CS499 Homework 2

Intersteller

1 Fibonacci Numbers and Other Recurrences

1.1 Identities among the Fibonacci Numbers

Exercise 2.1

Proof 1. We use induction on n.

(1) Basic step. $n = 1$, $F_1 = F_3 - 1$. It is obviously true.

(2) Induction hypothesis. Assume it holds for n $(n \geq 1)$. Then we have

$$F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$$

(3) Proof of induction step. Now let us prove that based on the induction hypothesis, we can get the same formula for $n + 1$.

$$F_1 + F_2 + F_3 + \cdots + F_n + F_{n+1} = F_{n+2} - 1 + F_{n+1} = F_{n+3} - 1 = F_{(n+1)+2} - 1$$

Proof 2. We use a combinatorial argument involving the sets A_i.

Since $F_n = \mid A_{n+2} \mid$, where $A_n = \{x \epsilon \{0,1\}^n \mid x$ does not contain $11\}$.We consider A_n.

Each element in A_n is a n-bit 01 string and each bit is 0 or 1. Thus, we have

$$A_n = 0A_{n-1} \cup 1B_{n-1}$$

where $B_{n-1} = 0A_{n-2}$. Obviously, $0A_{n-1} \cap 10A_{n-2} = \emptyset$ for every integer $n \geq 2$.

Then we find a way to partition A_n like this.

$$A_n = 0A_{n-1} \cup 1B_{n-1}$$
$$A_n = 00A_{n-2} \cup 010A_{n-3} \cup 10A_{n-2}$$
$$A_n = 000A_{n-3} \cup 0010A_{n-4} \cup 010A_{n-3} \cup 10A_{n-2}$$
$$A_n = 0000A_{n-4} \cup 00010A_{n-5} \cup 0010A_{n-4} \cup 010A_{n-3} \cup 10A_{n-2}$$
$$\cdots$$
$$\cdots$$

$$A_n = \{\underbrace{000\cdots00}_{\text{n}}, \underbrace{000\cdots0}_{\text{n-1}}1, \underbrace{000\cdots0}_{\text{n-2}}10\} \cup \underbrace{000\cdots10}_{\text{n-1}} A_1 \cup \cdots \cup 00010A_{n-5} \cup 0010A_{n-4} \cup 010A_{n-3} \cup 10A_{n-2}$$

The intersection of any two sets on the right side of the equation is an empty set.

So we have

$$\mid A_n \mid = 3 + \mid A_1 \mid + \cdots + \mid A_{n-4} \mid + \mid A_{n-3} \mid + \mid A_{n-2} \mid$$

Since $F_{n+2} = \mid A_n \mid$ for every integer $n \geq 1$ and $F_1 = 1$, we have

$$F_{n+2} - 1 = \sum_{i=3}^{n} F_i + F_1 + F_2$$

So we have

$$F_{n+2} - 1 = \sum_{i=1}^{n} F_i$$

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Exercise 2.2

Proof 1. We use induction on n.

(1) Basic step. $n = 1$, $F_1 + F_3 = 1 + 2 = 3 = F_4$. It is obviously true.

(2) Induction hypothesis. Assume it holds for n $(n \geq 1)$. Then we have

$$F_1 + F_3 + F_5 + \cdots + F_{2n+1} = F_{2n+2}$$

(3) Proof of induction step. Now let us prove that based on the induction hypothesis, we can get the same formula for $n + 1$.

$$F_1 + F_3 + F_5 + \cdots + F_{2n+1} + F_{2(n+1)+1} = F_{2n+2} + F_{2n+3} = F_{2(n+1)+2}$$

Proof 2. We use a combinatorial argument involving the sets A_i.

Since $F_{2n+2} =| A_{2n} |$, where $A_{2n} = \{x\epsilon\{0,1\}^{2n} \mid x$ does not contain $11\}$.We consider A_{2n}.

Each element in A_{2n} is a $2n$-bit 01 string and each bit is 0 or 1. Thus, we have

$$A_{2n} = 0A_{2n-1} \cup 1B_{2n-1}$$

where $B_{2n-1} = 0A_{2n-2}$. Obviously, $0A_{2n-1} \cap 10A_{2n-2} = \emptyset$ for every integer $n \geq 1$.

Then we find a way to partition A_{2n} like this.

$$A_{2n} = 0A_{2n-1} \cup 1B_{2n-1}$$
$$A_{2n} = 0A_{2n-1} \cup 100A_{2n-3} \cup 101B_{2n-3}$$
$$A_{2n} = 0A_{2n-1} \cup 100A_{2n-3} \cup 10100A_{2n-5} \cup 10101B_{2n-5}$$
$$\cdots$$
$$\cdots$$

$$A_{2n} = 0A_{2n-1} \cup 100A_{2n-3} \cup 10100A_{2n-5} \cup 1010100A_{2n-7} \cup \cdots \cup \underbrace{1010\cdots10}_{\text{2n-2}}0A_1 \cup \underbrace{1010\cdots1010}_{\text{2n}}$$

Since $F_{n+2} =| A_n |$ for every integer $n \geq 1$ and $F_1 = 1$, we have

$$F_{2n+2} = F_{2n+1} + F_{2n-1} + F_{2n-3} + \cdot + F_3 + F_1$$

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1.2 General Linear Recurrences

Exercise 2.3

solution 1:

For each eigenvalue of A, we have $|\lambda I - A| = 0$, which is

$$\begin{vmatrix} \lambda - a_1 & -a_2 & -a_3 & \cdots & -a_{k-1} & -a_k \\ -1 & \lambda & 0 & \cdots & 0 & 0 \\ 0 & -1 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 0 \\ 0 & 0 & 0 & \cdots & -1 & \lambda \end{vmatrix}$$

Let

$$column_{k-1} + \frac{1}{\lambda} column_k \to column_{k-1}$$

$$column_{k-2} + \frac{1}{\lambda} column_{k-1} \rightarrow column_{k-2}$$

$$\vdots$$

$$column_1 + \frac{1}{\lambda} column_2 \rightarrow column_1$$

We get

$$\begin{vmatrix} \lambda - a_1 - \frac{a_2}{\lambda} - \cdots - \frac{a_k}{\lambda^{k-1}} & \cdots & \cdots & \cdots & \cdots \\ 0 & \lambda & 0 & \cdots & \cdots \\ 0 & 0 & \lambda & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{vmatrix} = 0$$

$$\Rightarrow (\lambda - a_1 - \frac{a_2}{\lambda} - \cdots - \frac{a_k}{\lambda^{k-1}})\dot{\lambda}^{k-1} = 0$$

$$\Rightarrow \lambda^k = a_1\lambda^{k-1} + a_2\lambda^{k-2} + \cdots + a_{k-1}\lambda + a_k$$

solution 2:

We suppose that for each enigenvalue λ, the enigenvector is

$$\begin{pmatrix} \lambda^{k-1} \\ \lambda^{k-2} \\ \\ \vdots \\ 1 \end{pmatrix}$$

Thus,

$$A\alpha = \begin{pmatrix} a_1\lambda^{k-1} + a_2\lambda^{k-2} + \cdots + a_k \\ \lambda^{k-1} \\ \\ \vdots \\ \lambda \end{pmatrix}$$

$$\lambda\alpha = \begin{pmatrix} \lambda^k \\ \lambda^{k-1} \\ \\ \vdots \\ \lambda \end{pmatrix}$$

According to the definition of enigenvalue, $A\alpha = \lambda\alpha$. So, if $\lambda^k = a_1\lambda^{k-1} + a_2\lambda^{k-2} + \cdots + a_{k-1}\lambda + a_k$ **(1)**, λ is an eigenvalue of A, and $\alpha = \begin{pmatrix} \lambda^{k-1} \\ \lambda^{k-2} \\ \\ \vdots \\ 1 \end{pmatrix}$ is its corresponding eigenvector.

Also, n-dimension matrix has n eigenvalues, and equation **(1)** has n roots. So, λ is an eigenvalue of A if and only if equation **(1)** is satisfied.

Exercise 2.4

Proof

when λ is bigger than 0, We can rewrite the formula as follows

$$\frac{a_1}{\lambda} + \frac{a_2}{\lambda^2} + \cdots + \frac{a_{k-2}}{\lambda^{k-2}} + \frac{a_{k-1}}{\lambda^{k-1}} + \frac{a_k}{\lambda^k} - 1 = 0, a_i \geq 0$$

let $f(\lambda) = \frac{a_1}{\lambda} + \frac{a_2}{\lambda^2} + \cdots + \frac{a_{k-2}}{\lambda^{k-2}} + \frac{a_{k-1}}{\lambda^{k-1}} + \frac{a_k}{\lambda^k} - 1$

notice that in the area $(0, \inf)$,f is a monotone decreasing continuous function of λ (when $\lambda > 0$),and we have

$$f(1) = a_1 + a_2 + \cdots + a_k - 1 > 0$$

$$f(\infty) \to -1 < 0$$

according to the existence theorem of zero point of continuous function, there exists a $\lambda \in (1, \infty)$ such that $f(\lambda) = 0$
according to the monotonicity of the function, there exists exactly one such λ

2 Question

2.1

When we read the notes on discrete probability, we think about Exercise 3.1: when does the first 11/10 appear? We simulate the proof 3 and get the answer that averagely it takes 6 digits when first 11 appears and for 10 the answer is 4. Then we think further about when does the k^{th} 11/10 appear? For 10, obviously, each appearance of 10 is independent, so the answer should be 4k. But for 11, the continuous appearance may not be independent. For example, in 1111, 11 appears 3 times. So we want to know when the k^{th} 11 appears averagely?

2.2

In the previous lectures, we find that many proofs are based on visualized or concretized approach. Do we have some general thinking or method for discrete mathematics?