17. Para cualquier $u, v \in \mathbb{R}^n$, sea $A = uv^t$.

a) Hallar $Im(A)$ y $Nu(A)$.

b) Probar que $A^2 = tr(A) \cdot A$.

$$\begin{bmatrix} \\ u \\ \\ \end{bmatrix} \begin{bmatrix} v^t \\ \hline A \\ \\ \end{bmatrix} \cdot \begin{bmatrix} \\ x \\ \\ \end{bmatrix} = \begin{bmatrix} \\ y \\ \\ \end{bmatrix}$$

a)

$Im(A) = \{ Ax : x \in \mathbb{R}^n \}$

$$Ax = uv^t x = \begin{bmatrix} u_1 v^t \\ \vdots \\ u_n v^t \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} u_1 v^t x \\ \vdots \\ u_n v^t x \end{bmatrix} = \begin{bmatrix} u_j \sum_{i=1}^{n} v_i^t x_i \\ \\ \end{bmatrix}$$

$Nu(A) = \{ x \in \mathbb{R}^n : Ax = 0 \}$

$x \in Nu(A) \iff Ax = 0$

$\iff uv^t x = 0 \qquad u \in \mathbb{R}^{n \times 1} \quad v^t \in \mathbb{R}^{1 \times n} \quad x \in \mathbb{R}^{n \times 1}$

$\iff u\alpha = 0 \qquad \text{con } \alpha = v^t x \in \mathbb{R}^n$

Caso $u = 0$

$A = uv^t = 0$ (matriz nula) y $u\alpha = 0 \ \forall \alpha \implies Nu(A) = \mathbb{R}^n$

Caso $u \neq 0$

$u\alpha = 0 \iff \alpha = 0 \iff v^t x = 0 \iff Nu(v^t)$

$\implies Nu(A) = Nu(v^t)$

b)

QVQ: $A^2 = tr(A) \cdot A$

Ejemplo 2x2

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11}^2 + a_{12}a_{21} & a_{11}a_{12} + a_{12}a_{22} \\ a_{21}a_{11} + a_{22}a_{21} & a_{21}a_{12} + a_{22}^2 \end{bmatrix}$$

$$tr(A) = a_{11} + a_{22}$$