Supplementary Material for "Interconnected Active Distribution Networks: A Differentially Private Reconfiguration Approach "

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I. MIQP-based DNR Model

The interconnected active distribution networks (ADNs) are considered as a connected undirected tree $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, where $\mathcal{N}$ is the set of nodes and $\mathcal{E}$ is the set of branches. Suppose that an arbitrary branch $l := (m, n), \forall l \in \mathcal{E}$ is between nodes (m, n) and S denotes the set of root nodes regarded as multiple substations [1]. The simplified distribution network reconfiguration (DNR) model is based on the linearized DistFlow equations, which is cast as a mixed-integer quadratic programming (MIQP) problem [2]. The set of optimization variables involves a set of operational variables $\boldsymbol{x}^l := [\boldsymbol{P}^l, \boldsymbol{Q}^l, \boldsymbol{Q}^{cr}]^T$, switch status indicator variables $\boldsymbol{u}^l \in \mathbb{Z}^{|\mathcal{E}|}$ and continuous parent-child relationship variables $\boldsymbol{\beta}^l \in \mathbb{R}^{2|\mathcal{E}|}$. In this vein, $\boldsymbol{P}^l \in \mathbb{R}^{|\mathcal{E}|}$ and $\boldsymbol{Q}^l \in \mathbb{R}^{|\mathcal{E}|}$ refer to the vectors of sending-end active and reactive power flows. $\boldsymbol{Q}^{cr} \in \mathbb{R}^{|n_{cr}|}$ is the vector of nodal reactive power compensation and n_{cr} is the number of capacitors. $\boldsymbol{\beta}^l$ are constructed with spanning tree constraints with multiple sources. For convenience, we express the MIQP-based DNR model in following form:

$$\min_{\boldsymbol{x}^l, \boldsymbol{\beta}^l \in \mathbb{R}, \boldsymbol{u}^l \in \mathbb{Z}} \; f = \begin{bmatrix} \boldsymbol{P}^l \\ \boldsymbol{Q}^l \\ \boldsymbol{Q}^{cr} \end{bmatrix}^T \begin{bmatrix} \boldsymbol{D}_r & \mathbf{0}_{|\mathcal{E}|\times|\mathcal{E}|} & \mathbf{0}_{|\mathcal{E}|\times n_{cr}} \\ \mathbf{0}_{|\mathcal{E}|\times|\mathcal{E}|} & \boldsymbol{D}_r & \mathbf{0}_{|\mathcal{E}|\times n_{cr}} \\ \mathbf{0}_{n_{cr}\times|\mathcal{E}|} & \mathbf{0}_{n_{cr}\times|\mathcal{E}|} & \mathbf{0}_{n_{cr}\times n_{cr}} \end{bmatrix} \begin{bmatrix} \boldsymbol{P}^l \\ \boldsymbol{Q}^l \\ \boldsymbol{Q}^{cr} \end{bmatrix} \tag{A-1a}$$

$$\text{s.t.} \begin{bmatrix} \boldsymbol{A}_{\mathcal{G}}^T & \mathbf{0}_{|\mathcal{E}|\times|\mathcal{E}|} & \mathbf{0}_{|\mathcal{E}|\times n_{cr}} \\ \mathbf{0}_{|\mathcal{E}|\times|\mathcal{E}|} & \boldsymbol{A}_{\mathcal{G}}^T & \boldsymbol{A}_{cr} \end{bmatrix} \begin{bmatrix} \boldsymbol{P}^l \\ \boldsymbol{Q}^l \\ \boldsymbol{Q}^{cr} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{P}^g + \boldsymbol{P}^d \\ -\boldsymbol{Q}^g + \boldsymbol{Q}^d \end{bmatrix} \tag{A-1b}$$

$$\begin{bmatrix} 2\boldsymbol{D}_r & 2\boldsymbol{D}_x & \mathbf{0}_{|\mathcal{E}|\times n_{cr}} & M \\ -2\boldsymbol{D}_r & -2\boldsymbol{D}_x & \mathbf{0}_{|\mathcal{E}|\times n_{cr}} & -M \\ \mathbf{0}_{n_{cr}\times|\mathcal{E}|} & \mathbf{0}_{n_{cr}\times|\mathcal{E}|} & diag(\mathbb{1}_c) & \mathbf{0}_{n_{cr}\times|\mathcal{E}|} \\ \mathbf{0}_{n_{cr}\times|\mathcal{E}|} & \mathbf{0}_{n_{cr}\times|\mathcal{E}|} & -diag(\mathbb{1}_c) & \mathbf{0}_{n_{cr}\times|\mathcal{E}|} \end{bmatrix} \begin{bmatrix} \boldsymbol{P}^l \\ \boldsymbol{Q}^l \\ \boldsymbol{Q}^{cr} \\ \boldsymbol{u}^l \end{bmatrix} \leqslant \begin{bmatrix} \Delta\overline{\boldsymbol{v}} + M \\ -\Delta\underline{\boldsymbol{v}} + M \\ \overline{\boldsymbol{Q}}_{\boldsymbol{cr}} \\ -\underline{\boldsymbol{Q}}_{\boldsymbol{cr}} \end{bmatrix} \tag{A-1c}$$

$$\begin{bmatrix} diag(\mathbb{1}_N) & \mathbf{0}_{|\mathcal{E}|\times|\mathcal{E}|} & \mathbf{0}_{|\mathcal{E}|\times n_{cr}} & -\Gamma \\ -diag(\mathbb{1}_N) & \mathbf{0}_{|\mathcal{E}|\times|\mathcal{E}|} & \mathbf{0}_{|\mathcal{E}|\times n_{cr}} & -\Gamma \\ \mathbf{0}_{|\mathcal{E}|\times|\mathcal{E}|} & diag(\mathbb{1}_N) & \mathbf{0}_{|\mathcal{E}|\times n_{cr}} & -\Gamma \\ \mathbf{0}_{|\mathcal{E}|\times|\mathcal{E}|} & -diag(\mathbb{1}_N) & \mathbf{0}_{|\mathcal{E}|\times n_{cr}} & -\Gamma \end{bmatrix} \begin{bmatrix} \boldsymbol{P}^l \\ \boldsymbol{Q}^l \\ \boldsymbol{Q}^{cr} \\ \boldsymbol{u}^l \end{bmatrix} \leqslant \mathbf{0} \tag{A-1d}$$

$$\beta^l_{mn} + \beta^l_{nm} = u^l_{mn}, \quad \beta^l_{mn} = 0, \text{ if } m = S \tag{A-1e}$$

$$\sum_{n:(m,n)\in\mathcal{E}} \beta^l_{mn} = 1, \; \forall m \in \mathcal{N} \backslash S \tag{A-1f}$$

$$0 \leqslant \beta^l_{mn} \leqslant 1, \; \forall l \in \mathcal{E} \tag{A-1g}$$

where (A-1a) states the quadratic active power loss of DNs under the assumption of flat voltage profiles for all nodes. $\mathbb{1}_c$ and $\mathbb{1}_N$ are the $n_{cr} \times 1$ and $|\mathcal{E}| \times 1$ vectors with all ones, respectively. M refers to the big positive number and Γ denotes the branch capacity. $\boldsymbol{P}^g, \boldsymbol{Q}^g$ and $\boldsymbol{P}^d, \boldsymbol{Q}^d$ indicate the vectors of given nodal active and reactive power injections and active and reactive loads at nodes. $\boldsymbol{A}_{\mathcal{G}}$ is a $|\mathcal{E}|$ by $|\mathcal{N}|$ branch-node incidence matrix and $\boldsymbol{A}_{cr}$ is a diagonal matrix whose i-th diagonal element is equal to 1 if node i has the reactive compensation capacitors; otherwise this entry is zero. $\boldsymbol{D}_r$ and $\boldsymbol{D}_x$ indicate the diagonal matrices whose diagonal elements are the resistance and reactance vectors, respectively. $\underline{\boldsymbol{Q}}_{\boldsymbol{cr}}$, $\overline{\boldsymbol{Q}}_{\boldsymbol{cr}}$, $\Delta\underline{\boldsymbol{v}}$ and $\Delta\overline{\boldsymbol{v}}$ represent the boundaries of $\boldsymbol{Q}^{cr}$ and squared voltage profile deviation, respectively. (A-1e)-(A-1g) represents the spanning tree constraints with multiple sources for interconnected ADNs [2].

Therefore, we can summarize $\boldsymbol{c}$, $\boldsymbol{A}$, $\boldsymbol{G}_v$, $\boldsymbol{G}_{cr}$, $\boldsymbol{d}$, $\boldsymbol{b}_v$ and $\boldsymbol{b}_{cr}$ in (A-1b) as

$$\boldsymbol{c} = \begin{bmatrix} \boldsymbol{D}_r & \mathbf{0}_{|\mathcal{E}|\times|\mathcal{E}|} & \mathbf{0}_{|\mathcal{E}|\times n_{cr}} \\ \mathbf{0}_{|\mathcal{E}|\times|\mathcal{E}|} & \boldsymbol{D}_r & \mathbf{0}_{|\mathcal{E}|\times n_{cr}} \\ \mathbf{0}_{n_{cr}\times|\mathcal{E}|} & \mathbf{0}_{n_{cr}\times|\mathcal{E}|} & \mathbf{0}_{n_{cr}\times n_{cr}} \end{bmatrix} \tag{A-2a}$$

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_{\mathcal{G}}^T & \mathbf{0}_{|\mathcal{E}|\times|\mathcal{E}|} & \mathbf{0}_{|\mathcal{E}|\times n_{cr}} \\ \mathbf{0}_{|\mathcal{E}|\times|\mathcal{E}|} & \boldsymbol{A}_{\mathcal{G}}^T & \boldsymbol{A}_{cr} \end{bmatrix}, \; \boldsymbol{d} = \begin{bmatrix} -\boldsymbol{P}^g + \boldsymbol{P}^d \\ -\boldsymbol{Q}^g + \boldsymbol{Q}^d \end{bmatrix} \tag{A-2b}$$

$$\boldsymbol{G}_v = \begin{bmatrix} 2\boldsymbol{D}_r & 2\boldsymbol{D}_x & \mathbf{0}_{|\mathcal{E}|\times n_{cr}} & M \\ -2\boldsymbol{D}_r & -2\boldsymbol{D}_x & \mathbf{0}_{|\mathcal{E}|\times n_{cr}} & -M \end{bmatrix} \tag{A-2c}$$

$$\boldsymbol{G}_u = \begin{bmatrix} diag(\mathbb{1}_N) & \mathbf{0}_{|\mathcal{E}|\times|\mathcal{E}|} & \mathbf{0}_{|\mathcal{E}|\times n_{cr}} & -\Gamma \\ -diag(\mathbb{1}_N) & \mathbf{0}_{|\mathcal{E}|\times|\mathcal{E}|} & \mathbf{0}_{|\mathcal{E}|\times n_{cr}} & -\Gamma \\ \mathbf{0}_{|\mathcal{E}|\times|\mathcal{E}|} & diag(\mathbb{1}_N) & \mathbf{0}_{|\mathcal{E}|\times n_{cr}} & -\Gamma \\ \mathbf{0}_{|\mathcal{E}|\times|\mathcal{E}|} & -diag(\mathbb{1}_N) & \mathbf{0}_{|\mathcal{E}|\times n_{cr}} & -\Gamma \end{bmatrix} \tag{A-2d}$$

$$\boldsymbol{G}_{cr} = \begin{bmatrix} \mathbf{0}_{n_{cr}\times|\mathcal{E}|} & \mathbf{0}_{n_{cr}\times|\mathcal{E}|} & diag(\mathbb{1}_c) & \mathbf{0}_{n_{cr}\times|\mathcal{E}|} \\ \mathbf{0}_{n_{cr}\times|\mathcal{E}|} & \mathbf{0}_{n_{cr}\times|\mathcal{E}|} & -diag(\mathbb{1}_c) & \mathbf{0}_{n_{cr}\times|\mathcal{E}|} \end{bmatrix} \tag{A-2e}$$

$$\boldsymbol{b}_v = \begin{bmatrix} \Delta\overline{\boldsymbol{v}} + M \\ -\Delta\underline{\boldsymbol{v}} + M \end{bmatrix}, \; \boldsymbol{b}_{cr} = \begin{bmatrix} \overline{\boldsymbol{Q}}_{\boldsymbol{cr}} \\ -\underline{\boldsymbol{Q}}_{\boldsymbol{cr}} \end{bmatrix} \tag{A-2f}$$

$$\boldsymbol{b}_u = \mathbf{0}_{4|\mathcal{E}|\times 1} \tag{A-2g}$$

where $\boldsymbol{K}$ and $\boldsymbol{h}$ can be rewritten from (A-1e)-(A-1g).

II. Differential Privacy Proof of DP-DNR Mechanism

Theorem: The DP-DNR mechanism $\widetilde{\mathcal{M}}$ can output the obfuscated load flows as $\widetilde{\mathcal{M}} = \boldsymbol{x}^{l_k} + \boldsymbol{\alpha}^{l_k}\boldsymbol{\xi}^{l_k}$ for an arbitrary interconnected branch l_k subject to $(\boldsymbol{x}^l + \boldsymbol{\alpha}^l\boldsymbol{\xi}^l, \boldsymbol{\beta}^l, \boldsymbol{u}^l) \in \mathcal{X}$, which solution obeys $(\varepsilon, 0)$-differentially private.

Proof: According to the definition of Laplace mechanism [3], let the query output answers be $\mathcal{O} = \boldsymbol{x}^{l*} + \boldsymbol{\alpha}^{l^*}\boldsymbol{\xi}^l$, and we alternatively rewrite this proposition as

$$\mathbb{P}_{\xi}[\widetilde{\mathcal{M}}(\boldsymbol{d}) = \mathcal{O}] \leqslant \mathbb{P}_{\xi}[\widetilde{\mathcal{M}}(\boldsymbol{d}') = \mathcal{O}]e^{\varepsilon} \tag{A-4}$$

for any two ρ-neighborhood load datasets $\boldsymbol{d}$ and $\boldsymbol{d}'$ and output solutions $\mathcal{O}$.

Without loss of generality, we define $\boldsymbol{\xi}^l = [\xi^{l_1}, ..., \xi^{l_n}]$, $\boldsymbol{\alpha}^l = [\alpha^{l_1}, ..., \alpha^{l_n}]$, $\boldsymbol{x}^l = [x^{l_1}, ..., x^{l_n}]$, where n refers to the total number of interconnected branches $\{l_k\}$. Thus, the query output $\{\mathcal{O}_k\}$ for all interconnected branches $\{l_k\}$, $\forall k = 1, ..., n$, with the vectors of $\boldsymbol{\xi}^l$, $\boldsymbol{\alpha}^l$ and $\boldsymbol{x}^l$, can be written as

$$\mathcal{O}_k = \boldsymbol{x}^{l_k^*} + \boldsymbol{\alpha}^{l_k^*}\boldsymbol{\xi}^{l_k}, \quad \forall k \in [1, n] \tag{A-4}$$

Therefore, the ratio of probabilities on two ρ-indistinguishable load datasets $\boldsymbol{d}$ and $\boldsymbol{d}'$ can be bounded by

$$\begin{aligned}
\frac{\mathbb{P}_\xi[\widetilde{\mathcal{M}}(\boldsymbol{d}) = \mathcal{O}]}{\mathbb{P}_\xi[\widetilde{\mathcal{M}}(\boldsymbol{d}') = \mathcal{O}]} &= \frac{\mathbb{P}_\xi\left[\begin{bmatrix}\boldsymbol{x}^{l_1^*}(\boldsymbol{d})\\ \vdots \\ \boldsymbol{x}^{l_n^*}(\boldsymbol{d})\end{bmatrix} + \begin{bmatrix}\boldsymbol{\alpha}^{l_1^*}\\ \vdots \\ \boldsymbol{\alpha}^{l_n^*}\end{bmatrix}\begin{bmatrix}\boldsymbol{\xi}^{l_1^*}\\ \vdots \\ \boldsymbol{\xi}^{l_n^*}\end{bmatrix} = \begin{bmatrix}\mathcal{O}_1\\ \vdots \\ \mathcal{O}_n\end{bmatrix}\right]}{\mathbb{P}_\xi\left[\begin{bmatrix}\boldsymbol{x}^{l_1^*}(\boldsymbol{d}')\\ \vdots \\ \boldsymbol{x}^{l_n^*}(\boldsymbol{d}')\end{bmatrix} + \begin{bmatrix}\boldsymbol{\alpha}^{l_1^*}\\ \vdots \\ \boldsymbol{\alpha}^{l_n^*}\end{bmatrix}\begin{bmatrix}\boldsymbol{\xi}^{l_1^*}\\ \vdots \\ \boldsymbol{\xi}^{l_n^*}\end{bmatrix} = \begin{bmatrix}\mathcal{O}_1\\ \vdots \\ \mathcal{O}_n\end{bmatrix}\right]} \\
&= \frac{\mathbb{P}_\xi\left[\begin{bmatrix}\boldsymbol{\alpha}^{l_1^*}\boldsymbol{\xi}^{l_1^*}\\ \vdots \\ \boldsymbol{\alpha}^{l_n^*}\boldsymbol{\xi}^{l_n^*}\end{bmatrix} = \begin{bmatrix}\mathcal{O}_1 - \boldsymbol{x}^{l_1^*}(\boldsymbol{d})\\ \vdots \\ \mathcal{O}_n - \boldsymbol{x}^{l_n^*}(\boldsymbol{d})\end{bmatrix}\right]}{\mathbb{P}_\xi\left[\begin{bmatrix}\boldsymbol{\alpha}^{l_1^*}\boldsymbol{\xi}^{l_1^*}\\ \vdots \\ \boldsymbol{\alpha}^{l_n^*}\boldsymbol{\xi}^{l_n^*}\end{bmatrix} = \begin{bmatrix}\mathcal{O}_1 - \boldsymbol{x}^{l_1^*}(\boldsymbol{d}')\\ \vdots \\ \mathcal{O}_n - \boldsymbol{x}^{l_n^*}(\boldsymbol{d}')\end{bmatrix}\right]} \overset{(i)}{=} \frac{\prod_{k=1}^n \boldsymbol{\alpha}^{l_k^*} \exp\{-\frac{\varepsilon\|\mathcal{O}_k - \boldsymbol{x}^{l_k^*}(\boldsymbol{d})\|_2}{\Delta_\rho}\}}{\prod_{k=1}^n \boldsymbol{\alpha}^{l_k^*} \exp\{-\frac{\varepsilon\|\mathcal{O}_k - \boldsymbol{x}^{l_k^*}(\boldsymbol{d}')\|_2}{\Delta_\rho}\}} \\
&= \prod_{k=1}^n \exp\left(\frac{\varepsilon\|\mathcal{O}_k - \boldsymbol{x}^{l_k^*}(\boldsymbol{d}')\|_2 - \varepsilon\|\mathcal{O}_k - \boldsymbol{x}^{l_k^*}(\boldsymbol{d})\|_2}{\Delta_\rho}\right) \\
&\overset{(ii)}{\leqslant} \prod_{k=1}^n \exp\left(\frac{\varepsilon\|\boldsymbol{x}^{l_k^*}(\boldsymbol{d}) - \boldsymbol{x}^{l_k^*}(\boldsymbol{d}')\|_2}{\Delta_\rho}\right) = \exp\left(\frac{\varepsilon\|\boldsymbol{x}^{l^*}(\boldsymbol{d}) - \boldsymbol{x}^{l^*}(\boldsymbol{d}')\|_2}{\Delta_\rho}\right) \\
&\overset{(iii)}{\leqslant} \exp\left(\frac{\varepsilon\Delta_\rho}{\Delta_\rho}\right) = e^\varepsilon
\end{aligned} \tag{A-5}$$

where (i) comes from the definition of the probability density function of the Laplace distribution. In (ii) step, it is followed by the inequailty of norms, i.e., $|a| - |b| \leqslant |a - b|$. For the (iii) step, $\|\boldsymbol{x}^{l^*}(\boldsymbol{d}) - \boldsymbol{x}^{l^*}(\boldsymbol{d}')\|_2$ denotes the ℓ_2-sensitivity on ρ-indistinguishable input datasets $\boldsymbol{x}^{l^*}(\boldsymbol{d})$ and $\boldsymbol{x}^{l^*}(\boldsymbol{d}')$ subject to $\|\boldsymbol{x}^{l^*}(\boldsymbol{d}) - \boldsymbol{x}^{l^*}(\boldsymbol{d}')\|_2 \leqslant \Delta_\rho$.

Accordingly, it is clear that (A-4) holds based on (A-5), which proves the Theorem. ■

References

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