

I) Intercambio (\forall)

$$\begin{array}{c}
 \frac{}{ax} \\
 \frac{\Gamma_1 \vdash \forall x. \forall y. P(x, y)}{\Gamma_1 \vdash \forall y. P(x, y)} \forall_e \\
 \frac{\Gamma_1 \vdash \forall y. P(x, y)}{\Gamma_1 \vdash P(x, y)} \forall_i \\
 \frac{\Gamma_1 \vdash P(x, y)}{\Gamma_1 \vdash \forall x. P(x, y)} \forall_i \\
 \hline
 \Gamma_1: \forall x. \forall y. P(x, y) \vdash \forall y. \forall x. P(x, y) \Rightarrow_i \text{ simétrico} \\
 \hline
 \vdash \forall x. \forall y. P(x, y) \Rightarrow \forall y. \forall x. P(x, y) \wedge \vdash \forall y. \forall x. P(x, y) \Rightarrow \forall x. \forall y. P(x, y) \\
 \vdash \forall x. \forall y. P(x, y) \Leftrightarrow \forall y. \forall x. P(x, y)
 \end{array}$$

II) Intercambio (\exists)

$$\begin{array}{c}
 \frac{}{ax} \\
 \frac{\Gamma \vdash P(x, y)}{\Gamma \vdash \exists x. P(x, y)} \exists_i \\
 \frac{\Gamma \vdash \exists x. P(x, y)}{\Gamma: \sigma, \exists y. P(x, y), P(x, y) \vdash \exists y. \exists x. P(x, y)} \exists_e \\
 \frac{\sigma, \exists y. P(x, y) \vdash \exists y. P(x, y)}{\sigma \vdash \sigma} ax \\
 \frac{\sigma \vdash \sigma}{\sigma \vdash \exists y. \exists x. P(x, y)} \exists_e \\
 \frac{\sigma \vdash \exists y. \exists x. P(x, y)}{\vdash \exists x. \exists y. P(x, y) \Rightarrow \exists y. \exists x. P(x, y)} \Rightarrow_i \\
 \hline
 \vdash \exists x. \exists y. P(x, y) \Leftrightarrow \exists y. \exists x. P(x, y) \text{ La vuelta } \Leftarrow \text{ es simétrica} \\
 \vdash \exists x. \exists y. P(x, y) \Leftrightarrow \exists y. \exists x. P(x, y)
 \end{array}$$

$$\sigma: \exists x. \exists y. P(x, y)$$

III) Intercambio (\exists/\forall)

| | | | |
|--|-------------|--|-------------|
| $\Gamma \vdash \exists x. \forall y. P(x, y)$ | ax | $\Gamma, \forall y. P(x, y) \vdash \forall y. P(x, y)$ | ax |
| | | $\Gamma \vdash \forall y. P(x, y)$ | \forall_e |
| | | $\Gamma \vdash P(x, y)$ | \exists_i |
| | | $\Gamma \vdash \exists x. P(x, y)$ | \forall_i |
| $\Gamma : \exists x. \forall y. P(x, y) \vdash \forall y. \exists x. P(x, y)$ | | | |
| $\vdash \exists x. \forall y. P(x, y) \Rightarrow \forall y. \exists x. P(x, y)$ | | | |

Está mal aplicada la regla porque para sacar $\exists x$ necesitamos que $x \notin \text{Fr}(\forall y. P(x, y))$.

| | | | |
|--|-------------|--|-------------|
| | | $\Delta \vdash \forall y. P(x, y)$ | ax |
| | | $\Delta \vdash P(x, y)$ | \forall_e |
| | | $\Delta \vdash \exists x. P(x, y)$ | \exists_i |
| | | $\Delta \vdash \exists x. P(x, y)$ | \forall_i |
| $\Gamma \vdash \exists x. \forall y. P(x, y)$ | ax | $\Delta : \Gamma, \forall y. P(x, y) \vdash \forall y. \exists x. P(x, y)$ | \exists_e |
| $\Gamma : \exists x. \forall y. P(x, y) \vdash \forall y. \exists x. P(x, y)$ | | | |
| $\vdash \exists x. \forall y. P(x, y) \Rightarrow \forall y. \exists x. P(x, y)$ | | | |