13. Sea $G \in \overset{\mathbb{R}^{n\times n}}{\cancel{\mathbb{R}^n}}$ una matriz de rotación de Givens con un ángulo asociado $\theta \in [-\pi, \pi)$. Demostrar que G es definida positiva si y sólo si $|\theta| < \pi/2$.

$$G = \begin{array}{c} \left[\begin{array}{ccccccc} 1 & & & & & & \\ & 1 & & & & & \\ & & c & \cdots & s & & \\ & & \vdots & 1 & \vdots & & \\ & & -s & \cdots & c & & \\ & & & & & 1 & \\ & & & & & & 1 \end{array}\right] \begin{array}{c} \\ \\ i \\ \\ j \\ \\ \\ \end{array} \\ \begin{array}{ccccccc} \quad & \quad & i & \quad & j & \quad & \quad \end{array} \end{array} \qquad \begin{array}{l} c = \cos\theta \\ s = \operatorname{sen}\theta \end{array}$$

G dp $\iff x^T G x > 0 \quad \forall x \neq 0$

$$x^T G x$$

$$= x^T (x_1, \cdots, x_i \cos\theta + x_j \operatorname{sen}\theta, \cdots, -x_i \operatorname{sen}\theta + x_j \cos\theta, \cdots, x_n)$$

$$= \sum_{\substack{k=1 \\ k\neq i,j}}^{n} x_k^2 + x_i^2 \cos\theta + x_i x_j \operatorname{sen}\theta - x_i x_j \operatorname{sen}\theta + x_j^2 \cos\theta$$

$$\geq x_i^2 \cos\theta + x_i x_j \operatorname{sen}\theta - x_i x_j \operatorname{sen}\theta + x_j^2 \cos\theta \qquad x_k^2 \geq 0 \ \forall k \neq i,j$$

$$= (x_i^2 + x_j^2) \cos\theta$$

($\Rightarrow$)

Sabiendo que G es dp:

$x^T G x \geq (x_i^2 + x_j^2)\cos\theta > 0 \iff \cos\theta > 0 \iff |\theta| < \pi/2$

($\Leftarrow$)

Sabiendo que $|\theta| < \pi/2$:

$0 < \cos\theta \leq (x_i^2 + x_j^2)\cos\theta \leq x^T G x$