I) Intercambio ($\forall$)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma_1 \vdash \forall x.\forall y.P(x,y)}\,ax}{\Gamma_1 \vdash \forall y.P(x,y)}\,\forall e}{\Gamma_1 \vdash P(x,y)}\,\forall e}{\Gamma_1 \vdash \forall x.P(x,y)}\,\forall i}{\Gamma_1 \vdash \forall y.\forall x.P(x,y)}\,\forall i$$

Γ_1: $\forall x.\forall y.P(x,y) \vdash \forall y.\forall x.P(x,y)$ $\Rightarrow i$ simétrico

$\vdash \forall x.\forall y.P(x,y) \Rightarrow \forall y.\forall x.P(x,y) \quad \wedge \quad \vdash \forall y.\forall x.P(x,y) \Rightarrow \forall x.\forall y.P(x,y)$

$\vdash \forall x.\forall y.P(x,y) \Leftrightarrow \forall y.\forall x.P(x,y)$

II) Intercambio ($\exists$)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\sigma \vdash \sigma}\,ax \qquad \dfrac{\dfrac{}{\sigma, \exists y.P(x,y) \vdash \exists y.P(x,y)}\,ax \qquad \dfrac{\dfrac{\dfrac{}{\Gamma \vdash P(x,y)}\,\exists i}{\Gamma \vdash \exists x.P(x,y)}\,\exists i}{\Gamma: \sigma, \exists y.P(x,y), P(x,y) \vdash \exists y.\exists x.P(x,y)}\,\exists e}{\sigma, \exists y.P(x,y) \vdash \exists y.\exists x.P(x,y)}\,\exists e}{\sigma \vdash \exists y.\exists x.P(x,y)}\,\Rightarrow i}{\vdash \exists x.\exists y.P(x,y) \Rightarrow \exists y.\exists x.P(x,y)}$$

La vuelta $\Leftarrow$ es simétrica

$\vdash \exists x.\exists y.P(x,y) \Leftrightarrow \exists y.\exists x.P(x,y)$

$\sigma : \exists x.\exists y.P(x,y)$

III) Intercambio ($\exists/\forall$)

$$\dfrac{\dfrac{}{\Gamma \vdash \exists X.\forall Y.P(X,Y)}\,ax \qquad \dfrac{\dfrac{}{\Gamma, \forall Y.P(X,Y) \vdash \forall Y.P(x,Y)}\,ax}{\Gamma \vdash \forall Y.P(X,Y)}\,\forall_e}{\dfrac{\Gamma \vdash P(X,Y)}{\dfrac{\Gamma \vdash \exists X.P(X,Y)}{\dfrac{\Gamma : \exists X.\forall Y.P(X,Y) \vdash \forall Y.\exists X.P(X,Y)}{\vdash \exists X.\forall Y.P(X,Y) \Rightarrow \forall Y.\exists X.P(X,Y)}\,\Rightarrow_i}\,\forall_i}\,\exists_i}\,\exists_e$$

Está mal aplicada la regla porque para sacar $\exists X$ necesitamos que $X \notin Fv(\forall Y.P(X,Y))$.

$$\dfrac{\dfrac{}{\Gamma \vdash \exists X.\forall Y.P(X,Y)}\,ax \qquad \dfrac{\dfrac{\dfrac{}{\Delta \vdash \forall Y.P(X,Y)}\,\forall_e}{\Delta \vdash P(X,Y)}\,\exists_i}{\Delta \vdash \exists X.P(X,Y)}\,\forall_i}{\dfrac{\Delta : \Gamma, \forall Y.P(X,Y) \vdash \forall Y.\exists X.P(X,Y)}{\dfrac{\Gamma : \exists X.\forall Y.P(X,Y) \vdash \forall Y.\exists X.P(X,Y)}{\vdash \exists X.\forall Y.P(X,Y) \Rightarrow \forall Y.\exists X.P(X,Y)}\,\Rightarrow_i}\,\exists_e}$$

IV) Universal implica existencial

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x.\, P(x) \vdash \forall x.\, P(x)}\,\forall e}{\forall x.\, P(x) \vdash P(x)}\,\exists i}{\forall x.\, P(x) \vdash \exists x.\, P(x)}\,\Rightarrow i}{\vdash \forall x.\, P(x) \Rightarrow \exists x.\, P(x)}\,ax$$

V) Diagonal ($\forall$)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \forall x.\, \forall y.\, P(x,y)}\,\forall e}{\Gamma \vdash \forall y.\, P(x,y)}\,\forall e\ \{y := x\}}{\Gamma \vdash P(x,x)}\,\forall i}{\Gamma : \forall x.\, \forall y.\, P(x,y) \vdash \forall x.\, P(x,x)}\,\Rightarrow i}{\vdash \forall x.\, \forall y.\, P(x,y) \Rightarrow \forall x.\, P(x,x)}\,ax$$

VI) Diagonal ($\exists$)

Revisar

$$\dfrac{\dfrac{\dfrac{}{\exists x.\, P(x,x) \vdash \exists x.\, P(x,x)}\,ax \qquad \dfrac{\dfrac{\dfrac{}{\Gamma \vdash P(x,x)}\,ax}{\Gamma \vdash \exists y.\, P(x,y)}\,\exists i}{\Gamma : \exists x.\, P(x,x),\ P(x,x) \vdash \exists x.\, \exists y.\, P(x,y)}\,\exists i}{\exists x.\, P(x,x) \vdash \exists x.\, \exists y.\, P(x,y)}\,\exists e}{\vdash \exists x.\, P(x,x) \Rightarrow \exists x.\, \exists y.\, P(x,y)}\,\Rightarrow i$$

VII) De Morgan (I)

$\neg\exists x.P(x) \Leftrightarrow \forall x.\neg P(x)$

sii $(\neg\exists x.P(x) \Rightarrow \forall x.\neg P(x)) \wedge (\neg\exists x.P(x) \Leftarrow \forall x.\neg P(x))$

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \neg\exists x.P(x)}\,ax \quad \dfrac{\dfrac{}{\Gamma \vdash P(x)}\,ax}{\Gamma \vdash \exists x.P(x)}\,\exists i}{\Gamma : \neg\exists x.P(x),\ P(x) \vdash \bot}\,\neg e}{\neg\exists x.P(x) \vdash \neg P(x)}\,\neg i}{\neg\exists x.P(x) \vdash \forall x.\neg P(x)}\,\forall i}{\vdash \neg\exists x.P(x) \Rightarrow \forall x.\neg P(x)}\,\Rightarrow i$$

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \forall x.P(x)}\,ax \quad \dfrac{\dfrac{}{\Delta \vdash \forall x.\neg P(x)}\,ax}{\Delta \vdash \neg P(x)}\,\forall e}{\Delta : \Gamma, P(x) \vdash \bot}\,\neg e}{\Gamma : \forall x.\neg P(x),\ \exists x.P(x) \vdash \bot}\,\exists e}{\forall x.\neg P(x) \vdash \neg\exists x.P(x)}\,\neg i}{\vdash \forall x.\neg P(x) \Rightarrow \neg\exists x.P(x)}\,\Rightarrow i$$

VIII) De Morgan (II)

$\neg\forall x.P(x) \Leftrightarrow \exists x.\neg P(x)$

sii $(\neg\forall x.P(x) \Rightarrow \exists x.\neg P(x)) \wedge (\neg\forall x.P(x) \Leftarrow \exists x.\neg P(x))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Delta \vdash \neg\exists x.\neg P(x)}\,ax \qquad \dfrac{\dfrac{}{\Delta \vdash \neg P(x)}\,ax}{\Delta \vdash \exists x.\neg P(x)}\,\exists e}{\Delta : \Gamma, \neg P(x) \vdash \bot}\,\neg e}{\Gamma \vdash P(x)}\,PBC \qquad \dfrac{}{\Gamma \vdash \forall x.P(x)}\,ax}{\Gamma \vdash \forall x.P(x)}\,\forall e}{\Gamma : \neg\forall x.P(x), \neg\exists x.\neg P(x) \vdash \bot}\,\neg e}{\neg\forall x.P(x) \vdash \exists x.\neg P(x)}\,PBC \Rightarrow i}{\vdash \neg\forall x.P(x) \Rightarrow \exists x.\neg P(x)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\exists x.\neg P(x) \vdash \exists x.\neg P(x)}\,ax \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \forall x.P(x)}\,ax}{\Gamma \vdash P(x)}\,\forall e \qquad \dfrac{}{\Gamma \vdash \neg P(x)}\,ax}{\Gamma : \exists x.P(x), \neg P(x), \forall x.P(x) \vdash \bot}\,\neg e}{\exists x.P(x), \neg P(x) \vdash \neg\forall x.P(x)}\,\neg i}{\exists x.\neg P(x) \vdash \neg\forall x.P(x)}\,\exists e}{\vdash \exists x.\neg P(x) \Rightarrow \neg\forall x.P(x)}\,\Rightarrow i$$