On PDE approach to some learning problems

Hong Zhang

Applied Math, Brown University

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Partial differential equations (PDEs) are used to describe many phenomenons. Many famous PDEs are from physics.

- Heat: heat equation, v tempature

$$\partial_t u = \sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2} = \Delta u$$

- Fluid mechanics: Naiver-Stokes equation

$$\partial_t u + (u \cdot \nabla) u - \Delta u = -\nabla p.$$

- Quantum mechanics, electrodynamics, image processing,. . . .

Probability and PDEs

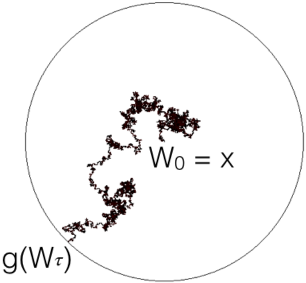

Figure: Brownian Motion

Let $g : S_1 \to \mathbb{R}$. Let W be a Brownian motion with $W_0 = x$. W hits $\partial\Omega$ at W_τ. Let $u(x) = \mathbb{E}^x[g(W_\tau)]$. Then

$$\Delta u = 0,$$

and

$$u = g \quad \text{on} \quad S_1.$$

Online Learning: Examples

Figure: Spam Detection

- emails arrive one by one
- spam detector makes prediction for each of them: spam or not
- sometimes get feedback from users
- make better detection next time

Online Learning: Example 2

Figure: Recommendation

- users visit Netflix /Amazon/...
- movies/products/... are recommended
- users click or not
- make better recommendation next time

Motivation

- PAC learning:
  - distribution fixed over time (training and test)
  - i.i.d. assumptions
  - a fixed predictor
- Online learning:
  - no distribution assumption
  - suitable for changing or adversarial environment
  - a natural model for many applications, mixing training and test
  - memory efficient

The expert problem

- Need: to make a decision every day
  - which stock to invest in
  - which route to drive home
  - ...
- Given: advice from a set of experts
  - financial adviser
  - Route 1, Route I-95, Route 295,...
  - ...
- Question: <u>How to combine advice and make smart choices?</u>

Gentle Start: a simple example

- Set up:
  - Predict an unknown sequence $y_1, y_2, \ldots,$ where $y_i \in \{0, 1\}$.
  - N experts provide advice $(f_{1,t}, f_{2,t}, \ldots, f_{N,t})$
  - A forecaster makes her guess $\hat{p}_t \in \{0, 1\}$ for y_t
  - True y_t is revealed and find if $\hat{p}_t = y_t$.
  - Know that one of the experts makes no mistakes.
- Goal: bound the number of the mistakes made by the forescaster

Halving Algorithm

Algorithm:

- at time t, expert i has weight w_i^t
- originally, $w_i^0 = 1, \forall i \in [1, N]$
- prediction according to the weighted majority
- update each wrong expert $w_i^t = 0$

Conclusion: the number of mistakes made by the forecaster is bounded by $\lfloor \log_2 N \rfloor$.

Proof

:

- let w_m be the sum of the weight of all experts after the forecaster has made m mistakes.
- Initially, $w_0 = N$.
- When the forecaster make her mth mistakes, at least half of the experts have been always correct make their first mistake

$$w_m \leq \frac{w_{m-1}}{2} \Longrightarrow w_m \leq \frac{w_0}{2^m}.$$

- $w_i = 1$ for some i

$$m \leq \lfloor \log_2 N \rfloor.$$

More general setting:

- Do not know there is one expert who makes no mistakes.
- Weighted Majority Algorithm:
  - at any time, expert i has weight w_i^t.
  - originally, $w_i^t = 1, \forall i \in [1, N]$.
  - prediction according to weighted majority.
  - weight of each wrong expert updated ($\beta \in (0, 1)$)

$$w_i^{t+1} \leftarrow w_i^t \beta.$$

- Theorem: let m_i^t be the number of mistakes made by expert i till time t and m^t be the number of mistakes made by the WM algorithm. Then

$$m_t \leq \frac{\log_2 N + m_i^t \log_2 \frac{1}{\beta}}{\log_2 \frac{2}{1+\beta}}.$$

  - $m_t \leq O(\log_2 N) + \text{constant} \times \text{best expert}.$

Remarks

:

- Linear dependence on the best expert
- Independent of the number of predictions
- Independent of the choice of sequence of outcomes
- Mild dependence on the number of the experts

General Setting

- For $t = 1$ to T do
  - receive instance $x_t \in X$ and advice $\hat{y}_{t,i} \in Y$, $i \in [1, N]$
  - predict $\hat{y}_t \in Y$
  - receive label $y_t \in Y$
  - incur loss $L(y_t, \hat{y}_t)$
- Objective: minimize regret, i.e., difference of total loss incurred and that of best exeprt:

$$Regret(T) = \sum_{t=1}^{T} L(y_t, \hat{y}_t) - \min_{i \in [1,N]} \sum_{t=1}^{T} L(y_t, \hat{y}_{t,i}).$$

- Goal: obtain sublinear regret to the best expert. (top 10 experts, any competitor, etc.)

$$\lim_{T \to \infty} \frac{Regret(T)}{T} = 0.$$

An Optimal Bound

- Exponential weighted average
  - weight update:
$$w_i^{t+1} \leftarrow w_i^t e^{-\eta_t L_{t,i}},$$
    where $L_{t,i}$ is total loss incurred by expert i till time t
  - prediction:
$$\hat{y}_t = \frac{\sum_{i=1}^N w_i^t y_{t,i}}{\sum_{i=1}^N w_i^t}.$$
- Theorem: Assume L is convex in its first argument and takes values in $[0, 1]$. Then for any T

$$Regret(T) \leq \frac{\sqrt{2}}{\sqrt{2}-1}\sqrt{(T/2)\log_2 N} + \sqrt{\log_2 N/2}.$$

Remarks on the theorem

- When the loss is between $[0, 1]$

$$Regret(T) = O(\sqrt{(\log N)T}).$$

- The bound can be improved if more information is known, e.g., if the loss functions are special, L_2 loss, L_1 loss,
- In general, the bound is asymptotically optimal.

Active research area, but connection to PDEs?

- ... who did what

A naive prediction problem (T.Cover, 1965)

- Predict $y_1, y_2, \ldots$, where $y_t \in \{-1, 1\}$ (stock goes up or down).
- Forecaster makes prediction $\hat{p}_t \in [-1, 1]$ (buy/sell $\hat{p}_1$ shares of stocks) according to algorithm $\hat{P}$.
- At each time t, investor loss (or gain) $-\hat{p}_t y_t$. Cumulative loss (or gain)

$$L(\hat{P}, y_T) := -\sum_{t=1}^{T} \hat{p}_t y_t$$

- Two naive experts: one always predicts 1 (always goes up); the other one always predicts -1(always goes down).

Cover proved

$$Regret(T) = \Theta(\sqrt{T}).$$

PDE approach

This problem was considered by Kohn using PDE technique.

- Instead of proposing an algorithm and then proving the asymptotic bound, PDE method focuses on the optimal strategy
- Typical Goal: minimize the regret with respect to the best performing expert.
- More general goal: minimize the worst case value of

  ϕ(regret with respect to expert "1", regret with respect to expert "-1")

  at time T. ("Typical goal" is $\phi(x_1, x_2) = \min\{x_1, x_2\}$.)

Two very simple experts

Essentially, the problem is an optimal control problem:

- state space:

$$(x_1, x_2) = (\text{regret w.r.t "1" expert, regret w.r.t "-1" expert}).$$

- control: prediction $\hat{p}_t \in [-1, 1]$ (investor's purchase bounded by 1)
- value function: $v(x, t) =$ optimal time T result, starting from relative regrets $x = (x_1, x_2)$ at time t.

Dynamical programming principle:

$$\begin{aligned} v(x_1, x_2, t) &= \min_{|p| \leq 1} \max_{b = \pm 1} v(\text{new position}, t+1) \\ &= \min_{|p| \leq 1} \max_{b = \pm 1} v(x_1 + b(1-p), x_2 - b(1+p), t+1) \end{aligned}$$

for $t < T$, with final-time condition $v(x, T) = \phi(x)$.

Dynamical Programming Principle

Recall: $(x_1, x_2) =$ (regret w.r.t. "'1" expert, regret w.r.t. "-1" expert), where regret = (investor's loss) - (expert's loss).

If the investor buys p shares and the market goes up, investor gains p, the "+" expert gains 1, the "-1" expert gains -1. So the state moves from (x_1, x_2) to $(x_1 + (1-p), x_2 + (-1-p))$.

Similarly, if investor buys p shares and market goes down, state move from (x_1, x_2) to $(x_1 - (1-p), x_2 - (-1-p))$.

Hence the dynamical programming principle:

$$v(x_1, x_2, t) = \min_{|p|\leq 1} \max_{b=\pm 1} v(x_1 + b(1-p), x_2 - b(1+p), t+1).$$

- Want to know how the regret accumulates as $T \to \infty$.
- To access this question, it is natural to rescale the problem and look for a continuum limit.
- Our scaling is like the passage from random walk to Brownian motion.
  - Let $\xi_1, \xi_2, \ldots$ be i.i.d. random variable with mean 0 and variance 1. For each n, define a continuous-time stochastic processes $\{W_n(t)\}$

$$W_n(t) = \frac{1}{\sqrt{n}} \sum_{1 \le j \le [nt]} \xi_j.$$

    Then $W_n(t) \to W_t$, where W_t is a standard Brownian motion.

- So we consider a scaled version of the problem: stock moves $\pm\epsilon$, time steps are ϵ^2, and the scaling of regret is ϵ^2. The value function is still the optimal time-T result. The dynamical programming principle becomes

$$w^{\epsilon}(x_1, x_2, t) = \min_{|p|\leq 1} \max_{b=\pm 1} w^{\epsilon}(x_1 + \epsilon b(1-p), x_2 - \epsilon b(1+p), t + \epsilon^2)$$

- We expect as $\epsilon \to 0$, $w^{\epsilon}(x,t) \to w(x,t)$ and w solves some PDEs.

The partial differential equation is the Hamilton-Jacobi-Bellman equation associate with the optimal control problem. Formal derivation

1. Use the Taylor's expansion to estimate

$$w(x_1 + \epsilon b(1-p), x_2 - \epsilon(1+p), t + \epsilon^2).$$

2. Investor choose p to make $O(\epsilon)$ terms vanishes; this gives

$$p = \frac{\partial_1 w - \partial_2 w}{\partial_1 w + \partial_2 w}.(\text{optimal strategy})$$

3. The $O(\epsilon^2)$ are insensitive to $b = \pm 1$; they give the nonlinear PDE

$$w_t + 2 < D^2 w \frac{\nabla^{\perp} w}{\partial_1 w + \partial_2 w}, \frac{\nabla^{\perp} w}{\partial_1 w + \partial_2 w} >= 0,$$

where $\nabla^{\perp} w = (\partial_2 w, -\partial_1 w)$. Prescribe the final value $w = \phi$ at T.

More detailed derivation of the PDE

Dynamical programming principle:

$$w^{\epsilon}(x_1, x_2, t) = \min_{|p|\le 1} \max_{b=\pm 1} w^{\epsilon}(x_1 + \epsilon b(1-p), x_2 - \epsilon b(1+p), t + \epsilon^2)$$

Taylor's expansion:

$$\begin{aligned}
&w(x_1 + \epsilon b(1-p), x_2 - \epsilon b(1+p), t + \epsilon^2) \\
&= w(x_1, x_2, t) + \epsilon b(1-p)\partial_1 w - \epsilon b(1+p)\partial_2 w + \partial_t w \epsilon^2 \\
&+ \frac{1}{2}\partial_{11} w \epsilon^2 b^2 (1-p)^2 - \partial_{12} w \epsilon^2 b^2 (1+p)(1-p) + \frac{1}{2}\partial_{22} w \epsilon^2 b^2 (1+p)^2.
\end{aligned}$$

After substitution and reorganization:

$$0 \approx \max_{|p|\leq 1} \min_{b=\pm 1} \{\epsilon b[(1-p)\partial_1 w - (1+p)\partial_2 w]$$

$$\epsilon^2 b^2 [\frac{1}{2}\partial_{11} w(1-p)^2 - \partial_{12} w(1-p)(1+p) + \frac{1}{2}\partial_{22} w(1+p)^2 + \partial_t w]\}$$

$O(\epsilon)$ terms vanishes and obtain

$$p = \frac{\partial_1 w - \partial_2 w}{\partial_1 w + \partial_2 w}.$$

Note: we expect $\partial_1 w > 0$ and $\partial_2 w > 0$, which implies $|p| \leq 1$.

Conclusion

Under some assumptions on the terminal data ϕ, we can show that

- solution exists under very general conditions on ϕ
- w is pretty regular or C^k for some $k \geq 0$
- $|\nabla w| \neq 0$
- $w_t < 0$

Convergence result:

- if ϕ smooth

$$\|w^\epsilon(x,t) - w(x,t)\| \leq C\epsilon,$$

  where C independent of t.

- if $\phi(x) = \max\{x_1, x_2\}$

$$\|w^\epsilon(x,t) - w(x,t)\| \leq C\epsilon \log \epsilon.$$

Application of the theory

Consider the terminal time is $t = 0$ and the equation is satisfied in $(-\infty, 0) \times \mathbb{R}^2$. Make an anstaz:

$$w(t,x,y) = \frac{x+y}{2} - \sqrt{-t}g(\frac{x-y}{\sqrt{-t}}).$$

Then g satisfies a Hermite ODE

$$g(z) - zg'(z) - 16g''(z) = 0,$$

which can be solved explicitly.

The optimal strategy

$$f = -2g'(\frac{x-y}{\sqrt{-t}}).$$

Discussion

- Stock prediction problem has a continuous-time limit. The PDE has a rather explicit solution.
- Problem is considered as a deterministic two-person game.
- One can consider more complex situation, e.g., more experts, more intelligent experts, etc.
- Dynamical programming principle can be hard to derive.

- The previous equation is the Hamilton-Jacobi-Bellman equation from stochastic control.
- In the discrete case, Markov decision process (MDP) is a discrete time stochastic control process. (Reinforcement Learning). The value function satisfies a discrete Bellman equation.
  - Given a policy π with infinite horizon for any state $x \in X$

$$V_\pi(x) = R(x, \pi(x)) + \gamma \sum_{x' \in X} P(x'|x, \pi(x)) V_\pi(x')$$

  - Optimal policy

$$V^*(x) = \max_{a \in A} \{R(x, a) + \gamma \sum_{x' \in X} P(x'|x, a) V_\pi(x')\}$$

- Continuous time and state: PDEs.
  - Dynamics of the state

$$dX_s = b(X_s, a_s)\, ds + \sigma(X_s, a_s)\, dW_s, \tag{1}$$

    where W_s is Brownian motion and a_s is control process.
  - Reward

$$R(t,x,a) = \mathbb{E}\Big[\int_t^T f(X_s^{t,x}, a_s)\, ds + g(X_T^{t,x})\Big],$$

    where $X^{t,x}$ is the solution of (1) starting from $X_t = x$.
  - Value function

$$v(t,x) = \sup_{a\in\mathcal{A}} R(t,x,a).$$

  - HJB equation

$$\frac{\partial v}{\partial t} + \sup_{a\in\mathcal{A}} (L_a v + f(x,a)) = 0 \quad \text{in}[0,T]\times\mathbb{R}^d,$$

    where L_a is the infinitesimal generator of the process (1).

Second-order Linear Elliptic PDEs

A classical example is

$$\Delta u(x) = \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} = 0.$$

Let W_t be a Brownian motion starting from x. Then

$$\lim_{t \to 0} \frac{E^x u(W_t) - u(x)}{t} = \frac{1}{2}\Delta u \quad \text{(by Itô's formula)}.$$

General diffusion process

$$L_a u = \frac{1}{2} Tr(\sigma \sigma^T D^2 u) = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij} D_{ij} u,$$

where $\{a_{ij}\}$ is some positive-definite matrix.

HJB equation

Fully nonlinear equation (v depends on (t, x))

$$\partial_t v + \sup_{a \in \mathcal{A}} (L_a v + f(x, a)) = 0,$$

where L_a is a second-order linear elliptic operator.

After deriving the HJB equation formally, the equation holds in a weak sense. The regularity of the solution is very low.

- Easy to find solution in the space of continuous function
- Equation holds in a very weak sense (viscosity solution).
- Important problem in the PDE community to prove the existence of classical solution.

Mathematical Result

Recall:

$$dX_s = b(X_s, a_s)\, ds + \sigma(X_s, a_s)\, dW_s.$$

Under some regularity assumption on the coefficients σ, b, then the solution to the HJB

$$\partial_t u + \sup_{a \in \mathcal{A}} \left(L_a u + f(x, a) \right) = 0,$$

is a classical solution. Rigorously, $u \in C^{2,\alpha}$ for some $\alpha > 0$.

The theoretical results helps to build numerical scheme to solve the HJB.

Jump processes and Nonlocal equations

Brownian motion is a continuous stochastic process. Consider more general process: Lévy processes (jump processes)

$$dX_s = b(X_s, a_s)\, ds + \sigma(X_s, a_s)\, dY_s,$$

where Y_s is a Lévy process. The infinitesimal generator of such processes is nonlocal

$$L_a u = \int_{\mathbb{R}^d} (u(x+y) + u(x-y) - 2u(x)) \frac{K_a(y)}{|y|^{d+\sigma}}\, dy,$$

and $\sigma \in (0, 2)$.

Theorem

Let v be a weak solution to

$$v_t + \sup_a (L_a v + f_a) = 0,$$

where L_a is a nonlocal operator defined previously. Then the solution v is a classical solution and derivatives of v are bounded.

Thank you.