1. Hallar los autovalores y autovectores de las siguientes matrices:

a) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ b) $\begin{pmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 1 \end{pmatrix}$ c) $\begin{pmatrix} -1 & -3 & -9 \\ 0 & 5 & 18 \\ 0 & -2 & -7 \end{pmatrix}$

$A-\lambda I$ es singular (no inversible).

Polinomio característico: $P(\lambda) = \det(A-\lambda I)$.

λ autovalor de A sii λ es raíz de $P(\lambda)$.

a)

$$A-\lambda I = \begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = (1-\lambda) \cdot \det\begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} - 1\cdot\det\begin{bmatrix} 1 & 1 \\ 1 & 1-\lambda \end{bmatrix} + 1\cdot\det\begin{bmatrix} 1 & 1 \\ 1-\lambda & 1 \end{bmatrix}$$

$$= (1-\lambda)\cdot\left[(1-\lambda)^2-1\right] - (1-\lambda-1) + \left[1-(1-\lambda)\right]$$

$$= (1-\lambda)\cdot(\lambda^2-2\lambda) + 2\lambda$$

$$= \lambda^2 - 2\lambda - \lambda^3 + 2\lambda^2 + 2\lambda$$

$$= 3\lambda^2 - \lambda^3$$

$P(\lambda) = 3\lambda^2 - \lambda^3$ $\quad P(\lambda) = 0 \iff \lambda = 0 \vee \lambda = 3$

$\lambda = 0$ $\quad Av = 0v \iff Av = 0 \iff v \in Nu(A)$

$Av = 0 \iff v_1 + v_2 + v_3 = 0 \iff v_1 = -v_2 - v_3$

$\therefore$ $v = (-2, 1, 1)$ asociado al autovalor $\lambda = 0$ con multiplicidad 2.

$\lambda = 3$ $\quad A v = 3v \iff (A-3I)v = 0$

$$A-3I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \begin{array}{c} \\ 2F_2+F_1 \\ 2F_3+F_1 \end{array} \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right] \begin{array}{c} \\ \\ F_3+F_2 \end{array} \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = 0$$

$\Rightarrow -3V_2 + 3V_3 = 0 \quad \Rightarrow V_2 = V_3$

$\Rightarrow -2V_1 + V_2 + V_3 = 0 \quad \Rightarrow V_1 = V_2$

$\therefore$ $v = (1,1,1)$ asociado al autovalor $\lambda = 3$.

b)

$$A - \lambda I = \begin{bmatrix} 1-\lambda & -1 & 0 \\ -2 & 4-\lambda & -2 \\ 0 & -1 & 1-\lambda \end{bmatrix}$$

$$\begin{aligned} \det(A-\lambda I) &= (1-\lambda)\cdot[(4-\lambda)(1-\lambda)-2] + 2[-(1-\lambda)] \\ &= (1-\lambda)\cdot(4-4\lambda-\lambda+\lambda^2-2) - 2 + 2\lambda \\ &= (1-\lambda)\cdot(\lambda^2-5\lambda+2) + 2\lambda - 2 \\ &= \lambda^2 - 5\lambda + 2 - \lambda^3 + 5\lambda^2 - 2\lambda + 2\lambda - 2 \\ &= -\lambda^3 + 6\lambda^2 - 5\lambda \\ &= \lambda(-\lambda^2 + 6\lambda - 5) \end{aligned}$$

$P(\lambda) = \lambda(-\lambda^2 + 6\lambda - 5) \qquad P(\lambda) = 0 \iff \lambda = 0 \lor \lambda = 1 \lor \lambda = 5$

$\lambda = 0 \qquad Av = \lambda v \iff Av = 0$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \xrightarrow{F_2+2F_1} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \xrightarrow{2F_3+F_2} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

$\Rightarrow \quad 2v_2 - 2v_3 = 0 \quad \Rightarrow v_2 = v_3$

$\Rightarrow \quad v_1 - v_2 = 0 \quad \Rightarrow v_1 = v_2$

$\therefore$ $v = (1,1,1)$ autovector asociado al autovalor $\lambda = 0$.

$\lambda = 1$ $\quad Av = \lambda v \iff (A - 1 \cdot I)v = 0 \iff (A - I)v = 0$

$$A - I = \begin{bmatrix} 0 & -1 & 0 \\ -2 & 3 & -2 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ -2 & 3 & -2 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \text{swap} \quad \left[\begin{array}{ccc|c} -2 & 3 & -2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \quad F_3 - F_2 \quad \left[\begin{array}{ccc|c} -2 & 3 & -2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \quad \begin{bmatrix} -2 & 3 & -2 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = 0$$

$\Rightarrow V_2 = 0$

$\Rightarrow -2V_1 - 2V_3 = 0 \Rightarrow V_3 = -V_1$

$\therefore$ $v = (1, 0, -1)$ autovector asociado al autovalor $\lambda = 1$.

$\lambda = 5$ $\quad Av = \lambda v \iff (A - 5I)v = 0$

$$A - 5I = \begin{bmatrix} -4 & -1 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & -4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -4 & -1 & 0 & 0 \\ -2 & -1 & -2 & 0 \\ 0 & -1 & -4 & 0 \end{array}\right] \xrightarrow{2F_2 - F_1} \left[\begin{array}{ccc|c} -4 & -1 & 0 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & -1 & -4 & 0 \end{array}\right] \xrightarrow{F_3 - F_2} \left[\begin{array}{ccc|c} -4 & -1 & 0 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} -4 & -1 & 0 \\ 0 & -1 & -4 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

$\Rightarrow -v_2 - 4v_3 = 0 \quad \Rightarrow \quad v_2 = -4v_3$

$\Rightarrow -4v_1 - v_2 = 0 \quad \Rightarrow \quad v_2 = -4v_1$

$\Rightarrow -4v_1 = -4v_3 \quad \Rightarrow \quad v_1 = v_3$

$\therefore$ $v = (1, -4, 1)$ autovector asociado al autovalor $\lambda = 5$.

c)

$$A-\lambda I = \begin{bmatrix} -1-\lambda & -3 & -9 \\ 0 & 5-\lambda & 18 \\ 0 & -2 & -7-\lambda \end{bmatrix}$$

$$\begin{aligned} \det(A-\lambda I) &= (-1-\lambda)[(5-\lambda)(-7-\lambda)-18\cdot(-2)] \\ &= (-1-\lambda)(-35-5\lambda+7\lambda+\lambda^2+36) \\ &= (-1-\lambda)(\lambda^2+2\lambda+1) \\ &= -(\lambda+1)(\lambda+1)^2 \\ &= -(\lambda+1)^3 \end{aligned}$$

$P(\lambda) = -(\lambda+1)^3 \qquad P(\lambda)=0 \iff \lambda=-1$

$\lambda=-1$ $\qquad Av=\lambda v \iff (A-\lambda I)v=0 \iff (A+I)v=0$

$$A+I = \begin{bmatrix} 0 & -3 & -9 \\ 0 & 6 & 18 \\ 0 & -2 & -6 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & -3 & -9 & 0 \\ 0 & 6 & 18 & 0 \\ 0 & -2 & -6 & 0 \end{array}\right] \begin{array}{l} \\ F_2+2F_1 \\ 3F_3-F_2 \end{array} \left[\begin{array}{ccc|c} 0 & -3 & -9 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$(A+I)\cdot(v_1,v_2,v_3)=0 \iff -3v_2-9v_3=0 \iff v_2=-3v_3$

$\therefore$ $v=(0,-3,1)$ autovector asociado al autovalor $\lambda=-1$ con multiplicidad 3.