# Supplementary Material for "Decentralized Optimal Power Flow for Multi-Agent Active Distribution Networks: A Differentially Private Consensus ADMM Algorithm "

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## I. Differential Privacy Proof

**Theorem:** Suppose random perturbations obey Laplace distribution $\boldsymbol{\xi}^{l_t} \sim \mathbb{P}_\xi$, $\mathbb{P}_\xi := Lap(\Delta_\rho/\varepsilon)$, for an arbitrary branch $l_t \in \mathcal{T}$ across agents. And $(\hat{\boldsymbol{x}}^l, \hat{\boldsymbol{x}}^g) \in \mathcal{X}$ is the optimal solution of non-private D-OPF model. Then, this differentially private D-OPF model can output a mixture of obfuscated-but-feasible load flow solution $\boldsymbol{x}^{l_t^*}$ for the branches across agents and realistic generator output solution $\boldsymbol{x}^{g*}$ , such that $\hat{\boldsymbol{x}}^{l_t} = \boldsymbol{x}^{l_t^*} + \boldsymbol{\alpha}^{l_t^*}\boldsymbol{\xi}^{l_t}$ for $\forall l_t \in \mathcal{T}$ and $\boldsymbol{x}^{g*} = \hat{\boldsymbol{x}}^g$.

**Proof:** According to the definition of Laplace mechanism, let the query output answers be $\mathcal{O} = \boldsymbol{x}^{l_t^*}$, and we alternatively rewrite this theorem in the definition of $\varepsilon$-differential privacy.

$$\mathbb{P}_\xi[\widetilde{\mathcal{M}}(\boldsymbol{d}) = \mathcal{O}] \leqslant \mathbb{P}_\xi[\widetilde{\mathcal{M}}(\boldsymbol{d}') = \mathcal{O}]e^{\varepsilon} \tag{A-1}$$

for any two $\rho$-neighborhood load datasets $\boldsymbol{d}$ and $\boldsymbol{d}'$ and output solutions $\mathcal{O}$.

For convenience, we define $\boldsymbol{\xi}^l = [\xi^{l_1}, ..., \xi^{l_n}]$, $\{\boldsymbol{\alpha}^{l_t}\} = [\alpha^{l_1}, ..., \alpha^{l_n}]$, $\{\boldsymbol{x}^{l_t}\} = [x^{l_1}, ..., x^{l_n}]$, where $n$ refers to the total number of tie-lines. Thus, the query output $\{\mathcal{O}^{l_t}\}$ for all tie-lines $\forall l_t \in \mathcal{T}$, $\forall t = 1, ..., n$, with the vectors of $\boldsymbol{\xi}^l$, can be written as

$$\mathcal{O}^{l_t} = \boldsymbol{x}^{l_t^*} = \hat{\boldsymbol{x}}^{l_t} - \boldsymbol{\alpha}^{l_t^*}\boldsymbol{\xi}^{l_t}, \quad \forall t \in [1, n] \tag{A-2}$$

Therefore, the ratio of probabilities on two $\rho$-indistinguishable load datasets $\boldsymbol{d}$ and $\boldsymbol{d}'$ can be bounded by

$$\begin{aligned}
\frac{\mathbb{P}_\xi[\widetilde{\mathcal{M}}(\boldsymbol{d}) = \mathcal{O}^{l_t}]}{\mathbb{P}_\xi[\widetilde{\mathcal{M}}(\boldsymbol{d}') = \mathcal{O}^{l_t}]} &= \frac{\mathbb{P}_\xi\left[\begin{bmatrix}\hat{\boldsymbol{x}}^{l_1}(\boldsymbol{d})\\ \vdots \\ \hat{\boldsymbol{x}}^{l_n}(\boldsymbol{d})\end{bmatrix} - \begin{bmatrix}\boldsymbol{\alpha}^{l_1^*}\\ \vdots \\ \boldsymbol{\alpha}^{l_n^*}\end{bmatrix}\begin{bmatrix}\boldsymbol{\xi}^{l_1}\\ \vdots \\ \boldsymbol{\xi}^{l_n}\end{bmatrix} = \begin{bmatrix}\mathcal{O}^{l_1}\\ \vdots \\ \mathcal{O}^{l_n}\end{bmatrix}\right]}{\mathbb{P}_\xi\left[\begin{bmatrix}\hat{\boldsymbol{x}}^{l_1}(\boldsymbol{d}')\\ \vdots \\ \hat{\boldsymbol{x}}^{l_n}(\boldsymbol{d}')\end{bmatrix} - \begin{bmatrix}\boldsymbol{\alpha}^{l_1^*}\\ \vdots \\ \boldsymbol{\alpha}^{l_n^*}\end{bmatrix}\begin{bmatrix}\boldsymbol{\xi}^{l_1}\\ \vdots \\ \boldsymbol{\xi}^{l_n}\end{bmatrix} = \begin{bmatrix}\mathcal{O}^{l_1}\\ \vdots \\ \mathcal{O}^{l_n}\end{bmatrix}\right]} \\
&= \frac{\mathbb{P}_\xi\left[\begin{bmatrix}\boldsymbol{\alpha}^{l_1^*}\boldsymbol{\xi}^{l_1}\\ \vdots \\ \boldsymbol{\alpha}^{l_n^*}\boldsymbol{\xi}^{l_n}\end{bmatrix} = \begin{bmatrix}\mathcal{O}^{l_1} - \hat{\boldsymbol{x}}^{l_1}(\boldsymbol{d})\\ \vdots \\ \mathcal{O}^{l_n} - \hat{\boldsymbol{x}}^{l_n}(\boldsymbol{d})\end{bmatrix}\right]}{\mathbb{P}_\xi\left[\begin{bmatrix}\boldsymbol{\alpha}^{l_1^*}\boldsymbol{\xi}^{l_1}\\ \vdots \\ \boldsymbol{\alpha}^{l_n^*}\boldsymbol{\xi}^{l_n}\end{bmatrix} = \begin{bmatrix}\mathcal{O}^{l_1} - \hat{\boldsymbol{x}}^{l_1}(\boldsymbol{d}')\\ \vdots \\ \mathcal{O}^{l_n} - \hat{\boldsymbol{x}}^{l_n}(\boldsymbol{d}')\end{bmatrix}\right]} \overset{(i)}{=} \frac{\prod_{t=1}^{n} \boldsymbol{\alpha}^{l_t^*} \exp\{-\frac{\varepsilon\|\mathcal{O}^{l_t} - \hat{\boldsymbol{x}}^{l_t}(\boldsymbol{d})\|_1}{\Delta_\rho}\}}{\prod_{t=1}^{n} \boldsymbol{\alpha}^{l_t^*} \exp\{-\frac{\varepsilon\|\mathcal{O}^{l_t} - \hat{\boldsymbol{x}}^{l_t}(\boldsymbol{d}')\|_1}{\Delta_\rho}\}} \\
&= \prod_{t=1}^{n} \exp\left(\frac{\varepsilon\|\mathcal{O}^{l_t} - \hat{\boldsymbol{x}}^{l_t}(\boldsymbol{d}')\|_1 - \varepsilon\|\mathcal{O}^{l_t} - \hat{\boldsymbol{x}}^{l_t}(\boldsymbol{d})\|_1}{\Delta_\rho}\right) \\
&\overset{(ii)}{\leqslant} \prod_{t=1}^{n} \exp\left(\frac{\varepsilon\|\hat{\boldsymbol{x}}^{l_t}(\boldsymbol{d}) - \hat{\boldsymbol{x}}^{l_t}(\boldsymbol{d}')\|_1}{\Delta_\rho}\right) = \exp\left(\frac{\varepsilon\|\hat{\boldsymbol{x}}^{l}(\boldsymbol{d}) - \hat{\boldsymbol{x}}^{l}(\boldsymbol{d}')\|_1}{\Delta_\rho}\right) \\
&\overset{(iii)}{\leqslant} \exp\left(\frac{\varepsilon\Delta_\rho}{\Delta_\rho}\right) = e^{\varepsilon}
\end{aligned} \tag{A-3}$$

where $(i)$ comes from the definition of the probability density function of the Laplace distribution. In $(ii)$ step, it is followed by the inequailty of norms, i.e., $|a| - |b| \leqslant |a - b|$ for any real-valued numbers $a$ and $b$. For the $(iii)$ step, $\|\hat{\boldsymbol{x}}^l(\boldsymbol{d}) - \hat{\boldsymbol{x}}^l(\boldsymbol{d}')\|_1$ denotes the $\ell_1$-sensitivity on $\rho$-indistinguishable input datasets $\hat{\boldsymbol{x}}^l(\boldsymbol{d})$ and $\hat{\boldsymbol{x}}^l(\boldsymbol{d}')$ subject to $\|\hat{\boldsymbol{x}}^l(\boldsymbol{d}) - \hat{\boldsymbol{x}}^l(\boldsymbol{d}')\|_1 \leqslant \Delta_\rho$.

Accordingly, it is clear that (A-1) holds based on (A-3), which proves this Theorem. ■