Homework 6

In collaboration with Yan and Amelia

Chapter 4.1

In [93]:
```
import numpy as np
import math
```

In [48]:
```
def forward_diff(f_1,f_0,h):
    f_prime = (f_1-f_0)/h
    return f_prime

def back_diff(f_1,f_0,h):
    f_prime = (f_0-f_1)/h
    return f_prime
```

In [49]:
```
# Exercise 1.
# (a)
x=[0.5,0.6,0.7]
f_x = [0.4794,0.5646,0.6442]

fprime = [forward_diff(f_x[1],f_x[0],0.1),forward_diff(f_x[2],f_x[1],0.1),back_diff(f_x[1],f_x[2],0.1)]
part_a = np.round(fprime,4)

print(f"f'(x) = {part_a}")
```

```
f'(x) = [0.852 0.796 0.796]
```

In [50]:
```
# (b)
x=[0.0,0.2,0.4]
f_x = [0.00000,0.74140,1.3718]

fprime = [forward_diff(f_x[1],f_x[0],0.2),forward_diff(f_x[2],f_x[1],0.2),back_diff(f_x[1],f_x[2],0.2)]
part_b = np.round(fprime,4)
print(f"f'(x) = {part_b}")
```

```
f'(x) = [3.707 3.152 3.152]
```

In [51]:
```
# Exercise 3
# (a)
x=[0.5,0.6,0.7]
f_x = [0.4794,0.5646,0.6442]
for i in range(len(x)):
    print(f'Actual error at x = {x[i]}: {np.abs(np.cos(x[i])-part_a[i]):.5f}.')
```

```
Actual error at x = 0.5: 0.02558.
Actual error at x = 0.6: 0.02934.
Actual error at x = 0.7: 0.03116.
```

Error bound:

We know $f''(x) = -\sin(x)$, so

$$\left|\frac{h}{2}f^{(2)}(\xi)\right| \leq 0.05\left(\max_{\xi\in[aa+h]}|-\sin(\xi)|\right).$$

Since $\sin(x)$ is increasing on $[0.5, 0.6]$ and $[0.6, 0.7]$, we take $\xi = 0.6$.

$$0.05\left(\max_{\xi\in[aa+h]}|-\sin(\xi)|\right) = 0.05\times 0.5646 = 0.02823.$$

For $x = 0.7$ we take $\xi = 0.7$

$$0.05\left(\max_{\xi\in[aa+h]}|-\sin(\xi)|\right) = 0.05\times 0.6442 = 0.03221.$$

In [52]:
```
# (b)
x=[0.0,0.2,0.4]
for i in range(len(x)):
    print(f'Actual error at x = {x[i]}: {(np.exp(x[i])-4*x[i]+3)-part_b[i]:.5f}.')
```

```
Actual error at x = 0.0: 0.29300.
Actual error at x = 0.2: 0.26940.
Actual error at x = 0.4: -0.26018.
```

Error bound:

We know $f''(x) = e^x - 4$. So

$$\left|\frac{h}{2}f^{(2)}(\xi)\right| \leq 0.05\left(\max_{\xi\in[aa+h]}|e^x - 4|\right)$$

Since e^x is increasing on $[0.0, 0.2]$ and $[0.2, 0.4]$, we take $\xi = 0.0$ to maximize

$$0.1\left(\max_{\xi\in[aa+h]}|e^x - 4|\right) = 0.1\times 3 = 0.3.$$

The bound associated with $x = 0.2$ and $x = 0.4$ will remain the same thus we take $\xi = 0.2$ to maximize

$$0.1\left(\max_{\xi\in[aa+h]}|e^x - 4|\right) = 0.1\times 0.5646 = 0.02823.$$

For $x = 0.7$ we take $\xi = 0.7$

$$0.1\left(\max_{\xi\in[aa+h]}|e^x - 4|\right) = 0.1\times 2.77859 = 0.277859.$$

In [53]:
```
# Exercise 5
# (a)
def three_endpoint(f_x0,f_x1,f_x2,h):
    f_prime = (1/(2*h))*(-3*(f_x0)+4*f_x1 -f_x2)
    return f_prime

def three_midpoint(f_x0,f_x1,h):
    f_prime = (1/(2*h))*((f_x1)-(f_x0))
    return f_prime

_5a = [three_endpoint(9.025013,11.02318,13.46374,0.1),three_midpoint(9.025013,13.46374,0.1),three_midpoint(11.02318,16.44465,0.1),three_endpoint(16.44465,13.46374 ,11.02318,-0.1
_5a = np.round(_5a,5)

print(f"f'(x) = {_5a}")
```

```
f'(x) = [17.76971 22.19364 27.10735 32.51085]
```

In [54]:
```
_5b = [three_endpoint(3.6887983,3.6905701,3.6688192,0.1),
       three_midpoint(3.6887983,3.6688192,0.1),
       three_midpoint(3.6905701,3.6245909,0.1),
       three_endpoint(3.6245909,3.6688192,3.6905701,-0.1)]
_5b = np.round(_5b,7)
print(f"f'(x) = {_5b}")
```

```
f'(x) = [ 0.1353315 -0.0998955 -0.329896  -0.55467  ]
```

In [55]:
```
# Exercise 7 (a)
x=[1.1,1.2,1.3,1.4]
for i in range(len(x)):
    print(f'Actual error at x = {x[i]}: {np.abs((2*np.exp(2*x[i]))-_5a[i]):.5f}')
```

```
Actual error at x = 1.1: 0.28032
Actual error at x = 1.2: 0.14729
Actual error at x = 1.3: 0.17987
Actual error at x = 1.4: 0.37844
```

For endpoints we have error $\frac{h^2}{3}f^{(3)}(\xi)$ the bound $\dfrac{Mh}{3}$ where M is a bound on $|f^{(3)}|$ and ξ is in between $x = 1.1$ and $x = 1.3$. $8e^{2x}$ is an increasing function. So we take 1.3 to maximize.

For $x = 1.1$

$$\frac{Mh}{3} = \frac{8e^{2(1.3)}(0.1)^2}{3} = 0.359033.$$

For midpoints we have error $\frac{h^2}{6}f^{(3)}(\xi)$, the bound $\dfrac{Mh}{6}$ where M is a bound on $|f^{(3)}|$ and ξ is in between $x = 1.2$ and $x = 1.3$. $8e^{2x}$ is an increasing function. So we take 1.3 to maximize.

For $x = 1.2$

$$\frac{Mh}{6} = \frac{8e^{2(1.3)}(0.1)^2}{6} = 0.17951.$$

For $x = 1.3$, we take $x = 1.4$ to maximize

$$\frac{Mh}{6} = \frac{8e^{2(1.4)}(0.1)^2}{6} = 0.219262.$$

For the last endpoint we take the endpoint bound with $[1.2, 1.4]$.

For $x = 1.4$ we take $x = 1.4$ to maximize

$$\frac{Mh}{3} = \frac{8e^{2(1.4)}(-0.1)^2}{3} = 0.4385239.$$

In [56]:
```
# Exercise 7 (d)
x=[2.0,2.1,2.2,2.3]
for i in range(len(x)):
    print(f'Actual error at x = {x[i]}: {np.abs((4*np.log(x[i])/x[i])+3*np.cos(x[i])-_5b[i]):.7f}')
```

```
Actual error at x = 2.0: 0.0025224
Actual error at x = 2.1: 0.0014288
Actual error at x = 2.2: 0.0020485
Actual error at x = 2.3: 0.0043795
```

For endpoints we have error $\dfrac{h^2}{3}f^{(3)}(\xi)$ the bound $\dfrac{Mh}{3}$ where M is a bound on $|f^{(3)}|$ and ξ is in between $x = 2.0$ and $x = 2.2$. $f^{(3)}(x) = \frac{-12}{x^3} + \frac{8\ln x}{x^3} - 3\cos x$ is an increasing function on $[2.0.2.3]$. So we take 2.2 to maximize. For $x = 2.0$

$$\frac{Mh}{3} = \left(\frac{\left(\frac{-12}{(2.2)^3} + \frac{8\ln(2.2)}{(2.2)^3} - 3\cos(2.2)\right)}{3}\right)\cdot(0.1)^2 = 0.004103036.$$

For $x = 2.1$, we take $x = 2.2$ to maximize using midpoint bound:

$$\frac{Mh}{6} = \left(\frac{\left(\frac{-12}{(2.2)^3} + \frac{8\ln(2.2)}{(2.2)^3} - 3\cos(2.2)\right)}{6}\right)\cdot(0.1)^2 = 0.00205152.$$

For $x = 2.2$ we take $x = 2.3$ to maximize using midpoint bound:

$$\frac{Mh}{6} = \left(\frac{\left(\frac{-12}{(2.3)^3} + \frac{8\ln(2.3)}{(2.3)^3} - 3\cos(2.3)\right)}{6}\right)\cdot(0.1)^2 = 0.0026003.$$

For endpoint $x = 2.3$ we have $x \in [2.1, 2.3]$, thus $x = 2.3$ to maximize:

$$\frac{Mh}{3} = \left(\frac{\left(\frac{-12}{(2.3)^3} + \frac{8\ln(2.3)}{(2.3)^3} - 3\cos(2.3)\right)}{3}\right)\cdot(0.1)^2 = 0.00520068.$$

In [60]:
```
# Exercise 22
# We will use 3-point formula:
#R = 0.142, L=0.98, h = 0.01

t_0=np.round(three_endpoint(3.10,3.12,3.14,0.01),5)  # t = 1.00
t_1=np.round(three_midpoint(3.10,3.14,.01))          # t = 1.01
t_2=np.round(three_midpoint(3.12,3.18,.01))          # t = 1.02
t_3=np.round(three_midpoint(3.14,3.24,.01))          # t = 1.03
t_4=np.round(three_endpoint(3.24,3.18,3.14,-0.01),5) # t = 1.04
```

In [65]:
```
t_list = [t_0,t_1,t_2,t_3,t_4]
t_list
```

Out[65]:
```
[2.0, 2.0, 3.0, 5.0, 7.0]
```

In [68]:
```
L = 0.98
R = 0.142
t = [1.00,1.01,1.02,1.03,1.04]
i=[3.10,3.12,3.14,3.18,3.24]
for j in range(len(t_list)):
    print(f"E({t[j]}) = {np.round(L*t_list[j]+ R*i[j],5)}.")
```

```
E(1.0) = 2.4002.
E(1.01) = 2.40304.
E(1.02) = 3.38588.
E(1.03) = 5.35156.
E(1.04) = 7.32008.
```

Problem 29

We have

$$e(h) = \frac{\varepsilon}{h} + \frac{h^2}{6}M,$$

where M is the bound for the third derivative of a function.

Taking the derivative gives

$$e'(h) = -\varepsilon/h^2 + hM/3.$$

If $e'(h) = 0$, then $h = \sqrt[3]{3\varepsilon/M}$. So, $e'(h) < 0$ if $h < \sqrt[3]{3\varepsilon/M}$ and $e'(h) > 0$ if $h > \sqrt[3]{3\varepsilon/M}$ Hence $e(h)$ has a minimum at $h = \sqrt[3]{3\varepsilon/M}$.

Chapter 4.3

In [101…
```
# Exercise 1 (a)
def trap(f,a,b,N=10):
    x = np.linspace(a,b,N+1)
    y = f(x)
    y_right = y[1:]
    y_left = y[:-1]
    dx = (b - a)/N
    T = (dx/2) * np.sum(y_right + y_left)
    return T

def f(x):
    return x ** 4
print(f"The integral approximation is: {trap(f,0.5,1,1)}.")
```

```
The integral approximation is: 0.265625.
```

In [102…
```
# Exercise 1 (d)
def f(x):
    return x**2 * np.exp(-x)
print(f"The integral approximation is: {trap(f,1,1.5,1)}.")
```

```
The integral approximation is: 0.21748057537635235.
```

In [104…
```
# Exercise 1 (g)
def f(x):
    return x * np.sin(x)
print(f"The integral approximation is: {trap(f,0,math.pi/4,1)}.")
```

```
The integral approximation is: 0.21808950623871498.
```

Exercise 3 (d)

$$\begin{aligned}\int_0^1 x^2e^{-x}dx &= -e^{-x}\left(x^2+2x+2\right)_0^1\\ &= 0.160602794.\end{aligned}$$

True error: $|0.160602794 - 0.21748057537635235| = 0.0568777813764.$

$f''(x) = e^{-x}(2 - 4x + x^2)$

We have error $\dfrac{h^3}{12}f''(\xi), \quad \text{where} \quad x_0 < \xi < x_1$, so $|f''(\xi)|$ is maximized at $x = 0$. Then

$$\frac{h^3}{12}|f''(\xi)| \leq 2\cdot\frac{1^3}{12} \approx .166666666667.$$

Exercise 3 (g)

$$\begin{aligned}\int_0^{\frac{\pi}{4}} x\sin x dx &= [-x\cos x + \sin x]_0^{\frac{\pi}{4}}\\ &= 0.152027958.\end{aligned}$$

True error: $|0.152027958 - 0.218089| \approx 0.066061$

$f''(x) = 2\cos(x) - x\sin(x)$

$|f''(x)|$ is maximized at at $x = 0$, so

$$\frac{h^3}{12}|f''(\xi)| \leq 2\cdot\frac{(\pi/4)^3}{12} \approx 0.0807455.$$

In [113…
```
# Exercise 5 (a)
def simpsons(f,a,b,n):
    if n % 2 == 1:
        raise ValueError("n must be an even integer.")
    dx = (b-a)/n
    x = np.linspace(a,b,n+1)
    y = f(x)
    S = dx/3 * np.sum(y[0:-1:2] + 4*y[1::2] + y[2::2])
    return S

def f(x):
    return x**4

print(f"The integral approximation is: {simpsons(f,0.5,1,2)}.")
```

```
The integral approximation is: 0.19401041666666666.
```

In [114…
```
# Exercise 5 (d)
def f(x):
    return x**2 * np.exp(-x)

print(f"The integral approximation is: {simpsons(f,0,1,2)}.")
```

```
The integral approximation is: 0.1624016834806793.
```