University of Zurich[UZH]

PHY401
Exercise Sheet 1

HS2025
Prof. Fabian Natterer
Prof. Marc Janoschek

Due on 23 Sept. 2025

Discussion on 23 Sept. 2025

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1 Hund's rules ❖ From 2024

Hidden due to University of Zurich policy.

2 Sign and magnitude of χ ★ New

The magnetic susceptibility χ is defined via $M = \chi H$.

(a) For a paramagnet, is χ positive or negative?

(b) For a diamagnet?

(c) Compare the typical magnitudes of χ for diamagnets and paramagnets.

3 Temperature dependence of χ ★ New

Consider a collection of localized magnetic moments in a crystal. The magnetization in weak fields follows Curie's law $\chi \propto 1/T$.

(a) Sketch qualitatively how $\chi(T)$ behaves for a paramagnet as T increases.

(b) Explain why χ decreases with T.

(c) What would the $\chi(T)$ curve look like for a diamagnet?

4 Reading assignment

Read Chapter 19.3 and 19.4 in Simon's Book (you can find the book on teams)

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PHY401
Exercise Sheet 2

HS2025
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Due on 30 Sept. 2025 — Discussion on 30 Sept. 2025

In this week's exercise, we set $\hbar = 1$ for convenience.

Notation. We use hats for operators and bold for vectors. The spin operators at site i are $\hat{\boldsymbol{S}}_i = (\hat{S}_i^x, \hat{S}_i^y, \hat{S}_i^z)$. The symbol S (no hat, not bold) denotes the *spin quantum number*. When we take the classical approximation later, we set $\hat{\boldsymbol{S}}_i \to \boldsymbol{S}_i = S\,\boldsymbol{n}_i$ with $|\boldsymbol{n}_i| = 1$ and then drop the hats.

1 Ferromagnet ★ New

We take the ferromagnetic Heisenberg Hamiltonian on a 1D lattice of N sites with spacing a:

$$\hat{H} \;=\; -J\sum_{i=1}^{N} \hat{\boldsymbol{S}}_i \cdot \hat{\boldsymbol{S}}_{i+1}, \qquad J > 0, \tag{1}$$

where the index i represents the i-th site, with periodic boundary conditions $\hat{\boldsymbol{S}}_{N+1} \equiv \hat{\boldsymbol{S}}_1$. Let S be the spin quantum number at each site. For electrons $S = \frac{1}{2}$.

(a) *A recap of spin operators*: For a spin-1/2 system, we have the following representation of the spin operators:

$$\hat{S}^x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{S}^y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{S}^z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

What are the eigenstates of the operator $\hat{S}^z$?[1] Let us denote these two eigenstates as $|\uparrow\rangle$ and $|\downarrow\rangle$. Calculate:

$$\hat{S}^x\,|\uparrow\rangle, \quad \hat{S}^y\,|\uparrow\rangle, \quad \hat{S}^x\,|\downarrow\rangle, \quad \hat{S}^y\,|\downarrow\rangle$$

and express the results in terms of the two states $|\downarrow\rangle$ and $|\uparrow\rangle$.

(b) Intuitively, the system has the lowest energy when all spins align. We use Dirac notation to denote this state:

$$|\phi\rangle \equiv |\uparrow_1\uparrow_2 \cdots \uparrow_N\rangle \tag{2}$$

where $\uparrow_i$ means the i-th spin is pointing up, i.e. the up-spin eigenstate of the $\hat{S}_i^z$ operator. Prove that this state $|\phi\rangle$ is indeed an eigenstate of the Hamiltonian (Eq. 1).[2]

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[1] Hint: An eigenstate $|\psi\rangle$ of an operator $\hat{O}$ satisfies the equation $\hat{O}\,|\psi\rangle = \lambda\,|\psi\rangle$, where λ is the eigen value of the state. In this case, you need to find the states that satisfy $\hat{S}^z\,|\psi\rangle = \lambda\,|\psi\rangle$. You should expect to find two eigenstates.

[2] Hint: To simplify the problem, you could try $N = 2$ first, then extend it to $N = 3$. Again, to prove that $|\phi\rangle$ is indeed an eigenstate of the Hamiltonian, you need to prove that $\hat{H}\,|\phi\rangle = \lambda\,|\phi\rangle$

2 Spin wave in ferromagnet ★ New

Take the Hamiltonian from the last exercise.

(a) The equation of motion of this system is

$$\dot{\hat{\boldsymbol{S}}}_i = J\hat{\boldsymbol{S}}_i \times \left(\hat{\boldsymbol{S}}_{i-1} + \hat{\boldsymbol{S}}_{i+1}\right), \tag{3}$$

where $\dot{\hat{\boldsymbol{S}}}$ is the time derivative of the operator $\hat{\boldsymbol{S}}$. This is a vector form of the equation of motion. Rewrite the equation in terms of $\hat{\boldsymbol{S}}$'s three components: $\hat{S}^x$, $\hat{S}^y$, and $\hat{S}^z$.

(b) We take a classical approximation, we treat the $\boldsymbol{S}$ as a classical variable, meaning that it is not a operator anymore. Each $\boldsymbol{S}$ can be simply written as a 3-dimensional vector (for example, $(0.1, 0.1, 0.2)$). Let's assume the system is sitting at its ground state where all spins are pointing in the z direction, i.e. $S_i^z = S$, and $S_i^x = S_i^y = 0$. A small perturbation will bring the state slightly away from this ground state. Try to argue, in a hand-wavy way, that a small transverse perturbation leads to

$$S^z \approx S, \quad S^x = \delta S^x, \quad S^y = \delta S^y,$$

up to first order, where $\delta S^{x,y}$ is a small value compared to S. This means that z component is almost unchanged under small perturbation. Therefore, we can treat it as a constant.

(c) Assuming a small perturbation is applied. Simplify the equation of motion Eq.2. You should get

$$\delta\dot{S}_i^x = -JS\left[\left(\delta S_{i+1}^y - \delta S_i^y\right) - \left(\delta S_i^y - \delta S_{i-1}^y\right)\right], \tag{4}$$

and similar expression for the y component. In fact, we can actually discard the δ sign in front of the S, because effectively $\delta S^x = S^x$, and $\delta S^y = S^y$.

(d) Instead of discretizing the system like we usually do in physics, we will do the opposite this time: let us "continu-nize" the system. This means that the lattice constant is so small that we can treat the system as a continuous system. Therefore, the spin operator now becomes a function of position, $S_i \Rightarrow S(x)$, where x is the coordinate. Prove that, with this approximation, the equation can be rewritten as

$$\frac{\partial S^x}{\partial t} = -JSa^2\frac{\partial^2 S^y}{\partial x^2}, \quad \frac{\partial S^y}{\partial t} = JSa^2\frac{\partial^2 S^x}{\partial x^2}, \tag{5}$$

where a is the lattice constant.

(e) Fourier transform the Eq. 5 to both frequency (energy) and k (momentum) space.[3] You should get the following equation:

$$\mathrm{i}\omega\tilde{S}^x(\omega, k) = -JSa^2k^2\tilde{S}^y(\omega, k) \tag{6}$$

and similar expression for the y component.

(f) Prove that this system has a dispersion of the form

$$\omega \propto (ka)^2$$

Explain why this LOOKS different from what you learned in the lecture, where $\omega \propto (1-\cos(ka))$.

3 Magnetic domain walls ❖ From 2024

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[3] Use $f(x,t) = \int \frac{dk}{2\pi}\frac{d\omega}{2\pi}\,\tilde{f}(\omega,k)\,e^{i(kx-\omega t)}$. Then $\partial_t \to -i\omega$, $\partial_x^2 \to -k^2$.

4 (Optional) We restore the $\hbar$ in this exercise. Spin waves in a two-dimensional antiferromagnet ❖ From 2024

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Due on 7 Oct. 2025 Discussion on 7 Oct. 2025

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1 NMR — spatial selectivity with a gradient ❖ From 2024

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2 Spin-wave Rabi oscillations & gate times ❖ From 2024

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3 Qubits — preparing and probing Bell states ❖ From 2024

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PHY401
Exercise Sheet 4

HS2025
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Due on 14 Oct. 2025 | Discussion on 14 Oct. 2025

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1 From second to first order transitions in Landau theory ★ New

Setup. Consider a uniform scalar order parameter m (e.g. magnetization) with $m \mapsto -m$ symmetry. The Landau free-energy density is

$$f(m;T,g,h) = f_0 + \frac{a(T)}{2}\,m^2 + \frac{b(g)}{4}\,m^4 + \frac{c}{6}\,m^6 - h\,m,$$

with $c \geq 0$ for stability, $a(T) = a_0\,(T - T_c)$ ($a_0 > 0$), and a non-thermal control parameter g that can tune $b(g)$. Unless stated otherwise, set $h = 0$.[1]

(a) Continuous (second-order) transition. Before doing the exercise, you can try out different combination of positive/negative a, b, and c and plot out the function $f(m)$. It is worth getting some feels of how the shape of the function changes with different signs of a, b, c. For all sub-exercises (a), we assume $c = 0$ for now.

(a) **Symmetry constraint.** *Prove* that odd powers of m are forbidden at $h = 0$ by the $m \to -m$ symmetry, so that the lowest non-trivial terms are m^2, m^4, m^6.

(b) **Equilibrium order parameter and β.** Assume $b > 0$. *Prove* that the global minimum satisfies

$$m_{\text{eq}}(T) = \begin{cases} 0, & T > T_c, \\ \pm\sqrt{-\,a(T)/b}, & T < T_c, \end{cases}$$

and hence the order-parameter critical exponent is $\beta = \frac{1}{2}$.[2]

(c) **Continuity of f and specific-heat jump.** Define $f_{\min}(T) = \min_m f(m;T,g,0)$.[3] *Prove* that $f_{\min}(T)$ is continuous at T_c, while the specific heat $C = -T\,\partial^2 f_{\min}/\partial T^2$ has a finite jump at T_c (no divergence). *Hint:* Evaluate $f_{\min}$ below T_c by inserting m_{eq} from (a2).

(d) (Optional: finish everything else before coming back for this)
Susceptibility and γ. Turn on a uniform field h (keep $b > 0$). *Prove* for $T > T_c$ that the linear susceptibility $\chi = \partial m/\partial h|_{h=0}$ obeys $\chi = 1/a(T)$, and infer $\gamma = 1$. *Prove* that the same exponent holds for $T < T_c$ when the response is computed around $m_{\text{eq}} \neq 0$. [4]

(e) (Optional: finish everything else before coming back for this)
Critical isotherm and δ. At $T = T_c$ and small h, *prove* that $m \propto h^{1/3}$, hence $\delta = 3$.[5]

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[1] It is worth reminding yourself that only a is T-dependent.

[2] β characterizes how the order parameter vanishes as the transition is approached from below: $m \sim (-t)^\beta$ for $t \to 0^-$ at $h = 0$ (equivalently $m \sim (-a)^\beta$ since $a \propto t$).

[3] Here $\min_m f$ means the *global* minimum of $f(m)$ over all real m. Practically: solve $\partial f/\partial m = 0$ for all stationary points m_i and compare the values $f(m_i)$. The phase realized in equilibrium is the one with the smallest f. If two minima have equal free energy the system is on a phase boundary (coexistence). Other local minima with larger f are *metastable*.

[4] γ governs the divergence of the (isothermal) susceptibility: $\chi \equiv \partial m/\partial h|_{h=0} \sim t^{-\gamma}$ as $t \to 0^+$ (and likewise from below when computed about $m_{\text{eq}} \neq 0$).

[5] δ is defined by the *critical isotherm* at $T = T_c$: $m \sim h^{1/\delta}$ for small h.

(b) Bridging to first order: softening the quartic term. Assume $b = b(g)$ can change sign under variation of the control parameter g (pressure, composition, ...), while $c > 0$ remains fixed. Note that you need to restore the m^6 to the free energy, since you have $c \neq 0$ now.

(a) **Emergence of extra extrema.** For $b < 0$ and small positive a, *prove* that $f(m)$ has multiple stationary points ($m = 0$ and nonzero m) if and only if $b^2 > 4ac$. (Optional) Prove that the $m = 0$ and the largest m are local minima by inspecting $\partial^2 f/\partial m^2$.

(b) **Coexistence condition.** Let $m_\star \neq 0$ denote a nonzero minimum when it exists. Coexistence (first-order transition) occurs when $f(m_\star) = f(0)$.[6] (Optional): Using the stationarity condition for $m_\star$, *prove* that coexistence requires

$$b \;=\; -4\sqrt{\frac{a\,c}{3}} \qquad (a > 0, b < 0),$$

You can try to examine the parameter space, i.e. the $a - b$ plane, and mark out the different phases (disordered phase $m = 0$ and ordered phase $m \neq 0$). You will find that the second-order line $a = 0$ ($b > 0$) joins a first-order line through a tricritical point at $(a, b) = (0, 0)$.

(c) **Order-parameter discontinuity on the first-order line.** Does the $m_\star$ undergo a jump across the first-order phase transition? What about the case of second order phase transition? Explain what makes first-order phase transition different than the second order one. For your information, the jump objeys

$$m_\star^2 = \sqrt{\frac{3a}{c}} = -\frac{3b}{4c}$$

(d) (Optional) **Latent heat along the first-order line.** Assume $a(T) = a_0(T - T_c)$ and b, c are T-independent near the transition. With $S = -\partial f_{\min}/\partial T$, *prove* that the latent heat (upon heating across the coexistence line at fixed g) is

$$L \;=\; T\left[S_{\text{disordered}} - S_{\text{ordered}}\right] \;=\; \frac{a_0 T}{2}\,m_\star^2 \;>\; 0,$$

[6] At a first-order transition two phases have equal Gibbs/Helmholtz free-energy density at the same control parameters; the equality $f(m_\star) = f(0)$ is precisely this condition at $h = 0$. The jump in m at coexistence is the order-parameter discontinuity.

1 From Landau to Ginzburg-Landau theory ★ New

Landau theory describes a homogeneous system with free energy

$$F_L = \frac{a(T)}{2}\eta^2 + \frac{b}{4}\eta^4 \tag{1}$$

where $a(T) \equiv a_0(T - T_c)$, $b > 0$, no linear coupling (magnetic field for example) is included. We ignore the F_0 term for now. The Ginzburg-Landau theory describes an inhomogeneous system with free energy written in the form of an integral

$$F_{GL} = \int d^3x \,\left\{\frac{a(T)}{2}\eta^2(x) + \frac{b}{4}\eta^4(x) + G\,(\nabla\eta)^2\right\} \tag{2}$$

We will now try to figure out why there should be a gradient term in the integral.

(a1) **Recap on Landau theory**. Get the equation of state (EOS) of Landau theory by minimizing Eq.1. You should get

$$a\eta + b\eta^3 = 0 \tag{3}$$

(a2) Let us discard the gradient term for now by setting $G = 0$. Find the EOS of Ginzburg-Landau theory Eq.2 by minimizing the integral in Eq.2.[1]

You should get the same result as Eq.3:

$$a\eta(x) + b\eta^3(x) = 0, \quad \forall x$$

This means that without the gradient term, both theories give the same EOS, thus the same result. It tells you that it is the gradient terms that makes the difference. But why does it have to be gradient (you don't have to answer this question)?

(b1) **Recap on Ising model**. Now let us look at the one-dimensional classical[2] Ising model, without the external field.

$$H = -J\sum_i s_i s_{i+1} \tag{4}$$

Prove that the Hamiltonian can be rewritten as

$$H = \frac{J}{2}\sum_i (s_i - s_{i+1})^2 + \text{constant}$$

[1] To do this, you should first assume a small variation on the order parameter $\eta(x) \to \eta(x) + \delta\eta(x)$. This variation will leads to a variation on the free energy as well: $F_{GL} \to F_{GL} + \delta F$. Plug this in the integral expression and you will end up with the following form of equation:

$$\delta F = \int d^3x \,\{\text{some expression}\}\,\delta\eta$$

To minimize the free energy, we requires $\{\text{some expression}\} = 0$. This will give you the EOS.

[2] by classical it means that each spin $s_i = \pm 1$ is a variable, not an operator.

where the constant term is independent of the spin configuration.[3]

(b2) Again Continous-nize the system and prove that the Hamiltonian can be written in terms of the derivatives of the spin:

$$H = \frac{Ja_{\text{lat}}}{2} \int dx \, \left(\frac{\partial s(x)}{\partial x}\right)^2 + \text{constant}$$

where a_{lat} is the lattice constant.

(c1) **Back to Ginzburg-Landau theory.** Now $G \neq 0$. Assuming we are still in one dimension. Then the gradient $\nabla\eta \to \partial\eta/\partial x$. Discretize the integral in Eq.2 and rewrite it in the summation form. Compare this with the Ising model, and interpret your results.

(c2) (Optional) Discretize the 3-dimensional version of Eq.2 and interpret your results.

(c3) Explain why this gradient term in the Ginzburg-Landau theory makes a difference in the EOS compared to the Landau theory.

(d) (**Optional**) Derive the EOS of Eq.2, with $G \neq 0$. You need to make use of the following formula

$$\int d^3x \; \delta(\nabla f) = \int d^3x \; \nabla(\delta f)$$

2 Correlation length and susceptibility ★ New

There is a continuous spin system in d dimension. The susceptibility χ is defined by

$$m(x) = \int d^dx \; \chi(x - x')\, h(x') \tag{5}$$

where $m(x)$ is the magnetization at x, and $h(x')$ is the magnetic field at x'. In an isotropic system, the susceptibility only depends on the distance r between the two points, and thus $\chi(x - x') = \chi(|x - x'|) \equiv \chi(r)$. This formula tells you how a magnetic field at x' can propagates its influence and induce a magnetization at another place x.

(a1) Let $d = 1$, i.e. a one dimensional system. Let us look at a simple case where the magnetic field is a pulse localized at $x = 0$. Therefore, $h \sim \delta(x)$, where $\delta(x)$ is the delta function. What is the magnetization $m(x)$ at x induced by this magnetic field?

(a2) If the total magnetic field is a summation of multiple small pulse fields, say $h(x') = \sum_i^n c_i\delta(x' - x_i)$, where c_i is a constant, and x_i is where the pulse is located. What is the magnetization $m(x)$ induced by this magnetic field? Try $n = 2$ first, i.e. two pulse fields.

(a3) In fact, every function $h(x)$ can be decomposed in a integral form:

$$h(x') = \int dy \; c(y)\, \delta(x' - y)$$

compare this form with the summation form in (a2). What is the magnetization $m(x)$ induced by this magnetic field?

(a4) Assuming now, in a 3-dimensional isotropic system, the susceptibility takes the form of

$$\chi(r) = \frac{c}{r} e^{-\xi^{-1}r} \tag{6}$$

where c is a constant and ξ is called the correlation length. Use again the pulse field $h = \delta(\boldsymbol{r})$ at $r = 0$, $\xi = 10\,\text{Å}$. Calculate the magnetization m_1 at a distance of $r = 10\,\text{Å}$ and m_2 at a further distance $r = 30\,\text{Å}$, in terms of c. How much smaller is m_2 than m_1. Explain why ξ is called *correlation length*.

[3]Hint: $(a - b)^2 = a^2 + b^2 - 2a \cdot b$

(b1) **From real space to k-space**. In a 1-dimensional system, where $d = 1$, The susceptibility take the form of

$$\chi(x) \propto e^{-\xi^{-1}|x|}$$

Fourier transform the susceptibility to k-space. Prove the following results:

$$\tilde{\chi}(k) \propto \frac{1}{\xi^{-2} + k^2}$$

The convention for the Fourier transform: $\tilde{f}(k) \equiv \int dx\, f(x) \exp(ikx)$. You might encounter a term like $e^{ik\cdot\infty}$, you can simply drop it and ignore this term.

(b2) In a 3-dimensional system, where $d = 3$, the susceptibility takes the form of

$$\chi(\boldsymbol{x}) \propto \frac{1}{|\boldsymbol{x}|} e^{-\xi^{-1}|\boldsymbol{x}|}$$

Fourier transform the susceptibility to k-space. Prove the following results[4]:

$$\tilde{\chi}(\boldsymbol{k}) \propto \frac{1}{\xi^{-2} + k^2}$$

(c1) **(Optional) From k-space to real space**. Again in an 1-dimensional isotropic system, the susceptibility takes the following form in k-space:

$$\tilde{\chi}(k) \propto \frac{1}{\xi^{-2} + k^2}$$

Fourier transform it back to real space and prove that you get the same result as in (b1)[5].

(c2) (Optional) Do the same for a 3-dimensional system.

(d) Use the following definition of free energy in Ginzburg-Landau theory (Following the lecture's convention)

$$F_{\mathrm{GL}} \equiv \int d^3x \left\{ A\eta^2 + B\eta^4 + G(\nabla\eta)^2 - h\eta \right\}$$

The corresponding susceptibility in k-space is (the momentum variable is now q following the notation in the lecture):

$$\tilde{\chi}(q) = \frac{1}{2A + 2Gq^2}$$

Let's now set $G = 0$. Prove that in real space the susceptibility is simply a delta function. This means that the magnetic field affects only locally, and its influence can no longer propagate. Explain why this is the case. What can you say about the correlation length?

[4] Hint: in 3 dimension, the Fourier transform is defined as

$$\tilde{f}(\boldsymbol{k}) \equiv \int d^3\boldsymbol{x}\, f(\boldsymbol{x}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

there are multiple ways to simplify this expression. (1) We set up a spherical coordinate system and put $\boldsymbol{x}$ to the z axis, and with this system the dot product becomes $\boldsymbol{k}\cdot\boldsymbol{x} = kr\cos(\theta)$. (2) In spherical coordinate system, the integral $\int d^3\boldsymbol{x} = \int r^2 \sin\theta\, d\phi d\theta dr$. Due to the isotropy, we can integrate out the ϕ and thus $\int d^3\boldsymbol{x} = 2\pi \int r^2 \sin\theta\, d\theta dr$

[5] You might need to make use of the Residue theorem:

$$\oint_\gamma f(z) dz = 2\pi i \sum_i \mathrm{Res}_i(f)$$

where $\mathrm{Res}_i(f)$ is the i^{th} residue inside the loop γ

PHY401
Exercise Sheet 6

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Due on 28 Oct. 2025

Discussion on 28 Oct. 2025

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1 Critical exponents ❖ From 2024

Hidden due to University of Zurich policy.

PHY401
Exercise Sheet 7

HS2025
Prof. Fabian Natterer
Prof. Marc Janoschek

Due on 4 Nov. 2025

Discussion on 4 Nov. 2025

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1 Approximating the electronic band structure of Si using a cubic lattice ❖ From 2024

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The exercises in the following pages are optional.

Preface

After the following two exercises, you should be able to: (1) Understand the usage of the matrix representation in solving a quantum mechanical problem; (2) Build up a connection between the spin system and the electron system. These two ideas can be generalized to more complex systems. Always try to analyze a system from different perspectives (Dirac notation, matrix representation, etc.). An analogy can sometimes help you understand a new system based on an old one.
The first exercise is a warm-up exercise that familiarizes you with the matrix representation in a spin system. The second exercise (electron system) is more related to the lecture but later you will find similarities between the the two systems.

2 Simple spin system ★ New

Setup A single spin-1/2 system. The Hamiltonian is given by $H = H_0 + \boldsymbol{\sigma} \cdot \boldsymbol{B}$[1] where

$$H_0 = E_0 \mathbb{I}_{2\times 2} = \begin{pmatrix} E_0 & 0 \\ 0 & E_0 \end{pmatrix}$$

is the free Hamiltonian where both spin-up and spin-down states have the same energy E_0. The magnetic field $\boldsymbol{B}$ is given by $\boldsymbol{B} = (B_x, B_y, B_z)$ and the Pauli matrices are given by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note: We omit physical constants and signs for simplicity.

(a1) Assuming $\boldsymbol{B} = 0$, write down the matrix form of the Hamiltonian in the basis of $|\uparrow\rangle$ and $|\downarrow\rangle$ (the eigenstates of σ_z).

(a2) Now we apply a magnetic field along the x axis. Write down the matrix form of the Hamiltonian in the same basis. You should get

$$H = \begin{pmatrix} E_0 & B_x \\ B_x & E_0 \end{pmatrix}$$

(a3) Apparently, with the magnetic field, the spin-up and spin-down states are no longer the eigenstates, meaning that when you are sitting on one state, the magnetic field will try to scatter it and put it into another state. In other words, the two states $|\uparrow\rangle$ and $|\downarrow\rangle$ are no longer stable. Denote $(1, 0)$ and $(0, 1)$ as the vector form of the $|\uparrow\rangle$ and $|\downarrow\rangle$ states, respectively. Prove that $(1, 1)$ and $(1, -1)$ are the eigenstates of the new Hamiltonian. Also prove that the new energy levels are

$$E_+ = E_0 + B_x, \quad E_- = E_0 - B_x.$$

Interestingly, $(1, 1)$ and $(1, -1)$ are also the eigenstates of σ_x. Try to explain your result. And think about why the magnetic field changes the energy?

(b) Now we want to understand the physical interpretation of the diagonal term in the Hamiltonian and the off-diagonal term in the Hamiltonian. You might need to recall the concept of *time-dependent perturbation theory* or *Fermi-Golden Rule*. Please You may use ChatGPT to explore this concept. Here is a prompt that you can use:

[1] Note: In this sheet we set $\hbar = 1$ and absorb the magnetic moment constant into the definition of $\mathbf{B}$, so the Zeeman term is written simply as $\boldsymbol{\sigma} \cdot \mathbf{B}$. The overall sign convention is unimportant for the purpose of this exercise.

Could you explain the physical meaning of the off-diagonal elements in the Hamiltonian using the time-dependent perturbation theory? You can explain the concept in the two-level system. For example the 2 dimensional Hamiltonian goes like

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E_1 & B \\ B & E_2
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(latex-like syntax). Please think before you answer. I need both relatively rigorous and intuitive answers.

In a word, the diagonal term is the "unperturbed energy" of the system and the off-diagonal term tells you how fast the system is hopping from one state to another.

(c) Now we are adding in a z-component of the magnetic field: now the Hamiltonian goes like

$$H = \begin{pmatrix} E_0 + B_z & B_x \\ B_x & E_0 - B_z \end{pmatrix}$$

Using the equation $\det(H - \lambda\mathbb{I}) = 0$, show that the eigenvalues are

$$E_{\pm} = E_0 \pm \sqrt{B_x^2 + B_z^2} = E_0 \pm |\boldsymbol{B}|$$

Discuss the two limiting cases where $B_z = 0, B_x \neq 0$, and $B_z \neq 0, B_x = 0$ separately. Explain why B_z and B_x have the same effect on the energy. Explain why the energy splitting depends only on the total field magnitude.

(d) Can you make a guess on the resulting energy levels when you add in all three components of the magnetic field, i.e. x, y and z-components. ? The Hamiltonian now is

$$H = \begin{pmatrix} E_0 + B_z & B_x - iB_y \\ B_x + iB_y & E_0 - B_z \end{pmatrix}$$

Take a minute to think about your result. Does it have anything to do with the rotational symmetry of the system?

3 Electron in a periodic potential ★ New

Setup A free electron in a periodic potential is described by the Hamiltonian $H = \frac{P^2}{2m} + V(x)$. For convenience, we set $\hbar = 1$ and $2m = 1$. In this convention the Hamiltonian is

$$H = P^2 + V(X)$$

In the case of $V = 0$, the eigenstates are denoted by $|k\rangle$ (Dirac notation) or $\phi_k(x) = e^{ikx}$ (wave function notation) where k is the wave vector. The energy of this state is k^2

(a1) Let us set $V(X) = 0$ for now. Sketch the free electron Hamiltonian $H = P^2$ in the $|k\rangle$ basis, assuming that k is discrete. You should get a diagonal matrix. You don't have to write the exact matrix, since there are infinitely many of such states. Just sketch the matrix.

(a2) What is the matrix element of $V(X)$ in the $|k\rangle$ basis? Can you sketch how the entire Hamiltonian $H = P^2 + V(X)$ looks in the $|k\rangle$ basis?[2]

(a3) Plot the free electron energy dispersion relation $\epsilon_k = k^2$. Explain that there is a one-to-one correspondence between the point on the dispersion and the matrix representation of the operator P^2.

(b1) Using the results obtained above, explain that within the first Brillouin zone, states cannot scatter into one another, except for the two states right at the boundary.[3]

(b2) Now let us look at the two states at the boundary of the first Brillouin zone. For convenience, we set $a = 1$. The two states are thus $|\pi\rangle$ and $|-\pi\rangle$. Here is the Hamiltonian concerning only these two states:

$$H = \begin{pmatrix} \epsilon & \tilde{V} \\ \tilde{V} & \epsilon \end{pmatrix}$$

where $\epsilon \equiv \pi^2$ is the energy of the two states. The off-diagonal element is the corresponding element of the Fourier transform of the potential $V(X)$. Review the class lecture note if you are not sure about how to get this matrix form. Compare this with the Hamiltonian in the spin system in the previous exercise (a2) with magnetic field along the x-axis. Directly write down the energy eigenvalues of this Hamiltonian.

(b3) Using the analogy between this system and the spin system in the previous exercise, try to explain why there is an energy split, intuitively.

(c1) Now consider a state a bit away from the boundary of the first Brillouin zone, say $|\pi - \delta\rangle$ where δ is small and represents the deviation from the boundary. Due to the periodic potential, this state will be mostly scattered into another state $|-\pi - \delta\rangle$ which is on the opposite side of the Brillouin zone. Prove that the Hamiltonian concerning only these two states is given by

$$H = \begin{pmatrix} \epsilon + 2\pi\delta & \tilde{V} \\ \tilde{V} & \epsilon - 2\pi\delta \end{pmatrix}$$

up to the first order in δ. Note that we also got similar result in the lecture, but in Dirac notation. Here in this exercise, we use another notation, the matrix representation, for the analysis.

(c2) Again try to build up an analogy between this Hamiltonian and the the Hamiltonian in the spin system defined in (c) in the previous exercise. Directly write down the new energy levels (or energy eigenvalues, in another word) of this Hamiltonian. Expand your result up to the second order in δ. You should get the same result as the result obtained in the lecture, just with different coefficients.

Spoiler: the energy level is quadratic in δ. Therefore you will have a parabolic-like dispersion near the boundary of the Brillouin zone.

[2] Hint1: the matrix element using Dirac notation: say $|1\rangle, |2\rangle, |3\rangle$ are three basis vectors. Now we have a 3 by 3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

as an example, the element $a_{12} = \langle 1|A|2\rangle$.
Hint2: there will be off-diagonal elements in the matrix.

[3] Hint: From the last exercise, you already know the physical interpretation of the off-diagonal elements in the Hamiltonian.

Bonus Exercise 1 Rewrite the analysis (purely Dirac notation) in the lecture note in the matrix representation.

Bonus Exercise 2 In exercise 2 (b1) and (c1), we only consider the matrix elements concerning the two states close to the boundary of the Brillouin zone. Why can we do this? What about the other states? Please justify this approximation.

The key takeaways from this sheet are: (1) matrix representation is a powerful tool to solve a quantum mechanical problem; (2) Setting up an analogy between a new system and a system you already know can be helpful.

1 Classical and quantum Hall effect ❖ From 2024

Hidden due to University of Zurich policy.

2 Landau gauge in Quantum Hall effect ★ New

Setup A 2D system in the x-y plane with magnetic field $\boldsymbol{B}$ in z direction. The vector potential $\boldsymbol{A}$ is defined by $\boldsymbol{B} \equiv \nabla \times \boldsymbol{A}$. In this exercise we use the Landau gauge: $\boldsymbol{A} \equiv xB\hat{\boldsymbol{y}}$, i.e. the vector potential is pointing in y direction and it increases with x.

(a1) Make a sketch/illustration/schematics of the system showing the magnetic field, vector potential, the coordinate system, possible electric field, and possible electric current.

(a2) The general form of the Hamiltonian in this system (without applying the gauge) is

$$H = \frac{(\hat{\boldsymbol{P}} - e\boldsymbol{A})^2}{2m}$$

Apply the Landau gauge now so that the vector potential can be rewritten in terms of the magnetic field (defined in the last exercise). Prove that the Hamiltonian can be written as

$$H = \frac{1}{2m}\left[\hat{P}_x^2 + \left(\hat{P}_y - eBx\right)^2\right]$$

(a3) In the lecture we made an ansatz: $\psi(x, y) \equiv \chi_k(x)e^{iky}$. This is because the Hamiltonian is independent of y, which means in y direction the particle behaves like a free particle without experiencing any potential or force. Of course in x direction it is a different story due to the dependence on x in the Hamiltonian. Show that the time-Independent Schrodinger equation can be written as

$$\left[\frac{\hat{P}_x^2}{2m} + \frac{m}{2}\omega_c^2\left(x - \frac{\hbar k}{eB}\right)^2\right]\chi_k(x) = E\chi_k(x)$$

where $\omega_c \equiv eB/m$. Explain why this represents a simple harmonic oscillator with a shifted center at $x_0 \equiv \hbar k/(eB)$

(a4) Show that the energy levels are given by

$$E_n = \hbar\omega_c\left(n + \frac{1}{2}\right),$$

where n is an integer. The energy is independent of k. Therefore, the degeneracy of the energy levels is given by the number of k values. *You can do this either with rigorous math or intuitive reasoning.*

(b1) **We will make an attempt** to understand the equations above. The first question is, how come the momentum in y direction creates a shift in the center of the harmonic oscillator in x direction? Let us consider a classical system where the particle behaves like a solid ball with charge e sitting right at the origin $(0,0)$. At time zero $t=0$, we kick off the ball with a velocity in y direction: $\boldsymbol{v}(t=0)=(0,v)$. A magnetic field B is applied in z direction. Prove that the ball will move in a circular orbit with radius[1]

$$R=\frac{mv}{eB}$$

(b2) Where is the center of the circular orbit? Make a sketch to depict the magnetic field, the velocity vector, the circular orbit, and the center of it. *Now, if we make the following naive substitution*

$$\hbar k \equiv p_y \equiv mv. \tag{1}$$

Prove that the center is

$$\boldsymbol{r_0}=\left(\frac{\hbar k}{eB},\ 0\right)$$

This coincides with our result in exercise (a)

(b3) So it looks like we should be satisfied with this interpretation: The shifted center x_0 is proportional to the *momentum* $\hbar k$ in y direction because the larger the momentum, the larger the circular orbit will be, and thus the further the center. But hold on, what about the energy? Calculate the kinetic energy of the particle with velocity $v \equiv \hbar k/m$ according to Eq. 1. Prove that the energy depends on k.

(b4) But how come in the exercise (a) the energy

$$E_n=\hbar\omega_c\left(n+\frac{1}{2}\right),$$

depends only on n but not on k? Is this conflicting? Another mysterious thing is: classically the electron goes in a circle, but with Landau gauge the electron behaves like a free particle in y direction. How come?

Please take a moment to think before moving on. You are encouraged to draw a sketch to help you visualize the system.

Starting from here, all exercises below are optional.

(c1) **Now we make a second attempt.** Would be great if you could make a recap on the canonical momentum and the mechanical momentum in Hamiltonian mechanics, and a bit of the gauge in electromagnetism. You could use the prompt below and put it in ChatGPT.

> In Hamiltonian mechanics, what is the difference between canonical momentum and mechanical momentum? You could use "electron in a magnetic field" as an example. The Hamiltonian is
>
> `H = (P - eA)^2/2m`
>
> Would be great if you could mention gauge invariance. Please think before you answer. Explain the concept like I am five.

[1] Hint: Use Newton's second law and the definition of the Lorentz force.

(c2) In a classical system, the momentum is a pure function. Use Hamiltonian equation[2] to prove that

$$\underbrace{m\boldsymbol{v}}_{\text{Mechanical Momentum}} = \underbrace{\boldsymbol{P}}_{\text{Canonical Momentum}} - \underbrace{e\boldsymbol{A}}_{\text{Vector Potential}}$$

In fact, this is also true in quantum mechanics. Please use Heisenberg equation get the same result. This is **the mistake** we made in (b). We should not interpret P as mechanical momentum. Therefore, the equations $\hbar k \equiv p_y \equiv mv$ in (b2) are *incorrect*!

(c3) Show that the canonical momentum is gauge dependent, while the mechanical momentum is gauge-invariant.[3] This means that the canonical momentum P is not even an observable.[4] Therefore, we also cannot see k as some kind of wave vector.

(c4) Plug in the Landau gauge and rewrite the above equation only in y direction. From classical considerations, we know that the electron is moving in a circle, therefore the time average of v_y will be zero. Take the time average, and prove

$$x_0 = \frac{\hbar k}{eB}$$

where x_0 is the center of the circular orbit.

(c5) Follow the argument, answer these questions:

(a) Why is the energy independent of k?

(b) Why is the center x_0 proportional to k, but not something like k^2? Could you explain this intuitively?

(c) Does the electron behave like a free particle in y direction?

[2] Hamiltonian equation: $\dot{q} = \frac{\partial H}{\partial p}$

[3] Hint: The Hamiltonian should also be gauge invariant while the vector potential has a gauge degree of freedom.

[4] Similarly, the gauge dependent $\boldsymbol{A}$ is also not an observable, while the gauge-invariant magnetic field $\boldsymbol{B}$ is an observable.

PHY401
Exercise Sheet 9

HS2025
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Prof. Marc Janoschek

Due on 18 Nov. 2025 Discussion on 18 Nov. 2025

Preface

After this exercise you will be able to construct the Landau theory for a superconductor from sketch (exercise 2) and explain why a persistent current is allowed, and thus zero resistance occurs (exercise 3), using London equation (exercise 1) and Bohr-Sommerfeld quantization (sheet 8).

1 Magnetic field penetration in a superconductor ❖ From 2024

Hidden due to University of Zurich policy.

2 Constructing the Landau Theory for a superconductor ★ New

In this exercise, you will be given a couple of assumptions on a superconductor, and you will be asked to construct the Landau theory based on those assumptions and derive the London equation

Setup We consider a spherical superconductor in a magnetic field with vector potential $\boldsymbol{A}(\boldsymbol{r})$, with a spatially uniform order parameter defined by $\psi(\boldsymbol{r}) = \psi$.

Assumptions

1. *The order parameter of a superconductor is a complex variable $\psi \in \mathbb{C}$.*
2. *The system has a spherical symmetry*
3. *Magnetic field will suppress the superconductivity (more like an observation)*

(a1) We can start with defining the free energy of the superconductor and the magnetic field, separately. Prove that the free energy of the superconductor

$$F_{\mathrm{SC}} = \int d^3r \left(\frac{a(T)}{2} |\psi|^2 + \frac{b}{4} |\psi|^4 \right)$$

and the free energy of the magnetic field

$$F_{\mathrm{mag}} = \int d^3r \left(\frac{1}{2\mu} \boldsymbol{B}^2 \right) \equiv \frac{1}{2\mu} \int d^3r \, (\nabla \times \boldsymbol{A})^2$$

both satisfy the requirements of: (0) the free energy being a scalar; (1) free energy being a real value; (2) the free energy being invariant under gauge transformations; and (3) the free energy being invariant under rotational transformations. [1]

[1] Sidenote: $|\psi|^2 \equiv \psi\psi^*$

(a2) Define the *naive* total free energy (pure summation, without interaction between the superconductor and the magnetic field) as

$$F_{\text{naive}} = F_{\text{SC}} + F_{\text{mag}}$$

Minimize the total free energy by taking the variational derivative with respect to the order parameter ψ and the vector potential $\boldsymbol{A}$ separately.[2][3] And find the equations of state (EOS) for both the order parameter ψ and the vector potential $\boldsymbol{A}$. Argue that they are independent of each other.

(b1) From exercise (a) we know that the naive summation of two free energies does not provide a good description of the system: the superconductor and the magnetic field are independent and they don't affect each other. Therefore, we need to add an *interaction term* that involves both the order parameter ψ and the vector potential $\boldsymbol{A}$ to the total free energy.

Here is a list of possible interaction terms:

$$\psi \boldsymbol{A}, \quad \psi \boldsymbol{e}_z \cdot \boldsymbol{A}, \quad \psi\psi^* \boldsymbol{A} \cdot \boldsymbol{e}_z, \quad \psi\psi^* \boldsymbol{A} \cdot \boldsymbol{A},$$

where $\boldsymbol{e}_z$ is the unit vector in the z-direction. Take a moment to think. Which one of these is the most reasonable interaction term[4]?

(b2) We can add in the interaction term now and the total free energy becomes

$$\begin{aligned} F_{\text{total}} &= F_{\text{SC}} + F_{\text{int}} + F_{\text{mag}} \\ &= \int d^3r \underbrace{\left(\frac{a(T)}{2} |\psi|^2 + \frac{b}{4} |\psi|^4 \right)}_{\text{superconductor}} + \underbrace{\frac{c}{2} |\psi|^2 |\boldsymbol{A}|^2}_{\text{interaction}} + \underbrace{\left(\frac{1}{2} (\nabla \times \boldsymbol{A})^2 \right)}_{\text{magnetic field}} \end{aligned} \tag{1}$$

where we set $\mu = 1$ for convenience, and this can be done by rescaling the free energy, which does not affect the physics at all. Argue that the coefficient c needs to be positive, i.e. $c > 0$.[5]

(b3) Derive the EOS by taking the variational derivative with respect to $\boldsymbol{A}$, and prove:

$$c\boldsymbol{A}|\psi|^2 + \nabla \times \boldsymbol{B} = 0.$$

And then use Maxwell's equations to further show

$$\boldsymbol{j} = -c|\psi|^2 \boldsymbol{A}. \tag{2}$$

This is nothing but the London equation in exercise 1. Comparing the two expressions gives $c = e^2/m$ and $|\psi|^2 = n_s$.

3 Persistent current in a superconductor ★ New

In this exercise, we will show that a persistent current can exist in a superconductor (and thus zero resistance), making use of the London equation and the Bohr–Sommerfeld quantization scheme. Let us consider a cylinder threaded by a magnetic field, and the thickness of this cylinder is much larger than the penetration depth λ. The figure below is a top-down view of the cylinder.

[2]Hint: Treat ψ and ψ^* as independent variables. Therefore, when you do the variational derivative wrt. ψ, the other variable ψ^* is considered a constant.

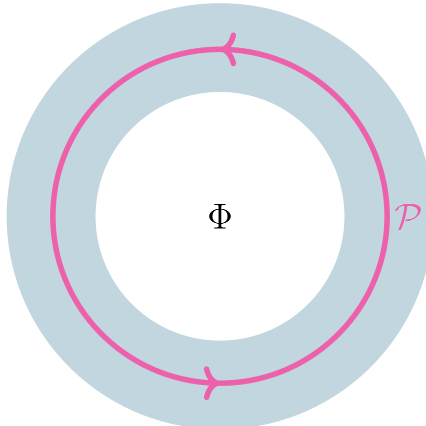
[3]Hint: If you don't want to do a bunch of vector calculus, you can make use of this identity:

$$\frac{\delta F_{\text{mag}}}{\delta \boldsymbol{A}} = 0 \qquad \Rightarrow \qquad \frac{\partial f_{\text{mag}}}{\partial \boldsymbol{A}} + \nabla \times \left(\frac{\partial f_{\text{mag}}}{\partial (\nabla \times \boldsymbol{A})} \right) = 0$$

where f_{mag} is the integrand of the magnetic field free energy. $\boldsymbol{A}$ and $(\nabla \times \boldsymbol{A})$ are treated as independent variables.

[4]Hint: Think about the symmetry and the assumptions we mentioned above

[5]Hint: Two perspectives can be considered: (1) The system always tries to minimize the free energy; (2) the coefficient $a(T)$ controls the superconductivity phase transition.



(a) The Bohr-Sommerfeld quantization scheme states that

$$\oint_{\mathcal{P}} \boldsymbol{p} \cdot d\boldsymbol{r} = 2\pi\hbar n, \quad n \in \mathbb{Z}$$

where $\boldsymbol{p}$ is the canonical momentum, n is an integer, and $\mathcal{P}$ is the circular integration path depicted in the figure above. Let us assume that the path is deep inside the superconductor, and thus there is no current along the path, according to exercise 1. Recall the definition of the canonical momentum, and prove that the magnetic flux through the cylinder is quantized in the following way[6][7]:

$$\Phi = \frac{h}{e} n, \quad n \in \mathbb{Z}$$

(b) Exercise (a) tells us that the magnetic flux is quantized and thus the magnetic field cannot change continuously, given that there is no external field. Since the magnetic field/flux can only be maintained by a loop current,

  (a) Can you explain why there has to be a persistent current in this superconductor?

  (b) Where is the current located?

[6]Hint: You might need to use the Stokes' theorem:

$$\oint_{\mathcal{P}} \boldsymbol{A} \cdot d\boldsymbol{r} = \int_{\mathcal{S}} (\nabla \times \boldsymbol{A}) \cdot d\boldsymbol{S}$$

[7]More rigrious derivation gives an extra 1/2 prefactor: $\Phi = \frac{h}{2e} n$

Setup In this exercise sheet we will explore the basic properties of two coupled superconductors. The order parameter ψ of each superconductor can be written as

$$\psi = |\psi| e^{i\phi} \equiv \sqrt{\rho_s} e^{i\phi} \tag{1}$$

where ρ_s is interpreted as the density of Cooper pairs (two-electron pairs) and ϕ is the phase of the order parameter. This order parameter behaves as a wave function and satisfies the time-dependent Schrödinger equation

$$i\dot{\psi} = H\psi$$

where we set $\hbar = 1$ for simplicity, $\dot{\psi} \equiv \partial\psi/\partial t$, and H is the Hamiltonian.

1 1 DC Josephson effect ★ New

If the two superconductors are separated, they evolve independently. The wave function of the two-superconductor system can be written as a vector $\Psi = (\psi_1, \psi_2)$, where ψ_i is the order parameter of the i-th superconductor. The Schrödinger equation for the two system is given by

$$i \begin{pmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \end{pmatrix} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

Now we bring the two superconductors closer and put a thin insulating layer between the two superconductors, which creates a coupling effect. As we discussed in exercise sheet 7, the coupling is characterized by the off-diagonal elements in the Hamiltonian. Let us add in an extra coupling constant κ to the Hamiltonian matrix:

$$i \begin{pmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \end{pmatrix} = \begin{pmatrix} E_1 & \kappa \\ \kappa & E_2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{2}$$

This off-diagonal term also encodes the tunneling effect.[1]

(a) Using Eq. (1), prove

$$\frac{\dot{\psi}_i}{\psi_i} = \frac{1}{2}\frac{\dot{\rho}_i}{\rho_i} + i\dot{\phi}_i, \quad i = 1, 2 \tag{3}$$

(b) Rewrite Eq. (2) and prove

$$i\frac{\dot{\psi}_1}{\psi_1} = E_1 + \kappa\sqrt{\frac{\rho_2}{\rho_1}} e^{i\delta}, \quad \delta \equiv \phi_2 - \phi_1 \tag{4}$$

similar for ψ_2.

[1] It is good to review the concept of coupling and the physical interpretation of the off-diagonal term in the Hamiltonian.

(c) Combine Eq. (3) and Eq. (4). Prove that from the real part of the equation, we have

$$-\dot{\phi}_1 = E_1 + \kappa\sqrt{\frac{\rho_2}{\rho_1}}\cos\delta \tag{5}$$

and from the imaginary part, we have

$$\dot{\rho}_1 = 2\kappa\sqrt{\rho_1\rho_2}\sin\delta \tag{6}$$

Similar for ψ_2 and ρ_2.

(d) Use Eq. (6) for ρ_1 and ρ_2 to prove that the Cooper pair density is conserved in the entire system, i.e. $\rho_1 + \rho_2 = \text{const.}$

(e) Prove that the current density is[2]

$$J = 4e\kappa\sqrt{\rho_1\rho_2}\sin\delta \equiv J_c\sin\delta \tag{7}$$

This means that as long as there is a phase difference, there is a current flowing between the two superconductors.

2 2 AC Josephson effect ★ New

Now we assume that the two superconductors have the same properties, i.e. same material, same transition temperature, and same Cooper pair density: $\rho_1 = \rho_2$.

(a) Use Eq. (2) to show that

$$\dot{\delta}(t) = E_1 - E_2 \equiv 2eV, \tag{8}$$

where V is the voltage across the junction. The extra factor of 2 again comes from the fact that each Cooper pair carries 2 electrons. Prove that this further leads to

$$J = J_c\sin(2eVt) \tag{9}$$

(b) Analyze the dimension, and resume the $\hbar$ in Eq. 9.

(c) Calculate the frequency of the AC Josephson effect when the voltage $V = 1\,\mathrm{V}$. How is this compared to the frequency of (1) the visible light, soft X-ray, microwave; (2) plasma frequency; (3) electron cyclotron frequency; (4) phonons?

3 3 SQUID ★ New

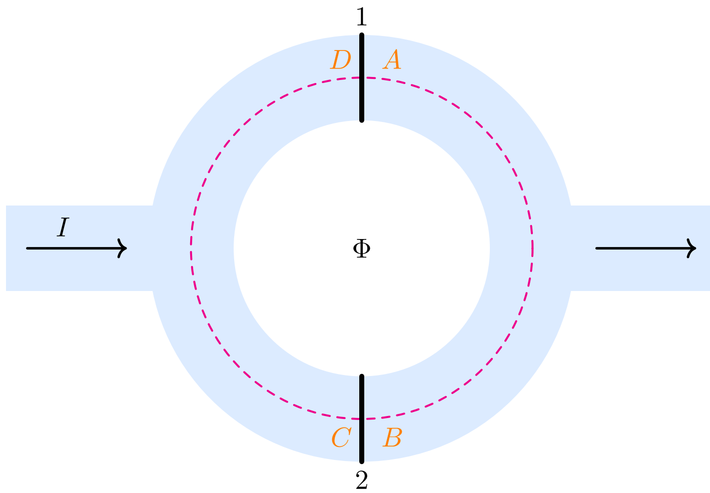
In exercise sheet 9, we derive the London equation when the order parameter ψ is uniform. However, if the phase is not uniform, the current density in the superconductor will not only depend on the vector potential A, but also depend on the gradient of the phase $\nabla\phi$:

$$J = \frac{\rho e}{m}(\nabla\phi - 2eA) \tag{10}$$

where m is the mass of the electron, and $\hbar = 1$.

Let us consider a superconducting ring with two junctions denoted by 1 and 2, shown in the figure below.

[2] The extra factor of 2 comes from the fact that each Cooper pair carries 2 electrons.



(a) Choose a path deep inside the ring (dashed line) where the current density is zero. Calculate the loop integral of both sides of Eq. (10). Prove that

$$2\pi \frac{\Phi}{\Phi_0} = \delta_1 - \delta_2, \tag{11}$$

where Φ is the magnetic flux, $\Phi_0 = \frac{h}{2e}$ is the flux quantum, $\delta_1 \equiv \phi_A - \phi_D$ and $\delta_2 \equiv \phi_B - \phi_C$.

(b) Assume that two junctions are identical, for simplicity. The current density flowing through the junction i is denoted by J_i. Use the relation ($J = J_c \sin\delta_1 + J_c \sin\delta_2$) to prove that[3]

$$J = 2J_c \sin\left(\frac{\delta_1 + \delta_2}{2}\right) \cos\left(\frac{\delta_1 - \delta_2}{2}\right) \tag{12}$$

which further leads to

$$J = 2J_c \sin\left(\frac{\delta_1 + \delta_2}{2}\right) \cos\left(\frac{\pi\Phi}{\Phi_0}\right)$$

(c) Depending on different circuit setup, δ_1 and δ_2 could evolves differently with time. In some cases, $\delta_1 + \delta_2$ leads to a high-frequency oscillation, while $\delta_1 - \delta_2$ gives a low-frequency oscillation. Therefore, the current density in Eq. 12 represents a modulated oscillation (high-frequency oscillation modulated by a low-frequency oscillation). Plot the function

$$f(t) = \sin(100t)\cos(5t)$$

and describe what it looks like.

---

[3]Hint:

$$\delta_1 = \frac{\delta_1 + \delta_2}{2} + \frac{\delta_1 - \delta_2}{2}, \quad \delta_2 = \frac{\delta_1 + \delta_2}{2} - \frac{\delta_1 - \delta_2}{2}$$

University of Zurich[UZH]

PHY401
Exercise Sheet 11

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Discussion on 9 Dec. 2025

1 Quantum Oscillation in Gold

Hidden due to University of Zurich policy.