4. Calcular la inversa A^{-1} de $A = \begin{pmatrix} 1 & -1 & 4 \\ 2 & -3 & 6 \\ 2 & -5 & 4 \end{pmatrix}$ de las siguientes maneras:

a) Resolviendo el sistema matricial $AX = I$ por pivoteo parcial.

b) Calculando la factorización LU de A, calculando las inversas de L y U, y aplicando la identidad $A^{-1} = U^{-1}L^{-1}$.

c) Calculando la factorización LU de A y resolviendo los sistemas $Ax_i = e_i$, $i = 1, \ldots, n$, con $n = 3$.

a) Me parece que no es lo que pedía el enunciado.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 4 & 1 & 0 & 0 \\ 2 & -3 & 6 & 0 & 1 & 0 \\ 2 & -5 & 4 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{c} \\ F_2 - 2F_1 \\ F_3 - 2F_1 \end{array} \quad \left[\begin{array}{ccc|ccc} 1 & -1 & 4 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & -3 & -4 & -2 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 4 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 2 & 4 & -3 & 1 \end{array}\right]$$

$$A \cdot col_1(X) = e_1 \iff \begin{cases} X_{11} - X_{21} + 4X_{31} = 1 \\ -X_{21} - 2X_{31} = -2 \\ 2X_{31} = 4 \end{cases}$$

$$\Rightarrow \quad X_{31} = 4/2 = 2$$

$$\Rightarrow \quad -X_{21} - 2 \cdot 2 = -2$$

$$\Rightarrow \quad -X_{21} = -2 + 4 \quad \Rightarrow \quad X_{21} = -2$$

$$\Rightarrow \quad X_{11} + 2 + 4 \cdot 2 = 1 \quad \Rightarrow \quad X_{11} = -9$$

$$\Rightarrow col_1(X) = (-9, -2, 2)$$

$$A \cdot col_2(X) = e_2 \quad \Leftrightarrow \quad \begin{cases} x_{12} - x_{22} + 4x_{32} = 0 \\ \quad\ \ - x_{22} - 2x_{32} = 1 \\ \qquad\qquad\ \ 2x_{32} = -3 \end{cases}$$

$$\Rightarrow \quad x_{32} = -3/2$$

$$\Rightarrow \quad -x_{22} - 2\left(-\tfrac{3}{2}\right) = 1 \quad \Rightarrow \quad x_{22} = 2$$

$$\Rightarrow \quad x_{12} - 2 + 4 \cdot \left(-\tfrac{3}{2}\right) = 0 \quad \Rightarrow \quad x_{12} = 8$$

$$\Rightarrow \quad col_2(X) = (8, 2, -3/2)$$

$$A \cdot col_3(x) = e_3 \quad \Leftrightarrow \quad \begin{cases} x_{13} - x_{23} + 4x_{33} = 0 \\ \quad\ \ - x_{23} - 2x_{33} = 0 \\ \qquad\qquad\ \ 2x_{33} = 1 \end{cases}$$

$$\Rightarrow \quad x_{33} = \tfrac{1}{2}$$

$$\Rightarrow \quad -x_{23} - 2 \cdot \tfrac{1}{2} = 0 \quad \Rightarrow \quad x_{23} = -1$$

$$\Rightarrow \quad x_{13} - (-1) + 4 \cdot \tfrac{1}{2} = 0 \quad \Rightarrow \quad x_{13} = -3$$

$$\Rightarrow \quad col_3(x) = (-3, -1, \tfrac{1}{2})$$

$$X = \begin{bmatrix} -9 & 8 & -3 \\ -2 & 2 & -1 \\ 2 & -3/2 & \tfrac{1}{2} \end{bmatrix} = A^{-1}$$

b)

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & -1 & 4 \\ 0 & -1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

Buscamos L^{-1}

$$Lx = e_1 \iff \begin{cases} x_1 = 1 \\ 2x_1 + x_2 = 0 \\ 2x_1 + 3x_2 + x_3 = 0 \end{cases}$$

$$\Rightarrow \quad 2 + x_2 = 0 \quad \Rightarrow x_2 = -2$$
$$\Rightarrow \quad 2 + 3(-2) + x_3 = 0 \quad \Rightarrow x_3 = 4$$

$\Rightarrow \operatorname{col}_1(L^{-1}) = (1,-2,4)$

$$Lx = e_2 \iff \begin{cases} x_1 = 0 \\ 2x_1 + x_2 = 1 \\ 2x_1 + 3x_2 + x_3 = 0 \end{cases}$$

$$\Rightarrow \quad x_1 = 0 \quad \Rightarrow x_2 = 1 \quad \Rightarrow x_3 = -3$$

$\Rightarrow \operatorname{col}_2(L^{-1}) = (0,1,-3)$

$$Lx = e_3 \iff \begin{cases} x_1 & = 0 \\ 2x_1 + x_2 & = 0 \\ 2x_1 + 3x_2 + x_3 & = 1 \end{cases}$$

$$\Rightarrow \quad x_1 = 0 \Rightarrow x_2 = 0 \Rightarrow x_3 = 1$$

$\Rightarrow \mathrm{col}_3(L^{-1}) = (0,0,1)$

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -3 & 1 \end{bmatrix}$$

Buscamos U^{-1}

$$Ux = e_1 \iff \begin{cases} x_1 - x_2 + 4x_3 & = 1 \\ -x_2 - 2x_3 & = 0 \\ 2x_3 & = 0 \end{cases}$$

$\Rightarrow \mathrm{cols}_1(U^{-1}) = (1,0,0)$

$$Ux = e_2 \iff \begin{cases} x_1 - x_2 + 4x_3 & = 0 \\ -x_2 - 2x_3 & = 1 \\ 2x_3 & = 0 \end{cases}$$

$\Rightarrow \mathrm{cols}_2(U^{-1}) = (-1,-1,0)$

$$Ux = e_3 \iff \begin{cases} x_1 - x_2 + 4x_3 = 0 \\ \quad\;\; - x_2 - 2x_3 = 0 \\ \qquad\qquad\; 2x_3 = 1 \end{cases}$$

$$\Rightarrow \text{cols}_3(U^{-1}) = (-3, -1, \tfrac{1}{2})$$

$$U^{-1} = \begin{bmatrix} 1 & -1 & -3 \\ 0 & -1 & -1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$A = LU \iff AU^{-1} = LUU^{-1} = L \iff AU^{-1}L^{-1} = LL^{-1} = I$$
$$\iff A^{-1}AU^{-1}L^{-1} = A^{-1} \iff U^{-1}L^{-1} = A^{-1}$$

$$A^{-1} = U^{-1}L^{-1} = \begin{bmatrix} 1 & -1 & -3 \\ 0 & -1 & -1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -3 & 1 \end{bmatrix} = \begin{bmatrix} -9 & 8 & -3 \\ -2 & 2 & -1 \\ 2 & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

c)

Resolvemos los sistemas:

$LUx = e_i \quad \forall i = 1,2,3 \quad \Longleftrightarrow \quad Ux = L^{-1}e_i \quad \forall i = 1,2,3$

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -3 & 1 \end{bmatrix} \qquad \begin{aligned} L^{-1}e_1 &= (1,-2,4) \\ L^{-1}e_2 &= (0,1,-3) \\ L^{-1}e_3 &= (0,0,1) \end{aligned}$$

$$U = \begin{bmatrix} 1 & -1 & 4 \\ 0 & -1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$Ux = L^{-1}e_1 \iff \begin{cases} x_1 - x_2 + 4x_3 = 1 \\ \quad -x_2 - 2x_3 = -2 \\ \quad\quad 2x_3 = 4 \end{cases}$$

$\Rightarrow x_3 = 2$

$\Rightarrow x_2 = -(-2+2\cdot 2) = -2$

$\Rightarrow x_1 = 1-2-4\cdot 2 = -9$

$$Ux = L^{-1}e_2 \iff \begin{cases} x_1 - x_2 + 4x_3 = 0 \\ \quad -x_2 - 2x_3 = 1 \\ \quad\quad 2x_3 = -3 \end{cases}$$

$\Rightarrow x_3 = -3/2$

$\Rightarrow x_2 = -(1+2(-3/2)) = 2$

$\Rightarrow x_1 = 2-4(-3/2) = 8$

$$Ux = L^{-1}e_3 \iff \begin{cases} x_1 - x_2 + 4x_3 = 0 \\ \quad -x_2 - 2x_3 = 0 \\ \quad\quad 2x_3 = 1 \end{cases}$$

$\Rightarrow x_3 = 1/2$

$\Rightarrow x_2 = -2\,\tfrac{1}{2} = -1$

$\Rightarrow x_1 = -1-4\,\tfrac{1}{2} = -3$

$LUx_1 = e_1 \iff x_1 = (-9, -2, 2)$

$LUx_2 = e_2 \iff x_2 = (8, 2, -\frac{3}{2})$

$LUx_3 = e_3 \iff x_3 = (-3, -1, \frac{1}{2})$

$$LUX = I \iff X = \begin{bmatrix} | & | & | \\ x_1 & x_2 & x_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} -9 & 8 & -3 \\ -2 & 2 & -1 \\ 2 & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} = A^{-1}$$