Métodos Numéricos
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Factorización QR

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Matrices Ortogonales

$Q \in \mathbb{R}^{n\times n}$ Q es ortogonal sii $QQ^t = Q^tQ = I$

- Columnas ortogonales de norma 2 igual a 1
- Filas ortogonales de norma 2 igual a 1
- $||Q||_2 = 1$
- $\kappa_2(Q) = 1$
- $||Qx||_2 = ||x||_2$
- Producto de ortogonales es ortogonal

Factorización QR

Sean $A \in \mathbb{R}^{n \times n}$, $Q \in \mathbb{R}^{n \times n}$ matriz ortogonal y $R \in \mathbb{R}^{n \times n}$ triangular superior tal que

$$A = QR$$

$$Ax = b$$

$$QRx = b$$

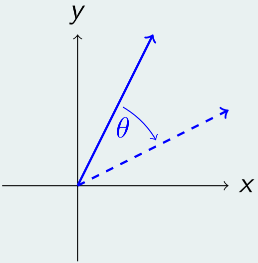
$$Q^t QRx = Q^t b$$

$$Rx = Q^t b$$

Sistema triangular superior, $\mathcal{O}(n^2)$

Método de Givens (rotaciones)

Dado un ángulo θ, sea la transformación lineal que rota a todo vector del plano en ángulo θ en sentido horario.



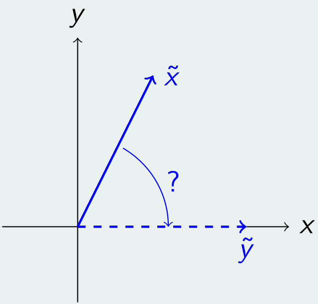
$$W = \begin{bmatrix} cos(\theta) & sen(\theta) \\ -sen(\theta) & cos(\theta) \end{bmatrix}$$

W es ortogonal y $||Wx||_2 = ||x||_2$

Método de Givens (rotaciones)

Dados $\tilde{x}, \tilde{y} \in R^2$, $\tilde{y} = \begin{pmatrix} ||\tilde{x}||_2 \\ 0 \end{pmatrix}$, se busca la rotación W tal que

$$W\tilde{x} = \tilde{y}$$



$$W = \begin{bmatrix} \frac{\tilde{x}_1}{||\tilde{x}||_2} & \frac{\tilde{x}_2}{||\tilde{x}||_2} \\ -\frac{\tilde{x}_2}{||\tilde{x}||_2} & \frac{\tilde{x}_1}{||\tilde{x}||_2} \end{bmatrix}$$

Método de Givens (rotaciones)

Sean $A \in \mathbb{R}^{2\times 2}$, $\tilde{x} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$ y $\tilde{y} = \begin{pmatrix} ||\tilde{x}||_2 \\ 0 \end{pmatrix}$.

Existe W tal que $W\tilde{x} = \tilde{y}$.

$$WA = \begin{bmatrix} ||\tilde{x}||_2 & * \\ 0 & * \end{bmatrix}$$

$$WA = R$$

$$W^t WA = W^t R$$

$$A = QR$$

Sean $A \in \mathbb{R}^{n\times n}$, $\tilde{x} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$ y $\tilde{y} = \begin{pmatrix} ||\tilde{x}||_2 \\ 0 \end{pmatrix}$.

Existe $W \in R^{2\times 2}$ tal que $W\tilde{x} = \tilde{y}$. Sea

$$W_{12} = \begin{bmatrix} w_{11} & w_{12} & 0 & \cdots & 0 \\ w_{21} & w_{22} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$W_{12}A = \begin{bmatrix} * & * & \cdots & * \\ {\color{red}0} & * & \cdots & * \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} a^1_{11} & a^1_{12} & \cdots & a^1_{1n} \\ {\color{red}0} & a^1_{22} & \cdots & a^1_{2n} \\ a^1_{31} & a^1_{32} & \cdots & a^1_{3n} \\ \vdots & \vdots & \cdots & \vdots \\ a^1_{n1} & a^1_{n2} & \cdots & a^1_{nn} \end{bmatrix}$$

Sean $\tilde{x} = \begin{pmatrix} a_{11}^1 \\ a_{31}^1 \end{pmatrix}$ y $\tilde{y} = \begin{pmatrix} ||\tilde{x}||_2 \\ 0 \end{pmatrix}$.

Existe $W \in R^{2\times 2}$ tal que $W\tilde{x} = \tilde{y}$. Sea

$$W_{13} = \begin{bmatrix} \mathbf{w_{11}} & 0 & \mathbf{w_{13}} & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \mathbf{w_{31}} & 0 & \mathbf{w_{33}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$W_{13}W_{12}A = \begin{bmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}^2 & a_{12}^2 & \cdots & a_{1n}^2 \\ 0 & a_{22}^2 & \cdots & a_{2n}^2 \\ 0 & a_{32}^2 & \cdots & a_{3n}^2 \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1}^2 & a_{n2}^2 & \cdots & a_{nn}^2 \end{bmatrix}$$

Método de Givens (rotaciones)

Sean $\tilde{x} = \begin{pmatrix} a_{11}^{i-1} \\ a_{i1}^{i-1} \end{pmatrix}$, $\tilde{y} = \begin{pmatrix} ||\tilde{x}||_2 \\ 0 \end{pmatrix}$ y $W \in R^{2\times 2}$ tal que $W\tilde{x} = \tilde{y}$. Sea

$$W_{1i} = \begin{bmatrix} \mathbf{w_{11}} & 0 & \cdots & \mathbf{w_{1i}} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & 0 \\ \mathbf{w_{i1}} & 0 & \cdots & \mathbf{w_{ii}} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$W_{1n}\cdots W_{13}W_{12}A = \begin{bmatrix} * & * & \cdots & * \\ \color{red}{0} & * & \cdots & * \\ \color{red}{0} & * & \cdots & * \\ \color{red}{\vdots} & \vdots & \cdots & \vdots \\ \color{red}{0} & * & \cdots & * \end{bmatrix} = \begin{bmatrix} a_{11}^{(n-1)} & a_{12}^{(n-1)} & \cdots & a_{1n}^{(n-1)} \\ 0 & a_{22}^{(n-1)} & \cdots & a_{2n}^{(n-1)} \\ 0 & a_{32}^{(n-1)} & \cdots & a_{3n}^{(n-1)} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & a_{n2}^{(n-1)} & \cdots & a_{nn}^{(n-1)} \end{bmatrix}$$

Método de Givens (rotaciones)

Para $i = 1, \ldots, n-1$, $j = i+1, \ldots, n$, sea

$$W_{ij} = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & \mathbf{w_{ii}} & \cdots & \mathbf{w_{ij}} & \cdots & 0 \\ \vdots & \cdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & \cdots & \mathbf{w_{ji}} & \cdots & \mathbf{w_{jj}} & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$$

$$W_{n-1n} W_{n-2n} W_{n-2n-1} \cdots W_{1n} \cdots W_{12} A = R$$

$$A = W_{12}^t \cdots W_{1n}^t \cdots W_{n-2n-1}^t W_{n-2n}^t W_{n-1n}^t R$$

$$\boxed{A = QR}$$

Método de Givens (rotaciones)

Costo

$$W_{n-1n}W_{n-2n-1}W_{n-2n}\dots W_{1n}\dots W_{13}W_{12}A = R$$

Calcular cada W_{ij}: 2 productos + 2 cocientes + 1 raiz

- Primera columna:
  W_{1j} actúa entre las filas 1 y j para $j = 2, \dots n$
  Costo: $4n$ productos $+2n$ sumas
  Costo total: $(n-1)(4n+2n+2+2+1)$
- i-ésima columna:
  W_{ij} actúa entre las filas i y j para $j = i+1, \dots n$
  Costo: $4(n-i+1)$ productos $+2(n-i+1)$ sumas
  Costo total $(n-i)(4(n-i+1)+2(n-i+1)+2+2+1)$

Costo total del algoritmo

$$\boxed{\sum_{i=1}^{n-1}(n-i)(4(n-i+1)+2(n-i+1)+2+2+1) = \mathcal{O}(\tfrac{4}{3}n^3)}$$

Método de Givens (rotaciones)

Ejemplo

$$A = \begin{bmatrix} 0 & -20 & -14 \\ 3 & 27 & -4 \\ 4 & 11 & -2 \end{bmatrix}$$

Sea $\tilde{x} = (0, 3)$, buscamos W, rotación en plano, tal que $W\tilde{x} = \tilde{y}$ con $\tilde{y} = (3, 0)$

$$W = \begin{bmatrix} \frac{\tilde{x}_1}{||\tilde{x}||_2} & \frac{\tilde{x}_2}{||\tilde{x}||_2} \\ -\frac{\tilde{x}_2}{||\tilde{x}||_2} & \frac{\tilde{x}_1}{||\tilde{x}||_2} \end{bmatrix} = \begin{bmatrix} \frac{0}{3} & \frac{3}{3} \\ -\frac{3}{3} & \frac{0}{3} \end{bmatrix} \Rightarrow W_{12} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ejemplo

$$W_{12}A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -20 & -14 \\ 3 & 27 & -4 \\ 4 & 11 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 27 & -4 \\ 0 & 20 & 14 \\ 4 & 11 & -2 \end{bmatrix}$$

Sea $\tilde{x} = (3, 4)$, buscamos W, rotación en el plano tal que $W\tilde{x} = \tilde{y}$ con $\tilde{y} = (5, 0)$

$$W = \begin{bmatrix} \frac{\tilde{x}_1}{||\tilde{x}||_2} & \frac{\tilde{x}_2}{||\tilde{x}||_2} \\ -\frac{\tilde{x}_2}{||\tilde{x}||_2} & \frac{\tilde{x}_1}{||\tilde{x}||_2} \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix} \Rightarrow W_{13} = \begin{bmatrix} \frac{3}{5} & 0 & \frac{4}{5} \\ 0 & 1 & 0 \\ -\frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix}$$

Método de Givens (rotaciones)

Ejemplo

$$W_{13}W_{12}A = \begin{bmatrix} \frac{3}{5} & 0 & \frac{4}{5} \\ 0 & 1 & 0 \\ -\frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 3 & 27 & -4 \\ 0 & 20 & 14 \\ 4 & 11 & -2 \end{bmatrix} = \begin{bmatrix} 5 & 25 & -4 \\ 0 & 20 & 14 \\ 0 & -15 & 2 \end{bmatrix}$$

Sea $\tilde{x} = (20, -15)$, buscamos W, rotación en el plano tal que $W\tilde{x} = \tilde{y}$ con $\tilde{y} = (25, 0)$

$$W = \begin{bmatrix} \frac{\tilde{x}_1}{||\tilde{x}||_2} & \frac{\tilde{x}_2}{||\tilde{x}||_2} \\ -\frac{\tilde{x}_2}{||\tilde{x}||_2} & \frac{\tilde{x}_1}{||\tilde{x}||_2} \end{bmatrix} = \begin{bmatrix} \frac{20}{25} & \frac{-15}{25} \\ \frac{15}{25} & \frac{20}{25} \end{bmatrix} \Rightarrow W_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{20}{25} & \frac{-15}{25} \\ 0 & \frac{15}{25} & \frac{20}{25} \end{bmatrix}$$

Método de Givens (rotaciones)

Ejemplo

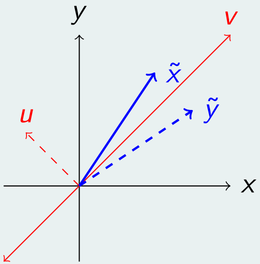
$$W_{23}W_{13}W_{12}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{20}{25} & \frac{-15}{25} \\ 0 & \frac{15}{25} & \frac{20}{25} \end{bmatrix} \begin{bmatrix} 5 & 25 & -4 \\ 0 & 20 & 14 \\ 0 & -15 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 25 & -4 \\ 0 & 25 & 10 \\ 0 & 0 & 10 \end{bmatrix}$$

$$Q = W_{12}^t W_{13}^t W_{23}^t = \begin{bmatrix} 0 & \frac{-20}{25} & \frac{-15}{25} \\ \frac{15}{25} & \frac{12}{25} & \frac{-16}{25} \\ \frac{20}{25} & \frac{-9}{25} & \frac{12}{25} \end{bmatrix} \qquad R = \begin{bmatrix} 5 & 25 & -4 \\ 0 & 25 & 10 \\ 0 & 0 & 10 \end{bmatrix}$$

$$A = QR$$

Método de Householder (reflexiones)

Una reflexión es una transformación lineal que *refleja* a todo vector respecto a un plano.



$$H\tilde{x} = \tilde{y}$$

$$Hu = -u$$

$$Hv = v$$

Método de Householder (reflexiones)

Como v y u forman una base, entonces $\tilde{x} = \alpha v + \beta u$. Además, la reflexión de $\tilde{x}$ es $\tilde{y} = \alpha v - \beta u$.

Entonces, buscamos H tal que $H\tilde{x} = \alpha v - \beta u$

$$\alpha v - \beta u = \alpha v + \beta u - 2\beta u$$

$$H\tilde{x} = I\tilde{x} - W\tilde{x} \text{ tal que } W\tilde{x} = \alpha Wv + \beta Wu = 2\beta u$$

Necesitamos que $Wv = 0$ y $Wu = 2u$

Método de Householder (reflexiones)

Sea $P = uu^t$ y asumamos $||u||_2 = 1$

- P es simétrica.
- $PP^t = P$
- $Pu = u$
- $Pv = 0$

Si definimos $W = 2P$

$$H = I - 2P$$

$$H\tilde{x} = (I - 2P)(\alpha v + \beta u) =$$

$$I(\alpha v + \beta u) - 2P(\alpha v + \beta u) =$$

$$\alpha v + \beta u - 2\beta u = \tilde{y}$$

Método de Householder (reflexiones)

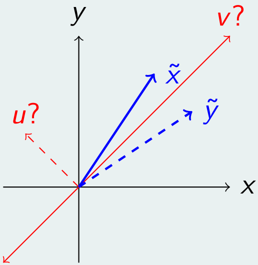
Propiedades de H

$$H = I - 2uu^t$$

- H es simétrica
- H es ortogonal

Método de Householder (reflexiones)

Sean $\tilde{x}, \tilde{y} \in \mathbb{R}^n$, $\tilde{x} \neq \tilde{y}$, $||\tilde{x}||_2 = ||\tilde{y}||_2$. Existe una transformación de Householder tal que $H\tilde{x} = \tilde{y}$.



$$v = \tilde{x} + \tilde{y}$$

$$u = \frac{\tilde{x} - \tilde{y}}{||\tilde{x} - \tilde{y}||_2}$$

$$H = I - 2\frac{(\tilde{x} - \tilde{y})(\tilde{x} - \tilde{y})^t}{||\tilde{x} - \tilde{y}||_2^2}$$

Método de Householder (reflexiones)

Sean $A \in \mathbb{R}^{2\times 2}$, $\tilde{x} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$ y $\tilde{y} = \begin{pmatrix} ||\tilde{x}||_2 \\ 0 \end{pmatrix}$.

Existe H tal que $H\tilde{x} = \tilde{y}$.

$$HA = \begin{bmatrix} ||\tilde{x}||_2 & * \\ 0 & * \end{bmatrix}$$

$$HA = R$$

$$H^t HA = H^t R$$

$$A = QR$$

Sean $A \in \mathbb{R}^{n \times n}$, $\tilde{x} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}$ y $\tilde{y} = \begin{pmatrix} ||\tilde{x}||_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\tilde{x}, \tilde{y} \in \mathbb{R}^n$

Existe $H_1 \in R^{n \times n}$ tal que $H_1 \tilde{x} = \tilde{y}$.

$H_1 = I - 2 u_1 u_1^t$ con $u_1 = \frac{\tilde{x} - \tilde{y}}{||\tilde{x} - \tilde{y}||_2}$

$$H_1 A = \begin{bmatrix} ||\tilde{x}||_2 & a_{12}^1 & \cdots & a_{1n}^1 \\ 0 & a_{22}^1 & \cdots & a_{2n}^1 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & a_{n2}^1 & \cdots & a_{nn}^1 \end{bmatrix} = A^1$$

Método de Householder (reflexiones)

Sean $\tilde{x} = \begin{pmatrix} a_{22}^1 \\ a_{32}^1 \\ \vdots \\ a_{n2}^1 \end{pmatrix}$ y $\tilde{y} = \begin{pmatrix} ||\tilde{x}||_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\tilde{x}, \tilde{y} \in \mathbb{R}^{n-1}$

Existe $H \in R^{(n-1)\times(n-1)}$ tal que $H\tilde{x} = \tilde{y}$.
$H = I - 2u_2u_2^t$ con $u_2 = \frac{\tilde{x}-\tilde{y}}{||\tilde{x}-\tilde{y}||_2}$, $u_2 \in \mathbb{R}^{n-1}$. Sea $H_2 \in R^{n\times n}$

$$H_2 = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & I - 2u_2u_2^t \end{bmatrix}$$

$$H_2A^1 = \begin{bmatrix} a_{12}^1 & a_{12}^1 & a_{13}^1 & \cdots & a_{1n}^1 \\ 0 & ||\tilde{x}||_2 & a_{23}^2 & \cdots & a_{2n}^2 \\ 0 & 0 & a_{33}^2 & \cdots & a_{3n}^2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & a_{n3}^2 & \cdots & a_{nn}^2 \end{bmatrix} = A^2$$

Método de Householder (reflexiones)

Sean $\tilde{x} = \begin{pmatrix} a_{ii}^{(i-1)} \\ a_{i+1i}^{(i-1)} \\ \vdots \\ a_{ni}^{(i-1)} \end{pmatrix}$ y $\tilde{y} = \begin{pmatrix} ||\tilde{x}||_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\tilde{x}, \tilde{y} \in \mathbb{R}^{n-i+1}$

Existe $H \in R^{(n-i+1)\times(n-i+1)}$ tal que $H\tilde{x} = \tilde{y}$. $H = I - 2u_i u_i^t$ con $u_i = \frac{\tilde{x}-\tilde{y}}{||\tilde{x}-\tilde{y}||_2}$, con $u_i \in \mathbb{R}^{n-i+1}$.

Sea $H_i \in R^{n\times n}$

$$H_i = \begin{bmatrix} I & 0 \\ 0 & I - 2u_i u_i^t \end{bmatrix}$$

con $I \in R^{i-1\times i-1}$

Método de Householder (reflexiones)

$$H_iA^{(i-1)} = \begin{bmatrix} a_{12}^{(i-1)} & a_{12}^{(i-1)} & \cdots & a_{1i}^{(i-1)} & a_{1i+1}^{(i-1)} & \cdots & a_{1n}^{(i-1)} \\ 0 & a_{22}^{(i-1)} & \cdots & a_{2i}^{(i-1)} & a_{2i+1}^{(i-1)} & \cdots & a_{2n}^{(i-1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & ||\tilde{x}||_2 & a_{ii+1}^{i} & \cdots & a_{in}^{i} \\ 0 & 0 & 0 & 0 & a_{i+1i+1}^{i} & \cdots & a_{i+1n}^{i} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & a_{ni+1}^{i} & \cdots & a_{nn}^{i} \end{bmatrix} = A^i$$

Método de Householder (reflexiones)

$$H_{n-1}H_{n-2}\dots H_1 A = R$$

$$A = H_1^t \dots H_{n-2}^t H_{n-1}^t R$$

$$A = QR$$

Costo

$$H_{n-1}H_{n-2}\dots H_1A = R$$

- Primera columna:
  $H_1 = I - 2u_1u_1^t \qquad\qquad H_1A = A - 2u_1u_1^tA$
  Costo de u_1^tA : n (n productos $+$ $(n-1)$ sumas)
  Costo de $u_1(u_1^tA)$ n^2 productos
  Costo total: $2n^2 + n(n-1)$
- i-ésima columna:
  $H_i = \begin{bmatrix} I & 0 \\ 0 & I - 2u_iu_i^t \end{bmatrix}$ actúa en matriz $(n-i+1)\times(n-i+1)$
  Costo total: $2(n-i+1)^2 + (n-i+1)(n-i)$

Costo total del algoritmo

$$\boxed{\sum_{i=1}^{n-1} 2(n-i+1)^2 + 2(n-i+1)(n-i) = \mathcal{O}(\tfrac{2}{3}n^3)}$$

Ejemplo

$$A = \begin{bmatrix} 1 & 19 & -34 \\ -2 & -5 & 20 \\ 2 & 8 & 37 \end{bmatrix}$$

Sea $\tilde{x} = (1, -2, 2)$. Buscamos H_1, reflexión, tal que $H\tilde{x} = \tilde{y}$ con $\tilde{y} = (3, 0, 0)$.

Definimos $u_1 = \frac{\tilde{x}-\tilde{y}}{||\tilde{x}-\tilde{y}||_2} = \frac{1}{\sqrt{12}}(-2, -2, 2)$

$$H_1 = I - 2u_1 u_1^t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{12} \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} \begin{bmatrix} -2 & -2 & 2 \end{bmatrix}$$

$$(I - 2uu^t)A = A - 2uu^t A$$

Método de Householder (reflexiones)

Ejemplo

$$(I - 2u_1u_1^t)A = A - 2uu^tA$$

$$\begin{bmatrix} 1 & 19 & -34 \\ -2 & -5 & 20 \\ 2 & 8 & 37 \end{bmatrix} - \frac{2}{12}\begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix}\begin{bmatrix} -2 & -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 19 & -34 \\ -2 & -5 & 20 \\ 2 & 8 & 37 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 19 & -34 \\ -2 & -5 & 20 \\ 2 & 8 & 37 \end{bmatrix} - \frac{2}{12}\begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix}\begin{bmatrix} 6 & -12 & 102 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 19 & -34 \\ -2 & -5 & 20 \\ 2 & 8 & 37 \end{bmatrix} - \begin{bmatrix} -2 & 4 & -34 \\ -2 & 4 & -34 \\ 2 & -4 & 34 \end{bmatrix} = \begin{bmatrix} 3 & 15 & 0 \\ 0 & -9 & 54 \\ 0 & 12 & 3 \end{bmatrix}$$

Método de Householder (reflexiones)

Ejemplo

$$A = \begin{bmatrix} 3 & 15 & 0 \\ \color{red}{0} & -9 & 54 \\ \color{red}{0} & 12 & 3 \end{bmatrix} \quad \tilde{A} = \begin{bmatrix} -9 & 54 \\ 12 & 3 \end{bmatrix}$$

Sea $\tilde{x} = (-9, 12)$. Buscamos H, reflexión, tal que $H\tilde{x} = \tilde{y}$ con $\tilde{y} = (15, 0)$.

Definimos $u_2 = \frac{\tilde{x}-\tilde{y}}{||\tilde{x}-\tilde{y}||_2} = \frac{1}{\sqrt{720}}(-24, 12)$

$$H = I - 2u_2u_2^t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{2}{720}\begin{bmatrix} -24 \\ 12 \end{bmatrix}\begin{bmatrix} -24 & 12 \end{bmatrix}$$

$$(I - 2u_2u_2^t)\tilde{A} = \tilde{A} - 2u_2u_2^t\tilde{A}$$

Método de Householder (reflexiones)

Ejemplo

$$(I - 2u_2u_2^t)\tilde{A} = \tilde{A} - 2u_2u_2^t\tilde{A}$$

$$\begin{bmatrix} -9 & 54 \\ 12 & 3 \end{bmatrix} - \frac{2}{720}\begin{bmatrix} -24 \\ 12 \end{bmatrix}\begin{bmatrix} -24 & 12 \end{bmatrix}\begin{bmatrix} -9 & 54 \\ 12 & 3 \end{bmatrix} =$$

$$\begin{bmatrix} -9 & 54 \\ 12 & 3 \end{bmatrix} - \frac{2}{720}\begin{bmatrix} -24 \\ 12 \end{bmatrix}\begin{bmatrix} 360 & -1260 \end{bmatrix} =$$

$$\begin{bmatrix} -9 & 54 \\ 12 & 3 \end{bmatrix} - \begin{bmatrix} -24 & 84 \\ 12 & -42 \end{bmatrix} = \begin{bmatrix} 15 & -30 \\ 0 & 45 \end{bmatrix}$$

Ejemplo

Definiendo $H_2 = \begin{bmatrix} 1 & 0 \\ 0 & H \end{bmatrix}$ resulta entonces que

$$H_2 H_1 A = \begin{bmatrix} 3 & 15 & 0 \\ \color{red}{0} & 15 & -30 \\ \color{red}{0} & \color{red}{0} & 45 \end{bmatrix}$$

$$A = H_1^t H_2^t \begin{bmatrix} 3 & 15 & 0 \\ \color{red}{0} & 15 & -30 \\ \color{red}{0} & \color{red}{0} & 45 \end{bmatrix}$$

$$A = [I - 2u_1 u_1^t] \begin{bmatrix} 1 & 0 \\ 0 & I - 2u_2 u_2^t \end{bmatrix} \begin{bmatrix} 3 & 15 & 0 \\ 0 & 15 & -30 \\ 0 & 0 & 45 \end{bmatrix}$$

Método de Householder (reflexiones)

Ejemplo

$$A = \left(\begin{bmatrix} 1 & 0 \\ 0 & I - 2u_2u_2^t \end{bmatrix} - 2u_1u_1^t \begin{bmatrix} 1 & 0 \\ 0 & I - 2u_2u_2^t \end{bmatrix} \right) \begin{bmatrix} 3 & 15 & 0 \\ 0 & 15 & -30 \\ 0 & 0 & 45 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 19 & -34 \\ -2 & -5 & 20 \\ 2 & 8 & 37 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{14}{15} & \frac{-2}{15} \\ \frac{-2}{3} & \frac{5}{15} & \frac{10}{15} \\ \frac{2}{3} & \frac{-2}{15} & \frac{11}{15} \end{bmatrix} \begin{bmatrix} 3 & 15 & 0 \\ 0 & 15 & -30 \\ 0 & 0 & 45 \end{bmatrix}$$

$$A = QR$$

Factorización QR

Sean $A \in \mathbb{R}^{n\times n}$, A no singular. Existen únicas $Q \in \mathbb{R}^{n\times n}$ matriz ortogonal y $R \in \mathbb{R}^{n\times n}$ triangular superior con $r_{ii} > 0$ para todo $i = 1, \dots, n$ tal que

$$A = QR$$

Factorización QR: bibliografía

Recomendamos consultar la numerosa bibliografía existente sobre el tema. Algunas sugerencias:

- Análisis numérico, Richard L. Burden, J. Douglas Faires, International Thomson Editores, 2002.
- Accuracy and Stability of Numerical Algorithms, Nicholas Higham, SIAM, 2002.
- Applied Linear Algebra, Peter J. Olver, Chehrzad Shakiban, Second Edition, Springer International Publishing, 2018.
- Numerical Linear Algebra, Lloyd N. Trefethen, SIAM, 1997.
- Fundamentals of Matrix Computations, David Watkins, Wiley, 2010.