7. Hallar la descomposición QR de la matriz A según los métodos de Givens y Householder, siendo

$$A = \begin{bmatrix} 12 & -51 & 4 \\ 6 & 167 & -68 \\ -4 & 24 & -41 \end{bmatrix}$$

Givens (rotaciones)

Paso 1

$\tilde{x} = (a_{11}, a_{21}) = (12, 6)$ $\qquad$ $\tilde{y} = (\|\tilde{x}\|_2, 0) = (\sqrt{12^2+6^2}, 0) = (6\sqrt{5}, 0)$

$\tilde{W} \in \mathbb{R}^{2\times 2} \,/\, \tilde{W}\tilde{x} = \tilde{y}$ $\qquad$ $\tilde{W} = \frac{1}{\|\tilde{x}\|_2} \cdot \begin{bmatrix} \tilde{x}_1 & \tilde{x}_2 \\ -\tilde{x}_2 & \tilde{x}_1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$

$$W_{12}A = \begin{bmatrix} \tilde{W}_{11} & \tilde{W}_{12} & 0 \\ \tilde{W}_{21} & \tilde{W}_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 12 & -51 & 4 \\ 6 & 167 & -68 \\ -4 & 24 & -41 \end{bmatrix} = \begin{bmatrix} 6\sqrt{5} & 13\sqrt{5} & 4 \\ 0 & 77\sqrt{5} & -68 \\ -4 & 24 & -41 \end{bmatrix} = A^{(1)}$$

Paso 2

$\tilde{x} = (a_{11}^{(1)}, a_{31}^{(1)}) = (6\sqrt{5}, -4)$ $\qquad$ $\tilde{y} = (\|\tilde{x}\|_2, 0) = (14, 0)$

$\tilde{W} \in \mathbb{R}^{2\times 2} \,/\, \tilde{W}\tilde{x} = \tilde{y}$ $\qquad$ $\tilde{W} = \frac{1}{\|x\|_2} \cdot \begin{bmatrix} \tilde{x}_1 & \tilde{x}_2 \\ -\tilde{x}_2 & \tilde{x}_1 \end{bmatrix} = \begin{bmatrix} \frac{3}{7}\sqrt{5} & -\frac{2}{7} \\ \frac{2}{7} & \frac{3}{7}\sqrt{5} \end{bmatrix}$

$$W_{13}\overbrace{W_{12}A}^{A^{(1)}} = \begin{bmatrix} \tilde{W}_{11} & 0 & \tilde{W}_{12} \\ 0 & 1 & 0 \\ \tilde{W}_{21} & 0 & \tilde{W}_{22} \end{bmatrix} \cdot \begin{bmatrix} 6\sqrt{5} & 13\sqrt{5} & 4 \\ 0 & 77\sqrt{5} & -68 \\ -4 & 24 & -41 \end{bmatrix} = \begin{bmatrix} 14 & 21 & * \\ 0 & 77\sqrt{5} & -68 \\ 0 & 14\sqrt{5} & * \end{bmatrix} = A^{(2)}$$

Paso 3

$\tilde{x} = (a_{22}^{(2)}, a_{32}^{(2)}) = (77\sqrt{5}, 14\sqrt{5})$ $\tilde{y} = (\|\tilde{x}\|_2, 0) = (175, 0)$

$\tilde{W} \in \mathbb{R}^{2\times 2} \ / \ \tilde{W}\tilde{x} = \tilde{y}$ $\tilde{W} = \frac{1}{\|x\|_2} \cdot \begin{bmatrix} \tilde{x}_1 & \tilde{x}_2 \\ -\tilde{x}_2 & \tilde{x}_1 \end{bmatrix} = \begin{bmatrix} \frac{11}{25}\sqrt{5} & \frac{2}{25}\sqrt{5} \\ \frac{-2}{25}\sqrt{5} & \frac{11}{25}\sqrt{5} \end{bmatrix}$

$$W_{23} \overbrace{W_{13} W_{12} A}^{A^{(2)}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \tilde{w}_{11} & \tilde{w}_{12} \\ 0 & \tilde{w}_{21} & \tilde{w}_{22} \end{bmatrix} \cdot \begin{bmatrix} 14 & 21 & * \\ 0 & 77\sqrt{5} & -68 \\ 0 & 14\sqrt{5} & * \end{bmatrix} = \begin{bmatrix} 14 & 21 & * \\ 0 & 175 & * \\ 0 & 0 & * \end{bmatrix} = A^{(3)}$$

$$W_{23} W_{13} W_{12} A = A^{(3)} \quad \Rightarrow \quad A = \underbrace{W_{12}^T W_{13}^T W_{23}^T}_{Q} \underbrace{A^{(3)}}_{R}$$

Householder (reflexiones)

Paso 1

$\tilde{x} = (a_{11}, a_{21}, a_{31}) = (12, 6, -4) \qquad \tilde{y} = (\|\tilde{x}\|_2, 0, 0) = (14, 0, 0)$

$$u_1 = \frac{\tilde{x} - \tilde{y}}{\|\hat{x} - \tilde{y}\|_2} = \frac{1}{14}(-2, 6, -4)$$

$H_1 \in \mathbb{R}^{3\times 3} \ / \ H_1 \hat{x} = \tilde{y} \qquad H_1 = I - 2u_1 u_1^T$

$H_1 A = A^{(1)} \quad \Rightarrow \quad A^{(1)} = (I - 2u_1 u_1^T)A = A - 2u_1 u_1^T A$

$$A^{(1)} = \begin{bmatrix} 12 & -51 & 4 \\ 6 & 167 & -68 \\ -4 & 24 & -41 \end{bmatrix} - \frac{2}{14^2} \begin{bmatrix} -2 \\ 6 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} -2 & 6 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & -51 & 4 \\ 6 & 167 & -68 \\ -4 & 24 & -41 \end{bmatrix} - \frac{2}{14^2} \begin{bmatrix} 4 & -12 & 8 \\ -12 & 36 & -24 \\ 8 & -24 & 16 \end{bmatrix} = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$$

Paso 2

$\hat{x} = (a_{22}^{(1)}, a_{32}^{(1)}) \in \mathbb{R}^2 \qquad \hat{y} = (\|\tilde{x}\|_2, 0)$ Me aburrí

$$u_2 = \frac{\hat{x} - \tilde{y}}{\|\hat{x} - \hat{y}\|_2}$$

$\tilde{H} \in \mathbb{R}^{2\times 2} \ / \ \hat{H}\hat{x} = \tilde{y} \qquad H_2 \in \mathbb{R}^{3\times 3} \quad H_2 = \begin{bmatrix} I & 0 \\ 0 & \tilde{H} \end{bmatrix}$