Supplementary Material for "Load Transfer Optimization with Graph Characterizations on Different Time-Scales for Multi-Voltage Distribution Networks Against Overload Cascades"

Chao Lei, *Student Member, IEEE*, Siqi Bu, *Senior Member, IEEE*, Qifan Chen and Qianggang Wang, *Senior Member, IEEE*

A. Linearized Formulation of DSC Graph

We define $\boldsymbol{x}_{dsc,linear}$ as $[\boldsymbol{P}_{g,A}, \boldsymbol{Q}_{g,A}, \boldsymbol{P}_{g,B}, \boldsymbol{Q}_{g,B}, \boldsymbol{v}_C]^T$, and we can derive the linearized form as

$$\boldsymbol{\Omega}_{dsc,linear}=\{\boldsymbol{x}_{dsc,linear}\in\mathbb{R}, u_1, u_2\in\mathbb{Z}|\text{(A-2a)} - \text{(A-2c)}\} \quad \text{(A-1)}$$

where the linearized power flow equations are

$$\begin{bmatrix}\boldsymbol{P}_{g,A}\\ \boldsymbol{Q}_{g,A}\\ \boldsymbol{P}_{g,B}\\ \boldsymbol{Q}_{g,B}\end{bmatrix}=\begin{bmatrix}\boldsymbol{P}_d & 0\\ \boldsymbol{Q}_d & 0\\ 0 & \boldsymbol{P}_d\\ 0 & \boldsymbol{Q}_d\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix} \quad \text{(A-2a)}$$

$$\boldsymbol{v}_C=\big[\boldsymbol{v}_A-2(\boldsymbol{P}_d r^l_{AC}+\boldsymbol{Q}_d x^l_{AC}) \;\; \boldsymbol{v}_B-2(\boldsymbol{P}_d r^l_{BC}+\boldsymbol{Q}_d x^l_{BC})\big]\begin{bmatrix}u_1\\ u_2\end{bmatrix} \quad \text{(A-2b)}$$

$$\underline{v}\leqslant \boldsymbol{v}_C\leqslant \overline{v},\; u_1+u_2=1, \quad u_1,u_2\in\mathbb{Z} \quad \text{(A-2c)}$$

B. Linearized Formulation of SSC Graph

We define $\boldsymbol{x}_{ssc,linear} := [\boldsymbol{P}_{g,A}, \boldsymbol{Q}_{g,A},\ \boldsymbol{P}_{g,B}, \boldsymbol{Q}_{g,B}, \widehat{\boldsymbol{P}}^l_{CD}, \widehat{\boldsymbol{Q}}^l_{CD}, \boldsymbol{v}_C, \boldsymbol{v}_D]^T$, and we can formulate the linearized expression as

$$\boldsymbol{\Omega}_{ssc,linear}:=\{\boldsymbol{x}_{ssc,linear}\in\mathbb{R}, u_1,u_2,u_3\in\mathbb{Z}|\text{(A-4a)} - \text{(A-4h)}\} \quad \text{(A-3)}$$

where the corresponding linearized power flow equations are

$$\begin{bmatrix}\boldsymbol{P}_{g,A}\\ \boldsymbol{Q}_{g,A}\\ \boldsymbol{P}_{g,B}\\ \boldsymbol{Q}_{g,B}\end{bmatrix}=\begin{bmatrix}\boldsymbol{P}_{d,1}+\boldsymbol{P}_{d,2} & \boldsymbol{P}_{d,2} & 0\\ \boldsymbol{Q}_{d,1}+\boldsymbol{Q}_{d,2} & \boldsymbol{Q}_{d,2} & 0\\ 0 & \boldsymbol{P}_{d,1} & \boldsymbol{P}_{d,1}+\boldsymbol{P}_{d,2}\\ 0 & \boldsymbol{Q}_{d,1} & \boldsymbol{Q}_{d,1}+\boldsymbol{Q}_{d,2}\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix}-\begin{bmatrix}\boldsymbol{P}_{d,2}\\ \boldsymbol{Q}_{d,2}\\ \boldsymbol{P}_{d,1}\\ \boldsymbol{Q}_{d,1}\end{bmatrix} \quad \text{(A-4a)}$$

$$\begin{bmatrix}\boldsymbol{v}_C\\ \boldsymbol{v}_D\end{bmatrix}=\begin{bmatrix}\Delta v_{AC} & 0\\ 0 & \Delta v_{BD}\end{bmatrix}\begin{bmatrix}u_1\\ u_3\end{bmatrix}-2\begin{bmatrix}r^l_{AC} & x^l_{AC}\\ r^l_{BC} & x^l_{BC}\end{bmatrix}\begin{bmatrix}\widehat{\boldsymbol{P}}^l_{CD}\\ \widehat{\boldsymbol{Q}}^l_{CD}\end{bmatrix} \quad \text{(A-4b)}$$

$$\Delta v_{AC}=\boldsymbol{v}_A-2[\boldsymbol{P}_{d,1}r^l_{AC}+\boldsymbol{Q}_{d,1}x^l_{AC}] \quad \text{(A-4c)}$$

$$\Delta v_{BD}=\boldsymbol{v}_B-2[\boldsymbol{P}_{d,2}r^l_{BC}+\boldsymbol{Q}_{d,2}x^l_{BC}] \quad \text{(A-4d)}$$

$$\begin{bmatrix}\boldsymbol{P}_{d,2}\\ \boldsymbol{Q}_{d,2}\end{bmatrix}+(u_1+u_2-2)B\leqslant\begin{bmatrix}\widehat{\boldsymbol{P}}^l_{CD}\\ \widehat{\boldsymbol{Q}}^l_{CD}\end{bmatrix}\leqslant\begin{bmatrix}\boldsymbol{P}_{d,2}\\ \boldsymbol{Q}_{d,2}\end{bmatrix}+(2-u_1-u_2)B \quad \text{(A-4e)}$$

$$\begin{bmatrix}\boldsymbol{P}_{d,1}\\ \boldsymbol{Q}_{d,1}\end{bmatrix}+(u_3+u_2-2)B\leqslant\begin{bmatrix}\widehat{\boldsymbol{P}}^l_{CD}\\ \widehat{\boldsymbol{Q}}^l_{CD}\end{bmatrix}\leqslant\begin{bmatrix}\boldsymbol{P}_{d,1}\\ \boldsymbol{Q}_{d,1}\end{bmatrix}+(2-u_3-u_2)B \quad \text{(A-4f)}$$

$$|\widehat{\boldsymbol{P}}^l_{CD}|, |\widehat{\boldsymbol{Q}}^l_{CD}|\leqslant u_2 B,\; \underline{v}\leqslant\boldsymbol{v}_C,\boldsymbol{v}_D\leqslant\overline{v} \quad \text{(A-4g)}$$

$$u_1+u_2+u_3=2, \quad u_1,u_2,u_3\in\mathbb{Z} \quad \text{(A-4h)}$$

C. Linearized Formulation of TSSC Graph

We define $\boldsymbol{x}_{tssc,linear}:=[\boldsymbol{P}_{g,A1}, \boldsymbol{Q}_{g,A1}, \boldsymbol{P}_{g,A2}, \boldsymbol{Q}_{g,A2}, \boldsymbol{P}_{g,B}, \boldsymbol{Q}_{g,B}, \boldsymbol{P}^l_{DC}, \boldsymbol{Q}^l_{DC}, \boldsymbol{v}_C, \boldsymbol{V}_D, \boldsymbol{v}_D, \widehat{\boldsymbol{P}}^l_{DC,A1}, \widehat{\boldsymbol{Q}}^l_{DC,A1}, \widehat{\boldsymbol{P}}^l_{DC,A2}, \widehat{\boldsymbol{Q}}^l_{DC,A2}]^T$, so that we write the linearized form of TSSC graph as

$$\boldsymbol{\Omega}_{tssc,linear}:=\{\boldsymbol{x}_{tssc,linear}\in\mathbb{R}, u_1,u_2,u_3,u_4\in\mathbb{Z}|\text{(A-6a)} - \text{(A-6j)}\} \quad \text{(A-5)}$$

where the corresponding linearized power flow equations are

$$\begin{bmatrix}\boldsymbol{P}^l_{DC}\\ \boldsymbol{Q}^l_{DC}\\ \boldsymbol{P}_{g,B}\\ \boldsymbol{Q}_{g,B}\end{bmatrix}=\begin{bmatrix}\boldsymbol{P}_{d,2} & 0\\ \boldsymbol{Q}_{d,2} & 0\\ 0 & \boldsymbol{P}_{d,2}\\ 0 & \boldsymbol{Q}_{d,2}\end{bmatrix}\begin{bmatrix}u_3\\ u_4\end{bmatrix} \quad \text{(A-6a)}$$

$$\boldsymbol{v}_C=\big[-2(\boldsymbol{P}_{d,2}r^l_{DC}+\boldsymbol{Q}_{d,2}x^l_{DC}) \;\; \boldsymbol{v}_B-2(\boldsymbol{P}_{d,2}r^l_{BC}+\boldsymbol{Q}_{d,2}x^l_{BC})\big]\begin{bmatrix}u_3\\ u_4\end{bmatrix} + \boldsymbol{V}_D \quad \text{(A-6b)}$$

$$\boldsymbol{v}_D+(u_3-1)\underline{v}\leqslant\boldsymbol{V}_D\leqslant\boldsymbol{v}_D+(1-u_3)\overline{v},\; u_3\underline{v}\leqslant\boldsymbol{V}_D\leqslant u_3\overline{v} \quad \text{(A-6c)}$$

$$\begin{bmatrix}\boldsymbol{P}_{g,A1}\\ \boldsymbol{Q}_{g,A1}\\ \boldsymbol{P}_{g,A2}\\ \boldsymbol{Q}_{g,A2}\end{bmatrix}=\begin{bmatrix}\boldsymbol{P}_{d,1}\\ \boldsymbol{Q}_{d,1}\\ \boldsymbol{P}_{d,1}\\ \boldsymbol{Q}_{d,1}\end{bmatrix}+\begin{bmatrix}\boldsymbol{P}^l_{DC}\\ \boldsymbol{Q}^l_{DC}\\ \boldsymbol{P}^l_{DC}\\ \boldsymbol{Q}^l_{DC}\end{bmatrix} \quad \text{(A-6d)}$$

$$\boldsymbol{v}_D= \big[\boldsymbol{v}_{A1}-2(\boldsymbol{P}_{d,1}r^l_{A1D}+\boldsymbol{Q}_{d,1}x^l_{A1D}) \;\; \boldsymbol{v}_{A2}-2(\boldsymbol{P}_{d,1}r^l_{A2D}+\boldsymbol{Q}_{d,1}x^l_{A2D})\big]\begin{bmatrix}u_1\\ u_2\end{bmatrix} - \big[2r^l_{A1D} \;\; 2x^l_{A1D}\big]\begin{bmatrix}\widehat{\boldsymbol{P}}^l_{DC,A1}\\ \widehat{\boldsymbol{Q}}^l_{DC,A1}\end{bmatrix}+\big[2r^l_{A2D} \;\; 2x^l_{A2D}\big]\begin{bmatrix}\widehat{\boldsymbol{P}}^l_{DC,A2}\\ \widehat{\boldsymbol{Q}}^l_{DC,A2}\end{bmatrix} \quad \text{(A-6e)}$$

$$|\widehat{\boldsymbol{P}}^l_{DC,A1}|, |\widehat{\boldsymbol{Q}}^l_{DC,A1}|\leqslant u_1 B,\; |\widehat{\boldsymbol{P}}^l_{DC,A2}|, |\widehat{\boldsymbol{Q}}^l_{DC,A2}|\leqslant u_2 B \quad \text{(A-6f)}$$

$$\begin{bmatrix}\boldsymbol{P}^l_{DC}\\ \boldsymbol{Q}^l_{DC}\end{bmatrix}+(u_1-1)B\leqslant\begin{bmatrix}\widehat{\boldsymbol{P}}^l_{DC,A1}\\ \widehat{\boldsymbol{Q}}^l_{DC,A1}\end{bmatrix}\leqslant\begin{bmatrix}\boldsymbol{P}^l_{DC}\\ \boldsymbol{Q}^l_{DC}\end{bmatrix}+(1-u_1)B \quad \text{(A-6g)}$$

$$\begin{bmatrix}\boldsymbol{P}^l_{DC}\\ \boldsymbol{Q}^l_{DC}\end{bmatrix}+(u_2-1)B\leqslant\begin{bmatrix}\widehat{\boldsymbol{P}}^l_{DC,A2}\\ \widehat{\boldsymbol{Q}}^l_{DC,A2}\end{bmatrix}\leqslant\begin{bmatrix}\boldsymbol{P}^l_{DC}\\ \boldsymbol{Q}^l_{DC}\end{bmatrix}+(1-u_2)B \quad \text{(A-6h)}$$

$$\underline{v}\leqslant\boldsymbol{v}_C,\boldsymbol{v}_D\leqslant\overline{v},\; |\boldsymbol{P}^l_{DC}|\leqslant u_3 B, |\boldsymbol{Q}^l_{DC}|\leqslant u_3 B \quad \text{(A-6i)}$$

$$u_1+u_2+u_3+u_4=2,\; u_3+u_4\leqslant 1,\; u_1,u_2,u_3,u_4\in\mathbb{Z} \quad \text{(A-6j)}$$

D. Linearized Formulation of MSSC Graph

We define $\boldsymbol{x}_{mssc,linear}:=[\boldsymbol{P}_{g,A1}, \boldsymbol{Q}_{g,A1}, \boldsymbol{P}_{g,A2}, \boldsymbol{Q}_{g,A2}, \boldsymbol{P}_{g,B}, \boldsymbol{Q}_{g,B}, \boldsymbol{P}^l_{CD}, \boldsymbol{Q}^l_{CD}, \boldsymbol{P}^l_{CE}, \boldsymbol{Q}^l_{CE}, \boldsymbol{v}_C, \boldsymbol{v}_D, \boldsymbol{v}_E, \boldsymbol{v}_\dagger, \widehat{\boldsymbol{P}}^l_{CE,A1}, \widehat{\boldsymbol{Q}}^l_{CE,A1}, \widehat{\boldsymbol{P}}^l_{CE,B}, \widehat{\boldsymbol{Q}}^l_{CE,B}]^T$, and then we write the linearized form of MSSC graph as

$$\boldsymbol{\Omega}_{mssc,linear}:=\left\{\begin{matrix}\boldsymbol{x}_{mssc,linear}\in\mathbb{R},\\ u_1,u_2,u_3,u_4,u_5\in\mathbb{Z}\end{matrix}\middle| \text{(A-8a)} - \text{(A-8q)}\right\} \quad \text{(A-7)}$$

where the corresponding linearized power flow equations are

$$\begin{bmatrix}\boldsymbol{P}^l_{CE}\\ \boldsymbol{Q}^l_{CE}\\ \boldsymbol{P}_{g,B}\\ \boldsymbol{Q}_{g,B}\end{bmatrix}=\begin{bmatrix}\boldsymbol{P}_{d,3} & 0\\ \boldsymbol{Q}_{d,3} & 0\\ 0 & \boldsymbol{P}_{d,3}\\ 0 & \boldsymbol{Q}_{d,3}\end{bmatrix}\begin{bmatrix}u_4\\ u_5\end{bmatrix} \quad \text{(A-8a)}$$

$$\underline{v}\leqslant\boldsymbol{v}_C,\boldsymbol{v}_D,\boldsymbol{v}_E\leqslant\overline{v} \quad \text{(A-8b)}$$

$$|\widehat{\boldsymbol{P}}^l_{CE,A1}|, |\widehat{\boldsymbol{Q}}^l_{CE,A1}|\leqslant u_1 B,\; |\widehat{\boldsymbol{P}}^l_{CE,B}|, |\widehat{\boldsymbol{Q}}^l_{CE,B}|\leqslant u_5 B \quad \text{(A-8c)}$$

$$\begin{bmatrix}\boldsymbol{P}_{g,A1}\\\boldsymbol{Q}_{g,A1}\\\boldsymbol{P}_{g,A2}\\\boldsymbol{Q}_{g,A2}\end{bmatrix}=\begin{bmatrix}\boldsymbol{P}_{d,1}+\boldsymbol{P}_{d,2} & \boldsymbol{P}_{d,2} & 0\\\boldsymbol{Q}_{d,1}+\boldsymbol{Q}_{d,2} & \boldsymbol{Q}_{d,2} & 0\\0 & \boldsymbol{P}_{d,1} & \boldsymbol{P}_{d,1}+\boldsymbol{P}_{d,2}\\0 & \boldsymbol{Q}_{d,1} & \boldsymbol{Q}_{d,1}+\boldsymbol{Q}_{d,2}\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}-\begin{bmatrix}\boldsymbol{P}_{d,2}\\\boldsymbol{Q}_{d,2}\\\boldsymbol{P}_{d,1}\\\boldsymbol{Q}_{d,1}\end{bmatrix}+\begin{bmatrix}\widehat{\boldsymbol{P}}^l_{CE,A1}\\\widehat{\boldsymbol{Q}^l}_{CE,A1}\\0\\0\end{bmatrix} \tag{A-8d}$$

$$\begin{bmatrix}\boldsymbol{P}^l_{CE}\\\boldsymbol{Q}^l_{CE}\end{bmatrix}+(u_1-1)B\leqslant\begin{bmatrix}\widehat{\boldsymbol{P}}^l_{CE,A1}\\\widehat{\boldsymbol{Q}}^l_{CE,A1}\end{bmatrix}\leqslant\begin{bmatrix}\boldsymbol{P}^l_{CE}\\\boldsymbol{Q}^l_{CE}\end{bmatrix}+(1-u_1)B \tag{A-8e}$$

$$\begin{bmatrix}\boldsymbol{P}_{g,B}\\\boldsymbol{Q}_{g,B}\end{bmatrix}=\begin{bmatrix}\boldsymbol{P}_{d,3}\\\boldsymbol{Q}_{d,3}\end{bmatrix}u_5+\begin{bmatrix}\widehat{\boldsymbol{P}}^l_{CE,B}\\\widehat{\boldsymbol{Q}}^l_{CE,B}\end{bmatrix} \tag{A-8f}$$

$$\begin{bmatrix}\boldsymbol{P}^l_{CE}\\\boldsymbol{Q}^l_{CE}\end{bmatrix}+(u_4-1)B\leqslant\begin{bmatrix}\widehat{\boldsymbol{P}}^l_{CE,B}\\\widehat{\boldsymbol{Q}}^l_{CE,B}\end{bmatrix}\leqslant\begin{bmatrix}\boldsymbol{P}^l_{CE}\\\boldsymbol{Q}^l_{CE}\end{bmatrix}+(1-u_4)B \tag{A-8g}$$

$$\begin{bmatrix}\boldsymbol{v}_C\\\boldsymbol{v}_D\end{bmatrix}=\begin{bmatrix}\Delta v_{A1C} & 0\\0 & \Delta v_{A2D}\end{bmatrix}\begin{bmatrix}u_1\\u_3\end{bmatrix}-2\begin{bmatrix}r^l_{A1C} & x^l_{A1C}\\0 & 0\end{bmatrix}\begin{bmatrix}\widehat{\boldsymbol{P}}^l_{CE,A1}\\\widehat{\boldsymbol{Q}}^l_{CE,A1}\end{bmatrix} \tag{A-8a}$$

$$-2\begin{bmatrix}r^l_{A1C} & x^l_{A1C}\\r^l_{A2D} & x^l_{A2D}\end{bmatrix}\begin{bmatrix}\widehat{\boldsymbol{P}}^l_{CD}\\\widehat{\boldsymbol{Q}}^l_{CD}\end{bmatrix} \tag{A-8b}$$

(A-8h)

$$\Delta v_{A1C}=\boldsymbol{v}_{A1}-2[\boldsymbol{P}_{d,1}r^l_{A1C}+\boldsymbol{Q}_{d,1}x^l_{A1C}] \tag{A-8i}$$

$$\Delta v_{A2D}=\boldsymbol{v}_{A2}-2[\boldsymbol{P}_{d,2}r^l_{A2D}+\boldsymbol{Q}_{d,2}x^l_{A2D}] \tag{A-8j}$$

$$\begin{bmatrix}\boldsymbol{P}_{d,2}\\\boldsymbol{Q}_{d,2}\end{bmatrix}+(u_1+u_2-2)B\leqslant\begin{bmatrix}\widehat{\boldsymbol{P}}^l_{CD}\\\widehat{\boldsymbol{Q}}^l_{CD}\end{bmatrix}\leqslant\begin{bmatrix}\boldsymbol{P}_{d,2}\\\boldsymbol{Q}_{d,2}\end{bmatrix}+(2-u_1-u_2)B \tag{A-8k}$$

$$\begin{bmatrix}\boldsymbol{P}_{d,1}\\\boldsymbol{Q}_{d,1}\end{bmatrix}+(u_3+u_2-2)B\leqslant\begin{bmatrix}\widehat{\boldsymbol{P}}^l_{CD}\\\widehat{\boldsymbol{Q}}^l_{CD}\end{bmatrix}\leqslant\begin{bmatrix}\boldsymbol{P}_{d,1}\\\boldsymbol{Q}_{d,1}\end{bmatrix}+(2-u_3-u_2)B \tag{A-8l}$$

$$\boldsymbol{v}_E=\begin{bmatrix}-2(\boldsymbol{P}_{d,3}r^l_{CE}+\boldsymbol{Q}_{d,3}x^l_{CE}) & \boldsymbol{v}_B-2(\boldsymbol{P}_{d,3}r^l_{BE}+\boldsymbol{Q}_{d,3}x^l_{BE})\end{bmatrix}\begin{bmatrix}u_4\\u_5\end{bmatrix}+\boldsymbol{v}_\dagger \tag{A-8m}$$

$$\boldsymbol{v}_C-(1-u_4)M\leqslant\boldsymbol{v}_\dagger\leqslant\boldsymbol{v}_C+(1-u_4)M,\ u_4\underline{v}\leqslant\boldsymbol{v}_\dagger\leqslant u_4\overline{v} \tag{A-8n}$$

$$|\widehat{\boldsymbol{P}}^l_{CD}|\leqslant u_2B,\ |\widehat{\boldsymbol{Q}}^l_{CD}|\leqslant u_2B \tag{A-8o}$$

$$u_1+u_2+u_3+u_4+u_5=3,\ u_1+u_2+u_3\leqslant 2 \tag{A-8p}$$

$$u_1+u_4+u_5\leqslant 2,\ u_2+u_4\leqslant 1,\ u_1,u_2,u_3,u_4,u_5\in\mathbb{Z} \tag{A-8q}$$