

Lecture 04 Exercise Solutions

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Exercise 1

- a) When offset 0 of the array a is equal to s .
- b) When the string s does not appear at all in a .
- c) Statement Counting Approach:
 - Let n be the length of the array a .
 - $T_{contains}(n) = 2q + 4$ where q is the number of times the “while” condition is true.
 - In the best case, $q = 0$ and $T_{contains}^B(n) = 4$
 - In the worst case, $q = n$ and $T_{contains}^W(n) = 2n + 4$

Exercise 2

Here is the analysis of insertion sort. We begin with the inner loop.

- Number of statements per inner loop iteration: 3
- Number of iterations of inner loop (worst case): i
- Plus one statement when the loop condition is false.
- Total: $3i + 1$;

Now the outer loop:

- Number of statements per outer loop iteration: $5 + \text{cost of inner loop} = 5 + 3i + 1$
- Since cost of inner loop depends on which iteration of the outer loop we are on, we must use summation notation.
- Cost for all iterations of outer loop: $\sum_{i=1}^{n-1} (5 + 3i + 1)$
- Plus one more for when the outer loop being false.
- Total: $\sum_{i=1}^{n-1} (5 + 3i + 1) + 1$

Now for the overall method:

- Total cost of outer loop, plus one initialization statement: $\sum_{i=1}^{n-1} (5 + 3i + 1) + 2$

Now it's just a matter of simplification:

$$\begin{aligned}
 \sum_{i=1}^{n-1} (5 + 3i + 1) + 2 &= 2 + \sum_{i=1}^{n-1} (6 + 3i) \\
 &= 2 + \sum_{i=1}^{n-1} 6 + \sum_{i=1}^{n-1} 3i \\
 &= 2 + 6(n-1) + 3 \sum_{i=1}^{n-1} i \\
 &= 2 + 6n - 6 + 3 \frac{(n-1)(n)}{2} \\
 &= 2 + 6n - 6 + \frac{3}{2}n^2 - \frac{3}{2}n \\
 &= 1.5n^2 + 4.5n - 4
 \end{aligned}$$

Exercise 3

Analysis of `contains()` method from Exercise 1 using active operation approach:

The active operation is the loop condition which, in the worst case, will execute $n + 1$ times, where n is the length of the array `a`. No other statement executes more often.

Analysis of `insertionSort()` method from Exercise 1 using active operation approach:

The active operation for the insertion sort method is the condition of the inner while loop: `j >= 0 && temp.compareTo(a[j]) < 0`. How many times does it execute? Each time the inner loop is executed by the outer loop, the active operation is executed $i + 1$ times in the worst case. To analyze the outer loop we must use summation notation because the number of active operations in each iteration of the outer loop depends on which iteration is executing. The number of active operations in the outer loop is:

$$\sum_{i=1}^{n-1} (i + 1)$$

And now we just need to simplify, and substitute the closed-form of the summation

$$\begin{aligned}
 \sum_{i=1}^{n-1} (1 + i) &= (n-1) + \frac{(n-1)(n)}{2} \\
 &= n^2/2 + n/2 - 1
 \end{aligned}$$

Exercise 4

- $n \in O(n)$
- $47n \log n + 10000n \in O(n \log n)$
- $100n + 500 \log n + 1000 \in O(n)$
- $\log n + 100\sqrt{n} + 76 \in O(\sqrt{n})$
- $n^2 \in O(n^2)$
- $2^n + n + \log n \in O(2^n)$
- $5 \in O(1)$
- $T_{count}^B(n) = 3n + 4 \in O(n)$
- $T_{count}^W(n) = 4n + 4 \in O(n)$
- $T_{contains}^B(n) = 7 \in O(1)$
- $T_{contains}^W(n) = 3n + 4 \in O(n)$
- $T_{insort}^W(n) = 1.5n^2 + 4.5n - 4 \in O(n^2)$

Exercise 5

Analysis of method s:

- active operation: $x = x + k(i) * k(2n - i)$
- number of active operations executed: $n + 1$
- cost of active operation: $2 \times O(\log m)$.
- total cost: $(n + 1) \cdot 2 \cdot O(\log m) = O(n \log m)$.

Exercise 6

- How many times is the loop body executed?
- Let r be the number of times the loop body is executed.
- Consider the values of p when the loop condition is true:

$$n, n/2, n/4, \dots, n/2^{r-1}$$

All these values (there are r of them) must be greater than 1, or the loop would stop.

- This implies: $n/2^r \leq 1 < n/2^{r-1}$. Now we need to solve for r .

- If the previous inequality is true, it means that $n \leq 2^r$ and $2^{r-1} < n$
- Taking logarithm of each inequality, we get $\log n \leq r$ and $r - 1 < \log n$, or, equivalently:

$$r - 1 < \log n \leq r$$

- Since r must be a non-negative integer (it's the number of loop iterations), $r = \lceil \log n \rceil$ for $n \geq 1$.
- Therefore, the number of times the loop body executes is $O(\lceil \log n \rceil)$.
- Using the active operation approach: the active operation is the loop condition, which, as we just demonstrated, executes $O(\log n)$ times. Therefore the entire algorithm is $O(\log n)$.

Exercise 7

It is not immediately obvious which statement should be the active operation because there are three statements that call methods, all of different cost. So we consider all three possible active operations and their total cost. The first two are fairly straightforward:

- Active operation: $q()$
 - number of times executed: 1
 - cost per execution: $O(n \log m)$
 - total cost: $O(n \log m)$
- Active operation: $s()$
 - number of times executed: 1
 - cost per execution: $O(m \log m)$
 - total cost: $O(m \log m)$

The third one is trickier, because it's not immediately obvious how many times the loop containing the active operation executes:

- Active operation: $p()$
 - number of times executed: ??
 - cost per execution: $O(n^2)$
 - total cost: $?? \cdot O(n^2)$

In order to find out how many times the active operation $p()$ is executed, we must determine how many times the loop body executes.

- Let x = number of times loop in method r is executed.
- Consider the values of i in each loop iteration (there are x of them): $1, 2, 4, 8, 16, \dots, 2^{x-1}$.

- When loop stops, $i \geq m$, i.e. when $2^{x-1} \geq m$. Thus:

$$\begin{aligned} 2^{x-2} &< m \leq 2^{x-1} \\ x-2 &< \log m \leq x-1 \\ x &= \lceil \log m \rceil + 1 \end{aligned}$$

- Result follows since x must either be a positive integer at least as big as $\log m + 1$.

Now we can plug our value for x back into the original analysis:

- Active operation: $p()$
 - number of times executed: $\log m + 1$
 - cost per execution: $O(n^2)$
 - total cost: $(\log m + 1) \cdot O(n^2)$ or $O(n^2 \log m)$

So now we have three active operations costing $O(n \log m)$, $O(m \log m)$, and $O(n^2 \log m)$, respectively. Our final answer for the time complexity of method `r` should be whichever of these grows the fastest. But which grows fastest? Clearly $O(n \log m)$ can be eliminated since $O(n^2 \log m)$ definitely grows faster. But which of $O(n^2 \log m)$ and $O(m \log m)$ grows faster? The answer is: **we don't know!**. There is no way of knowing. All we can do is write our final answer as: $O(n^2 \log m) + O(m \log m)$ because we don't know if one of these terms grows more quickly than the other – we do not know the relationship (if any) between m and n . The best we can do to simplify this further is to use our rules for combining Big-Oh expressions:

$$O(n^2 \log m) + O(m \log m) = O(\max(n^2, m) \cdot \log m)$$

Exercise 8

Time Complexity of `LinkedList` Class Methods

The following methods are $O(1)$ because they contain no loops, and no method calls that are slower than $O(1)$. Some of these call methods from the `LinkedList` class, but these are also $O(1)$ because they are just accessor and mutator (a.k.a. getters and setter) methods.

<code>itemExists</code>	<code>item</code>	<code>goFirst</code>
<code>goForth</code>	<code>after</code>	<code>before</code>
<code>goBefore</code>	<code>isEmpty</code>	<code>insertFirst</code>
<code>deleteFirst</code>	<code>firstItem</code>	

The method `goLast` contains a loop. We analyze this method with statement counting. Starting with the inner loop:

- Number of statements per loop iteration: 2
- Number of loop iterations: $n - 1$, where n is the size of the array. The cursor starts on the first element, and the loop stops when the cursor is on the last element, so the loop body has to move the cursor forward $n - 1$ times to get it to the last item.

- Total statements for the loop: $2(n - 1) + 1$.

In the worst case, the list is non-empty, so the if condition results in just 1 statement (the exception does not occur), then the statement before the loop is another 1 statement which makes the total number of statements: $2 + 2(n - 1) + 1 = 2n + 1$, which is $O(n)$.

The method `goAfter` just calls two methods, `goLast` and `goForth`. We've already analyzed these methods, so we know that `goLast` is $O(n)$ and `goForth` is $O(1)$. Using our rules to combine Big-Oh expressions we know that $O(n) + O(1) = O(\max(n, 1)) = O(n)$.

Time Complexity of ArrayList Class Methods

None of the methods in `ArrayList` contain any loops or call any methods that are not, themselves $O(1)$, so every method in the `ArrayList` class is $O(1)$ – not bad!!

Exercise 9

Part a): we need to show that there are numbers c and n_0 such that $5n^3 + 4\log n + 11n \leq cn^3$ for all $n \geq n_0$. We know that $\log n \leq n^3$ for all $n \geq 2$. Then:

$$\begin{aligned} 4\log n &\leq 4n^3 \\ 4\log n + 11n &\leq 4n^3 + 11n^3 \\ 4\log n + 11n &\leq 15n^3 \\ 5n^3 + 4\log n + 11n &\leq 20n^3 \end{aligned}$$

Thus, for $c = 20$ and $n_0 = 2$, this statement $5n^3 + 4\log n + 11n \leq cn^3$ is true for all $n > n_0$. Therefore $5n^3 + 4\log n + 11n$ is $O(n^3)$.

Part b): we need to show that there are numbers c and n_0 such that $2n\log n + 4\log n \leq cn\log n$ for all $n \geq n_0$. It is clear that $4\log n \leq n\log n$ for $n \geq 4$. If that's true, then:

$$2n\log n + 4\log n \leq 3n\log n$$

Thus for $c = 3$ and $n_0 = 4$, the statement $2n\log n + 4\log n \leq cn\log n$ is true for all $n \geq n_0$. Therefore $2n\log n + 4\log n \leq cn\log n$ is $O(n\log n)$.