3. Sea $A \in \mathbb{R}^{2\times 2}$. Llamamos $d = \det(A)^2$ y $f = \|A\|_F^2$, siendo $\|A\|_F = \sqrt{\sum_{i,j} |a_{ij}|^2}$ la norma Frobenius de la matriz A. Demostrar que los valores singulares de A son de la forma:

$$\sqrt{\frac{f \pm \sqrt{f^2 - 4d}}{2}}$$

Los valores singulares de A son la raíz cuadrada de los autovalores de AA^T.

Por hipótesis

$$\det(AA^T) = \det(A)\cdot\det(A^T) = \det(A)\cdot\det(A) = \det(A)^2 \overset{\uparrow}{=} d$$

Notemos que AA^T es simétrica $\Rightarrow (AA^T)_{12} = (AA^T)_{21}$

$$\begin{aligned}
\det(AA^T - \lambda I) &= [(AA^T)_{11} - \lambda]\cdot[(AA^T)_{22} - \lambda] - (AA^T)_{12}^2 \\
&= \lambda^2 - \lambda\cdot[(AA^T)_{11} + (AA^T)_{22}] + (AA^T)_{11}\cdot(AA^T)_{22} - (AA^T)_{12}^2 \\
&= \lambda^2 - \lambda\cdot[(AA^T)_{11} + (AA^T)_{22}] + \det(AA^T) \\
&= \lambda^2 - \lambda\cdot[(AA^T)_{11} + (AA^T)_{22}] + d
\end{aligned}$$

Sea $b = (AA^T)_{11} + (AA^T)_{22}$.

$$\det(AA^T - \lambda I) = 0 \iff \lambda = \frac{1}{2}\cdot\left(b \pm \sqrt{b^2 - 4d}\right)$$

Luego los valores singulares de A son:

$$\sigma^2 = \lambda \iff \sigma = \sqrt{\lambda} \iff \sigma = \sqrt{\frac{1}{2}\cdot\left(b \pm \sqrt{b^2-4d}\right)}$$

Basta ver que $b = f \overset{?}{\iff} (AA^T)_{11} + (AA^T)_{22} = \|A\|_F^2$

$$AA^T = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11}^2 + a_{12}^2 & * \\ * & a_{21}^2 + a_{22}^2 \end{bmatrix}$$

$$\left.\begin{aligned} (AA^T)_{11} + (AA^T)_{22} &= a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2 \\ \|A\|_F^2 = \sum_{i,j} |a_{ij}|^2 &= a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2 \end{aligned}\right\} (AA^T)_{11} + (AA^T)_{22} = \|A\|_F^2$$