

Supplementary Material for “Decentralized Optimal Power Flow for Multi-Agent Active Distribution Networks: A Differentially Private Consensus ADMM Algorithm ”

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I. DIFFERENTIAL PRIVACY PROOF

Theorem: Suppose random perturbations obey Laplace distribution $\xi^{l_t} \sim \mathbb{P}_\xi$, $\mathbb{P}_\xi := \text{Lap}(\Delta_\rho/\varepsilon)$, for an arbitrary branch $l_t \in \mathcal{T}$ across agents. And $(\hat{\mathbf{x}}^l, \hat{\mathbf{x}}^g) \in \mathcal{X}$ is the optimal solution of non-private D-OPF model. Then, this differentially private D-OPF model can output a mixture of obfuscated-but-feasible load flow solution \mathbf{x}^{l_t*} for the branches across agents and realistic generator output solution \mathbf{x}^{g*} , such that $\hat{\mathbf{x}}^{l_t} = \mathbf{x}^{l_t*} + \alpha^{l_t*} \xi^{l_t}$ for $\forall l_t \in \mathcal{T}$ and $\mathbf{x}^{g*} = \hat{\mathbf{x}}^g$.

Proof: According to the definition of Laplace mechanism, let the query output answers be $\mathcal{O} = \mathbf{x}^{l_t*}$, and we alternatively rewrite this theorem in the definition of ε -differential privacy.

$$\mathbb{P}_\xi[\widetilde{\mathcal{M}}(\mathbf{d}) = \mathcal{O}] \leq \mathbb{P}_\xi[\widetilde{\mathcal{M}}(\mathbf{d}') = \mathcal{O}]e^\varepsilon \quad (\text{A-1})$$

for any two ρ -neighborhood load datasets \mathbf{d} and \mathbf{d}' and output solutions \mathcal{O} .

For convenience, we define $\xi^l = [\xi^{l_1}, \dots, \xi^{l_n}]$, $\{\alpha^{l_t}\} = [\alpha^{l_1}, \dots, \alpha^{l_n}]$, $\{\mathbf{x}^{l_t}\} = [x^{l_1}, \dots, x^{l_n}]$, where n refers to the total number of tie-lines. Thus, the query output $\{\mathcal{O}^{l_t}\}$ for all tie-lines $\forall l_t \in \mathcal{T}$, $\forall t = 1, \dots, n$, with the vectors of ξ^l , can be written as

$$\mathcal{O}^{l_t} = \mathbf{x}^{l_t*} = \hat{\mathbf{x}}^{l_t} - \alpha^{l_t*} \xi^{l_t}, \quad \forall t \in [1, n] \quad (\text{A-2})$$

Therefore, the ratio of probabilities on two ρ -indistinguishable load datasets \mathbf{d} and \mathbf{d}' can be bounded by

$$\begin{aligned} \frac{\mathbb{P}_\xi[\widetilde{\mathcal{M}}(\mathbf{d}) = \mathcal{O}^{l_t}]}{\mathbb{P}_\xi[\widetilde{\mathcal{M}}(\mathbf{d}') = \mathcal{O}^{l_t}]} &= \frac{\mathbb{P}_\xi \left[\begin{bmatrix} \hat{\mathbf{x}}^{l_1}(\mathbf{d}) \\ \vdots \\ \hat{\mathbf{x}}^{l_n}(\mathbf{d}) \end{bmatrix} - \begin{bmatrix} \alpha^{l_1*} \\ \vdots \\ \alpha^{l_n*} \end{bmatrix} \begin{bmatrix} \xi^{l_1} \\ \vdots \\ \xi^{l_n} \end{bmatrix} = \begin{bmatrix} \mathcal{O}^{l_1} \\ \vdots \\ \mathcal{O}^{l_n} \end{bmatrix} \right]}{\mathbb{P}_\xi \left[\begin{bmatrix} \hat{\mathbf{x}}^{l_1}(\mathbf{d}') \\ \vdots \\ \hat{\mathbf{x}}^{l_n}(\mathbf{d}') \end{bmatrix} - \begin{bmatrix} \alpha^{l_1*} \\ \vdots \\ \alpha^{l_n*} \end{bmatrix} \begin{bmatrix} \xi^{l_1} \\ \vdots \\ \xi^{l_n} \end{bmatrix} = \begin{bmatrix} \mathcal{O}^{l_1} \\ \vdots \\ \mathcal{O}^{l_n} \end{bmatrix} \right]} \\ &= \frac{\mathbb{P}_\xi \left[\begin{bmatrix} \alpha^{l_1*} \xi^{l_1} \\ \vdots \\ \alpha^{l_n*} \xi^{l_n} \end{bmatrix} = \begin{bmatrix} \mathcal{O}^{l_1} - \hat{\mathbf{x}}^{l_1}(\mathbf{d}) \\ \vdots \\ \mathcal{O}^{l_n} - \hat{\mathbf{x}}^{l_n}(\mathbf{d}) \end{bmatrix} \right]}{\mathbb{P}_\xi \left[\begin{bmatrix} \alpha^{l_1*} \xi^{l_1} \\ \vdots \\ \alpha^{l_n*} \xi^{l_n} \end{bmatrix} = \begin{bmatrix} \mathcal{O}^{l_1} - \hat{\mathbf{x}}^{l_1}(\mathbf{d}') \\ \vdots \\ \mathcal{O}^{l_n} - \hat{\mathbf{x}}^{l_n}(\mathbf{d}') \end{bmatrix} \right]} \stackrel{(i)}{=} \frac{\prod_{t=1}^n \alpha^{l_t*} \exp \left\{ -\frac{\varepsilon \|\mathcal{O}^{l_t} - \hat{\mathbf{x}}^{l_t}(\mathbf{d})\|_2}{\Delta_\rho} \right\}}{\prod_{t=1}^n \alpha^{l_t*} \exp \left\{ -\frac{\varepsilon \|\mathcal{O}^{l_t} - \hat{\mathbf{x}}^{l_t}(\mathbf{d}')\|_2}{\Delta_\rho} \right\}} \\ &= \prod_{t=1}^n \exp \left(\frac{\varepsilon \|\mathcal{O}^{l_t} - \hat{\mathbf{x}}^{l_t}(\mathbf{d}')\|_2 - \varepsilon \|\mathcal{O}^{l_t} - \hat{\mathbf{x}}^{l_t}(\mathbf{d})\|_2}{\Delta_\rho} \right) \\ &\stackrel{(ii)}{\leq} \prod_{t=1}^n \exp \left(\frac{\varepsilon \|\hat{\mathbf{x}}^{l_t}(\mathbf{d}) - \hat{\mathbf{x}}^{l_t}(\mathbf{d}')\|_2}{\Delta_\rho} \right) = \exp \left(\frac{\varepsilon \|\hat{\mathbf{x}}^l(\mathbf{d}) - \hat{\mathbf{x}}^l(\mathbf{d}')\|_2}{\Delta_\rho} \right) \\ &\stackrel{(iii)}{\leq} \exp \left(\frac{\varepsilon \Delta_\rho}{\Delta_\rho} \right) = e^\varepsilon \end{aligned} \quad (\text{A-3})$$

where (i) comes from the definition of the probability density function of the Laplace distribution. In (ii) step, it is followed by the inequality of norms, i.e., $|a| - |b| \leq |a - b|$ for any real-valued numbers a and b . For the (iii) step, $\|\hat{\mathbf{x}}^l(\mathbf{d}) - \hat{\mathbf{x}}^l(\mathbf{d}')\|_2$ denotes the ℓ_2 -sensitivity

on ρ -indistinguishable input datasets $\hat{\mathbf{x}}^l(\mathbf{d})$ and $\hat{\mathbf{x}}^l(\mathbf{d}')$ subject to $\|\hat{\mathbf{x}}^l(\mathbf{d}) - \hat{\mathbf{x}}^l(\mathbf{d}')\|_2 \leq \Delta_\rho$.

Accordingly, it is clear that (A-1) holds based on (A-3), which proves this Theorem. ■