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# Newton update in $L_2$ -norm random tree approximation

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# Preliminaries

- ▶  $\mathcal{X}$  is an arbitrary input space,  $\mathbf{x} \in \mathcal{X}$ .
- ▶  $\mathcal{Y}$  is an output space of a set of  $\ell$ -dimensional *multilabels*

$$\mathbf{y} = (y_1, \dots, y_\ell) \in \mathcal{Y}.$$

- ▶  $y_i$  is a *microlabel* and  $y_i \in \{1, \dots, r_i\}$ ,  $r_i \in \mathbb{Z}$ .
- ▶ For example, multilabel binary classification  $y_i \in \{-1, +1\}$ .
- ▶ Training examples are sampled from  $(\mathbf{x}, \mathbf{y}) \in \mathcal{X} \times \mathcal{Y}$ .
- ▶ Each example  $(\mathbf{x}, \mathbf{y})$  is mapped into a joint feature space  $\phi(\mathbf{x}, \mathbf{y})$ .
- ▶  $\mathbf{w}$  is the weight vector in the joint feature space.
- ▶ Define a linear score function  $F(\mathbf{w}, \mathbf{x}, \mathbf{y}) = \langle \mathbf{w}, \phi(\mathbf{x}, \mathbf{y}) \rangle$ .
- ▶ The prediction  $\mathbf{y}_{\mathbf{w}}(\mathbf{x})$  of an input  $\mathbf{x}$  is the multilabel  $\mathbf{y}$  that maximizes the score function

$$\mathbf{y}_{\mathbf{w}}(\mathbf{x}) = \underset{\mathbf{y} \in \mathcal{Y}}{\operatorname{argmax}} \langle \mathbf{w}, \phi(\mathbf{x}, \mathbf{y}) \rangle. \quad (1)$$

- ▶ (1) is called *inference* problem which is  $\mathcal{NP}$ -hard for most output feature maps.

# Markov network

- ▶ We assume that the output feature map  $\phi$  is a potential function on a Markov network  $G = (E, V)$ .
- ▶  $G$  is a complete graph with  $|V| = \ell$  nodes and  $|E| = \frac{\ell(\ell-1)}{2}$  undirected edges.
- ▶  $\varphi(\mathbf{x})$  is the input feature map, e.g., bag-of-words feature of an example  $\mathbf{x}$ .
- ▶  $\psi(\mathbf{y})$  is the output feature map which is a collection of edges and labels

$$\varphi(\mathbf{y}) = (u_e)_{e \in E}, u_e \in \{-1, +1\}^2.$$

- ▶ The joint feature is the Kronecker product of  $\varphi(\mathbf{x})$  and  $\psi(\mathbf{y})$

$$\phi(\mathbf{x}, \mathbf{y}) = (\phi_e(\mathbf{x}, \mathbf{y}))_{e \in E} = (\varphi(\mathbf{x}) \otimes \psi_e(\mathbf{y}_e))_{e \in E}.$$

- ▶ The score function is

$$F(\mathbf{w}, \mathbf{x}, \mathbf{y}) = \langle \mathbf{w}, \phi(\mathbf{x}, \mathbf{y}) \rangle = \sum_{e \in E} \langle \mathbf{w}_e, \phi_e(\mathbf{x}, \mathbf{y}_e) \rangle.$$

# Inference in terms of spanning trees

- Solving the following inference problem on a complete graph is  $\mathcal{NP}$ -hard

$$\mathbf{y}_{\mathbf{w}}(\mathbf{x}) = \operatorname{argmax}_{\mathbf{y} \in \mathcal{Y}} F(\mathbf{w}, \mathbf{x}, \mathbf{y}) = \operatorname{argmax}_{\mathbf{y} \in \mathcal{Y}} \sum_{e \in E} \langle \mathbf{w}_e, \phi_e(\mathbf{x}, \mathbf{y}_e) \rangle. \quad (2)$$

- For a complete graph, there are  $\ell^{\ell-2}$  unique spanning trees.
- We can write  $F(\mathbf{w}, \mathbf{x}, \mathbf{y})$  as a conic combination of all spanning trees

$$F(\mathbf{w}, \mathbf{x}, \mathbf{y}) = \sum_{T \in U(G)} a_T \langle \mathbf{w}_T, \phi_T(\mathbf{x}, \mathbf{y}) \rangle$$

# Multilabel classification

- ▶ Multilabel classification is an important research field in machine learning.
  - ▶ For example, a document can be classified as “science”, “genomics”, and “drug discovery”.
  - ▶ Each input variable  $\mathbf{x} \in \mathcal{X}$  is associated with multiple output variables  $\mathbf{y} \in \mathcal{Y}$ ,  $\mathcal{Y} = \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_l$ ,  $\mathcal{Y}_i = \{+1, -1\}$ .
  - ▶ The goal is to find a mapping function that predicts the best values of an output given an input  $f \in \mathcal{H} : \mathcal{X} \rightarrow \mathcal{Y}$ .
- ▶ The central problems of multilabel classification:
  - ▶ The size of the output space  $\mathcal{Y}$  is exponential in the number of microlabels.
  - ▶ The dependency of microlabels needs to be exploited to improve the prediction performance.

# Structured output learning

- ▶ There is an *output graph* connecting multiple labels.
  - ▶ A set of nodes represents multiple labels.
  - ▶ A set of edges represents the correlation between labels.
- ▶ Hierarchical classification:
  - ▶ The output graph is a rooted tree or a directed graph defining different levels of granularities.
  - ▶ For example, SSVM, ...
- ▶ Graph labeling:
  - ▶ The output graph often takes a general form (e.g., a tree, a chain).
  - ▶ For example,  $M^3N$ , CRF, MMCRF, ...
- ▶ The output graph is assumed to be known *a priori*.

# Research question

- ▶ The output graph is hidden in many applications.
  - ▶ For example, a surveillance photo can be tagged with “building”, “road”, “pedestrian”, and “vehicle”.
- ▶ We study the problem in structured output learning when the output graph is not observed.
- ▶ In particular:
  - ▶ Assume the dependency can be expressed by a complete set of pairwise correlations.
  - ▶ Build a structured output learning model with a complete graph as the output graph.
  - ▶ Solve the optimization problem and the inference problem ( $\mathcal{NP}$ -hard).

# Today

- ▶ A structured prediction model which performs max-margin learning on a random sample of spanning tree.
- ▶ Two ways to combine the set of random spanning trees
  - ▶ conical combination in NIPS paper.
  - ▶ convex combination as future work.
- ▶ Derivations and the corresponding optimization problems.



# Model

- ▶ Training examples comes in pair  $S = \{(\mathbf{x}_i, \mathbf{y}_i)\}_{i=1}^m \in \mathcal{X} \times \mathcal{Y}$ .
- ▶ A complete graph  $G = (E, V)$  is used as the output graph.
- ▶  $\varphi(\mathbf{x})$  is the input feature map, e.g., a feature vector of  $d$  dimension.
- ▶  $\Gamma_G(\mathbf{y})$  is the output feature map of  $\mathbf{y}$  on  $G$  of  $4 \times |E|$  dimension

$$\begin{aligned}\Gamma_G(\mathbf{y}) &= \{\Gamma_e(\mathbf{y}_e)\}_{e \in G}, \\ \Gamma_e(\mathbf{y}_e) &= [\mathbf{1}_{\mathbf{y}_e=00}, \mathbf{1}_{\mathbf{y}_e=01}, \mathbf{1}_{\mathbf{y}_e=10}, \mathbf{1}_{\mathbf{y}_e=11}]\end{aligned}$$

- ▶ A joint feature map of  $(\mathbf{x}_i, \mathbf{y}_i)$

$$\phi_G(\mathbf{x}_i, \mathbf{y}_i) = \varphi(\mathbf{x}_i) \otimes \Gamma_G(\mathbf{y}_i) = \{\phi_e(x_i, \mathbf{y}_{i,e})\}_{e \in G}.$$

- ▶ A compatibility score is defined as

$$F(\mathbf{x}, \mathbf{y}; \mathbf{w}_G) = \langle \mathbf{w}_G, \phi_G(\mathbf{x}, \mathbf{y}) \rangle = \sum_{e \in G} \langle \mathbf{w}_{G,e}, \phi_e(\mathbf{x}, \mathbf{y}_e) \rangle$$

# Model (cont.)

- ▶  $\mathbf{w}$  ensures an input  $\mathbf{x}_i$  with a correct multilabel  $\mathbf{y}_i$  achieves a higher score than with any incorrect multilabel  $\mathbf{y} \in \mathcal{Y}$ .
- ▶ The predicted output  $\mathbf{y}(\mathbf{x})$  for a given input  $\mathbf{x}$  is computed by

$$\mathbf{y}(\mathbf{x}) = \operatorname{argmax}_{\mathbf{y} \in \mathcal{Y}} F(\mathbf{x}, \mathbf{y}; \mathbf{w}_G) = \operatorname{argmax}_{\mathbf{y} \in \mathcal{Y}} \sum_{e \in G} \langle \mathbf{w}_{G,e}, \phi_{G,e}(\mathbf{x}, \mathbf{y}_e) \rangle,$$

which is called *inference problem*.

- ▶ The inference problem is  $\mathcal{NP}$ -hard for most joint feature maps on the complete graph.

# How to learn $w$ on a complete graph?

- ▶ The *margin* of an example  $\mathbf{x}_i$  is

$$\gamma_G(\mathbf{x}_i; \mathbf{w}_G) = F(\mathbf{x}_i, \mathbf{y}_i; \mathbf{w}_G) - \max_{\mathbf{y} \in \mathcal{Y}/\mathbf{y}_i} F(\mathbf{x}_i, \mathbf{y}; \mathbf{w}_G).$$

- ▶  $\mathbf{w}$  is solved by *max-margin principle* which aims to maximize  $\gamma(\mathbf{x}_i; \mathbf{w}_G)$  over all training example  $\mathbf{x}_i, i \in \{1, \dots, m\}$ .
- ▶ The inference problem on a complete graph is  $\mathcal{NP}$ -hardness.
- ▶ The parameter space is quadratic in the number of microlabels  $k$ .
- ▶ We aim to use a joint feature map that allows the inference problem be solved in polynomial time.

# Superposition of random trees

- ▶  $S(G)$  is a complete set of spanning tree generate from  $G$ ,  $|S(G)| = \ell^{\ell-2}$ .
- ▶ Recall  
 $\phi_G(\mathbf{x}, \mathbf{y}) = \{\phi_{G,e}(\mathbf{x}, \mathbf{y}_e)\}_{e \in G}$ ,  $\mathbf{w}_G = \{\mathbf{w}_{G,e}\}_{e \in G}$ ,  $\|\phi_G(\mathbf{x}, \mathbf{y})\| = \|\mathbf{w}_G\| = 1$ .
- ▶  $\phi_T(\mathbf{x}, \mathbf{y}) = \{\phi_e(\mathbf{x}, \mathbf{y})\}_{e \in T}$  is the projection of  $\phi_G(\mathbf{x}, \mathbf{y})$  on  $T \in S(G)$ .
- ▶  $\mathbf{w}_T = \{\mathbf{w}_{G,e}\}_{e \in T}$  is the projection of  $\mathbf{w}_G$  on  $T \in S(G)$ .
- ▶ Rewrite

$$\begin{aligned} F(\mathbf{x}, \mathbf{y}, \mathbf{w}_G) &= \sum_{e \in G} \langle \mathbf{w}_{G,e}, \phi_{G,e}(\mathbf{x}, \mathbf{y}_e) \rangle \\ &= \frac{1}{\ell^{\ell-2}} \sum_{T \in S(G)} \sqrt{\frac{\ell}{2}} \langle \mathbf{w}_T, \phi_T(\mathbf{x}, \mathbf{y}_e) \rangle \\ &= \frac{1}{n} \sum_{i=1}^n a_{T_i} \langle \hat{\mathbf{w}}_{T_i}, \hat{\phi}_{T_i}(\mathbf{x}, \mathbf{y}_e) \rangle, \end{aligned}$$

$$\|\hat{\phi}_T(\mathbf{x}, \mathbf{y})\| = \|\hat{\mathbf{w}}_T\| = 1, \frac{1}{n} \sum_{i=1}^n a_{T_i}^2 = 1, \frac{1}{n} \sum_{i=1}^n a_{T_i} \leq 1, a_{T_i} \geq 0, n = \ell^{\ell-2}.$$

# How many trees?

- ▶ If there is a predictor  $\mathbf{w}_G$  on complete graph achieves a margin on some training data, with high probability we need  $n$  spanning tree predictors  $\{\mathbf{w}_{T_i}\}_{i=1}^n$  to achieve a close margin.  $n$  is quadratic in terms of  $\ell$ .
- ▶ Recall

$$F(\mathbf{x}, \mathbf{y}, \mathbf{w}_T) = \frac{1}{n} \sum_{i=1}^n a_{T_i} \underbrace{\langle \hat{\mathbf{w}}_{T_i}, \hat{\phi}_{T_i}(\mathbf{x}, \mathbf{y}_e) \rangle}_{F(\mathbf{x}, \mathbf{y}, \mathbf{w}_{T_i})},$$

$$\|\hat{\phi}_T(\mathbf{x}, \mathbf{y})\| = \|\hat{\mathbf{w}}_T\| = 1, \frac{1}{n} \sum_{i=1}^n a_{T_i}^2 = 1, \frac{1}{n} \sum_{i=1}^n a_{T_i} \leq 1, a_{T_i} \geq 0, \cancel{n = \ell^2}.$$

# Conical combination

- ▶ A sample  $\mathcal{T} = \{T_1, \dots, T_n\}$  of  $n$  spanning trees drawn from  $G$ .
- ▶ Normalized feature vectors  $\hat{\phi}_{T_i}(\mathbf{x}, \mathbf{y}) = \frac{\phi_{T_i}(\mathbf{x}, \mathbf{y})}{\|\phi_{T_i}(\mathbf{x}, \mathbf{y})\|}$ ,  $T_i \in \mathcal{T}$ .
- ▶ Normalized feature weights  $\hat{\mathbf{w}}_{T_i} = \frac{\mathbf{w}_{T_i}}{\|\mathbf{w}_{T_i}\|}$ ,  $T_i \in \mathcal{T}$ .
- ▶ Conical combination of spanning trees

$$F(\mathbf{x}, \mathbf{y}, \mathbf{w}_{\mathcal{T}}) = \frac{1}{\sqrt{n}} \sum_{i=1}^n q_i \underbrace{\langle \hat{\mathbf{w}}_{T_i}, \hat{\phi}_{T_i}(\mathbf{x}, \mathbf{y}) \rangle}_{F(\mathbf{x}, \mathbf{y}, \mathbf{w}_{T_i})}$$

$$\sum_{i=1}^n q_i^2 = 1, q_i \geq 0, \forall i \in \{1, \dots, n\}.$$

## Conical combination (cont.)

- To solve  $\{\mathbf{w}_{T_i}\}_{T_i \in \mathcal{T}}$ , we need to work on the optimization problem

$$\begin{aligned} \min_{\xi, \gamma, \mathbf{q}, \mathcal{W}} \quad & \frac{1}{2\gamma^2} + \frac{C}{\gamma} \sum_{k=1}^m \xi_k \\ \text{s.t.} \quad & \frac{1}{\sqrt{n}} \sum_{i=1}^n q_i \langle \hat{\mathbf{w}}_{T_i}, \hat{\phi}_{T_i}(\mathbf{x}_k, \mathbf{y}_k) \rangle - \max_{\mathbf{y} \in \mathcal{Y}} \frac{1}{\sqrt{n}} \sum_{i=1}^n q_i \langle \hat{\mathbf{w}}_{T_i}, \hat{\phi}_{T_i}(\mathbf{x}_k, \mathbf{y}) \rangle \\ & \geq \gamma - \xi_k, \xi_k \geq 0, \forall k \in \{1, \dots, m\}, \sum_{i=1}^n q_i^2 = 1, q_i \geq 0, \forall i \in \{1, \dots, n\}. \end{aligned}$$

- This is equivalent to

$$\begin{aligned} \min_{\mathbf{w}_{T_i}, \xi_i} \quad & \frac{1}{2} \sum_{i=1}^n \|\mathbf{w}_{T_i}\|^2 + C \sum_{k=1}^m \xi_k \\ \text{s.t.} \quad & \frac{1}{\sqrt{n}} \sum_{i=1}^n \langle \mathbf{w}_{T_i}, \phi_{T_i}(\mathbf{x}_k, \mathbf{y}_k) \rangle - \max_{\mathbf{y} \neq \mathbf{y}_k} \frac{1}{\sqrt{n}} \sum_{i=1}^n \langle \mathbf{w}_{T_i}, \phi_{T_i}(\mathbf{x}_k, \mathbf{y}) \rangle \geq 1 - \xi_k, \\ & \xi_k \geq 0, \forall k \in \{1, \dots, m\}. \end{aligned}$$

# Inference Problem

- ▶ The inference problem of RTA is defined as finding the multilabel  $\mathbf{y}_{\mathcal{T}}(\mathbf{x})$  that maximizes the sum of scores over a collection of trees

$$\mathbf{y}_{\mathcal{T}}(\mathbf{x}) = \operatorname{argmax}_{\mathbf{y} \in \mathcal{Y}} F_{\mathcal{T}}(\mathbf{x}, \mathbf{y}; \mathbf{w}_{\mathcal{T}}) = \operatorname{argmax}_{\mathbf{y} \in \mathcal{Y}} \sum_{t=1}^n \langle \mathbf{w}_{T_t}, \phi_{T_t}(\mathbf{x}, \mathbf{y}) \rangle.$$

- ▶ The inference problem on each individual spanning tree can be solve efficiently in  $\Theta(l)$  by *dynamic programming*

$$\mathbf{y}_{T_t}(\mathbf{x}) = \operatorname{argmax}_{\mathbf{y} \in \mathcal{Y}} F_{T_t}(\mathbf{x}, \mathbf{y}; \mathbf{w}_{T_t}) = \operatorname{argmax}_{\mathbf{y} \in \mathcal{Y}} \langle \mathbf{w}_{T_t}, \phi_{T_t}(\mathbf{x}, \mathbf{y}) \rangle.$$

- ▶ There is no guarantee that there exists a tree  $T_t \in \mathcal{T}$  in which the maximizer of  $F_{T_t}$  is the maximizer of  $F_{\mathcal{T}}$ .



# Fast Inference Over a Collection of Trees

- ▶ For each tree  $T_t$ , instead of computing the best multilabel  $\mathbf{y}_{T_t}$ , we compute  $K$ -best multilabels in  $\Theta(KI)$  time

$$\mathcal{Y}_{T_t, K} = \{\mathbf{y}_{T_t, 1}, \dots, \mathbf{y}_{T_t, K}\}.$$

- ▶ Performing the same computation on all trees gives a candidate list of  $n \times K$  multilabels in  $\Theta(nKI)$  time

$$\mathcal{Y}_{\mathcal{T}, K} = \mathcal{Y}_{T_1, K} \cup \dots \mathcal{Y}_{T_n, K}.$$

- ▶ For now, we assume the best scoring multilabel of a collection of trees exists in the list  $\mathcal{Y}_{\mathcal{T}, K}$ .
- ▶ We proved that with a high probability  $\mathbf{y}_{\mathcal{T}}$  will appear in  $\mathcal{Y}_{\mathcal{T}, K}$ .
- ▶ We can identify  $\mathbf{y}_{\mathcal{T}}$  from  $\mathcal{Y}_{\mathcal{T}, K}$ .

# Convex combination

- ▶ A sample  $\mathcal{T}$  of  $n$  spanning trees drawn from  $G$ .
- ▶ Normalized feature weights  $\hat{\mathbf{w}}_{T_i} = \frac{\mathbf{w}_{T_i}}{\|\mathbf{w}_{T_i}\|}$ ,  $T_i \in \mathcal{T}$ .
- ▶ Normalized feature vectors  $\hat{\phi}_{T_i}(\mathbf{x}, \mathbf{y}) = \frac{\phi_{T_i}(\mathbf{x}, \mathbf{y})}{\|\phi_{T_i}(\mathbf{x}, \mathbf{y})\|}$ ,  $T_i \in \mathcal{T}$ .
- ▶ Convex combination of spanning trees

$$F(\mathbf{x}, \mathbf{y}, \mathbf{w}_{\mathcal{T}}) = \frac{1}{n} \sum_{i=1}^n q_i \langle \hat{\mathbf{w}}_{T_i}, \hat{\phi}_{T_i}(\mathbf{x}, \mathbf{y}) \rangle$$
$$\sum_{i=1}^n q_i = 1, q_i \geq 0, \forall i \in \{1, \dots, n\}.$$

## Convex combination (cont.)

- To solve  $\{\mathbf{w}_{T_i}\}_{T_i \in \mathcal{T}}$ , we need to work on the optimization problem

$$\begin{aligned} \min_{\xi, \gamma, \mathbf{q}, \mathcal{W}} \quad & \frac{1}{2\gamma^2} + \frac{C}{\gamma} \sum_{k=1}^m \xi_k \\ \text{s.t.} \quad & \frac{1}{n} \sum_{i=1}^n q_i \langle \hat{\mathbf{w}}_{T_i}, \hat{\phi}_{T_i}(\mathbf{x}_k, \mathbf{y}_k) \rangle - \max_{\mathbf{y} \in \mathcal{Y}} \frac{1}{n} \sum_{i=1}^n q_i \langle \hat{\mathbf{w}}_{T_i}, \hat{\phi}_{T_i}(\mathbf{x}_k, \mathbf{y}) \rangle \\ & \geq \gamma - \xi_k, \xi_k \geq 0, \forall k \in \{1, \dots, m\}, \sum_{i=1}^n q_i = 1, q_i \geq 0, \forall i \in \{1, \dots, n\}. \end{aligned}$$

- This is equivalent to

$$\begin{aligned} \min_{\mathbf{w}_{T_i}, \xi_i} \quad & \frac{1}{2} \left( \sum_{i=1}^n \|\mathbf{w}_{T_i}\| \right)^2 + C \sum_{k=1}^m \xi_k \\ \text{s.t.} \quad & \frac{1}{n} \sum_{i=1}^n \langle \mathbf{w}_{T_i}, \phi_{T_i}(\mathbf{x}_k, \mathbf{y}_k) \rangle - \max_{\mathbf{y} \neq \mathbf{y}_k} \frac{1}{n} \sum_{i=1}^n \langle \mathbf{w}_{T_i}, \phi_{T_i}(\mathbf{x}_k, \mathbf{y}) \rangle \geq 1 - \xi_k, \\ & \xi_k \geq 0, \forall k \in \{1, \dots, m\}. \end{aligned}$$

# Convex combination (cont.)

- This can be expressed equivalently as

$$\begin{aligned} \min_{\mathbf{w}_{T_i}, \xi_i, \lambda_i} \quad & \frac{1}{2} \sum_{i=1}^n \frac{1}{\lambda_i} \|\mathbf{w}_{T_i}\|^2 + C \sum_{k=1}^m \xi_k \\ \text{s.t.} \quad & \frac{1}{n} \sum_{i=1}^n \langle \mathbf{w}_{T_i}, \phi_{T_i}(\mathbf{x}_k, \mathbf{y}_k) \rangle - \max_{\mathbf{y} \neq \mathbf{y}_k} \frac{1}{n} \sum_{i=1}^n \langle \mathbf{w}_{T_i}, \phi_{T_i}(\mathbf{x}_k, \mathbf{y}) \rangle \geq 1 - \xi_k, \\ & \xi_k \geq 0, \forall k \in \{1, \dots, m\}, \sum_{i=1}^n \lambda_i = 1, \lambda_i \geq 0, \forall i \in \{1, \dots, n\}. \end{aligned}$$

# Conclusions

- ▶ We show that if there is a learner  $\mathbf{w}_G$  defined on a complete graph achieves a margin on some training data, then with a random collection of spanning tree learners  $\{\mathbf{w}_{T_i}\}_{i=1}^n$  we can achieve a similar margin with high probability. Besides,  $n$  is polynomial in  $k$ .
- ▶ We propose two methods to combine the random collection of trees, namely, convex combination and conical combination.