

4. Calcular la inversa A^{-1} de $A = \begin{pmatrix} 1 & -1 & 4 \\ 2 & -3 & 6 \\ 2 & -5 & 4 \end{pmatrix}$ de las siguientes maneras:

- Resolviendo el sistema matricial $AX = I$ por pivoteo parcial.
- Calculando la factorización LU de A , calculando las inversas de L y U , y aplicando la identidad $A^{-1} = U^{-1}L^{-1}$.
- Calculando la factorización LU de A y resolviendo los sistemas $Ax_i = e_i$, $i = 1, \dots, n$, con $n = 3$.

a) No es lo que pedía el enunciado. Hay que armar el sistema de 9×9 .

$$\begin{bmatrix} 1 & -1 & 4 & | & 1 & 0 & 0 \\ 2 & -3 & 6 & | & 0 & 1 & 0 \\ 2 & -5 & 4 & | & 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} \\ F_2 - 2F_1 \\ F_3 - 2F_1 \end{matrix} \quad \begin{bmatrix} 1 & -1 & 4 & | & 1 & 0 & 0 \\ 0 & -1 & -2 & | & -2 & 1 & 0 \\ 0 & -3 & -4 & | & -2 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 4 & | & 1 & 0 & 0 \\ 0 & -1 & -2 & | & -2 & 1 & 0 \\ 0 & 0 & 2 & | & 4 & -3 & 1 \end{bmatrix}$$

$$A \cdot \text{col}_1(x) = e_1 \Leftrightarrow \begin{cases} x_{11} - x_{21} + 4x_{31} = 1 \\ -x_{21} - 2x_{31} = -2 \\ 2x_{31} = 4 \end{cases}$$

$$\Rightarrow x_{31} = \frac{4}{2} = 2$$

$$\Rightarrow -x_{21} - 2 \cdot 2 = -2$$

$$\Rightarrow -x_{21} = -2 + 4 \Rightarrow x_{21} = -2$$

$$\Rightarrow x_{11} + 2 + 4 \cdot 2 = 1 \Rightarrow x_{11} = -9$$

$$\Rightarrow \text{col}_1(x) = (-9, -2, 2)$$

$$A \cdot \text{col}_2(X) = e_2 \quad \Leftrightarrow \quad \begin{cases} x_{12} - x_{22} + 4x_{32} = 0 \\ -x_{22} - 2x_{32} = 1 \\ 2x_{32} = -3 \end{cases}$$

$$\Rightarrow x_{32} = -\frac{3}{2}$$

$$\Rightarrow -x_{22} - 2\left(-\frac{3}{2}\right) = 1 \quad \Rightarrow x_{22} = 2$$

$$\Rightarrow x_{12} - 2 + 4 \cdot \left(-\frac{3}{2}\right) = 0 \quad \Rightarrow x_{12} = 8$$

$$\Rightarrow \text{col}_2(X) = (8, 2, -\frac{3}{2})$$

$$A \cdot \text{col}_3(X) = e_3 \quad \Leftrightarrow \quad \begin{cases} x_{13} - x_{23} + 4x_{33} = 0 \\ -x_{23} - 2x_{33} = 0 \\ 2x_{33} = 1 \end{cases}$$

$$\Rightarrow x_{33} = \frac{1}{2}$$

$$\Rightarrow -x_{23} - 2 \cdot \frac{1}{2} = 0 \quad \Rightarrow x_{23} = -1$$

$$\Rightarrow x_{13} - (-1) + 4 \cdot \frac{1}{2} = 0 \quad \Rightarrow x_{13} = -3$$

$$\Rightarrow \text{col}_3(X) = (-3, -1, \frac{1}{2})$$

$$X = \begin{bmatrix} -9 & 8 & -3 \\ -2 & 2 & -1 \\ 2 & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} = A^{-1}$$

b)

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & -1 & 4 \\ 0 & -1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

Buscamos L^{-1}

$$Lx = e_1 \quad \Leftrightarrow \quad \begin{cases} x_1 & = 1 \\ 2x_1 + x_2 & = 0 \\ 2x_1 + 3x_2 + x_3 & = 0 \end{cases}$$

$$\Rightarrow 2 + x_2 = 0 \quad \Rightarrow x_2 = -2$$

$$\Rightarrow 2 + 3(-2) + x_3 = 0 \quad \Rightarrow x_3 = 4$$

$$\Rightarrow \text{col}_1(L^{-1}) = (1, -2, 4)$$

$$Lx = e_2 \quad \Leftrightarrow \quad \begin{cases} x_1 & = 0 \\ 2x_1 + x_2 & = 1 \\ 2x_1 + 3x_2 + x_3 & = 0 \end{cases}$$

$$\Rightarrow x_1 = 0 \quad \Rightarrow x_2 = 1 \quad \Rightarrow x_3 = -3$$

$$\Rightarrow \text{col}_2(L^{-1}) = (0, 1, -3)$$

$$LX = e_3 \quad \Leftrightarrow \quad \begin{cases} x_1 & = 0 \\ 2x_1 + x_2 & = 0 \\ 2x_1 + 3x_2 + x_3 & = 1 \end{cases}$$

$$\Rightarrow x_1 = 0 \Rightarrow x_2 = 0 \Rightarrow x_3 = 1$$

$$\Rightarrow \text{col}_3(L^{-1}) = (0, 0, 1)$$

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -3 & 1 \end{bmatrix}$$

Buscamos U^{-1}

$$UX = e_1 \quad \Leftrightarrow \quad \begin{cases} x_1 - x_2 + 4x_3 & = 1 \\ -x_2 - 2x_3 & = 0 \\ 2x_3 & = 0 \end{cases}$$

$$\Rightarrow \text{col}_1(U^{-1}) = (1, 0, 0)$$

$$UX = e_2 \quad \Leftrightarrow \quad \begin{cases} x_1 - x_2 + 4x_3 & = 0 \\ -x_2 - 2x_3 & = 1 \\ 2x_3 & = 0 \end{cases}$$

$$\Rightarrow \text{col}_2(U^{-1}) = (-1, -1, 0)$$

$$UX = e_3 \Leftrightarrow \begin{cases} x_1 - x_2 + 4x_3 = 0 \\ -x_2 - 2x_3 = 0 \\ 2x_3 = 1 \end{cases}$$

$$\Rightarrow \text{cols}_3(U^{-1}) = (-3, -1, \frac{1}{2})$$

$$U^{-1} = \begin{bmatrix} 1 & -1 & -3 \\ 0 & -1 & -1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$\begin{aligned} A = LU &\Leftrightarrow AU^{-1} = LUU^{-1} = L \Leftrightarrow AU^{-1}L^{-1} = LL^{-1} = I \\ &\Leftrightarrow A^{-1}AU^{-1}L^{-1} = A^{-1} \Leftrightarrow U^{-1}L^{-1} = A^{-1} \end{aligned}$$

$$A^{-1} = U^{-1}L^{-1} = \begin{bmatrix} 1 & -1 & -3 \\ 0 & -1 & -1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -3 & 1 \end{bmatrix} = \begin{bmatrix} -9 & 8 & -3 \\ -2 & 2 & -1 \\ 2 & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

c)

Resolvemos los sistemas:

$$LUX = e_i \quad \forall i=1,2,3 \quad \Leftrightarrow \quad UX = L^{-1}e_i \quad \forall i=1,2,3$$

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -3 & 1 \end{bmatrix} \quad \begin{aligned} L^{-1}e_1 &= (1, -2, 4) \\ L^{-1}e_2 &= (0, 1, -3) \\ L^{-1}e_3 &= (0, 0, 1) \end{aligned}$$

$$U = \begin{bmatrix} 1 & -1 & 4 \\ 0 & -1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$UX = L^{-1}e_1 \Leftrightarrow \begin{cases} x_1 - x_2 + 4x_3 = 1 \\ -x_2 - 2x_3 = -2 \\ 2x_3 = 4 \end{cases} \Rightarrow \begin{aligned} x_3 &= 2 \\ x_2 &= -(-2 + 2 \cdot 2) = -2 \\ x_1 &= 1 - 2 - 4 \cdot 2 = -9 \end{aligned}$$

$$UX = L^{-1}e_2 \Leftrightarrow \begin{cases} x_1 - x_2 + 4x_3 = 0 \\ -x_2 - 2x_3 = 1 \\ 2x_3 = -3 \end{cases} \Rightarrow \begin{aligned} x_3 &= -\frac{3}{2} \\ x_2 &= -(1 + 2(-\frac{3}{2})) = 2 \\ x_1 &= 2 - 4(-\frac{3}{2}) = 8 \end{aligned}$$

$$UX = L^{-1}e_3 \Leftrightarrow \begin{cases} x_1 - x_2 + 4x_3 = 0 \\ -x_2 - 2x_3 = 0 \\ 2x_3 = 1 \end{cases} \Rightarrow \begin{aligned} x_3 &= \frac{1}{2} \\ x_2 &= -2 \cdot \frac{1}{2} = -1 \\ x_1 &= -1 - 4 \cdot \frac{1}{2} = -3 \end{aligned}$$

$$LUX_1 = e_1 \quad \Leftrightarrow \quad X_1 = (-9, -2, 2)$$

$$LUX_2 = e_2 \quad \Leftrightarrow \quad X_2 = (8, 2, -\frac{3}{2})$$

$$LUX_3 = e_3 \quad \Leftrightarrow \quad X_3 = (-3, -1, \frac{1}{2})$$

$$LUX = I \quad \Leftrightarrow \quad X = \begin{bmatrix} | & | & | \\ X_1 & X_2 & X_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} -9 & 8 & -3 \\ -2 & 2 & -1 \\ 2 & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} = A^{-1}$$