# Fermat-Toricelli Problem with Weights

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(For Math 583C Project and the computer programming exam)

This is a popular problem in location science: Given $n$ points $x_1, \cdots, x_n$ in $\mathbb{R}^d$ and positive weights $\eta_1, \cdots, \eta_n$. Define $F : \mathbb{R}^n \to [0, \infty)$ given by $F(x) = \sum_{j=1}^{n} \eta_j \|x - a_j\|$ where $\|\cdot\|$ is the standard Euclidean norm. Fermat-Toricelli Problem is the minimization of $F(x)$ over $\mathbb{R}^d$. The unique minimizer of this problem is called the Fermat-Toricelli point. The goal is to use `Sage` to compute such point.

In the case $d = 2$, $n = 3$ and $\eta_1 = \eta_2 = \eta_3 = 1$, the problem was proposed by Pierre de Fermat in the early 17th century and solved by Evangelista Toricelli. This justifies the name of the problem. We refer readers to [2] and the references therein for the extensions and applications of this problem.

We shall solve this problem using three methods and then compare their performance. Our algorithm takes $\{a_1, \cdots, a_n\}$, $\{\eta_1, \cdots, \eta_n\}$ and the method as input. It outputs the Fermat-Torricelli point and the time it takes.

The first method is called modified Weiszfeld algorithm described in [4]. The second method is the interior point method which solves

$$\min \left\{ \sum_{j=1}^{n} \|z_j\| : x \in \mathbb{R}^{d_1}, z_j \in \mathbb{R}^{d_2}, A_j^T x + z_j = a_j, j = 1, \cdots, n \right\}.$$

Setting $d_1 = d_2 = d$ and all $A_j = I_d$ it becomes our problem. The desciption of the interior point algorithm can be found in [1]. Thirdly the problem with $d = 2$ can be solved by formulating it as a semidefinite programming (SDP) problem [3].

For methodology, the `cvxopt` package will be used to solve the problem. Also a good understanding of doing numerical linear algebra using `numpy` is required.

# References

[1] Andersen, K.D., Christiansen, A., Conn, A.R., Overton, M.L.: *An efficient primal-dual interior- point method for minimizing a sum of Euclidean norms.* SIAM J. Scient. Comp. 22 (2000), 243-262.

[2] Mordukhovich, B.S., Nam, N.M.: *Applications of variational analysis to a generalized Fermat-Torricelli problem.* J. Optim. Theory Appl., 148 (2011), 431-454.

[3] Nie, J., Parrilo, P. A..Sturmfels, B. : *Semidefinite Representation of the k-Ellipse. In Algorithms in Algebraic Geometry.* Springer New York (2008), 117-132.

[4] Vardi, Y., Zhang, C-H.: *A modified Weiszfeld algorithm for the Fermat-Weber location problem.* Math. Program., Ser. A 90 (2001), 559-566.