I) Modus ponens relativizado

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash P \Rightarrow (Q \Rightarrow R)}\,ax \quad \dfrac{}{\Gamma \vdash P}\,ax}{\Gamma \vdash Q \Rightarrow R}\,\Rightarrow e \quad \dfrac{\dfrac{}{\Gamma \vdash P \Rightarrow Q}\,ax \quad \dfrac{}{\Gamma \vdash P}\,ax}{\Gamma \vdash Q}\,\Rightarrow e}{\Gamma = (P \Rightarrow Q \Rightarrow R), (P \Rightarrow Q), P \vdash R}\,\Rightarrow e}{(P \Rightarrow Q \Rightarrow R), (P \Rightarrow Q) \vdash (P \Rightarrow R)}\,\Rightarrow i}{(P \Rightarrow Q \Rightarrow R) \vdash (P \Rightarrow Q) \Rightarrow (P \Rightarrow R)}\,\Rightarrow i}{\vdash (P \Rightarrow Q \Rightarrow R) \Rightarrow ((P \Rightarrow Q) \Rightarrow (P \Rightarrow R))}\,\Rightarrow i$$

Ojo con la asociatividad: $P \Rightarrow Q \Rightarrow R \equiv P \Rightarrow (Q \Rightarrow R)$

II) Reducción al absurdo

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash P \Rightarrow \bot}\,ax \quad \dfrac{}{\Gamma \vdash P}\,ax}{\Gamma = (P \Rightarrow \bot), P \vdash \bot}\,\Rightarrow e}{(P \Rightarrow \bot) \vdash \neg P}\,\neg i}{\vdash (P \Rightarrow \bot) \Rightarrow \neg P}\,\Rightarrow i$$

III) Introducción de la doble negación

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P, \neg P \vdash P}\,ax \quad \dfrac{}{P, \neg P \vdash \neg P}\,ax}{P, \neg P \vdash \bot}\,\neg e}{P \vdash \neg\neg P}\,\neg i}{\vdash P \Rightarrow \neg\neg P}\,\Rightarrow i$$

IV) Eliminación de la triple negación

$$\dfrac{\dfrac{}{\neg\neg\neg P \vdash \neg\neg\neg P}\,ax}{\dfrac{\neg\neg\neg P \vdash \neg P}{\vdash \neg\neg\neg P \Rightarrow \neg P}\,\Rightarrow_i}\,\neg\neg e$$

No se podían usar principios clásicos

$$\dfrac{\dfrac{}{\Gamma \vdash \neg\neg\neg P}\,ax \qquad \dfrac{\dfrac{}{\Gamma \vdash P}\,ax}{\Gamma \vdash \neg\neg P}\,\neg\neg i}{\dfrac{\dfrac{\Gamma = \neg\neg\neg P, P \vdash \bot}{\neg\neg\neg P \vdash \neg P}\,\neg i}{\vdash \neg\neg\neg P \Rightarrow \neg P}\,\Rightarrow_i}\,\neg e$$

V) Contraposición

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash P \Rightarrow Q}\,ax \quad \dfrac{}{\Gamma \vdash P}\,ax}{\Gamma \vdash Q}\,\Rightarrow e \qquad \dfrac{}{\Gamma \vdash \neg Q}\,ax}{\Gamma = (P \Rightarrow Q), \neg Q, P \vdash \bot}\,\neg e}{\dfrac{(P \Rightarrow Q), \neg Q \vdash \neg P}{(P \Rightarrow Q) \vdash (\neg Q \Rightarrow \neg P)}\,\Rightarrow_i}\,\Rightarrow_i}{\vdash (P \Rightarrow Q) \Rightarrow (\neg Q \Rightarrow \neg P)}\,\Rightarrow_i$$

Otra versión con MT

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash P \Rightarrow Q}\,ax \qquad \dfrac{}{\Gamma \vdash \neg Q}\,ax}{\Gamma = (P \Rightarrow Q), \neg Q \vdash \neg P}\,MT}{\dfrac{(P \Rightarrow Q) \vdash (\neg Q \Rightarrow \neg P)}{\vdash (P \Rightarrow Q) \Rightarrow (\neg Q \Rightarrow \neg P)}\,\Rightarrow_i}\,\Rightarrow_i$$

VI) Adjunción

$$\dfrac{\dfrac{}{\Gamma \vdash (P \wedge Q) \Rightarrow R}\,ax \quad \dfrac{\dfrac{}{\Gamma \vdash P}\,ax \quad \dfrac{}{\Gamma \vdash Q}\,ax}{\Gamma \vdash P \wedge Q}\,\wedge_i}{\dfrac{\dfrac{\dfrac{\Gamma = ((P \wedge Q) \Rightarrow R), P, Q \vdash R}{((P \wedge Q) \Rightarrow R), P \vdash Q \Rightarrow R}\,\Rightarrow_i}{(P \wedge Q) \Rightarrow R) \vdash (P \Rightarrow Q \Rightarrow R)}\,\Rightarrow_i}{\vdash ((P \wedge Q) \Rightarrow R) \Rightarrow (P \Rightarrow Q \Rightarrow R)}\,\Rightarrow_i}$$

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash P \Rightarrow Q \Rightarrow R}\,ax \quad \dfrac{\dfrac{}{\Gamma \vdash P \wedge Q}\,ax}{\Gamma \vdash P}\,\wedge e_1}{\Gamma \vdash Q \Rightarrow R}\,\Rightarrow e \quad \dfrac{\dfrac{}{\Gamma \vdash P \wedge Q}\,ax}{\Gamma \vdash Q}\,\wedge e_2}{\dfrac{\dfrac{\Gamma = (P \Rightarrow Q \Rightarrow R), (P \wedge Q) \vdash R}{(P \Rightarrow Q \Rightarrow R) \vdash ((P \wedge Q) \Rightarrow R)}\,\Rightarrow_i}{\vdash (P \Rightarrow Q \Rightarrow R) \Rightarrow ((P \wedge Q) \Rightarrow R)}\,\Rightarrow_i}$$

$$\dfrac{\vdash ((P \wedge Q) \Rightarrow R) \Rightarrow (P \Rightarrow Q \Rightarrow R) \qquad \vdash (P \Rightarrow Q \Rightarrow R) \Rightarrow ((P \wedge Q) \Rightarrow R)}{\vdash ((P \wedge Q) \Rightarrow R) \Leftrightarrow (P \Rightarrow Q \Rightarrow R)}\,\wedge_i$$

VII) De Morgan (I)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma_1 \vdash P}\,ax}{\Gamma_1 \vdash P \vee Q}\,\vee e_1 \quad \dfrac{}{\Gamma_1 \vdash \neg(P \vee Q)}\,ax}{\Gamma_1 = \neg(P \vee Q), P \vdash \bot}\,\neg e}{\neg(P \vee Q) \vdash \neg P}\,\neg i \quad \dfrac{\dfrac{\dfrac{}{\Gamma_2 \vdash Q}\,ax}{\Gamma_2 \vdash P \vee Q}\,\vee e_2 \quad \dfrac{}{\Gamma_2 \vdash \neg(P \vee Q)}\,ax}{\Gamma_2 = \neg(P \vee Q), Q \vdash \bot}\,\neg e}{\neg(P \vee Q) \vdash \neg Q}\,\neg i}{\neg(P \vee Q) \vdash \neg P \wedge \neg Q}\,\wedge i}{\vdash \neg(P \vee Q) \Rightarrow (\neg P \wedge \neg Q)}\,\Rightarrow i$$

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash P \vee Q}\,ax \quad \dfrac{\dfrac{}{\Gamma, P \vdash P}\,ax \quad \dfrac{}{\Gamma, P \vdash \neg P \wedge \neg Q}\,ax}{\Gamma, P \vdash \neg P}\,\wedge e_1}{\Gamma, P \vdash \bot}\,\neg e \quad \dfrac{\dfrac{}{\Gamma, Q \vdash Q}\,ax \quad \dfrac{}{\Gamma, Q \vdash \neg P \wedge \neg Q}\,ax}{\Gamma, Q \vdash \neg Q}\,\wedge e_2}{\Gamma, Q \vdash \bot}\,\neg e}{\Gamma = (\neg P \wedge \neg Q), (P \vee Q) \vdash \bot}\,\vee e}{(\neg P \wedge \neg Q) \vdash \neg(P \vee Q)}\,\neg i}{\vdash (\neg P \wedge \neg Q) \Rightarrow \neg(P \vee Q)}\,\Rightarrow i$$

$$\dfrac{\vdash \neg(P \vee Q) \Rightarrow (\neg P \wedge \neg Q) \qquad \vdash (\neg P \wedge \neg Q) \Rightarrow \neg(P \vee Q)}{\vdash \neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)}\,\wedge i$$

VIII) De Morgan (II)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \neg(\neg P \vee \neg Q)}\,ax}{\Gamma \vdash (\neg\neg P \wedge \neg\neg Q)}\,DM_1}{\Gamma \vdash \neg\neg P}\,\wedge e_1}{\Gamma \vdash P}\,\neg\neg e \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \neg(\neg P \vee \neg Q)}\,ax}{\Gamma \vdash (\neg\neg P \wedge \neg\neg Q)}\,DM_1}{\Gamma \vdash \neg\neg Q}\,\wedge e_2}{\Gamma \vdash Q}\,\neg\neg e$$

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \neg(P \wedge Q)}\,ax \qquad \dfrac{\Gamma \vdash P \qquad \Gamma \vdash Q}{\Gamma \vdash (P \wedge Q)}\,\wedge i}{\Gamma = \neg(P \wedge Q), \neg(\neg P \vee \neg Q) \vdash \bot}\,\neg e}{\neg(P \wedge Q) \vdash (\neg P \vee \neg Q)}\,PBC}{\vdash \neg(P \wedge Q) \Rightarrow (\neg P \vee \neg Q)}\,\Rightarrow i$$

DM_1: De Morgan (I) Probado en el ejercicio anterior.

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \neg P \vee \neg Q}\,ax \qquad \dfrac{\dfrac{}{\Gamma, \neg P \vdash \neg P}\,ax \qquad \dfrac{}{\Gamma, \neg P \vdash P \wedge Q}\,ax}{\Gamma, \neg P \vdash P}\,\wedge e_1}{\Gamma, \neg P \vdash \bot}\,\neg e \qquad \dfrac{\dfrac{}{\Gamma, \neg Q \vdash \neg Q}\,ax \qquad \dfrac{}{\Gamma, \neg Q \vdash P \wedge Q}\,ax}{\Gamma, \neg Q \vdash Q}\,\wedge e_2}{\Gamma, \neg Q \vdash \bot}\,\neg e}{\Gamma = (\neg P \vee \neg Q), (P \wedge Q) \vdash \bot}\,\vee e$$

$$\dfrac{\dfrac{\Gamma = (\neg P \vee \neg Q), (P \wedge Q) \vdash \bot}{(\neg P \vee \neg Q) \vdash \neg(P \wedge Q)}\,\neg i}{\vdash (\neg P \vee \neg Q) \Rightarrow \neg(P \wedge Q)}\,\Rightarrow i$$

$$\dfrac{\vdash \neg(P \wedge Q) \Rightarrow (\neg P \vee \neg Q) \qquad \vdash (\neg P \vee \neg Q) \Rightarrow \neg(P \wedge Q)}{\vdash \neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)}\,\wedge i$$

IX) Conmutatividad ($\wedge$)

$$\dfrac{\dfrac{\dfrac{}{(P\wedge Q)\vdash(P\wedge Q)}\,ax}{(P\wedge Q)\vdash Q}\,\wedge e_2 \qquad \dfrac{\dfrac{}{(P\wedge Q)\vdash(P\wedge Q)}\,ax}{(P\wedge Q)\vdash P}\,\wedge e_1}{(P\wedge Q)\vdash(Q\wedge P)}\,\wedge i}{\vdash(P\wedge Q)\Rightarrow(Q\wedge P)}\,\Rightarrow i$$

X) Asociatividad ($\wedge$)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma_1\vdash(P\wedge Q)\wedge R}\,ax}{\Gamma_1\vdash P\wedge Q}\,\wedge e_1}{\Gamma_1\vdash P}\,\wedge e_1 \qquad \dfrac{\dfrac{\dfrac{}{\Gamma_1\vdash(P\wedge Q)\wedge R}\,ax}{\Gamma_1\vdash P\wedge Q}\,\wedge e_1}{\Gamma_1\vdash Q}\,\wedge e_2 \qquad \dfrac{\dfrac{}{\Gamma_1\vdash(P\wedge Q)\wedge R}\,ax}{\Gamma_1\vdash R}\,\wedge e_2}{\Gamma_1\vdash Q\wedge R}\,\wedge i}{\Gamma_1=(P\wedge Q)\wedge R\vdash P\wedge(Q\wedge R)}\,\Rightarrow i}{\vdash((P\wedge Q)\wedge R)\Rightarrow(P\wedge(Q\wedge R))}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma_2\vdash P\wedge(Q\wedge R)}\,ax}{\Gamma_2\vdash P}\,\wedge e_1 \qquad \dfrac{\dfrac{\dfrac{}{\Gamma_2\vdash P\wedge(Q\wedge R)}\,ax}{\Gamma_2\vdash Q\wedge R}\,\wedge e_2}{\Gamma_2\vdash Q}\,\wedge e_1 \qquad \dfrac{\dfrac{}{\Gamma_2\vdash P\wedge(Q\wedge R)}\,ax}{\Gamma_2\vdash Q\wedge R}\,\wedge e_2}{\Gamma_2\vdash R}\,\wedge e_2}{\Gamma_2\vdash P\wedge Q}\,\wedge i}{\Gamma_2=P\wedge(Q\wedge R)\vdash(P\wedge Q)\wedge R}\,\Rightarrow i}{\vdash(P\wedge(Q\wedge R))\Rightarrow((P\wedge Q)\wedge R)}$$

$$\dfrac{\vdash((P\wedge Q)\wedge R)\Rightarrow(P\wedge(Q\wedge R)) \qquad \vdash(P\wedge(Q\wedge R))\Rightarrow((P\wedge Q)\wedge R)}{\vdash((P\wedge Q)\wedge R)\iff(P\wedge(Q\wedge R))}\,\wedge i$$