JULY 29, 2025

# Numerical details for AxionDarkPhotonSimulator


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## Contents



## 1 Introduction

In this note, I provide some details for the numerical algorithm used in the python package AxionDarkPhotonSimulator, which is released along with [1]. The metric signature is $(-,+,+,+)$. The natural units $c = \hbar = \varepsilon_0 = k_{\mathrm{B}} = 1$ are adopted.

## 2 Equations

Consider the matter action

$$S_{\mathrm{M}} = \int d^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - U(\phi) - \frac{1}{4}X_{\mu\nu}X^{\mu\nu} - \frac{m_X^2}{2}X_\mu X^\mu - \frac{\alpha}{4f_\phi}\phi X_{\mu\nu}\widetilde{X}^{\mu\nu}\right] \ , \tag{2.1}$$

where $\alpha$ is the coupling constant, $f_\phi$ is the decay constant of axions, $X_{\mu\nu} = \partial_\mu X_\nu - \partial_\nu X_\mu$ is the electromagnetic tensor, $\widetilde{X}^{\mu\nu} = \mathcal{E}^{\mu\nu\rho\sigma}X_{\rho\sigma}/2$ is the dual tensor, $\mathcal{E}^{\mu\nu\rho\sigma} = \varepsilon^{\mu\nu\rho\sigma}/\sqrt{-g}$ is the Levi-Civita tensor, and $\varepsilon^{\mu\nu\rho\sigma}$ is the the Levi-Civita symbol with the convention $\varepsilon^{0123} = 1$. With the Robertson-Walker metric,

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j \ , \tag{2.2}$$

the field equations for $\phi$ and $X_\mu$ are given by (2.2)–(2.4) in [1]. It is useful to rewrite the equations in conformal time $d\tau = dt/a$, and one obtains

$$\varphi'' - \frac{a''}{a}\varphi - \nabla^2\varphi + a^3\partial_\phi U - \frac{\alpha}{af_\phi}(\nabla\mathcal{X}_0 - \boldsymbol{X}')\cdot(\nabla\times\boldsymbol{X}) = 0 \ , \tag{2.3}$$

$$\nabla\cdot\boldsymbol{X}' - \nabla^2\mathcal{X}_0 + a^2m_X^2\mathcal{X}_0 - \frac{\alpha}{af_\phi}(\nabla\varphi)\cdot(\nabla\times\boldsymbol{X}) = 0 \ , \tag{2.4}$$

$$\boldsymbol{X}'' - \nabla^2\boldsymbol{X} + 2\mathcal{H}\nabla\mathcal{X}_0 + a^2m_X^2\boldsymbol{X} - \frac{\alpha}{af_\phi}\left[(\varphi' - \mathcal{H}\varphi)(\nabla\times\boldsymbol{X}) + \nabla\varphi\times(\nabla\mathcal{X}_0 - \boldsymbol{X}')\right] = 0 \ , \tag{2.5}$$

where $\varphi = a\phi$, $\mathcal{X}_0 = aX_0$, and $X_\mu = (X_0, \boldsymbol{X})$. An additional constraint equation is given by

$$-\mathcal{X}_0' - 2\mathcal{H}\mathcal{X}_0 + \nabla \cdot \boldsymbol{X} = 0 \; . \tag{2.6}$$

The energy densities for each field are

$$\rho_\phi = \frac{1}{2a^4}(\varphi' - \mathcal{H}\varphi)^2 + \frac{1}{2a^4}(\nabla\varphi)^2 + U(\phi) \; , \tag{2.7}$$

$$\rho_X = \frac{1}{2a^4}(\nabla\mathcal{X}_0 - \boldsymbol{X}')^2 + \frac{1}{2a^4}(\nabla \times \boldsymbol{X})^2 + \frac{m_X^2}{2a^2}\left(\mathcal{X}_0^2 + \boldsymbol{X}^2\right) \; . \tag{2.8}$$

In an expanding universe, the scale factor and conformal Hubble parameter can be written as

$$a(\tau) = \left(\frac{\tau}{\tau_i}\right)^{2/(1+3w)} \; , \quad \mathcal{H}(\tau) = \frac{2}{\tau(1+3w)} \; , \tag{2.9}$$

where $w$ is the equation of state parameter and the initial scale factor is set to $a_i = 1$. For a radiation- or matter-dominated universe, we have $w = 1/3$ or $0$. For a nonexpanding universe, $w = \infty$. The axion field starts to oscillate when $H = \mathcal{H}/a \sim m_\phi$. If the universe undergoes a transition from radiation to matter domination, we may parameterize the scale factor as

$$a(\tau) = \frac{a_{\rm t}}{2}\left[\left(\frac{\tau}{\tau_{\rm t}}\right)^2 + \left(\frac{\tau}{\tau_{\rm t}}\right)\right] \; , \tag{2.10}$$

where $\tau_{\rm t}$ is the transition time and $a_{\rm t} = 2/[(\tau_i/\tau_{\rm t})^2 + (\tau_i/\tau_{\rm t})]$. If the universe undergoes a transition from matter to radiation domination, we may parameterize the scale factor as

$$a(\tau) = \frac{2a_{\rm t}}{(\tau_{\rm t}/\tau)^2 + (\tau_{\rm t}/\tau)} \; , \tag{2.11}$$

where $a_{\rm t} = [(\tau_{\rm t}/\tau_i)^2 + (\tau_{\rm t}/\tau_i)]/2$. The conformal Hubble parameter can be calculated using the finite difference method, i.e., $\mathcal{H}(\tau) = [a(\tau + 10^{-8}) - a(\tau - 10^{-8})]/(2 \times 10^8)/a(\tau)$.

# 3 Numerical setup

## 3.1 Numerical equations

To discretize the equations, I will nondimensionalize the variables by making the following replacement

$$x_\mu \to \frac{1}{m_\phi}x_\mu \; , \quad \varphi \to f_\phi\varphi \; , \quad \phi \to f_\phi\phi \; , \quad \boldsymbol{X} \to \frac{m_X}{\sqrt{\lambda_s}}\boldsymbol{X} \; , \quad \mathcal{X}_0 \to \frac{m_X}{\sqrt{\lambda_s}}\mathcal{X}_0 \; , \tag{3.1}$$

where $\lambda_s$ is an arbitrary positive constant. The resulting variables are dimensionless. In what follows, I will always use the dimensionless variables unless otherwise stated. The field equations (2.3)–(2.6) now become

$$\varphi'' - \frac{a''}{a}\varphi - \nabla^2\varphi + a^3\frac{\partial_\phi U}{m_\phi^2 f_\phi^2} - \frac{\alpha\beta^2 m_\phi^2}{a\lambda_s f_\phi^2}(\nabla\mathcal{X}_0 - \boldsymbol{X}') \cdot (\nabla \times \boldsymbol{X}) = 0 \; , \tag{3.2}$$

$$\nabla \cdot \boldsymbol{X}' - \nabla^2\mathcal{X}_0 + a^2\beta^2\mathcal{X}_0 - \frac{\alpha}{a}(\nabla\varphi) \cdot (\nabla \times \boldsymbol{X}) = 0 \; , \tag{3.3}$$

$$\boldsymbol{X}'' - \nabla^2\boldsymbol{X} + 2\mathcal{H}\nabla\mathcal{X}_0 + a^2\beta^2\boldsymbol{X} - \frac{\alpha}{a}\left[(\varphi' - \mathcal{H}\varphi)(\nabla \times \boldsymbol{X}) + (\nabla\varphi) \times (\nabla\mathcal{X}_0 - \boldsymbol{X}')\right] = 0 \; , \tag{3.4}$$

$$-\mathcal{X}_0' - 2\mathcal{H}\mathcal{X}_0 + \nabla \cdot \boldsymbol{X} = 0 \; , \tag{3.5}$$

where $\beta = m_X/m_\phi$. The energy densities (2.7) and (2.8) become

$$\rho_\phi = m_\phi^2 f_\phi^2 \left[ \frac{1}{2a^4}(\varphi' - \mathcal{H}\varphi)^2 + \frac{1}{2a^4}(\nabla\varphi)^2 + \frac{U(\phi)}{m_\phi^2 f_\phi^2} \right] , \tag{3.6}$$

$$\rho_X = \frac{\beta^2 m_\phi^4}{\lambda_s} \left[ \frac{1}{2a^4}(\nabla\mathcal{X}_0 - \boldsymbol{X}')^2 + \frac{1}{2a^4}(\nabla\times\boldsymbol{X})^2 + \frac{\beta^2}{2a^2}(\mathcal{X}_0^2 + \boldsymbol{X}^2) \right] . \tag{3.7}$$

The potential terms may need a little more clarification. For example, if the original dimensional axion potential is $U(\phi) = m_\phi^2 f_\phi^2[1 - \cos(\phi/f_\phi)]$, then the nondimensionalized potential term in (3.2) is $\partial_\phi U/(m_\phi^2 f_\phi^2) = \sin\phi = \sin(\varphi/a)$ and the one in (3.6) becomes $U(\phi)/(m_\phi^2 f_\phi^2) = 1 - \cos(\phi)$.

### 3.2 Numerical algorithm

Here I outline the numerical algorithm to evolve the numerical field equations (3.2)–(3.5), following [2]. For notation simplicity I define $\pi \equiv \varphi'$, $P_0 \equiv \mathcal{X}_0'$, and $\boldsymbol{P} \equiv \boldsymbol{X}'$.

Given the fields at time $\tau_m = \tau_i + m d\tau$, labeled as $\varphi^m, \pi^m, \mathcal{X}_0^m, \boldsymbol{X}^m, P_0^m, \boldsymbol{P}^m$, where $\tau_i$ is the initial conformal time, $m$ is the time index, and $d\tau$ is the time step, the time evolution is carried out in the following way:

1. Evolve $\varphi, \mathcal{X}_0, \boldsymbol{X}$ by half a step using the forward difference

$$\varphi^{m+1/2} = \varphi^m + \frac{d\tau}{2}\pi^m , \quad \mathcal{X}_0^{m+1/2} = \mathcal{X}_0^m + \frac{d\tau}{2}P_0^m , \quad \boldsymbol{X}^{m+1/2} = \boldsymbol{X}^m + \frac{d\tau}{2}\boldsymbol{P}^m . \tag{3.8}$$

2. Evolve $\pi$ and $\boldsymbol{P}$ by a full step using the centered difference

$$\pi^{m+1} = \pi^m + d\tau\ \pi'^{m+1/2} , \quad \boldsymbol{P}^{m+1} = \boldsymbol{P}^m + d\tau\ \boldsymbol{P}'^{m+1/2} . \tag{3.9}$$

However, $\pi'^{m+1/2}, \boldsymbol{P}'^{m+1/2}$ depend on $\pi^{m+1/2}, \boldsymbol{P}^{m+1/2}$ according to (3.2) and (3.4). To deal with this we can replace them by their average

$$\pi^{m+1/2} \to \frac{\pi^m + \pi^{m+1}}{2} , \quad \boldsymbol{P}^{m+1/2} \to \frac{\boldsymbol{P}^m + \boldsymbol{P}^{m+1}}{2} . \tag{3.10}$$

This procedure makes (3.9) an implicit method in solving $\pi^{m+1}, \boldsymbol{P}^{m+1}$, which typically has the advantage of strong stability. Now (3.9) can be rewritten in a matrix form

$$\begin{pmatrix} 1 & b_1^{m+1/2} & b_2^{m+1/2} & b_3^{m+1/2} \\ -b_1^{m+1/2} & 1 & -c_3^{m+1/2} & c_2^{m+1/2} \\ -b_2^{m+1/2} & c_3^{m+1/2} & 1 & -c_1^{m+1/2} \\ -b_3^{m+1/2} & -c_2^{m+1/2} & c_1^{m+1/2} & 1 \end{pmatrix} \begin{pmatrix} \pi^{m+1} \\ P_1^{m+1} \\ P_2^{m+1} \\ P_3^{m+1} \end{pmatrix} = \begin{pmatrix} d_0^{m+1/2} \\ d_1^{m+1/2} \\ d_2^{m+1/2} \\ d_3^{m+1/2} \end{pmatrix} , \tag{3.11}$$

where

$$\boldsymbol{b}^{m+1/2} = \frac{d\tau}{2}\frac{\alpha\beta^2 m_\phi^2}{\lambda_s f_\phi^2}\left(\frac{1}{a}\nabla\times\boldsymbol{X}\right)^{m+1/2} , \quad \boldsymbol{c}^{m+1/2} = \frac{d\tau}{2}\alpha\left(\frac{1}{a}\nabla\varphi\right)^{m+1/2} , \tag{3.12}$$

and

$$d_0^{m+1/2} = \pi^m + d\tau\left(\frac{a''}{a}\varphi + \nabla^2\varphi - \frac{a^3\partial_\phi U}{m_\phi^2 f_\phi^2}\right)^{m+1/2} - \boldsymbol{b}^{m+1/2}\cdot\left(\boldsymbol{P}^m - 2\nabla\mathcal{X}_0^{m+1/2}\right) , \tag{3.13}$$

$$\begin{aligned} \boldsymbol{d}^{m+1/2} = \boldsymbol{P}^m &+ d\tau\left[\nabla^2\boldsymbol{X} - 2\mathcal{H}\nabla\mathcal{X}_0 - a^2\beta^2\boldsymbol{X}\right]^{m+1/2} \\ &+ \boldsymbol{b}^{m+1/2}\left(\pi^m - 2\mathcal{H}^{m+1/2}\varphi^{m+1/2}\right) + (\boldsymbol{P}^m - 2\nabla\mathcal{X}_0^{m+1/2})\times\boldsymbol{c}^{m+1/2} . \end{aligned} \tag{3.14}$$

The matrix equation (3.11) can be solved to obtain $\pi^{m+1}$ and $P_i^{m+1}$.

3. Evolve $\varphi, \boldsymbol{X}$ by another half a step using the backward difference

$$\varphi^{m+1} = \varphi^{m+1/2} + \frac{d\tau}{2}\pi^{m+1} \ , \quad \boldsymbol{X}^{m+1} = \boldsymbol{X}^{m+1/2} + \frac{d\tau}{2}\boldsymbol{P}^{m+1} \ . \tag{3.15}$$

4. Evolve $\mathcal{X}_0^{m+1}$ by another half step by approximating the left hand side of (3.5) with the backward difference in time, which leads to

$$\mathcal{X}_0^{m+1} = \frac{\mathcal{X}_0^{m+1/2} + (d\tau/2)\nabla\cdot\boldsymbol{X}^{m+1}}{1 + d\tau\mathcal{H}^{m+1}} \ . \tag{3.16}$$

5. Solve $P_0^{m+1}$ using (3.5).

One then repeats the steps 1–5 to evolve the system in time. The spatial derivatives are approximated using the centered difference method. The step size is determined by the Nyquist frequency or the axion/dark photon mass, with $d\tau = 0.25\ \mathrm{min}[dx, m_\phi^{-1}, m_X^{-1}]/a(\tau)$.

During the simulation, the numerical accuracy can be monitored via (3.3) by defining

$$\delta \equiv \mathrm{RMS}\left\{\frac{1}{\mathcal{N}}\left[\nabla\cdot\boldsymbol{P} - \nabla^2\mathcal{X}_0 + a^2\beta^2\mathcal{X}_0 - \frac{\alpha}{a}(\nabla\varphi)\cdot(\nabla\times\boldsymbol{X})\right]\right\} = 0 \ , \tag{3.17}$$

where $\mathcal{N}$ is a normalization factor given by the sum of absolute values of each term in the bracket at each point of the spatial grid, and $\mathrm{RMS}\{\cdots\}$ represents the operator of taking the root mean square over the whole grid. Numerically we expect $\delta \sim \mathcal{O}(dx^2, d\tau^2)$.

### 3.3 Initial conditions

Initially, the axion field is homogeneous with small fluctuations, while the dark photon field is unexcited and can be regarded as small fluctuations. The homogeneous value of the axion field can be chosen by hand. If the initial velocity of the axion field is zero, i.e., $\phi' = 0$, the velocity field $\pi$ can be determined through $\pi = \mathcal{H}\varphi$.

Assuming that the fluctuations are Gaussian, the initial fluctuations can be specified through

$$\delta\varphi(t_i, \boldsymbol{x}) = \frac{1}{\sqrt{L^3}}\sum_{\boldsymbol{k}}\delta\varphi(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \ , \quad \delta\pi(t_i, \boldsymbol{x}) = \frac{1}{\sqrt{L^3}}\sum_{\boldsymbol{k}}\delta\pi(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \ , \tag{3.18}$$

where the real and imaginary part of the Fourier mode $\delta\varphi(\boldsymbol{k})$ and $\delta\pi(\boldsymbol{k})$ are independently drawn from a Gaussian distribution. Assuming harmonic oscillators, the Gaussian distribution has zero mean with the variance given by[1]

$$\langle|\mathrm{Re}\{\delta\varphi(\boldsymbol{k})\}|^2\rangle = \langle|\mathrm{Im}\{\delta\varphi(\boldsymbol{k})\}|^2\rangle = \frac{m_\phi^2}{4\omega_\phi f_\phi^2} \ , \tag{3.19}$$

$$\langle|\mathrm{Re}\{\delta\pi(\boldsymbol{k})\}|^2\rangle = \langle|\mathrm{Im}\{\delta\pi(\boldsymbol{k})\}|^2\rangle = \frac{m_\phi^2\omega_\phi}{4f_\phi^2} \ , \tag{3.20}$$

$$\langle|\mathrm{Re}\{X_i(\boldsymbol{k})\}|^2\rangle = \langle|\mathrm{Im}\{X_i(\boldsymbol{k})\}|^2\rangle = \frac{\lambda_s}{4\omega_X\beta^2} \ , \tag{3.21}$$

$$\langle|\mathrm{Re}\{P_i(\boldsymbol{k})\}|^2\rangle = \langle|\mathrm{Im}\{P_i(\boldsymbol{k})\}|^2\rangle = \frac{\lambda_s\omega_X}{4\beta^2} \ , \tag{3.22}$$

[1] In terms of dimensional variables, the power spectrum of axion fluctuations is $P_{\delta\varphi}(\tau,\boldsymbol{k}) = (2\omega_\phi)^{-1}$ and $P_{\delta\varphi'}(\tau,\boldsymbol{k}) = \omega_\phi/2$, where $\omega_\phi^2 = k^2 + m_\phi^2a^2$, and that of dark photon fields is $P_{X_i}(\tau,\boldsymbol{k}) = (2\omega_X)^{-1}$ and $P_{X_i'}(\tau,\boldsymbol{k}) = \omega_X/2$, where $\omega_X^2 = k^2 + m_X^2a^2$. In terms of dimensionless variables, which are defined through the rescaling relation (3.1), the power spectrum of axion fluctuations becomes $P_{\delta\varphi}(\tau,\boldsymbol{k}) \to m_\phi^2/(2\omega_\phi f_\phi^2)$ and $P_{\delta\varphi'}(\tau,\boldsymbol{k}) \to m_\phi^2\omega_\phi/(2f_\phi^2)$, where $\omega_\phi \to \sqrt{k^2+a^2}$, and that of dark photon fields becomes $P_{X_i}(\tau,\boldsymbol{k}) \to \lambda_s/(2\omega\beta^2)$ and $P_{X_i'}(\tau,\boldsymbol{k}) \to \lambda_s\omega/(2\beta^2)$, where $\omega \to \sqrt{k^2+a^2\beta^2}$.

where $\omega_\phi = \sqrt{k^2 + a^2}$ and $\omega_X = \sqrt{k^2 + a^2\beta^2}$. The reality of $\delta\varphi(\boldsymbol{x})$ requires $\delta\varphi(\boldsymbol{k}) = \delta\varphi^*(-\boldsymbol{k})$, equivalently $\mathrm{Re}\{\delta\varphi(-\boldsymbol{k})\} = \mathrm{Re}\{\delta\varphi(\boldsymbol{k})\}$ and $\mathrm{Im}\{\delta\varphi(-\boldsymbol{k})\} = -\mathrm{Im}\{\delta\varphi(\boldsymbol{k})\}$, hence we only need to specify $\delta\varphi(\boldsymbol{k})$ over half of the Fourier space. This also applies for $\delta\pi(\boldsymbol{k})$, $X_i(\boldsymbol{k})$, and $P_i(\boldsymbol{k})$. Once $X_i$ and $P_i$ have been initialized, we can calculate $\mathcal{X}_0$ using (3.3) in the Fourier space

$$\mathcal{X}_0(\boldsymbol{k}) = -\int \frac{d^3k}{(2\pi)^3}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \frac{ik_i P_i(\boldsymbol{k})}{k^2 + a^2\beta^2}\,, \tag{3.23}$$

where the term with the axion field is higher-order and thus has been neglected. Finally $P_0$ can be calculated using (3.5).

In python, the numpy module provides fast Fourier transform methods. Technically, in the grid of the Fourier space, the momentum coordinates on each axis correspond to

$$k_m = \begin{cases} \frac{2\pi}{L}\left[0, 1, \cdots, \frac{N}{2} - 1, -\frac{N}{2}, \cdots, -1\right] & \text{for even } N \\ \frac{2\pi}{L}\left[0, 1, \cdots, \frac{N-1}{2}, -\frac{N-1}{2}, \cdots, -1\right] & \text{for odd } N \end{cases}, \tag{3.24}$$

which can be created using `2*np.pi*np.fft.fftfreq(N, dx)`. The fast Fourier transform and its inverse can be realized using `np.fft.fftn()` and `np.fft.ifftn()`. Denoting the Fourier mode obtained through this method as $\delta\varphi_\mathrm{d}$, one should note that it is related to the one in (3.18) through $\delta\varphi_\mathrm{d}(\boldsymbol{k}) \equiv (\sqrt{L^3}/dx^3)\delta\varphi(\boldsymbol{k})$. This also applies for $\delta\pi(\boldsymbol{k})$, $X_i(\boldsymbol{k})$, and $P_i(\boldsymbol{k})$.

## References


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