5. Sea A una matriz simétrica. Probar que la función $f(x) = \frac{(x^t Ax)^{\frac{1}{2}}}{2}$ es una norma vectorial en $\mathbb{R}^n$ si y sólo si A es definida positiva.

($\Leftarrow$) QVQ: A sdp $\Rightarrow$ f es norma vectorial

- $f(x) > 0 \quad \forall x \neq 0 \iff \frac{1}{2}(x^T A x)^{\frac{1}{2}} > 0 \quad \forall x \neq 0$

  $\downarrow$

  A sdp $\Rightarrow x^T A x > 0 \quad \forall x \neq 0$

- $x^T A x = 0 \iff x = 0$ porque A sdp

  $\Rightarrow (f(x) = 0 \iff x = 0)$

- $f(\alpha x) = \frac{1}{2}(\alpha x^T A \alpha x)^{\frac{1}{2}} = \frac{1}{2}(\alpha^2 x^T A x)^{\frac{1}{2}} = \frac{1}{2}(\alpha^2)^{\frac{1}{2}}(x^T A x)^{\frac{1}{2}}$

  $= |\alpha| \frac{1}{2}(x^T A x)^{\frac{1}{2}} = |\alpha| f(x)$

- QVQ: $f(x+y) \leq f(x) + f(y) \iff f(x+y)^2 \leq (f(x)+f(y))^2$

$$
\begin{aligned}
& f(x+y)^2 \\
&= \tfrac{1}{4}\,|(x+y)^T A (x+y)| \\
&= \tfrac{1}{4}\,|(x^T + y^T)(Ax + Ay)| \\
&= \tfrac{1}{4}\,|x^T A x + x^T A y + y^T A x + y^T A y| \\
&= \tfrac{1}{4}\,|x^T A x + 2x^T A y + y^T A y| \\
&= \tfrac{1}{4} x^T A x + \tfrac{1}{2}|x^T A y| + \tfrac{1}{4} y^T A y
\end{aligned}
$$

$$
\begin{aligned}
& (f(x)+f(y))^2 \\
&= f(x)^2 + 2f(x)f(y) + f(y)^2 \\
&= \tfrac{1}{4} x^T A x + \tfrac{1}{2}(x^T A x)^{\frac{1}{2}}(y^T A y)^{\frac{1}{2}} + \tfrac{1}{4} y^T A y
\end{aligned}
$$

$\frac{1}{4} x^T A x + \frac{1}{2}|x^T A y| + \frac{1}{4} y^T A y \leq \frac{1}{4} x^T A x + \frac{1}{2}(x^T A x)^{\frac{1}{2}}(y^T A y)^{\frac{1}{2}} + \frac{1}{4} y^T A y$

$\iff |x^T A y| \leq (x^T A x)^{\frac{1}{2}}(y^T A y)^{\frac{1}{2}}$

$\downarrow$

ejercicio 4

$\therefore$ f es una norma vectorial

$(\Rightarrow)$ QVQ: f es norma vectorial $\Rightarrow$ A es dp

A es dp $\Leftrightarrow$ $x^T A x > 0 \quad \forall x \neq 0$

$f(x) = \frac{1}{2}(x^T A x)^{\frac{1}{2}} > 0 \quad \forall x \neq 0$ porque f es una norma

$\Leftrightarrow \quad (x^T A x)^{\frac{1}{2}} > 0 \quad \Rightarrow \quad x^T A x > 0 \quad \forall x \neq 0$

Elevamos al cuadrado ambos lados que son ≥ 0

$\therefore$ A es dp.