 = \frac{20}{11} \\ E_4 = \frac{16}{11} \end{cases}$$

$$\Rightarrow E[T] = \frac{1}{4}(E_1 + E_2 + E_3 + E_4) = \frac{384}{121}$$

2.

(a) Similar to the solution above, assuming the prabability of the event ”10 appears in x before 11 appears in y” under each state is $Pra_1, Pra_2, Pra_3, Pra_4$, we have

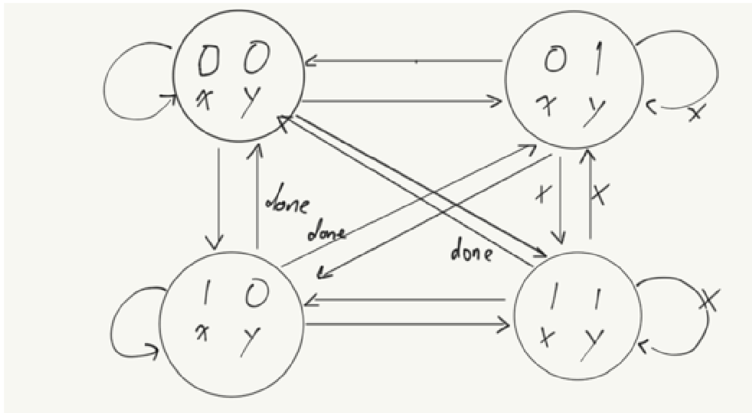

Figure 3: The state machine 3.3.2

$$\begin{cases} Pra_1 = \frac{1}{4}Pra_1 + \frac{1}{4}Pra_2 + \frac{1}{4}Pra_3 + \frac{1}{4}Pra_4 \\ Pra_2 = \frac{1}{2} + \frac{1}{4}Pra_2 + \frac{1}{4}Pra_4 \\ Pra_3 = \frac{1}{4}Pra_1 + \frac{1}{4}Pra_2 \\ Pra_4 = \frac{1}{4} + \frac{1}{4}Pra_2 \end{cases}$$

$$\Rightarrow \begin{cases} Pra_1 = \frac{65}{121} \\ Pra_2 = \frac{9}{11} \\ Pra_3 = \frac{41}{121} \\ Pra_4 = \frac{5}{11} \end{cases}$$

$$\Rightarrow Pra = \frac{1}{4}(Pra_1 + Pra_2 + Pra_3 + Pra_4) = \frac{65}{121}$$

(b) Similar to the solution above, assuming the prabability of the event ”10 appears in x at the same time with 11 appears in y” under each state is $Prb_1, Prb_2, Prb_3, Prb_4$, we have

$$\begin{cases} Prb_1 = \frac{1}{4}Prb_1 + \frac{1}{4}Prb_2 + \frac{1}{4}Prb_3 + \frac{1}{4}Prb_4 \\ Prb_2 = \frac{1}{4}Prb_2 + \frac{1}{4}Prb_4 \\ Prb_3 = \frac{1}{4}Prb_1 + \frac{1}{4}Prb_2 \\ Prb_4 = \frac{1}{4} + \frac{1}{4}Prb_2 \end{cases}$$

$$\Rightarrow \begin{cases} Prb_1 = \frac{17}{121} \\ Prb_2 = \frac{1}{11} \\ Prb_3 = \frac{7}{121} \\ Prb_4 = \frac{3}{11} \end{cases}$$

$$\Rightarrow Prb = \frac{1}{4}(Prb_1 + Prb_2 + Prb_3 + Prb_4) = \frac{17}{121}$$

(c) Similar to the solution above, assuming the prabability of the event "10 appears in x after 11 appears in y" under each state is $Prc_1, Prc_2, Prc_3, Prc_4$, we have

$$\begin{cases} Prc_1 = \frac{1}{4}Prc_1 + \frac{1}{4}Prc_2 + \frac{1}{4}Prc_3 + \frac{1}{4}Prc_4 \\ Prc_2 = \frac{1}{4}Pra_2 + \frac{1}{4}Pra_4 \\ Prc_3 = \frac{1}{2} + \frac{1}{4}Prc_1 + \frac{1}{4}Prc_2 \\ Prc_4 = \frac{1}{4} + \frac{1}{4}Prc_2 \end{cases}$$

$$\Rightarrow \begin{cases} Prc_1 = \frac{39}{121} \\ Prc_2 = \frac{1}{11} \\ Prc_3 = \frac{73}{121} \\ Prc_4 = \frac{3}{11} \end{cases}$$

$$\Rightarrow Prc = \frac{1}{4}(Prc_1 + Prc_2 + Prc_3 + Prc_4) = \frac{39}{121}$$

Exercise 3.4

1.

We have $Pr(T = n) = p(1-p)^{n-1}$,$n = 1, 2, \cdots$ Thus, T obeys geometric distribution.

Then

$$E(T^2) = \sum_{n=1}^{\infty} n^2 p(1-p)^{n-1}$$

.

Here, we have

$$\sum_{n=1}^{\infty} nx^{n-1} = (\sum_{n=1}^{\infty} x^n)' = (\frac{x}{1-x})' = \frac{1}{(1-x)^2}$$

and

$$\sum_{n=1}^{\infty} n^2 x^{n-1} = (\sum_{n=1}^{\infty} nx^n)' = (x\sum_{n=1}^{\infty} nx^{n-1})' = (\frac{x}{(1-x)^2})' = \frac{1+x}{(1-x)^3}$$

Then we order $x = 1 - p$. We have

$$E(T^2) = \sum_{n=1}^{\infty} n^2 p(1-p)^{n-1} = p\sum_{n=1}^{\infty} n^2 (1-p)^{n-1} = p \times \frac{2-p}{p^3} = \frac{2-p}{p^2}$$

2.

$$
\begin{aligned}
E(\frac{1}{T}) &= \sum_{n=1}^{\infty} \frac{1}{n} p(1-p)^{n-1} \\
&= \frac{p}{1-p} \sum_{n=1}^{\infty} \frac{(1-p)^n}{n} \\
&= \frac{p}{1-p} \sum_{n=1}^{\infty} \int_1^p -(1-p)^{n-1}\, dp \\
&= \frac{p}{1-p} \int_1^p \{\sum_{n=1}^{\infty} -(1-p)^{n-1}\}\, dp \\
&= \frac{p}{1-p} \int_1^p - \lim_{n\to\infty} \frac{1-(1-p)^n}{p}\, dp \\
&= \frac{p}{1-p} \int_1^p -\frac{1}{p}\, dp \\
&= -\frac{p \ln p}{1-p}
\end{aligned}
$$

Question

1. In Exercise 3.3 of Homework 3, we are required to solve E[T], where $T := min(T_{10}(x), T_{11}(y))$. We use transformation equation of state to solve the problem. However, if we change this question to $T := max(T_{10}(x), T_{11}(y))$, it is rather difficult to set up transformation equation because we do not know whether 10 has existed in X or 11 has existed in Y, or neither has happened. So we want to know how to solve the problem if $T := max(T_{10}(x), T_{11}(y))$.

2. Consider that in one infinite sequence the same as Exercise 3.2, we let $T := T_{11} - T_{10}$. It means the number of bits from the first appearance of 10 to the first appearance of 11. How can we calculate E[T]?