16. Sean x, y dos vectores de $\mathbb{R}^n$ tales que $\|x\|_2 = \|y\|_2$. Demostrar que la elección $v := x - y$ conduce a una transformación de Householder $H := I - \frac{2vv^t}{\|v\|^2}$ tal que $Hx = y$ y $Hy = x$.

QVQ: $\begin{cases} Hx = y \\ Hy = x \end{cases}$ con $H = I - \dfrac{2vv^T}{\|v\|^2}$, $v = x - y$

Veamos $H(x+y)$ y $H(x-y)$.

$$\begin{aligned}
H(x+y) &= \left(I - \frac{2vv^T}{\|v\|^2}\right)(x+y) \\
&= (x+y) - \frac{2v}{\|v\|^2} \cdot (x-y)^T(x+y) \\
&= (x+y) - \frac{2v}{\|v\|^2} \cdot (x^Tx + x^Ty - y^Tx - y^Ty) \\
&= (x+y) - \frac{2v}{\|v\|^2} \cdot (\|x\|_2^2 + x^Ty - x^Ty - \|y\|_2^2) \\
&= (x+y) - \frac{2v}{\|v\|^2} \cdot 0 \\
&= x + y
\end{aligned}$$

$x^Ty = y^Tx \in \mathbb{R}$

$\|x\|_2 = \|y\|_2$

$$\begin{aligned}
H(x-y) &= \left(I - \frac{2vv^T}{\|v\|^2}\right)(x-y) \\
&= (x-y) - \frac{2v}{\|v\|^2} \cdot (x-y)^T(x-y) \\
&= (x-y) - \frac{2v}{\|v\|^2} \cdot \|x-y\|^2 \\
&= (x-y) - \frac{2v}{\|v\|^2} \cdot \|v\|^2 \\
&= (x-y) - 2(x-y) = x - y - 2x + 2y = -x + y
\end{aligned}$$

$$\begin{cases} H(X+Y) = X+Y \\ H(X-Y) = -X+Y \end{cases}$$

$$\Leftrightarrow \begin{cases} HX + HY = X+Y & 1 \\ HX - HY = -X+Y & 2 \end{cases}$$

$$\Leftrightarrow \begin{cases} HX + HY + HX - HY = 2HX = X+Y-X+Y = 2Y & 1+2 \\ HX + HY - HX + HY = 2HY = X+Y+X-Y = 2X & 1-2 \end{cases}$$

$$\Leftrightarrow \begin{cases} 2HX = 2Y \\ 2HY = 2X \end{cases}$$

$$\Leftrightarrow \begin{cases} HX = Y \\ HY = X \end{cases}$$