CS499 Homework 8

Intersteller

Exercise 8.1

1.

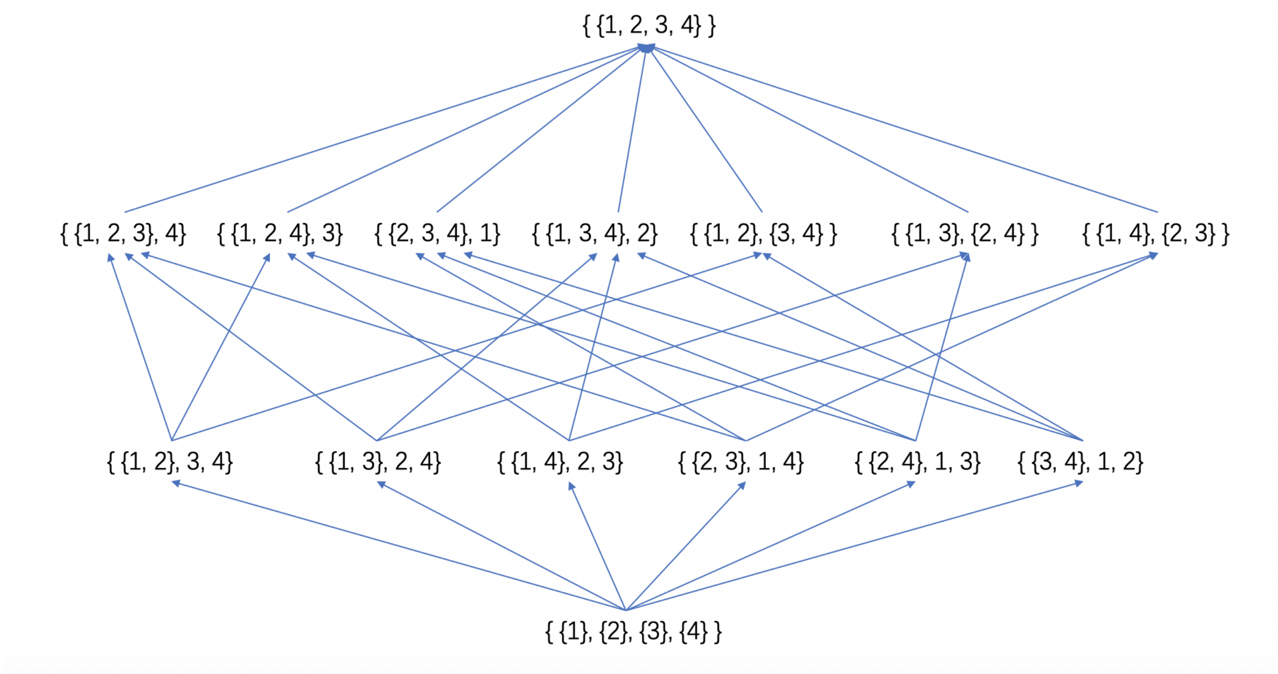

Figure 1:

2. The size of the largest chain is 4.
3. The size of the largest antichain is 7.

Exercise 8.2

1.$(0, 0, 0, 0, 0, \cdots)$ are minimal. There is not a maximum.

2.There is a minimum, but there is not a maximum.

3.Yes. For example,

$$(1, 1, 1, 1, \cdots)$$
$$(2, 1, 1, 1, \cdots)$$
$$(3, 1, 1, 1, \cdots)$$

$$(4, 1, 1, 1, \cdots)$$

$$\vdots$$

4.

When $n = 1$, we can't find an antichain.

When $n \geq 2$, we can find an antichain. For example,

$$\begin{gathered}(0, k, 0, \cdots, 0)\\(1, k-1, 0, \cdots, 0)\\(2, k-2, 0, \cdots, 0)\\\vdots\\(k-1, 1, 0, \cdots, 0)\end{gathered}$$

Exercise 8.3

Yes. We prove it by mathematical induction.

(1)When $n = 1$, obviously, S is an ordered set. We can sort elements from small to large to get an infinite chain.

(2)We suppose every infinite subset $S \subseteq N_0^n$ contain an infinite chain, then when it comes to n+1, according to our hypothesis, ignoring the last element in each vector in S, the first n elements of each vector form a chain named T. Then we consider the last element of each vector. We define sequence A is the sequence that consists of the last element in each vector in S, by the order in T.

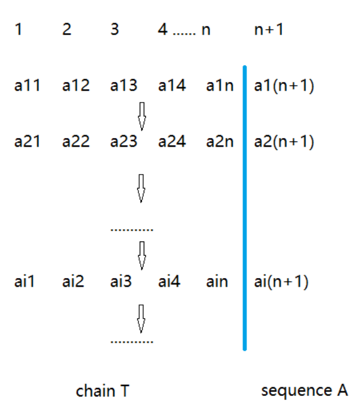

Figure 2:

There are two situations:

1.Sequence A is bounded.

Obviously, at least one number (we call it x) appears infinite times in sequence A. We select those vectors in chain T whose last element equals to x. Obviously they form an infinite chain of N_0^{n+1}.

2.Sequence A is unbounded.

Since each element is non-negative integer and the sequence is unbounded, we can find a subsequence A' of sequence A which is monotonically increasing. We select those sets whose last element in A'. Then, they form an infinite chain of N_0^{n+1}.

So, every infinite subset $S \subseteq N_0^n$ contains an infinite chain.

Exercise 8.4

We assume that $(\mathbb{N}_0^n, \leq)$ has an infinite antichain T. Obviously, $T \subseteq (\mathbb{N}_0^n, \leq)$. According to **exercise 8.3**, we get T contains an infinite chain, which contradicts T is an infinite antichain. Thus $(\mathbb{N}_0^n, \leq)$ has no infinite antichain.

Exercise 8.5

$n = 2$

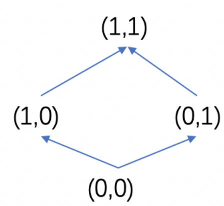

Figure 3:

$n = 3$

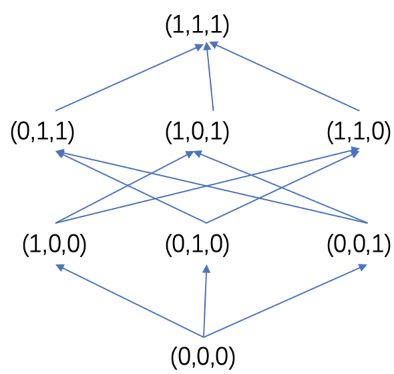

Figure 4:

Exercise 8.6

Maximum is $(\underbrace{1, 1, 1, \cdots, 1}_{n})$

Minimum is $(\underbrace{0, 0, 0, \cdots, 0}_{n})$

Maximal is $(\underbrace{1, 1, 1, \cdots, 1}_{n})$

Minimal is $(\underbrace{0, 0, 0, \cdots, 0}_{n})$

Exercise 8.7

We construct the Hasse diagrams like **Exercise 8.5**. The first layer is an element with n one. The second layer is n elements with $n-1$ one. The third layer is $\binom{n}{2}$ elements with $n-2$ one......The k layer is $\binom{n}{k-1}$ elements with $n+1-k$ one, where $1 \leq k \leq n+1$.

Obviously elements in the same layer is antichain, since elements in chain must have different numbers of one and in the same layer the number of one is equivalent. Thus it has $n+1$ antichain partition.

According to Mirsky's Theorem, max size of chain=min size of antichain partition. It means max size of chain must less than or equal to the size of each antichain partition. Thus the longest chain of $\{0,1\}^n \le n+1$. Since we get $n+1$ antichain partition in above Hasse diagrams, then the longest chain of $\{0,1\}^n$ is $n+1$. For example, $(0,0,...,0),(0,0,...,1),(0,0,...,0,1),...,(1,1,...,1)$ is a chain with $n+1$ elements.

Exercise 8.8

The largest antichain of $\{0,1\}^n$ is $\binom{n}{\lfloor n/2 \rfloor}$.According to the Dilworth Theorem,the largest antichain equals to the minimum size of chain partition.

We define a layer as a set of strings containing same number of $'1'$ and is sorted by how many $'1'$ a string in this layer contains.

1.There are $\binom{n}{\lfloor n/2 \rfloor}$ strings in the middle layer, which has the most strings. Since any two strings from the same layer are not comparable, there are at least $\binom{n}{\lfloor n/2 \rfloor}$ chain partitions.

2. All strings in any layer except the middle one can form chains with unique strings in its adjacent layer with the following method:

Assuming there are more $'1'$ than $'0'$ in this layer, we calculate a strings score by the following rules: scan the string from the beginning and the initial score is 0, add one if current digit is 1,minus one otherwise. Find the digit where the first highest score appears (which must be a $'1'$),change it to 0. Then we get a string belongs to its adjacent layer and these two strings can form a chain(they are comparable).Now we prove that this string is unique:

Assume that there are two different strings that transform into a same string. Assume that the first string changes the i-th digit, and the other changes the j-th digit (with no loss of generality, assume $i<j$). Then the i-th digit of the second string and the j-th digit of the first string are 0, whereas other digits are the same. Assume that the score of the $(i-1)$-th digit is k.Then the score of the i-th digit is $k+1$ for the first string and $(k-1)$ for the second. Assume that the score of the $(j-1)$-th digit for the first string is $k+1+p$,then the score of the $(j-1)$-th digit for the second string is $k-1+p$.The score of the j-th digit for the second string is $k+p$. Since the changing digit is where the first largest score occurs, we have

$$k+1 \ge k+1+p$$

$$k < k+p$$

where we get $p \ge 0$ and $p > 0$ which contradict each other. So different strings cannot transform into a same string by the method. Similarly, if there are more $'0'$ than $'1'$ in the layer,we can use a similar method to form chains with unique strings in its adjacent layer.

Question

1How to prove the proposition that the set of integer has the smallest cardinality among all the infinite sets?

2How to handle the Russell's paradox?