SDP Formulation of the Fermat-Toricelli Problem with Weights

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We describe here the idea of SDP formulation in [1]. Our problem ($d = 2$ case) is to minimize over $x \in \mathbb{R}^2$, $F(x) = \sum_{j=1}^{n} \eta_j \|x - a_j\|$ where all η_j is positive and $a_j \in \mathbb{R}^2$. Let x and a_j have coordinates (x, y) and (a_j, b_j) respectively (some abuse of notation here). Then the objective function becomes

$$F(x, y) = \sum_{j=1}^{n} \eta_j \sqrt{(x - a_j)^2 + (y - b_j)^2}.$$

Let the minimization problem be $(\mathcal{P})$.

Introduce slack variables $d_j = \sqrt{(x - a_j)^2 + (y - b_j)^2}$ for all j. Then $d_j^2 = (x - a_j)^2 + (y - b_j)^2$. One can use this to show that:

Theorem 1. *The problem $(\mathcal{P})$ is equivalent to the semidefinite program (SDP) $(\mathcal{P}_2)$:*

$$\textit{minimize } F(x) = \sum_{j=1}^{n} \eta_j d_j \textit{ subject to } \begin{bmatrix} d_j + x - a_j & y - b_j \\ y - b_j & d_j - x + a_j \end{bmatrix} \succeq 0 \textit{ for all } j = 1, \cdots, n$$

(Here $A \succeq 0$ means the matrix A is positive semidefinite; Notice the decision variables of $(\mathcal{P}_2)$ are $d_1, \cdots, d_n, x, y$).

Proof. □

We remark that for two square matrices A and B, $A \oplus B$ is formed by stacking A and B on the diagonal:

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

The problem $(\mathcal{P}_2)$ is an SDP because the constraints of the problem is equivalent to

$$\oplus_{j=1}^{n} \begin{bmatrix} d_j + x - a_j & y - b_j \\ y - b_j & d_j - x + a_j \end{bmatrix} \succeq 0.$$

This condition is equivalent to the linear matrix inequality (LMI)

$$\sum_{j=1}^{n} d_j B_j + xC + yD \succeq 0$$

where B_j, C, D are $2n \times 2n$ matrices of which definitions are obvious.

References

[1] Nie, J., Parrilo, P. A..Sturmfels, B. : *Semidefinite Representation of the k-Ellipse. In Algorithms in Algebraic Geometry.* Springer New York (2008), 117-132.