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Newton update in L_2 -norm random tree approximation

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Preliminaries

- ▶ \mathcal{X} is an arbitrary input space, $\mathbf{x} \in \mathcal{X}$.
- ▶ \mathcal{Y} is an output space of a set of ℓ -dimensional *multilabels*

$$\mathbf{y} = (y_1, \dots, y_\ell) \in \mathcal{Y}.$$

- ▶ y_i is a *microlabel* and $y_i \in \{1, \dots, r_i\}$, $r_i \in \mathbb{Z}$.
- ▶ For example, multilabel binary classification $y_i \in \{-1, +1\}$.
- ▶ Training examples are sampled from $(\mathbf{x}, \mathbf{y}) \in \mathcal{X} \times \mathcal{Y}$.
- ▶ Each example (\mathbf{x}, \mathbf{y}) is mapped into a joint feature space $\phi(\mathbf{x}, \mathbf{y})$.
- ▶ \mathbf{w} is the weight vector in the joint feature space.
- ▶ Define a linear score function $F(\mathbf{w}, \mathbf{x}, \mathbf{y}) = \langle \mathbf{w}, \phi(\mathbf{x}, \mathbf{y}) \rangle$.
- ▶ The prediction $\mathbf{y}_{\mathbf{w}}(\mathbf{x})$ of an input \mathbf{x} is the multilabel \mathbf{y} that maximizes the score function

$$\mathbf{y}_{\mathbf{w}}(\mathbf{x}) = \underset{\mathbf{y} \in \mathcal{Y}}{\operatorname{argmax}} \langle \mathbf{w}, \phi(\mathbf{x}, \mathbf{y}) \rangle. \quad (1)$$

- ▶ (1) is called *inference* problem which is \mathcal{NP} -hard for most output feature maps.

Markov network

- ▶ We assume that the joint feature map ϕ is a potential function on a Markov network $G = (E, V)$.
- ▶ G is a complete graph with $|V| = \ell$ nodes and $|E| = \frac{\ell(\ell-1)}{2}$ undirected edges.
- ▶ G models all pairwise correlations.
- ▶ $\varphi(\mathbf{x})$ is the input feature map, e.g., bag-of-words feature of an example \mathbf{x} .
- ▶ $\psi(\mathbf{y})$ is the output feature map which is a collection of edges and labels

$$\varphi(\mathbf{y}) = (u_e)_{e \in E}, u_e \in \{-1, +1\}^2.$$

- ▶ The joint feature is the Kronecker product of $\varphi(\mathbf{x})$ and $\psi(\mathbf{y})$

$$\phi(\mathbf{x}, \mathbf{y}) = (\phi_e(\mathbf{x}, \mathbf{y}))_{e \in E} = (\varphi(\mathbf{x}) \otimes \psi_e(\mathbf{y}_e))_{e \in E}.$$

- ▶ The score function can be factorized by the complete graph G

$$F(\mathbf{w}, \mathbf{x}, \mathbf{y}) = \langle \mathbf{w}, \phi(\mathbf{x}, \mathbf{y}) \rangle = \sum_{e \in E} \langle \mathbf{w}_e, \phi_e(\mathbf{x}, \mathbf{y}_e) \rangle.$$

Inference in terms of all spanning trees

- ▶ Solving the following inference problem on a complete graph is \mathcal{NP} -hard

$$\mathbf{y}_{\mathbf{w}}(\mathbf{x}) = \operatorname{argmax}_{\mathbf{y} \in \mathcal{Y}} F(\mathbf{w}, \mathbf{x}, \mathbf{y}) = \operatorname{argmax}_{\mathbf{y} \in \mathcal{Y}} \sum_{e \in E} \langle \mathbf{w}_e, \phi_e(\mathbf{x}, \mathbf{y}_e) \rangle.$$

- ▶ For a complete graph, there are $\ell^{\ell-2}$ unique spanning trees.
- ▶ We can write $F(\mathbf{w}, \mathbf{x}, \mathbf{y})$ as a conic combination of all spanning trees

$$F(\mathbf{w}, \mathbf{x}, \mathbf{y}) = \mathbf{E}_{T \in U(G)} a_T \langle \mathbf{w}_T, \phi_T(\mathbf{x}, \mathbf{y}) \rangle$$
$$\mathbf{E}_{T \in U(G)} a_T^2 = 1, \quad \mathbf{E}_{T \in U(G)} a_T < 1.$$

- ▶ $U(G)$ is the uniform distribution over $\ell^{\ell-2}$ spanning trees.
- ▶ The number of spanning trees is exponentially dependent on the number of nodes ℓ .

A sample of n spanning trees

- ▶ Instead of using all spanning trees, we can just use n spanning trees

$$F_{\mathcal{T}}(\mathbf{w}, \mathbf{x}, \mathbf{y}) = \frac{1}{n} \sum_{i=1}^n a_{T_i} \langle \mathbf{w}_{T_i}, \phi_{T_i}(\mathbf{x}, \mathbf{y}) \rangle$$
$$\frac{1}{n} \sum_{i=1}^n a_{T_i}^2 = 1, \quad \frac{1}{n} \sum_{i=1}^n a_{T_i} < 1.$$

- ▶ When

$$n \geq \frac{\ell^2}{\epsilon^2} \left(\frac{1}{16} + \frac{1}{2} \ln \frac{8\sqrt{n}}{\delta} \right),$$

we have $|F_{\mathcal{T}}(\mathbf{w}, \mathbf{x}, \mathbf{y}) - F(\mathbf{w}, \mathbf{x}, \mathbf{y})| \leq \epsilon$, with high probability.

- ▶ A sample of $n \in \Theta(\ell^2/\delta^2)$ random spanning tree is sufficient to estimate the score function.
- ▶ Margin achieved by $F(\mathbf{w}, \mathbf{x}, \mathbf{y})$ is also preserved by the sample of n random spanning trees $F_{\mathcal{T}}(\mathbf{w}, \mathbf{x}, \mathbf{y})$.

Optimization problem

- The primal optimization problem is defined as

$$\begin{aligned} \min_{\mathbf{w}_{T_i}, \xi_i} \quad & \frac{1}{2} \sum_{i=1}^n \|\mathbf{w}_{T_i}\|^2 + C \sum_{k=1}^m \xi_k \\ \text{s.t.} \quad & \sum_{i=1}^n \langle \mathbf{w}_{T_i}, \Delta \phi_{T_t}(\mathbf{x}_k, \mathbf{y}_k) \rangle \geq \ell_{T_i, k} - \xi_k, \\ & \xi_k \geq 0, \forall k \in \{1, \dots, m\}. \end{aligned}$$

- The marginalized dual problem is defined as

$$\begin{aligned} \max_{\mu \in \mathcal{M}} \quad & \sum_{i=1}^n \left(\mu_{T_i} \ell_{T_i} - \frac{1}{2} \mu_{T_i} K_{T_i}^{\Delta \phi} \mu_{T_i} \right) \\ \text{s.t.} \quad & \sum_{u_e} \mu_{T_i, e}(u_e) \leq C. \end{aligned}$$

Optimization on a single random spanning tree T_i

- ▶ The optimization problem can be solved efficiently on a single tree T_i .
- ▶ The algorithm iterates over all training examples until convergence.
- ▶ We drop the index momentarily $\mu \leftarrow \mu(j)$, $g \leftarrow g(j)$, $\ell \leftarrow \ell(j)$.
- ▶ For the k th iteration:
 1. Obtain the current solution of the j th example $\mu_{T_i}^k$.
 2. Compute the current gradient on $\mu_{T_i}^k$, $g_{T_i}^k = \ell_{T_i} - K_{T_i}^{\Delta\phi} \mu_{T_i}^k$.
 3. Compute a feasible solution $\mu_{T_i}^{k,*}$ as an update direction (**efficiently**)

$$\mu_{T_i}^{k,*} = \underset{\mu \in \mathcal{M}}{\operatorname{argmax}} \mu^\top g_{T_i}^k. \quad (2)$$

4. Compute a stationary point (τ) and perform the update

$$\mu_{T_i}^{k+1} = \mu_{T_i}^k + \tau \Delta \mu_{T_i}^k = \mu_{T_i}^k + \tau (\mu_{T_i}^{k,*} - \mu_{T_i}^k).$$

- ▶ Line search gives the optimal feasible solution as a stationary point (τ).

$$\max_{\tau} f(\mu_{T_i}^k + \tau \Delta \mu_{T_i}^k)$$

$$\text{s.t. } 0 \leq \tau \leq 1.$$

- ▶ Feasible maximum update is achieved at $\tau = 1$.

Optimization on a collection of n spanning trees

- ▶ The algorithm iterates over all training examples until convergence.
- ▶ We drop the index momentarily $\mu \leftarrow \mu(j)$, $g \leftarrow g(j)$, $\ell \leftarrow \ell(j)$.
- ▶ For the k th iteration:
 1. Obtain the current solutions over all spanning trees $(\mu_{T_i}^k)_{i=1}^n$.
 2. Compute the gradients over all trees $(g_{T_i}^k)_{i=1}^n$.
 3. Compute a feasible solution for each individual spanning tree

$$\mu_{T_i}^{k,*} = \underset{\mu \in \mathcal{M}}{\operatorname{argmax}} \mu^\top g_{T_i}^k, \forall i.$$

4. Compute the best direction

$$\mu_T^{k,*} = \underset{\mu \in (\mu_{T_i}^{k,*})_{i=1}^n}{\operatorname{argmax}} \sum_{i=1}^n \mu^\top g_{T_i}^k$$

5. Compute the difference $\Delta \mu_{T_i}^k = \mu_{T_i}^k - \mu_{T_i}^{k,*}$, $\forall i$.
6. Compute the step size τ .
7. Perform the update $\mu_{T_i}^{k+1} = \mu_{T_i}^k + \tau \Delta \mu_{T_i}^k$, $\forall i$.

Line search to get the step size

- The step size along the update direction τ is given by the exact line search.

$$\begin{aligned} \max_{\tau} \quad & \sum_{i=1}^n f(\mu_{T_i}^k + \tau \Delta \mu_{T_i}^k) \\ \text{s.t.} \quad & 0 \leq \tau \leq 1. \end{aligned}$$

Update with multiple directions

- ▶ The algorithm iterates over all training example until convergence.
- ▶ $\mu : \mu(j)$, and $g : g(j)$.
- ▶ For the k th iteration:
 1. Obtain the solutions of the j th example over all trees $(\mu_{T_i}^k)_{i=1}^n$.
 2. Compute the gradients over all trees $(g_{T_i}^k)_{i=1}^n$.
 3. Compute local update direction from each spanning tree

$$\mu_{T_i}^{k,*} = \underset{\mu \in \mathcal{M}}{\operatorname{argmax}} \mu^\top g_{T_i}^k, \forall i.$$

4. Project local directions into global directions

$$\mu_{T_i}^{G,k,*} \leftarrow \mu_{T_i}^{k,*}, \forall i.$$

5. Define a conic combination of update directions

$$\Delta \mu^{G,k} = \sum_{i=1}^n \tau_i \Delta \mu_{T_i}^{G,k,*} = \sum_{i=1}^n \tau_i \left(\mu^{G,k} - \mu_{T_i}^{G,k,*} \right), 0 \leq \tau_i \leq 1, \forall i.$$

6. Perform the update $\mu^{G,k+1} = \mu^{G,k} + \Delta \mu^{G,k+1}$.
7. Project the global solution on spanning trees $(\mu_{T_i}^{k+1})_{i=1}^n \leftarrow \mu^{G,k+1}$.

Newton method to compute τ

- ▶ We want to find τ that maximize the objective function given the update

$$\begin{aligned} \max_{\tau} \quad & f(\mu^{G,k} + \Delta\mu^{G,k+1}) \\ \text{s.t.} \quad & 0 \leq \tau_i \leq 1. \end{aligned}$$

- ▶ The objective is quadratic with respect to τ .
- ▶ We use Newton method to find τ that maximize the objective.
- ▶ τ is projected into the feasible region.

Compute duality gap

- ▶ We use duality gap to measure the progress of the optimization.
- ▶ Primal and dual objective function

$$f(\mathbf{w}) = \frac{1}{2} \|\mathbf{w}\|^2 + C \sum_{i=1}^m (\ell_i - \langle \mathbf{w}, \Delta\phi(\mathbf{x}_i, \mathbf{y}_i) \rangle)$$

$$g(\boldsymbol{\alpha}) = \sum_{i=1}^m \alpha_i \ell_i - \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m \alpha_i K^{\Delta\phi}(\mathbf{x}_i, \mathbf{y}_i; \mathbf{x}_j, \mathbf{y}_j) \alpha_j$$

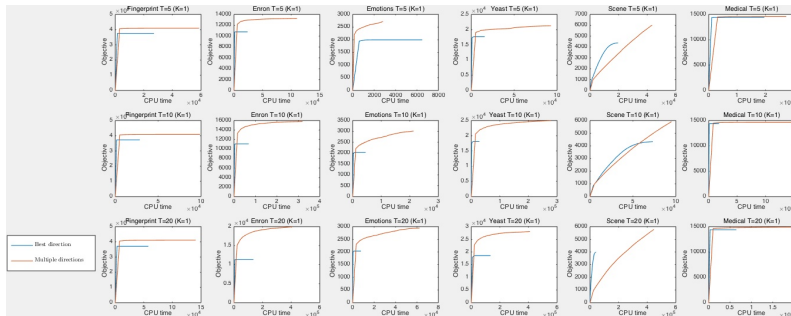
- ▶ $\max_{\boldsymbol{\alpha}} g(\boldsymbol{\alpha}) \leq \min_{\mathbf{w}} f(\mathbf{w})$, minimum gap when optimal.
- ▶ Duality gap at $\boldsymbol{\alpha}^k$

$$\begin{aligned} f(\mathbf{w}^k) - g(\boldsymbol{\alpha}^k) &= C \left(\ell - K^{\Delta\phi} \boldsymbol{\alpha}^k \right) - \boldsymbol{\alpha}^k \left(\ell - K^{\Delta\phi} \boldsymbol{\alpha}^k \right) \\ &= C^T \nabla g(\boldsymbol{\alpha}^k) - \boldsymbol{\alpha}^{kT} \nabla g(\boldsymbol{\alpha}^k) \end{aligned}$$

1. Estimate the dual objective function using a linear approximation ∇g .
2. Dual objective value at $\boldsymbol{\alpha}^k$ is computed by $\boldsymbol{\alpha}^{kT} \nabla g(\boldsymbol{\alpha}^k)$.
3. Primal objective value is estimate by $C^T \nabla g(\boldsymbol{\alpha}^k)$.

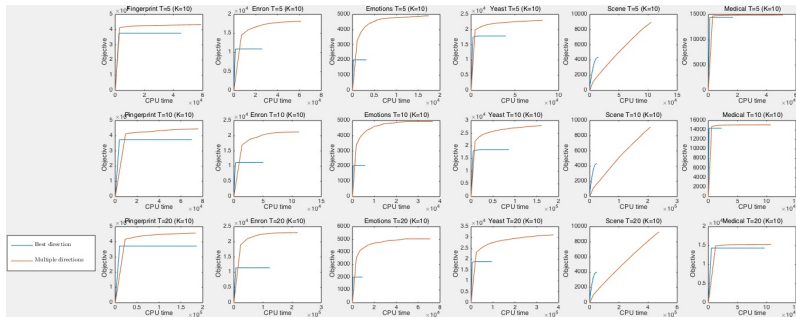
Experimental results - Objective value

- ▶ Compare update with the best direction v.s. update with multiple directions
- ▶ Number of spanning trees $|T| = \{5, 10, 20\}$.
- ▶ Each spanning tree outputs top $\kappa = 1$ best direction.



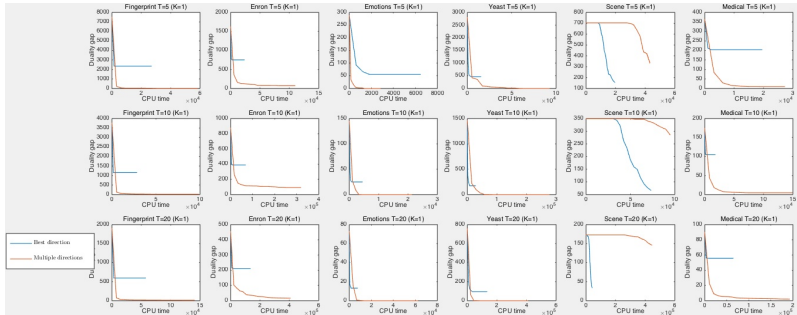
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- ▶ Compare update with the best direction v.s. update with multiple directions
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- ▶ Each spanning tree outputs top $\kappa = 10$ best directions.



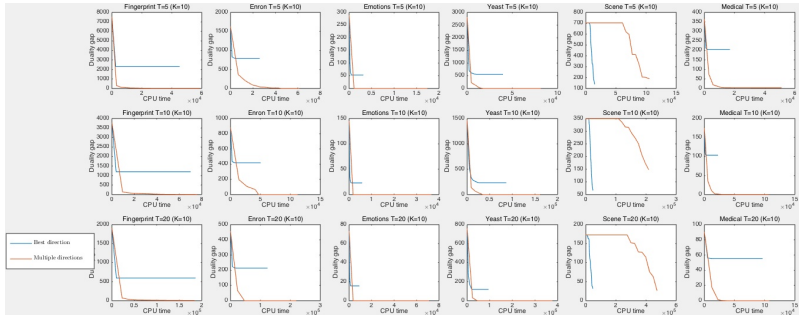
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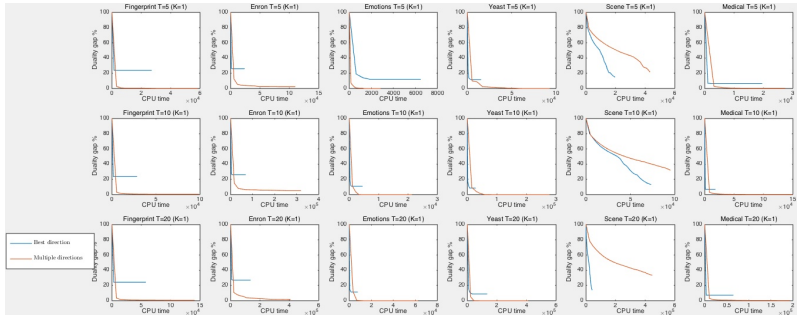
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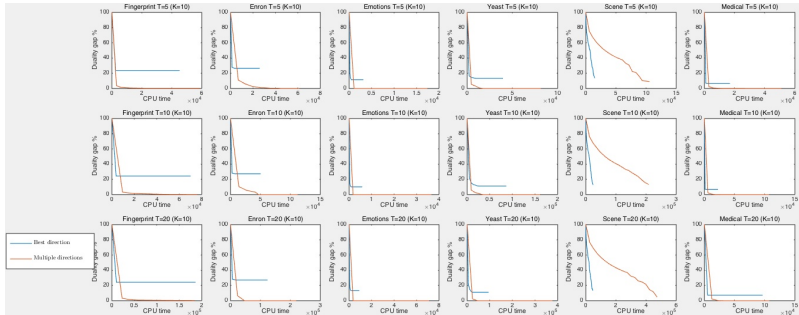
Experimental results - Relative duality gap

- ▶ Compare update with the best direction v.s. update with multiple directions
- ▶ Number of spanning trees $|T| = \{5, 10, 20\}$.
- ▶ Each spanning tree outputs top $\kappa = 1$ best direction.



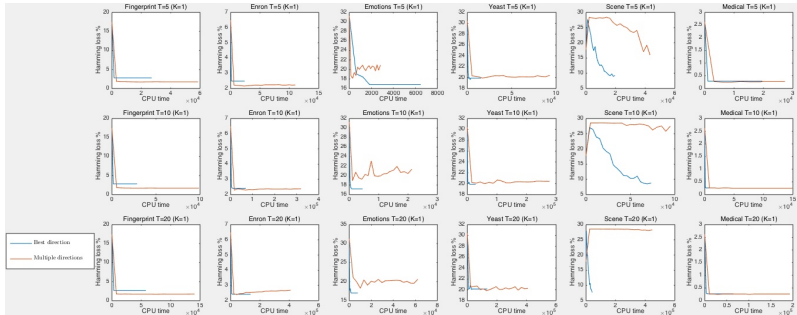
Experimental results - Relative duality gap

- ▶ Compare update with the best direction v.s. update with multiple directions
- ▶ Number of spanning trees $|T| = \{5, 10, 20\}$.
- ▶ Each spanning tree outputs top $\kappa = 10$ best directions.



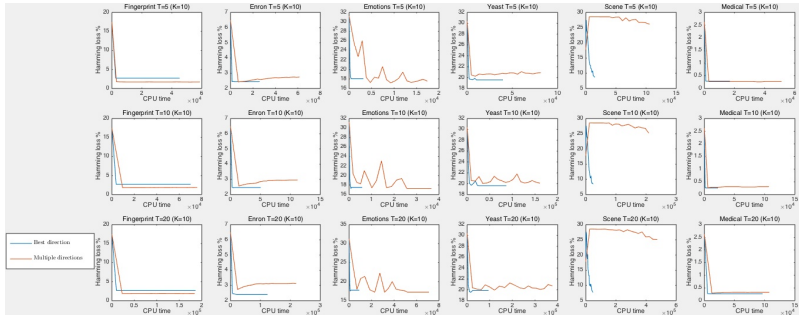
Experimental results - Hamming loss training

- ▶ Compare update with the best direction v.s. update with multiple directions
- ▶ Number of spanning trees $|T| = \{5, 10, 20\}$.
- ▶ Each spanning tree outputs top $\kappa = 1$ best direction.



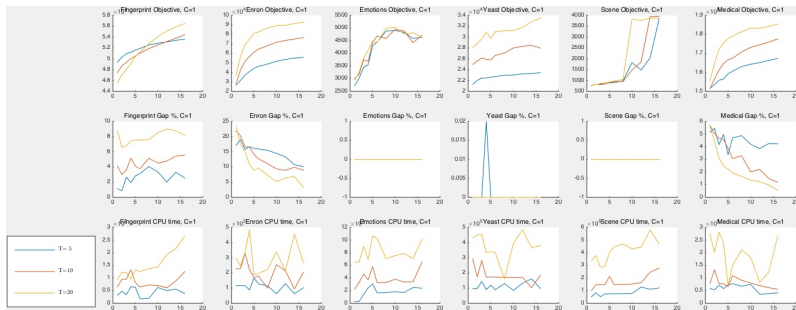
Experimental results - Hamming loss training

- ▶ Compare update with the best direction v.s. update with multiple directions
- ▶ Number of spanning trees $|T| = \{5, 10, 20\}$.
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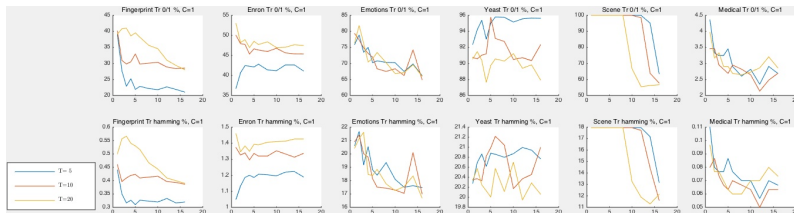
Experimental results

- ▶ Number of spanning trees $|T| = \{5, 10, 20\}$.
- ▶ Each spanning tree outputs $\kappa = \{1, \dots, 16\}$ best directions.



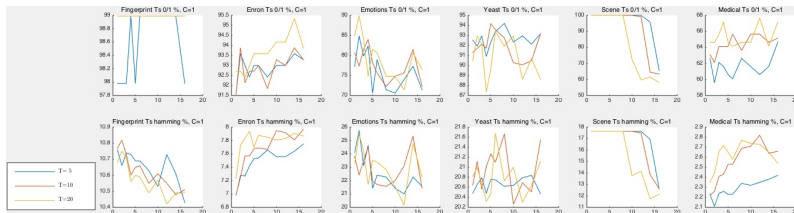
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Conclusions

