Theorem 2.4.1 General solutions of $AX = B$

➔ Let P be any solution of $AX = B$. Then every solution X of $AX = B$ is of the form $X = H + P$,

where H is a solution of $AX = O_{n1}$.

Nullspace

Proof: Let W be any solution of $AX = B$. Then $W - P$ is a solution of $AX = O_{n1}$

Because $A(W - P) = AW - AP = B - B = O_{n1}$.

O_{n1}

Thus $W - P$ is a solution of $AX = O_{n1}$. Therefore $W - P = H$, i.e., $W = H + P$.

Thus any solution W of $AX = B$ is of the form $W = H + P$.

Q.E.D.

Note:

- $N(A)$ is the Nullspace of A and also the general solution of $AX = O_{n1}$.
- $\boxed{X = N(A) + P}$ is called the general solution of $AX = B$ because we get all the solutions.

➔ the general solution $N(A)$ of $AX = O_{n1}$ plus the particular solution P of $AX = B$.

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Example 2.4.1 Find the general solution

of the nonhomogeneous system

leading entry corresponding column

non-leading entry corresponding column

$$AX = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightarrow A_R X = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 + \alpha_3 \\ \alpha_2 + \alpha_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} -\alpha_3 + 1 \\ -\alpha_3 + 1 \end{bmatrix}$$

Answer

$\{N(A)\}$ ξ

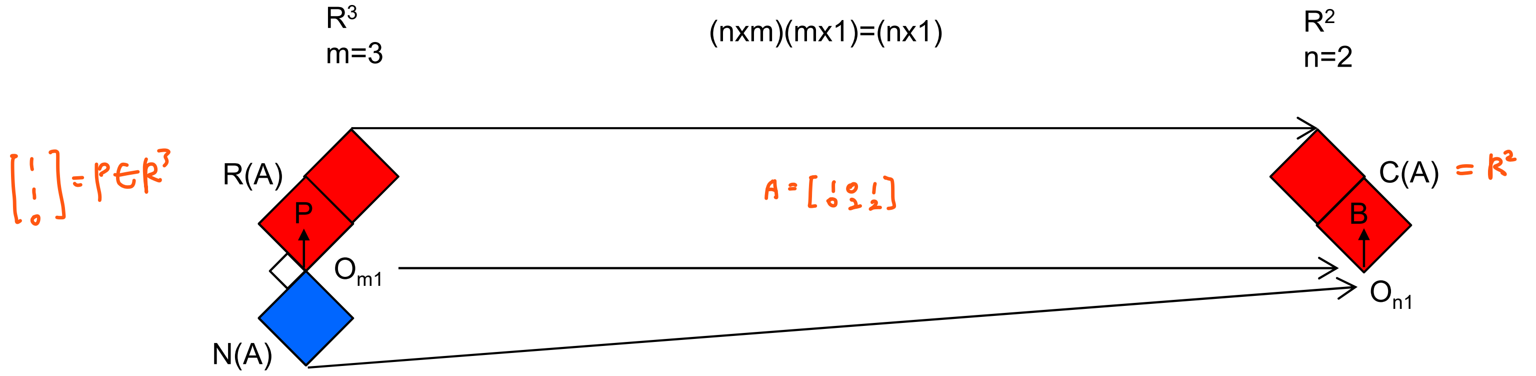
$$\therefore X = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} -\alpha_3 + 1 \\ -\alpha_3 + 1 \\ \alpha_3 \end{bmatrix} = \alpha_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Express leading entry corresponding variables

in terms of non-leading entry corresponding variables

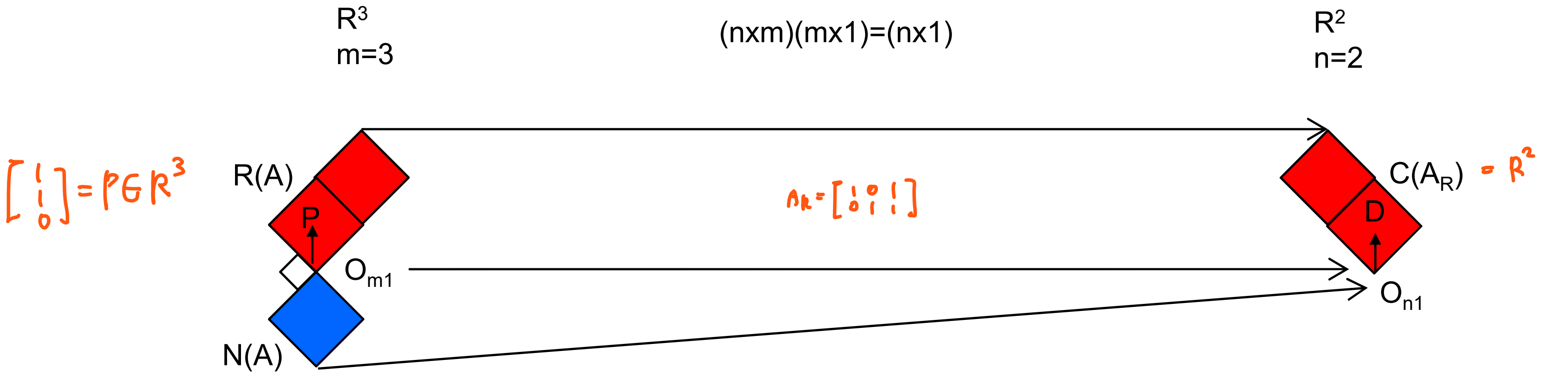
AP=B $\begin{bmatrix}1\\2\end{bmatrix}$

(nxm)(mx1)=(nx1)



$A_RP=D$ $=\begin{bmatrix}1\\1\end{bmatrix}$

(nxm)(mx1)=(nx1)



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Example 2.4.2 Find the general solution

of the nonhomogeneous system

$$AX = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightarrow$$

leading entry corresponding column

non-leading entry corresponding column

$$\begin{array}{cccc} & \downarrow & \downarrow & \downarrow \end{array}$$

$$A_R X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0+0 \\ 0+1 \end{bmatrix}$$

$0 \quad 0 \quad -1$

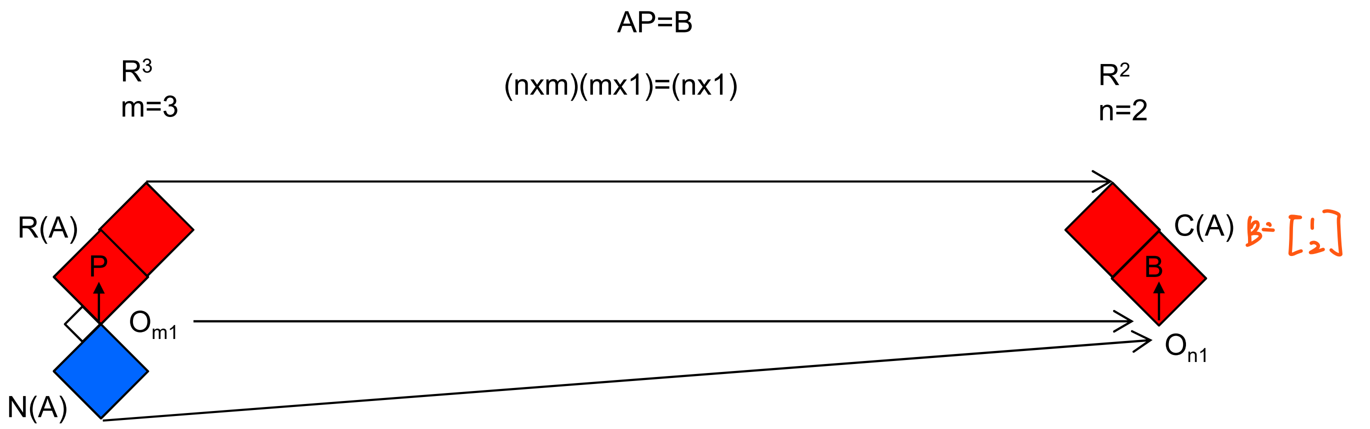
$\begin{matrix} 0 & 0 & -1 \\ 0 & 0 & -1 \end{matrix}$

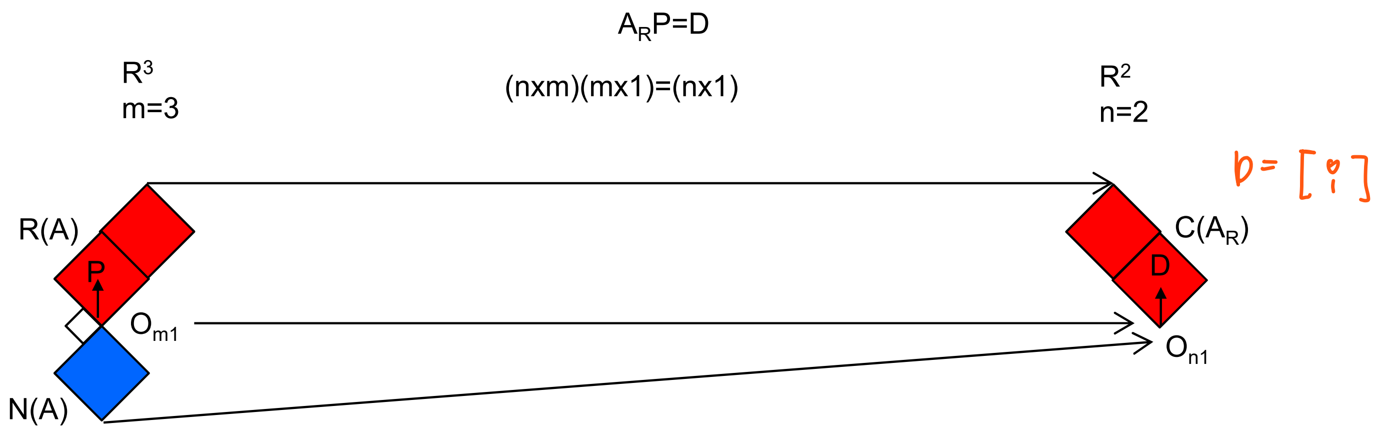
$$\therefore X = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0+0 \\ \alpha_2 \\ 0+1 \end{bmatrix} = \alpha_2 \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

N(A) P

Express leading entry corresponding variables

in terms of non-leading entry corresponding variables


AP=B
(nxm)(mx1)=(nx1)
R3
m=3
R2
n=2
R(A)
P
Om1
N(A)
C(A)
B
On1
B= [1 2]


ARP=D
(nxm)(mx1)=(nx1)
R3
m=3
R2
n=2
R(A)
P
Om1
N(A)
C(AR)
D
On1
D= [0 1]

Candidate (20f) Final Q.4

Example 2.4.3 Find the general solution of the nonhomogeneous system

$$AX = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \rightarrow$$

leading entry corresponding column

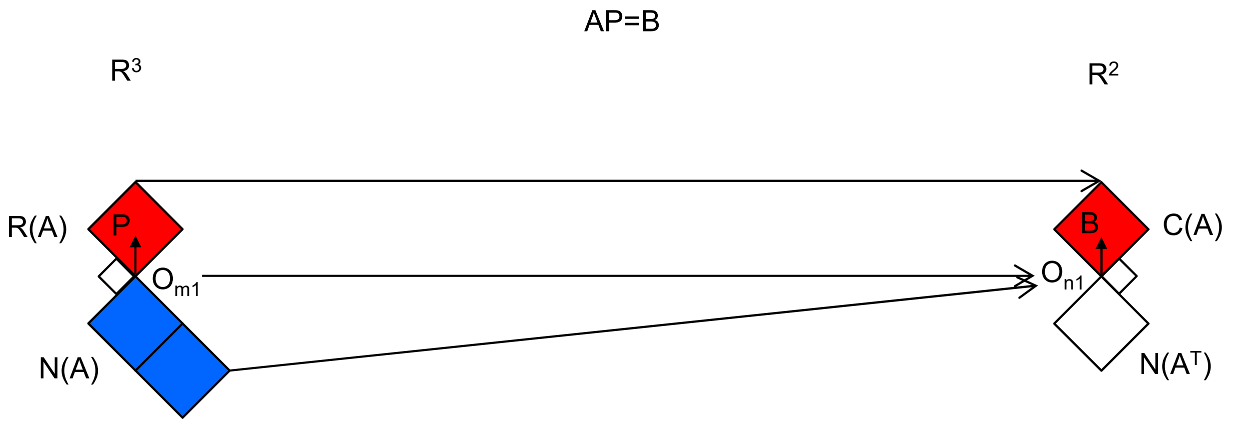
non-leading entry corresponding column

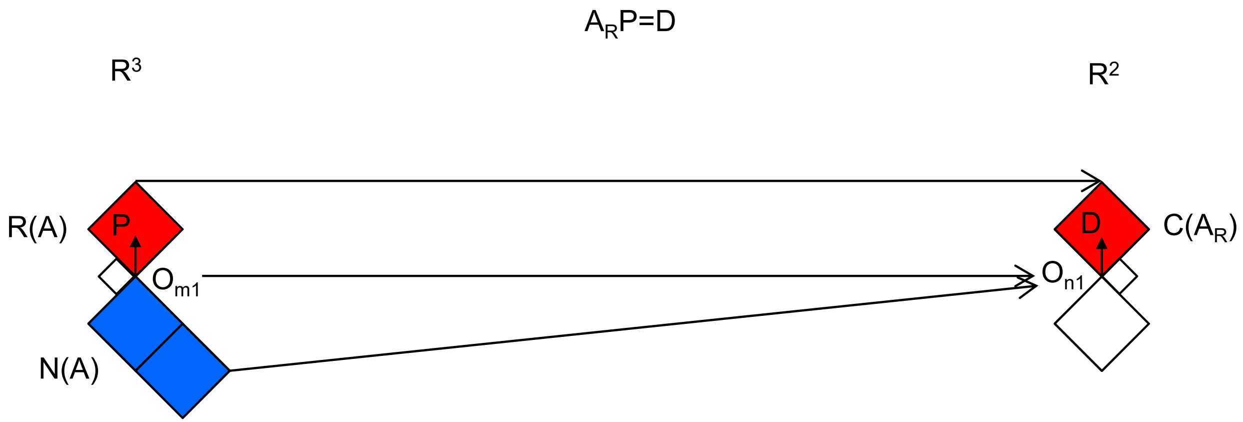
$$\begin{array}{cccc} & \downarrow & \downarrow & \downarrow \\ A_R X = & \left[\begin{matrix} 1 \\ 0 \end{matrix}\right. & \begin{matrix} 0 \\ 0 \end{matrix} & \left.\begin{matrix} 0 \\ 0 \end{matrix}\right] \end{array} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 + 2 \\ 0 \end{bmatrix}$$

$$\therefore X = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 + 2 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Express leading entry corresponding variables in terms of non-leading entry corresponding variables


AP=B
R3
R2
R(A)
P
B
C(A)
Om1
On1
N(A)
N(AT)


ARP=D
R3
R2
R(A)
P
D
C(AR)
Om1
On1
N(A)

Candidate (4 of 4) Final Q.4

Example 2.4.4 Find the general solution of the nonhomogeneous system

$$AX = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \rightarrow$$

leading entry corresponding column

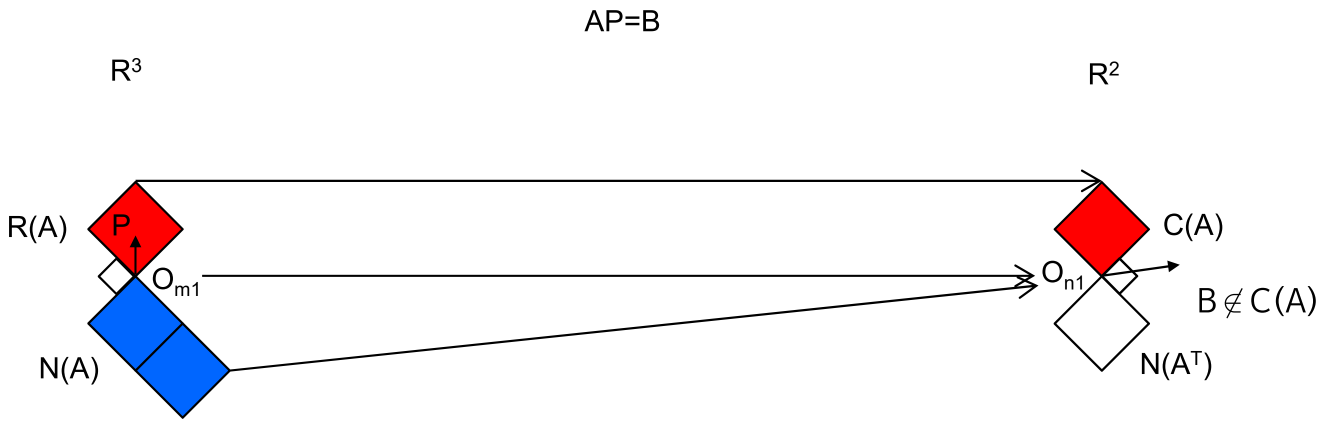
non-leading entry corresponding column

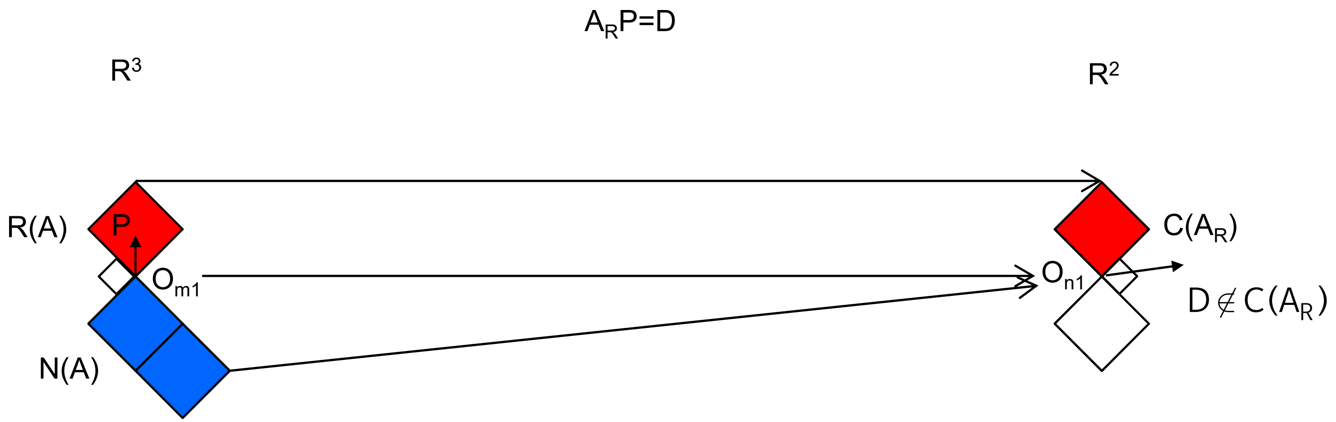
$$A_R X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0+2 \\ 1 \end{bmatrix}$$

"0=1" inconsistent

answer $\boxed{\therefore \text{ This nonhomogeneous system has no solution.}}$


AP=B
R3
R2
R(A)
P
Om1
N(A)
C(A)
On1
B ∉ C(A)
N(AT)


ARP=D
R3
R2
R(A)
P
Om1
N(A)
C(AR)
On1
D ∉ C(AR)