10. Para las siguientes matrices $A \in \mathbb{R}^{m\times n}$ hallar $Nu(A)$, $Im(A)$, su rango fila, su rango columna y comprobar que $n = dim(Nu(A)) + dim(Im(A))$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & 3 & 4 \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{4\times 3} \quad A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 7 \\ 0 & 6 & 30 \end{pmatrix} \in \mathbb{R}^{3\times 3}$$

$A^{m\times n}$

$Nu(A) = \{ X \in \mathbb{R}^n : Ax = 0 \}$

$Im(A) = \{ Y \in \mathbb{R}^m : \exists x \in \mathbb{R}^n \text{ tq } Ax = Y \}$

$A \in \mathbb{R}^{4\times 3}$

Buscamos $x \in \mathbb{R}^3$ tal que $Ax = 0$.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 1 & 2 & 0 \\ 1 & 3 & 4 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\substack{F_2 - 2F_1 \\ F_3 - F_1}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{3F_3 + F_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -4 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{F_4 + F_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -4 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ -3x_2 - 4x_3 &= 0 \\ -x_3 &= 0 \end{aligned}$$

$$\Downarrow$$

$$x_1 = x_2 = x_3 = 0$$

$Ax = 0 \iff x = 0 \quad \therefore \quad Nu(A) = \{0\}$

Buscamos $Y \in \mathbb{R}^4$ en función de $X \in \mathbb{R}^3$ tal que $Ax = Y$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = Y \iff \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 2x_1 + x_2 + 2x_3 \\ x_1 + 3x_2 + 4x_3 \\ x_3 \end{bmatrix} = Y \iff$$

$$\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 2 \\ 1 \\ 3 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 3 \\ 2 \\ 4 \\ 1 \end{bmatrix} x_3 = Y \iff$$

$$Y = x_1 \cdot (1,2,1,0) + x_2 \cdot (2,1,3,0) + x_3 \cdot (3,2,4,1)$$

$\therefore$ $Im(A) = \langle (1,2,1,0), (2,1,3,0), (3,2,4,1) \rangle$

$Nu(A) = \{0\} \iff \#cols(A)$ LI $\Rightarrow$ $Rango_{col}(A) = 3$

$Rango_{Fila}(A) = 3$ pues filas 1,2 y 3 son LI

$\dim(Nu(A)) = 0$

$\dim(Im(A)) = 3$

$n = 0 + 3 = 3$

$A \in \mathbb{R}^{3\times 3}$

Buscamos $x \in \mathbb{R}^3$ tal que $Ax = 0$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 3 & 7 & 0 \\ 0 & 6 & 30 & 0 \end{array}\right] \xrightarrow{F_2 - 2F_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 6 & 30 & 0 \end{array}\right] \xrightarrow{F_3 - 6F_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 + x_2 + x_3 = 0$$
$$x_2 + 5x_3 = 0 \quad \Rightarrow \quad x_2 = -5x_3$$
$$\Rightarrow \quad x_1 - 5x_3 + x_3 = 0 \quad \Leftrightarrow \quad x_1 = 4x_3$$

$$Ax = 0 \quad \Leftrightarrow \quad x = \alpha \cdot (4, -5, 1) \quad \alpha \in \mathbb{R}$$

$\therefore$ $Nu(A) = \langle (4, -5, 1) \rangle$

Buscamos $y \in \mathbb{R}^3$ en función de $x \in \mathbb{R}^3$ tal que $Ax = y$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 7 \\ 0 & 6 & 30 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = y \quad \Leftrightarrow$$

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 7 \\ 30 \end{bmatrix} x_3 = y \quad \Leftrightarrow$$

$$y = x_1 \cdot (1, 2, 0) + x_2 \cdot (1, 3, 6) + x_3 \cdot (1, 7, 30)$$

Veamos si esos 3 vectores son LI.

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 1 & 3 & 6 & 0 \\ 1 & 7 & 30 & 0 \end{array}\right] \xrightarrow[F_2-F_1]{F_3-F_1} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 5 & 30 & 0 \end{array}\right] \xrightarrow{F_3-5F_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

No eran LI. Pero $(1,2,0)$ y $(1,3,6)$ sí son LI.

$\therefore \mathrm{Im}(A) = \langle (1,2,0), (1,3,6) \rangle$

$\dim(\mathrm{Nu}(A)) = 1$

$\dim(\mathrm{Im}(A)) = 2$

$n = 1 + 2 = 3$