Supplementary Material for "Interconnected Active Distribution Networks: A Differentially Private Reconfiguration Approach "

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I. MIQP-BASED DNR MODEL

The interconnected active distribution networks (ADNs) are considered as a connected undirected tree $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, where $\mathcal{N}$ is the set of nodes and $\mathcal{E}$ is the set of branches. Suppose that an arbitrary branch $l := (m, n), \forall l \in \mathcal{E}$ is between nodes (m, n) and S denotes the set of root nodes regarded as multiple substations [1]. The simplified distribution network reconfiguration (DNR) model is based on the linearized DistFlow equations, which is cast as a mixed-integer quadratic programming (MIQP) problem [2]. The set of optimization variables involves a set of operational variables $\boldsymbol{x}^l := [\boldsymbol{P}^l, \boldsymbol{Q}^l, \boldsymbol{Q}^{cr}]^T$, switch status indicator variables $\boldsymbol{u}^l \in \mathbb{Z}^{|\mathcal{E}|}$ and continuous parent-child relationship variables $\boldsymbol{\beta}^l \in \mathbb{R}^{2|\mathcal{E}|}$. In this vein, $\boldsymbol{P}^l \in \mathbb{R}^{|\mathcal{E}|}$ and $\boldsymbol{Q}^l \in \mathbb{R}^{|\mathcal{E}|}$ refer to the vectors of sending-end active and reactive power flows. $\boldsymbol{Q}^{cr} \in \mathbb{R}^{|n_{cr}|}$ is the vector of nodal reactive power compensation and n_{cr} is the number of capacitors. $\boldsymbol{\beta}^l$ are constructed with spanning tree constraints with multiple sources. For convenience, we express the MIQP-based DNR model in following form:

$$\min_{\boldsymbol{x}^l, \boldsymbol{\beta}^l \in \mathbb{R}, \boldsymbol{u}^l \in \mathbb{Z}} \quad f = \begin{bmatrix} \boldsymbol{P}^l \\ \boldsymbol{Q}^l \\ \boldsymbol{Q}^{cr} \end{bmatrix}^T \begin{bmatrix} \boldsymbol{D}_r & \boldsymbol{0}_{|\mathcal{E}|\times|\mathcal{E}|} & \boldsymbol{0}_{|\mathcal{E}|\times n_{cr}} \\ \boldsymbol{0}_{|\mathcal{E}|\times|\mathcal{E}|} & \boldsymbol{D}_r & \boldsymbol{0}_{|\mathcal{E}|\times n_{cr}} \\ \boldsymbol{0}_{n_{cr}\times|\mathcal{E}|} & \boldsymbol{0}_{n_{cr}\times|\mathcal{E}|} & \boldsymbol{0}_{n_{cr}\times n_{cr}} \end{bmatrix} \begin{bmatrix} \boldsymbol{P}^l \\ \boldsymbol{Q}^l \\ \boldsymbol{Q}^{cr} \end{bmatrix} \tag{A-1a}$$

$$\text{s.t.} \begin{bmatrix} \boldsymbol{A}_{\mathcal{G}}^T & \boldsymbol{0}_{|\mathcal{E}|\times|\mathcal{E}|} & \boldsymbol{0}_{|\mathcal{E}|\times n_{cr}} \\ \boldsymbol{0}_{|\mathcal{E}|\times|\mathcal{E}|} & \boldsymbol{A}_{\mathcal{G}}^T & \boldsymbol{A}_{cr} \end{bmatrix} \begin{bmatrix} \boldsymbol{P}^l \\ \boldsymbol{Q}^l \\ \boldsymbol{Q}^{cr} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{P}^g + \boldsymbol{P}^d \\ -\boldsymbol{Q}^g + \boldsymbol{Q}^d \end{bmatrix} \tag{A-1b}$$

$$\begin{bmatrix} 2\boldsymbol{D}_r & 2\boldsymbol{D}_x & \boldsymbol{0}_{|\mathcal{E}|\times n_{cr}} & M \\ -2\boldsymbol{D}_r & -2\boldsymbol{D}_x & \boldsymbol{0}_{|\mathcal{E}|\times n_{cr}} & -M \\ \boldsymbol{0}_{n_{cr}\times|\mathcal{E}|} & \boldsymbol{0}_{n_{cr}\times|\mathcal{E}|} & diag(\mathbb{1}_c) & \boldsymbol{0}_{n_{cr}\times|\mathcal{E}|} \\ \boldsymbol{0}_{n_{cr}\times|\mathcal{E}|} & \boldsymbol{0}_{n_{cr}\times|\mathcal{E}|} & -diag(\mathbb{1}_c) & \boldsymbol{0}_{n_{cr}\times|\mathcal{E}|} \end{bmatrix} \begin{bmatrix} \boldsymbol{P}^l \\ \boldsymbol{Q}^l \\ \boldsymbol{Q}^{cr} \\ \boldsymbol{u}^l \end{bmatrix} \leqslant \begin{bmatrix} \Delta\overline{\boldsymbol{v}} + M \\ -\Delta\underline{\boldsymbol{v}} + M \\ \overline{\boldsymbol{Q}}_{\boldsymbol{cr}} \\ -\underline{\boldsymbol{Q}}_{\boldsymbol{cr}} \end{bmatrix} \tag{A-1c}$$

$$\begin{bmatrix} diag(\mathbb{1}_N) & \boldsymbol{0}_{|\mathcal{E}|\times|\mathcal{E}|} & \boldsymbol{0}_{|\mathcal{E}|\times n_{cr}} & -\Gamma \\ -diag(\mathbb{1}_N) & \boldsymbol{0}_{|\mathcal{E}|\times|\mathcal{E}|} & \boldsymbol{0}_{|\mathcal{E}|\times n_{cr}} & -\Gamma \\ \boldsymbol{0}_{|\mathcal{E}|\times|\mathcal{E}|} & diag(\mathbb{1}_N) & \boldsymbol{0}_{|\mathcal{E}|\times n_{cr}} & -\Gamma \\ \boldsymbol{0}_{|\mathcal{E}|\times|\mathcal{E}|} & -diag(\mathbb{1}_N) & \boldsymbol{0}_{|\mathcal{E}|\times n_{cr}} & -\Gamma \end{bmatrix} \begin{bmatrix} \boldsymbol{P}^l \\ \boldsymbol{Q}^l \\ \boldsymbol{Q}^{cr} \\ \boldsymbol{u}^l \end{bmatrix} \leqslant \boldsymbol{0} \tag{A-1d}$$

$$\beta_{mn}^l + \beta_{nm}^l = u_{mn}^l, \quad \beta_{mn}^l = 0, \text{ if } m = S \tag{A-1e}$$

$$\sum_{n:(m,n)\in\mathcal{E}} \beta_{mn}^l = 1, \ \forall m \in \mathcal{N}\backslash S \tag{A-1f}$$

$$0 \leqslant \beta_{mn}^l \leqslant 1, \ \forall l \in \mathcal{E} \tag{A-1g}$$

where (A-1a) states the quadratic active power loss of DNs under the assumption of flat voltage profiles for all nodes. $\mathbb{1}_c$ and $\mathbb{1}_N$ are the $n_{cr} \times 1$ and $|\mathcal{E}| \times 1$ vectors with all ones, respectively. M refers to the big positive number and Γ denotes the branch capacity. $\boldsymbol{P}^g, \boldsymbol{Q}^g$ and $\boldsymbol{P}^d, \boldsymbol{Q}^d$ indicate the vectors of given nodal active and reactive power injections and active and reactive loads at nodes. $\boldsymbol{A}_{\mathcal{G}}$ is a $|\mathcal{E}|$ by $|\mathcal{N}|$ branch-node incidence matrix and $\boldsymbol{A}_{cr}$ is a diagonal matrix whose i-th diagonal element is equal to 1 if node i has the reactive compensation capacitors; otherwise this entry is zero. $\boldsymbol{D}_r$ and $\boldsymbol{D}_x$ indicate the diagonal matrices whose diagonal elements are the resistance and reactance vectors, respectively. $\underline{\boldsymbol{Q}}_{\boldsymbol{cr}}$, $\overline{\boldsymbol{Q}}_{\boldsymbol{cr}}$, $\Delta\underline{\boldsymbol{v}}$ and $\Delta\overline{\boldsymbol{v}}$ represent the boundaries of $\boldsymbol{Q}^{cr}$ and squared voltage profile deviation, respectively. (A-1e)-(A-1g) represents the spanning tree constraints with multiple sources for interconnected ADNs [2].

Therefore, we can summarize $\boldsymbol{c}$, $\boldsymbol{A}$, $\boldsymbol{G}_v$, $\boldsymbol{G}_{cr}$, $\boldsymbol{d}$, $\boldsymbol{b}_v$ and $\boldsymbol{b}_{cr}$ in (A-1b) as

$$\boldsymbol{c} = \begin{bmatrix} \boldsymbol{D}_r & \boldsymbol{0}_{|\mathcal{E}|\times|\mathcal{E}|} & \boldsymbol{0}_{|\mathcal{E}|\times n_{cr}} \\ \boldsymbol{0}_{|\mathcal{E}|\times|\mathcal{E}|} & \boldsymbol{D}_r & \boldsymbol{0}_{|\mathcal{E}|\times n_{cr}} \\ \boldsymbol{0}_{n_{cr}\times|\mathcal{E}|} & \boldsymbol{0}_{n_{cr}\times|\mathcal{E}|} & \boldsymbol{0}_{n_{cr}\times n_{cr}} \end{bmatrix} \tag{A-2a}$$

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_{\mathcal{G}}^T & \boldsymbol{0}_{|\mathcal{E}|\times|\mathcal{E}|} & \boldsymbol{0}_{|\mathcal{E}|\times n_{cr}} \\ \boldsymbol{0}_{|\mathcal{E}|\times|\mathcal{E}|} & \boldsymbol{A}_{\mathcal{G}}^T & \boldsymbol{A}_{cr} \end{bmatrix}, \ \boldsymbol{d} = \begin{bmatrix} -\boldsymbol{P}^g + \boldsymbol{P}^d \\ -\boldsymbol{Q}^g + \boldsymbol{Q}^d \end{bmatrix} \tag{A-2b}$$

$$\boldsymbol{G}_v = \begin{bmatrix} 2\boldsymbol{D}_r & 2\boldsymbol{D}_x & \boldsymbol{0}_{|\mathcal{E}|\times n_{cr}} & M \\ -2\boldsymbol{D}_r & -2\boldsymbol{D}_x & \boldsymbol{0}_{|\mathcal{E}|\times n_{cr}} & -M \end{bmatrix} \tag{A-2c}$$

$$\boldsymbol{G}_u = \begin{bmatrix} diag(\mathbb{1}_N) & \boldsymbol{0}_{|\mathcal{E}|\times|\mathcal{E}|} & \boldsymbol{0}_{|\mathcal{E}|\times n_{cr}} & -\Gamma \\ -diag(\mathbb{1}_N) & \boldsymbol{0}_{|\mathcal{E}|\times|\mathcal{E}|} & \boldsymbol{0}_{|\mathcal{E}|\times n_{cr}} & -\Gamma \\ \boldsymbol{0}_{|\mathcal{E}|\times|\mathcal{E}|} & diag(\mathbb{1}_N) & \boldsymbol{0}_{|\mathcal{E}|\times n_{cr}} & -\Gamma \\ \boldsymbol{0}_{|\mathcal{E}|\times|\mathcal{E}|} & -diag(\mathbb{1}_N) & \boldsymbol{0}_{|\mathcal{E}|\times n_{cr}} & -\Gamma \end{bmatrix} \tag{A-2d}$$

$$\boldsymbol{G}_{cr} = \begin{bmatrix} \boldsymbol{0}_{n_{cr}\times|\mathcal{E}|} & \boldsymbol{0}_{n_{cr}\times|\mathcal{E}|} & diag(\mathbb{1}_c) & \boldsymbol{0}_{n_{cr}\times|\mathcal{E}|} \\ \boldsymbol{0}_{n_{cr}\times|\mathcal{E}|} & \boldsymbol{0}_{n_{cr}\times|\mathcal{E}|} & -diag(\mathbb{1}_c) & \boldsymbol{0}_{n_{cr}\times|\mathcal{E}|} \end{bmatrix} \tag{A-2e}$$

$$\boldsymbol{b}_v = \begin{bmatrix} \Delta\overline{\boldsymbol{v}} + M \\ -\Delta\underline{\boldsymbol{v}} + M \end{bmatrix}, \ \boldsymbol{b}_{cr} = \begin{bmatrix} \overline{\boldsymbol{Q}}_{\boldsymbol{cr}} \\ -\underline{\boldsymbol{Q}}_{\boldsymbol{cr}} \end{bmatrix} \tag{A-2f}$$

$$\boldsymbol{b}_u = \boldsymbol{0}_{4|\mathcal{E}|\times 1} \tag{A-2g}$$

where $\boldsymbol{K}$ and $\boldsymbol{h}$ can be rewritten from (A-1e)-(A-1g).

II. ε-DIFFERENTIALLY PRIVATE PROOF OF DP-DNR MECHANISM

Theorem: The DP-DNR mechanism $\widetilde{\mathcal{M}}$ can output the obfuscated load flows as $\widetilde{\mathcal{M}} = \boldsymbol{x}^{l_k} + \boldsymbol{\alpha}^{l_k}\boldsymbol{\xi}^{l_k}$ for an arbitrary interconnected branch l_k subject to $(\boldsymbol{x}^l + \boldsymbol{\alpha}^l\boldsymbol{\xi}^l, \boldsymbol{\beta}^l, \boldsymbol{u}^l) \in \mathcal{X}$, which solution obeys ε-differentially private.

Proof: According to the definition of DP [3], let the query output answers be $\mathcal{O} = \boldsymbol{x}^{l*} + \boldsymbol{\alpha}^{l^*}\boldsymbol{\xi}^l$, and we alternatively rewrite this proposition as

$$\mathbb{P}_\xi[\widetilde{\mathcal{M}}(\boldsymbol{d}) = \mathcal{O}] \leqslant \mathbb{P}_\xi[\widetilde{\mathcal{M}}(\boldsymbol{d}') = \mathcal{O}]e^\varepsilon \tag{A-4}$$

for any two ρ-neighborhood load datasets $\boldsymbol{d}$ and $\boldsymbol{d}'$ and output solutions $\mathcal{O}$.

Without loss of generality, we define $\boldsymbol{\xi}^l = [\xi^{l_1}, ..., \xi^{l_n}]$, $\boldsymbol{\alpha}^l = [\alpha^{l_1}, ..., \alpha^{l_n}]$, $\boldsymbol{x}^l = [x^{l_1}, ..., x^{l_n}]$, where n refers to the total number of interconnected branches $\{l_k\}$. Thus, the query output $\{\mathcal{O}_k\}$ for all interconnected branches $\{l_k\}$, $\forall k = 1, ..., n$, with the vectors of $\boldsymbol{\xi}^l$, $\boldsymbol{\alpha}^l$ and $\boldsymbol{x}^l$, can be written as

$$\mathcal{O}_k = \boldsymbol{x}^{l_k^*} + \boldsymbol{\alpha}^{l_k^*}\boldsymbol{\xi}^{l_k}, \quad \forall k \in [1, n] \tag{A-4}$$

With achievable ℓ_1-sensitivity Δ_ρ value from given measurable quantity ρ, the ratio of probabilities on two ρ-indistinguishable load datasets $\boldsymbol{d}$ and $\boldsymbol{d}'$ can be bounded by

$$\begin{aligned}
&\frac{\mathbb{P}_\xi[\widetilde{\mathcal{M}}(\boldsymbol{d}) = \mathcal{O}]}{\mathbb{P}_\xi[\widetilde{\mathcal{M}}(\boldsymbol{d}') = \mathcal{O}]} = \frac{\mathbb{P}_\xi\left[\begin{bmatrix}\boldsymbol{x}^{l_1^*}(\boldsymbol{d})\\ \vdots \\ \boldsymbol{x}^{l_n^*}(\boldsymbol{d})\end{bmatrix} + \begin{bmatrix}\boldsymbol{\alpha}^{l_1^*}\\ \vdots \\ \boldsymbol{\alpha}^{l_n^*}\end{bmatrix}\begin{bmatrix}\boldsymbol{\xi}^{l_1^*}\\ \vdots \\ \boldsymbol{\xi}^{l_n^*}\end{bmatrix} = \begin{bmatrix}\mathcal{O}_1\\ \vdots \\ \mathcal{O}_n\end{bmatrix}\right]}{\mathbb{P}_\xi\left[\begin{bmatrix}\boldsymbol{x}^{l_1^*}(\boldsymbol{d}')\\ \vdots \\ \boldsymbol{x}^{l_n^*}(\boldsymbol{d}')\end{bmatrix} + \begin{bmatrix}\boldsymbol{\alpha}^{l_1^*}\\ \vdots \\ \boldsymbol{\alpha}^{l_n^*}\end{bmatrix}\begin{bmatrix}\boldsymbol{\xi}^{l_1^*}\\ \vdots \\ \boldsymbol{\xi}^{l_n^*}\end{bmatrix} = \begin{bmatrix}\mathcal{O}_1\\ \vdots \\ \mathcal{O}_n\end{bmatrix}\right]} \\
&= \frac{\mathbb{P}_\xi\left[\begin{bmatrix}\boldsymbol{\alpha}^{l_1^*}\boldsymbol{\xi}^{l_1^*}\\ \vdots \\ \boldsymbol{\alpha}^{l_n^*}\boldsymbol{\xi}^{l_n^*}\end{bmatrix} = \begin{bmatrix}\mathcal{O}_1 - \boldsymbol{x}^{l_1^*}(\boldsymbol{d})\\ \vdots \\ \mathcal{O}_n - \boldsymbol{x}^{l_n^*}(\boldsymbol{d})\end{bmatrix}\right]}{\mathbb{P}_\xi\left[\begin{bmatrix}\boldsymbol{\alpha}^{l_1^*}\boldsymbol{\xi}^{l_1^*}\\ \vdots \\ \boldsymbol{\alpha}^{l_n^*}\boldsymbol{\xi}^{l_n^*}\end{bmatrix} = \begin{bmatrix}\mathcal{O}_1 - \boldsymbol{x}^{l_1^*}(\boldsymbol{d}')\\ \vdots \\ \mathcal{O}_n - \boldsymbol{x}^{l_n^*}(\boldsymbol{d}')\end{bmatrix}\right]} \overset{(i)}{=} \frac{\prod_{k=1}^n \boldsymbol{\alpha}^{l_k^*} \exp\{-\frac{\varepsilon\|\mathcal{O}_k - \boldsymbol{x}^{l_k^*}(\boldsymbol{d})\|_1}{\Delta_\rho}\}}{\prod_{k=1}^n \boldsymbol{\alpha}^{l_k^*} \exp\{-\frac{\varepsilon\|\mathcal{O}_k - \boldsymbol{x}^{l_k^*}(\boldsymbol{d}')\|_1}{\Delta_\rho}\}} \\
&= \prod_{k=1}^n \exp\left(\frac{\varepsilon\|\mathcal{O}_k - \boldsymbol{x}^{l_k^*}(\boldsymbol{d}')\|_1 - \varepsilon\|\mathcal{O}_k - \boldsymbol{x}^{l_k^*}(\boldsymbol{d})\|_1}{\Delta_\rho}\right) \\
&\overset{(ii)}{\leqslant} \prod_{k=1}^n \exp\left(\frac{\varepsilon\|\boldsymbol{x}^{l_k^*}(\boldsymbol{d}) - \boldsymbol{x}^{l_k^*}(\boldsymbol{d}')\|_1}{\Delta_\rho}\right) = \exp\left(\frac{\varepsilon\|\boldsymbol{x}^{l^*}(\boldsymbol{d}) - \boldsymbol{x}^{l^*}(\boldsymbol{d}')\|_1}{\Delta_\rho}\right) \\
&\overset{(iii)}{\leqslant} \exp\left(\frac{\varepsilon\Delta_\rho}{\Delta_\rho}\right) = e^\varepsilon
\end{aligned} \tag{A-5}$$

where (i) comes from the definition of the probability density function of the Laplace distribution. In (ii) step, it is followed by the reverse inequailty of norms. For the (iii) step, $\|\boldsymbol{x}^{l^*}(\boldsymbol{d}) - \boldsymbol{x}^{l^*}(\boldsymbol{d}')\|_1$ denotes the ℓ_1-sensitivity on ρ-indistinguishable input datasets $\boldsymbol{x}^{l^*}(\boldsymbol{d})$ and $\boldsymbol{x}^{l^*}(\boldsymbol{d}')$ subject to $\|\boldsymbol{x}^{l^*}(\boldsymbol{d}) - \boldsymbol{x}^{l^*}(\boldsymbol{d}')\|_1 \leqslant \Delta_\rho$.

Accordingly, it is clear from (A-6) that (A-4) holds, which proves the Theorem. ■

References

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