Métodos Numéricos
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Factorización LU

DEPARTAMENTO DE COMPUTACION

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Factorización LU

Resolver varios sistemas de ecuaciones con Eliminación Gaussiana tiene un costo de $\mathcal{O}(n^3)$ por cada uno. ¿Cómo evitarlo?

$A \in R^{n \times n}$, $L \in R^{n \times n}$ triang. inferior, $U \in R^{n \times n}$ triang. superior

$$A = LU$$

¿Para qué resulta útil?

$$\begin{aligned} Ax &= b \\ LUx &= b \end{aligned}$$

Resolvemos en dos etapas:

$$\begin{aligned} Ly &= b \\ Ux &= y \end{aligned}$$

Dos sistemas triangulares con costo $\mathcal{O}(n^2)$!

Sea $A \in R^{n \times n}$. Supongamos que aplicamos eliminación Gaussiana y se verifica que $a_{ii}^{i-1} \neq 0$ para todo $i = 1, \ldots, n-1$

Sea la matriz elemental (tipo 2)

$$E = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -m_{21} & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
-m_{21} & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
a_{11}^0 & a_{12}^0 & \cdots & a_{1j}^0 & \cdots & a_{1n}^0 \\
a_{21}^0 & a_{22}^0 & \cdots & a_{2j}^0 & \cdots & a_{2n}^0 \\
a_{31}^0 & a_{32}^0 & \cdots & a_{3j}^0 & \cdots & a_{3n}^0 \\
\vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
a_{n1}^0 & a_{n2}^0 & \cdots & a_{nj}^0 & \cdots & a_{nn}^0
\end{bmatrix}
=
$$

$$
\begin{bmatrix}
a_{11}^0 & a_{12}^0 & \cdots & a_{1j}^0 & \cdots & a_{1n}^0 \\
a_{21}^0 - m_{21}a_{11}^0 & a_{22}^0 - m_{21}a_{12}^0 & \cdots & a_{2j}^0 - m_{21}a_{1j}^0 & \cdots & a_{2n}^0 - m_{21}a_{1n}^0 \\
a_{31}^0 & a_{32}^0 & \cdots & a_{3j}^0 & \cdots & a_{3n}^0 \\
\vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
a_{n1}^0 & a_{n2}^0 & \cdots & a_{nj}^0 & \cdots & a_{nn}^0
\end{bmatrix}
$$

$$F_2 - m_{21}F_1!$$

Factorización LU

$$M^1 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ -m_{21} & 1 & 0 & \cdots & 0 & \cdots & 0 \\ -m_{31} & 0 & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ -m_{i1} & 0 & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots & \vdots \\ -m_{n1} & 0 & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$$

$$
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & \cdots & 0\\
-m_{21} & 1 & 0 & \cdots & 0 & \cdots & 0\\
-m_{31} & 0 & 1 & \cdots & 0 & \cdots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots\\
-m_{i1} & 0 & 0 & \cdots & 1 & \cdots & 0\\
\vdots & \vdots & \vdots & \cdots & \vdots & \ddots & \vdots\\
-m_{n1} & 0 & 0 & \cdots & 0 & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
a_{11}^0 & a_{12}^0 & \cdots & a_{1j}^0 & \cdots & a_{1n}^0\\
a_{21}^0 & a_{22}^0 & \cdots & a_{2j}^0 & \cdots & a_{2n}^0\\
a_{31}^0 & a_{32}^0 & \cdots & a_{3j}^0 & \cdots & a_{3n}^0\\
\vdots & \vdots & \cdots & \vdots & \cdots & \vdots\\
a_{i1}^0 & a_{i2}^0 & \cdots & a_{ij}^0 & \cdots & a_{in}^0\\
\vdots & \vdots & \cdots & \vdots & \cdots & \vdots\\
a_{n1}^0 & a_{n2}^0 & \cdots & a_{nj}^0 & \cdots & a_{nn}^0
\end{bmatrix}
$$

$$
= \begin{bmatrix}
a_{11}^0 & a_{12}^0 & \cdots & a_{1j}^0 & \cdots & a_{1n}^0\\
a_{21}^0 - m_{21}a_{11}^0 & a_{22}^0 - m_{21}a_{12}^0 & \cdots & a_{2j}^0 - m_{21}a_{1j}^0 & \cdots & a_{2n}^0 - m_{21}a_{1n}^0\\
a_{31}^0 - m_{31}a_{11}^0 & a_{32}^0 - m_{31}a_{12}^0 & \cdots & a_{3j}^0 - m_{31}a_{1j}^0 & \cdots & a_{3n}^0 - m_{31}a_{1n}^0\\
\vdots & \vdots & \cdots & \vdots & \cdots & \vdots\\
a_{i1}^0 - m_{i1}a_{11}^0 & a_{i2}^0 - m_{i1}a_{12}^0 & \cdots & a_{ij}^0 - m_{i1}a_{1j}^0 & \cdots & a_{in}^0 - m_{i1}a_{1n}^0\\
\vdots & \vdots & \cdots & \vdots & \cdots & \vdots\\
a_{n1}^0 - m_{n1}a_{11}^0 & a_{n2}^0 - m_{n1}a_{12}^0 & \cdots & a_{nj}^0 - m_{n1}a_{1j}^0 & \cdots & a_{nn}^0 - m_{n1}a_{1n}^0
\end{bmatrix}
$$

Primer paso de Eliminación Gaussiana

$$M^i = \begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & -m_{i+1i} & 1 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -m_{ni} & 0 & \cdots & 1 \end{bmatrix}$$

Factorización LU

$$
\begin{bmatrix}
1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & & \vdots & \cdots & \vdots \\
0 & \cdots & 1 & 0 & \cdots & 0 \\
0 & \cdots & -m_{i+1i} & 1 & \cdots & 0 \\
\vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & -m_{ni} & 0 & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
a_{11}^{i-1} & \cdots & a_{1i}^{i-1} & \cdots & a_{1n}^{i-1} \\
\vdots & \cdots & \vdots & \cdots & \vdots \\
0 & \cdots & a_{ii}^{i-1} & \cdots & a_{in}^{i-1} \\
0 & \cdots & a_{i+1i}^{i-1} & \cdots & a_{i+1n}^{i-1} \\
\vdots & \cdots & \vdots & \cdots & \vdots \\
0 & \cdots & a_{ni}^{i-1} & \cdots & a_{nn}^{i-1}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
a_{11}^{i-1} & a_{12}^{i-1} & \cdots & a_{1i}^{i-1} & \cdots & a_{1n}^{i-1} \\
0 & a_{22}^{i-1} & \cdots & a_{2i}^{i-1} & \cdots & a_{2n}^{i-1} \\
\vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & a_{ii}^{i-1} & \cdots & a_{in}^{i-1} \\
0 & 0 & \cdots & a_{i+1i}^{i-1} - m_{i+1i} a_{ii}^{i-1} & \cdots & a_{i+1n}^{i-1} - m_{i+1i} a_{in}^{i-1} \\
\vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & a_{ni}^{i-1} - m_{ni} a_{ii}^{i-1} & \cdots & a_{nn}^{i-1} - m_{ni} a_{in}^{i-1}
\end{bmatrix}
$$

i-ésimo paso de Eliminación Gaussiana!

Factorización LU

Método de Eliminación Gaussiana

$$M^{n-1}M^{n-2}\cdots M^{1}A = U \text{ con } U \text{ triangular superior}$$

$$\text{con } M^{i} = \begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & -m_{i+1i} & 1 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -m_{ni} & 0 & \cdots & 1 \end{bmatrix}$$

IMPORTANTE: asumimos $a_{ii}^{i-1} \neq 0 \forall i = 1, \ldots, n$

Factorización LU

Propiedades de M^i

$$M^i = \begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & -m_{i+1i} & 1 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -m_{ni} & 0 & \cdots & 1 \end{bmatrix} = I - \begin{bmatrix} 0 \\ \vdots \\ m_{i+1i} \\ \vdots \\ m_{ni} \end{bmatrix} \begin{bmatrix} 0 & \cdots & 1 & \cdots & 0 \end{bmatrix}$$

$$M^i = I - m_i^t e_i$$

con $m_i = (0, \ldots, m_{i+1i}, \cdots, m_{ni})$ y e_i el i-ésimo vector canónico.

Propiedades de M^i

- M^i es triangular inferior.
- M^i es inversible.

$$(I - m_i^t e_i)(I + m_i^t e_i) = I + m_i^t e_i - m_i^t e_i - m_i^t e_i m_i^t e_i = I - m_i^t e_i m_i^t e_i$$

pero $e_i m_i^t = \begin{bmatrix} 0 & \cdots & 1 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ m_{i+1i} \\ \vdots \\ m_{ni} \end{bmatrix} = 0$

Entonces $(I - m_i^t e_i)(I + m_i^t e_i) = I \Longrightarrow (I - m_i^t e_i)^{-1} = I + m_i^t e_i$

Factorización LU

$$M^{n-1}M^{n-2}\cdots M^{1}A = U$$

$$A = (M^{1})^{-1}(M^{2})^{-1}\dots(M^{n-1})^{-1}U$$

$$A = (I + m_1^t e_1)(I + m_2^t e_2)\cdots(I + m_{n-1}^t e_{n-1})U$$

Observando que $m_i^t e_i m_j^t e_j = 0$ si $i < j$

$$A = (I + m_1^t e_1 + m_2^t e_2 + \cdots + m_{n-1}^t e_{n-1})U$$

$$
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1i} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2i} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\
a_{i1} & a_{i2} & \cdots & a_{ii} & \cdots & a_{in} \\
\vdots & \vdots & \cdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{ni} & \cdots & a_{nn}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
1 & 0 & \cdots & 0 & \cdots & 0 \\
m_{21} & 1 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\
m_{i1} & m_{i2} & \cdots & 1 & \cdots & 0 \\
\vdots & \vdots & \cdots & \vdots & \ddots & \vdots \\
m_{n1} & m_{n2} & \cdots & m_{ni} & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
u_{11} & u_{12} & \cdots & u_{1i} & \cdots & u_{1n} \\
0 & u_{22} & \cdots & u_{2i} & \cdots & u_{2n} \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & u_{ii} & \cdots & u_{in} \\
\vdots & \vdots & \cdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & u_{nn}
\end{bmatrix}
$$

Ejemplo

$$\begin{bmatrix} 2 & 1 & -1 & 3 \\ -2 & 0 & 0 & 0 \\ 4 & 1 & -2 & 4 \\ -6 & -1 & 2 & -3 \end{bmatrix} \longrightarrow \begin{matrix} F_2 - (-1)F_1 \\ F_3 - (2)F_1 \\ F_4 - (-3)F_1 \end{matrix} \longrightarrow \begin{bmatrix} 2 & 1 & -1 & 3 \\ \color{red}{-1} & 1 & -1 & 3 \\ \color{red}{2} & -1 & 0 & -2 \\ \color{red}{-3} & 2 & -1 & 6 \end{bmatrix}$$

$$\longrightarrow \begin{matrix} F_3 - (-1)F_2 \\ F_4 - (2)F_2 \end{matrix} \longrightarrow \begin{bmatrix} 2 & 1 & -1 & 3 \\ \color{red}{-1} & 1 & -1 & 3 \\ \color{red}{2} & \color{red}{-1} & -1 & 1 \\ \color{red}{-3} & \color{red}{2} & 1 & 0 \end{bmatrix} \longrightarrow F_4 - (-1)F_3 \longrightarrow \begin{bmatrix} 2 & 1 & -1 & 3 \\ \color{red}{-1} & 1 & -1 & 3 \\ \color{red}{2} & \color{red}{-1} & -1 & 1 \\ \color{red}{-3} & \color{red}{2} & \color{red}{-1} & 1 \end{bmatrix}$$

$$A = LU$$

$$\begin{bmatrix} 2 & 1 & -1 & 3 \\ -2 & 0 & 0 & 0 \\ 4 & 1 & -2 & 4 \\ -6 & -1 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ -3 & 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 & 3 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Factorización LU

- LU asociada a Eliminación Gaussiana sin necesidad de intercambio de filas. Se verifica que $l_{ii} = 1$ para todo $i = 1, \ldots, n$.
- No toda matriz tiene factorización LU. Por ejemplo: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- Si $A \in R^{n \times n}$ es no singular y existe factorización LU, entonces es única.
- Si $A \in R^{n \times n}$ tiene todas sus submatrices principales no singulares, entonces tiene factorización LU.
- Si $A \in R^{n \times n}$ es estrictamente diagonal dominante, entonces tiene factorización LU.

Factorización PLU

Existe algún $a_{ii}^{i-1} = 0$

- La Eliminación Gaussiana puede continuar mediante permutaciones de filas.
- Se obtiene factorización LU de la matriz original *permutada*.

Factorización PLU

Ejemplo

$$\begin{bmatrix} 0 & 2 & 6 & 4 \\ 4 & 6 & 1 & 5 \\ 2 & 3 & 6 & 4 \\ 6 & 8 & -7 & 2 \end{bmatrix} \text{ Inicialamos } P = I = [1,2,3,4]$$

$a_{11}^0 = 0.$ Permutamos $P = [3,2,1,4]$

$$\longrightarrow \begin{bmatrix} 2 & 3 & 6 & 4 \\ 4 & 6 & 1 & 5 \\ 0 & 2 & 6 & 4 \\ 6 & 8 & -7 & 2 \end{bmatrix} \longrightarrow \begin{matrix} F_2 - (2)F_1 \\ F_3 - (0)F_1 \\ F_4 - (3)F_1 \end{matrix} \longrightarrow \begin{bmatrix} 2 & 3 & 6 & 4 \\ \color{red}{2} & 0 & -11 & -3 \\ \color{red}{0} & 2 & 6 & 4 \\ \color{red}{3} & -1 & -25 & -10 \end{bmatrix}$$

$a_{22}^1 = 0.$ Permutamos $P = [3,1,2,4]$

$$\longrightarrow \begin{bmatrix} 2 & 3 & 6 & 4 \\ \color{red}{0} & 2 & 6 & 4 \\ \color{red}{2} & 0 & -11 & -3 \\ \color{red}{3} & -1 & -25 & -10 \end{bmatrix} \longrightarrow \begin{matrix} F_3 - (0)F_2 \\ F_4 - (-\frac{1}{2})F_1 \end{matrix} \longrightarrow \begin{bmatrix} 2 & 3 & 6 & 4 \\ \color{red}{0} & 2 & 6 & 4 \\ \color{red}{2} & \color{red}{0} & -11 & -3 \\ \color{red}{3} & \color{red}{-\frac{1}{2}} & -22 & -8 \end{bmatrix}$$

$$\longrightarrow F_4 - (2)F_3 \longrightarrow \begin{bmatrix} 2 & 3 & 6 & 4 \\ \color{red}{0} & 2 & 6 & 4 \\ \color{red}{2} & \color{red}{0} & -11 & -3 \\ \color{red}{3} & \color{red}{-\frac{1}{2}} & \color{red}{2} & -2 \end{bmatrix}$$

Factorización PLU-Ejemplo

$$P = [3, 1, 2, 4] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & -\frac{1}{2} & 2 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 2 & 3 & 6 & 4 \\ 0 & 2 & 6 & 4 \\ 0 & 0 & -11 & -3 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$$PA = LU$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 6 & 4 \\ 4 & 6 & 1 & 5 \\ 2 & 3 & 6 & 4 \\ 6 & 8 & -7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & -\frac{1}{2} & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 6 & 4 \\ 0 & 2 & 6 & 4 \\ 0 & 0 & -11 & -3 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Factorización LU: bibliografía

Recomendamos consultar la numerosa bibliografía existente sobre el tema. Algunas sugerencias:

- Análisis numérico, Richard L. Burden, J. Douglas Faires, International Thomson Editores, 2002.
- Matrix Computations, Gene H. Golub, Charles F. Van Loan, JHU Press, 2013.
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- Numerical Linear Algebra, Lloyd N. Trefethen, SIAM, 1997.