16

Overview of Solution Methods

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In previous Chapters we have covered the governing equations of geometrically nonlinear structural analysis and the discretization of those equations by finite element methods. The result is a set of parametrized nonlinear algebraic equations called *residual force equations*.

The solution of these equations as the control parameters are varied varied provides the *equilibrium response* of the structure. In this Chapter we begin the coverage of solution methods suitable for digital computation.

§16.1. Introduction

It was noted in Chapter 1 that all solution procedures of practical importance are strongly rooted in the idea of "advancing the solution" by *continuation*. The basic idea is to follow the equilibrium response of the structure as the control and state parameters vary by small amounts. The motivation in terms of circumventing the "solution morass" is described in that Chapter.

This general idea gives rise to many variants called solution schemes. But a common feature is that continuation is a *multilevel* process, as illustrated in Figure 16.1. The process involves a hierarchical breakdown into stages, incremental steps, and iterative steps. The middle level: incrementation, is always present.

In the present Chapter multilevel continuation is described in general terms, with the goal of maintaining independence from specific solution schemes. The final subsections describe how the general procedure is adapted to the analysis of problems encountered in engineering practice.

§16.1.1. Stages, Increments and Iterations

As discussed in Chapter 4, processing a complex nonlinear problem generally involves performing a series of *analysis stages*. Multiple control parameters are not varied independently in each stage and may therefore be characterized by a single stage control parameter λ. Stages are only weakly coupled in the sense that the end solution of one may provide the starting point for another. Throughout this and following Chapters attention is focused on a *generic stage* and there is no need to use an identifying index for it.

To advance the solution, the stage is broken down into *incremental steps*, or *increments* for short. If necessary incremental steps will be identified by the subscript n; for example, the state vector after the n^{th} increment is $\mathbf{u}_n$ and the state vector before any increment (at stage start) is $\mathbf{u}_0$. Over each incremental step the state vector $\mathbf{u}$ and stage control parameter λ undergo finite changes denoted by $\Delta\mathbf{u}$ and $\Delta\lambda$, respectively.

Incremental solution methods can be divided into two broad classes:

1. *Purely incremental* methods, also called *predictor-only* methods.
2. *Corrective* methods, also called *predictor-corrector* or *incremental-iterative* methods.

In purely incremental methods the iteration level is missing. In corrective methods a predictor step is followed by one or more iteration steps. The set of iterations is called the *corrective phase*. Its purpose is to eliminate or reduce the so called *drifting error*, discussed in §16.4, which plagues purely incremental methods.

Iteration steps will be usually identified by the superscript k. This superscript is enclosed in parentheses if there is potential confusion with exponents. Iterative changes in $\Delta\mathbf{u}$ are often denoted by $\mathbf{d}$.

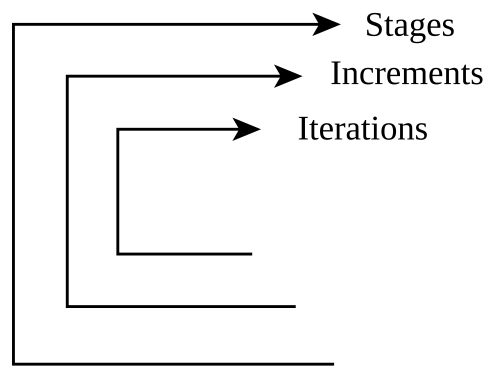

Figure 16.1. Nested hierarchy in nonlinear solution methods: stages, increments and iterations.

Remark 16.1. Solutions accepted after each increment after a corrective phase, are often of interest to users because they represent approximations to equilibrium states. They are therefore saved as they are computed. On the other hand, intermediate results of iterative processes are rarely of interest unless one is studying the "inside" of solution processes. Hence most production programs discard them.

Remark 16.2. The terminology of nonlinear static analysis is far from standardized. Despite their practical importance, few authors recognize the existence of stages. Many use the term *step* to mean *incremental step* whereas the terms *substep*, *subincrement* and *cycle* are used for the iteration level. There is more uniformity in dynamic analysis, possibly because there is only one advancing level: *step* is universally used to denote the change over a time increment.

§16.1.2. Why Incrementation?

The use of increments may seem at first sight unnecessary if one is interested primarily in the final solution. But breaking up a stage into increments may serve other purposes:

Helping convergence. Success in the correction phase described below may hinge on having a good initial guess supplied by the predictor. The quality of this guess can be improved by reducing the increment.

Avoiding extraneous roots. Incrementation helps the solution procedure from falling into the "root morass" discussed in Chapter 1.

Insight into structural behavior. As noted in Remark 16.1, programs often save converged solutions after each increment and for a good reason: a response plot such as that in can teach the engineer more about the structural behavior than simply knowing the final solution.

Surprises. Critical points may occur before the stage end. There are problems in which such points, especially bifurcation, may be masked if coarse increments are taken.

Path dependence. Although the focus of this course is on path-independent problems, it should be noted that the presence of path-dependent effects severely restricts increment sizes because of history-tracing constraints. For example, in plasticity analysis stress states must not be allowed to stray too far outside the yield surface.

§16.2. Advancing the Solution: Increment Control

A nonlinear analysis program is "marching" along a stage. Assume that n incremental steps have been completed. The last accepted solution is $\mathbf{u}_n$, which corresponds to λ_n. Performing the $(n+1)^{th}$ step entails the calculation of the increments

$$\Delta\mathbf{u}_n = \mathbf{u}_{n+1} - \mathbf{u}_n, \qquad \Delta\lambda_n = \lambda_{n+1} - \lambda_n, \tag{16.1}$$

that satisfy the residual equilibrium equations $\mathbf{r}(\mathbf{u}, \lambda) = \mathbf{0}$ to requested accuracy. As stated the task is not fully defined because there are less equations than unknowns, which makes the increment sizes indeterminate. The problem is closed by adopting an *increment control strategy*. The strategy may be expressed in general form as a constraint condition:

$$c(\Delta\mathbf{u}_n, \Delta\lambda_n) = 0, \tag{16.2}$$

which equalizes the number of equations to the number of unknowns. Expressing the constraint (16.2) in terms of the increments (16.1) helps attaining invariance with respect to the origin chosen for $\mathbf{u}$ and λ.

A rate form of the constraint equation (16.2) is obtained by differentiating with respect to t:

$$\mathbf{a}^T\dot{\mathbf{u}} + g\dot{\lambda} = 0, \tag{16.3}$$

where

$$\mathbf{a}^T = \frac{\partial c}{\partial \mathbf{u}}, \qquad g = \frac{\partial c}{\partial \lambda}. \tag{16.4}$$

Remark 16.3. The addition of the constraint equation serves two purposes: it makes the algebraic problem determinate, and it can be used to control the increment size directly or indirectly to enhance robustness and convergence.

Remark 16.4. Specific choices for (16.4) are discussed in §16.4 below but for some developments it is possible to keep c arbitrary. Furthermore, it is also possible to specify the constraint directly in the rate form (16.3) without an explicit integral. An example is Fried's orthogonal trajectory accession method.[1]

§16.3. Advancing the Solution: Prediction

Having decided upon an increment control strategy, to start up the $(n+1)^{th}$ incremental step, an initial approximation

$$\Delta\mathbf{u}_n^0, \;\; \Delta\lambda_n^0, \tag{16.5}$$

to the increments (16.1) is calculated by a *prediction* step. These values are called the *predicted increments* and the formula used is called a *predictor* or *extrapolator*.

Most predictors are based on the first-order path equation derived in Chapter 4 and repeated here for convenience:

$$\dot{\mathbf{r}} = \mathbf{0}, \qquad \text{or} \qquad \mathbf{K}\dot{\mathbf{u}} = \mathbf{q}\,\dot{\lambda}, \tag{16.6}$$

[1] I. Fried, Orthogonal Trajectory Accession to the Nonlinear Equilibrium Curve, *Comp. Meth. Appl. Mech. Engrg.*, **47**, 283–297, (1984).

Assuming $\mathbf{K}$ to be nonsingular, the forward Euler method furnishes the simplest predictor:

$$\Delta\mathbf{u}_n^0 = \mathbf{K}_n^{-1}\mathbf{q}_n\Delta\lambda_n^0 = \mathbf{v}_n\,\Delta\lambda_n^0, \tag{16.7}$$

in which $\mathbf{v}$ is the incremental velocity vector defined in Chapter 4. The process is completed by selecting an increment control strategy through the constraint (16.2). Two examples follow.

Example 16.1. For the *prescribed-load-value* strategy in which $\Delta\lambda_n$ is specified to be ℓ_n (positive or negative), the constraint is

$$c(\Delta u_n, \Delta\lambda_n) = \Delta\lambda_n - \ell_n = 0. \tag{16.8}$$

Then the increments are directly given by (16.6), *i.e.*

$$\Delta\mathbf{u}_n^0 = \mathbf{v}_n\ell_n, \qquad \Delta\lambda_n = \ell_n. \tag{16.9}$$

This formula obviously fails when $\mathbf{K}_n$ is singular, *i.e.* at critical points, because there $\mathbf{v}_n$ becomes either infinite (at limit points) or nonunique (at bifurcation points). This suggests that the solution process will break down at those points.

Example 16.2. For the *arclength strategy* in which the absolute value of the distance (4.23) is specified to be $\ell_n > 0$, the constraint is

$$c(\Delta u_n, \Delta\lambda_n) = |\Delta s_n| - \ell_n = \frac{1}{f_n}\Big|\mathbf{v}_n^T\Delta\mathbf{u}_n + \Delta\lambda_n\Big| - \ell_n = 0, \tag{16.10}$$

where $f_n = +\sqrt{1+\mathbf{v}_n^T\mathbf{v}_n}$. Substitution into (16.6) yields

$$\Delta\lambda_n^0 = \frac{\ell_n f_n}{\pm(\mathbf{v}_n^T\mathbf{v}_n+1)} = \frac{\ell_n}{\pm\sqrt{\mathbf{v}_n^T\mathbf{v}_n+1}} = \pm\frac{\ell_n}{f_n}, \qquad \Delta\mathbf{u}_n^0 = \pm\frac{\mathbf{v}_n\ell_n}{f_n}. \tag{16.11}$$

In this case two signs for the increment are obtained. The proper one is obtained by applying one of the "path advancing" criteria discussed below.

Note also that (16.10) does not fail at *isolated limit points* if one properly passes to the limit $\mathbf{v}/|\mathbf{v}| \to \mathbf{z}$, as per Remark 4.2. This limit process yields

$$\Delta\lambda_n^0 = 0, \qquad \Delta\mathbf{u}_n^0 = \pm\ell_n\mathbf{z} \tag{16.12}$$

The normalized $\mathbf{v}$ near the limit point serves as a good approximation for $\mathbf{z}$. It should be noted, however, that the formula fails at multiple limit points and at bifurcation points; thus the arclength strategy is no panacea.

Both of the foregoing examples above contain a specified length ℓ_n. For the first step, ℓ_0 is normally chosen by the user. If the predictor is followed by a corrective process, in subsequent steps ℓ_n may be roughly adjusted according to the "last iteration count" rule of Crisfield[2] which works well in practice. If no corrective phase follows, the proper selection of ℓ_n is discussed later in the section dealing with purely incremental methods.

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[2] M. A. Crisfield, An Incremental-Iterative Algorithm that Handles Snap-Through, *Computer & Structures*, **16**, 55–62 (1981)
M. A. Crisfield, An Arc-Length Method Including Line Searches and Accelerations, *Int. J. Num. Meth. Engrg.*, **19**, 1269–1289 (1983).

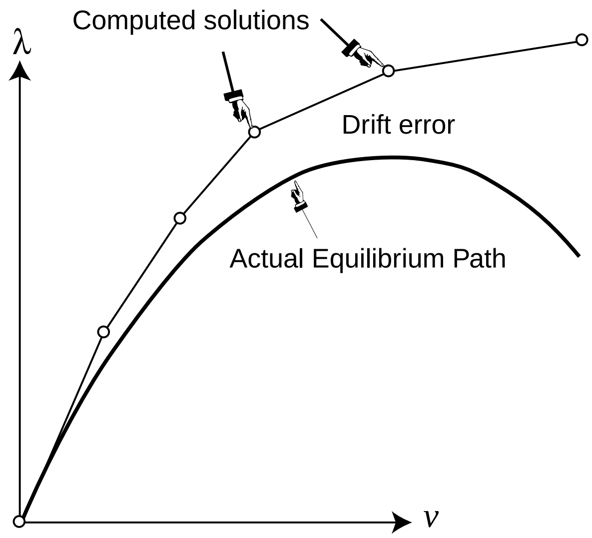

Figure 16.2. Drift error in purely incremental solution procedure.

§16.4. Advancing the Solution: Correction

If the predicted increments (16.5) are inserted in the residual equation $\mathbf{r}(\mathbf{u},\lambda) = \mathbf{0}$, there will generally be a *departure* from equilibrium:

$$\mathbf{r}_n^0 = \mathbf{r}\big(\mathbf{u}_n + \Delta\mathbf{u}_n^0, \lambda_n + \Delta\lambda_n^0\big) \neq \mathbf{0}. \tag{16.13}$$

This departure is called *drift error*. A *corrective process* is an iterative scheme that eliminates, or at least reduces, the drift error by producing a sequence of values

$$\Delta\mathbf{u}_n^k,\ \Delta\lambda_n^k, \tag{16.14}$$

that as $k \to \infty$ hopefully tend to the increments (16.2) that satisfy equilibrium and meet increment control specifications. Popular corrective methods are studied in subsequent Chapters.

As previously noted, there are *purley incremental* methods that omit the corrective phase. They are covered in following Chapters. See Figure 16.2 for an illustration of the drift error phenomenon that occurs when a corrector is not applied.

Remark 16.5. An even simpler predictor consists of setting $\Delta\mathbf{u}_n^0 = \mathbf{0}$, $\Delta\lambda_n^0 = 0$. The corrective process then starts from the previous solution. This overcautious approach is rarely used in practice.

§16.5. Traversing Equilibrium Path in Positive Sense

In Example 16.2 two signs were obtained for the predicted $\Delta\lambda_n^0$ and $\Delta\mathbf{u}_n^0$. This is typical of constraints that are *symmetric* about the last solution point (that is, reversing the signs of both $\Delta\mathbf{u}$ and $\Delta\lambda$ satisfies $c = 0$). In that case the resulting algebraic system usually provides two solutions:

$$\pm\Delta\lambda_n^0, \quad \pm\Delta\mathbf{u}_n^0. \tag{16.15}$$

Even in Example 16.1 there is an ambiguity because the specified ℓ_n may be positive or negative.

The sign ambiguity arises because, as explained in Chapter 4, the tangent at regular points of an equilibrium path has two possible directions, which generally intersect the constraint hypersurface in at least two points. Thus it becomes necessary to chose the direction corresponding to a positive path traversal. Two rules for chosing the proper sign are described below.

§16.5.1. Positive External Work

The simplest rule requires that the external work expenditure over the predictor step be positive:

$$\Delta W = \mathbf{q}^T \Delta \mathbf{u}_n^0 = \mathbf{q}^T \mathbf{v}_n \Delta\lambda_n > 0. \tag{16.16}$$

That is, $\Delta\lambda$ should have the sign of $\mathbf{q}^T\mathbf{v} = \mathbf{q}^T\mathbf{K}^{-1}\mathbf{q}$.

This condition works well when "the structure follows the load" and is particularly effective at limit points. It fails if $\mathbf{q}$ and $\mathbf{v}$ are orthogonal:

$$\mathbf{q}^T\mathbf{v} = 0, \tag{16.17}$$

because then the condition (16.16) is vacuous. This happens in the following cases.

Bifurcation points. As a bifurcation point B is approached, $\mathbf{v}/|\mathbf{v}| \to \mathbf{z}$, achieving equality at B. Since $\mathbf{q}^T\mathbf{z} = 0$, it follows that (16.16) fails at B.

Incremental velocity reversal. If the structure becomes "infinitely stiff" at a point in the equilibrium path $\mathbf{v}$ vanishes. This case is rarer than the previous one, but may arise in the vicinity of turning points.

Bifurcation points demand special treatment and cannot be easily passed through simple predictor methods. One way out is to insert artificial purtuebations that transformperturbations are inserted. However, the case $\mathbf{v} \to \mathbf{0}$ can be overcome by a modification of the previous rule.

§16.5.2. Angle Criterion

There are problems in which the structure gains suddenly stiffness, as for example in the vicinity of a turning point T. If the positive work criterion is used eventually the solution process "turns back" and begins retracing the equilibrium path. When it reaches the high stiffness point again it does another U-turn and so on. The net result of this "ping pong" effect is that the solution process gets stuck. Physically a positive work rule is incorrect because the structure needs to *release* external work to continue along the equilibrium path.

To get over this difficulty a condition on the angle of the prediction vector is more effective. Let $\mathbf{t}_{n-1}$ be the tangent at the previous solution. Then chose the positive sense so that

$$\mathbf{t}_n^T\mathbf{t}_{n-1} > 0. \tag{16.18}$$

Once the "ping-pong" region is crossed, the work criterion should be reversed so the external work is negative.

Remark 16.6. Other geometric criteria are given by Crisfield (*loc. cit.* in footnote 2) and Skeie and Felippa[3]

[3] G. Skeie and C. A. Felippa, A Local Hyperelliptic Constraint for Nonlinear Analysis, Proceedings of NUMETA'90 Conference, Swansea, Wales, Elsevier Sci. Pubs, 1990.

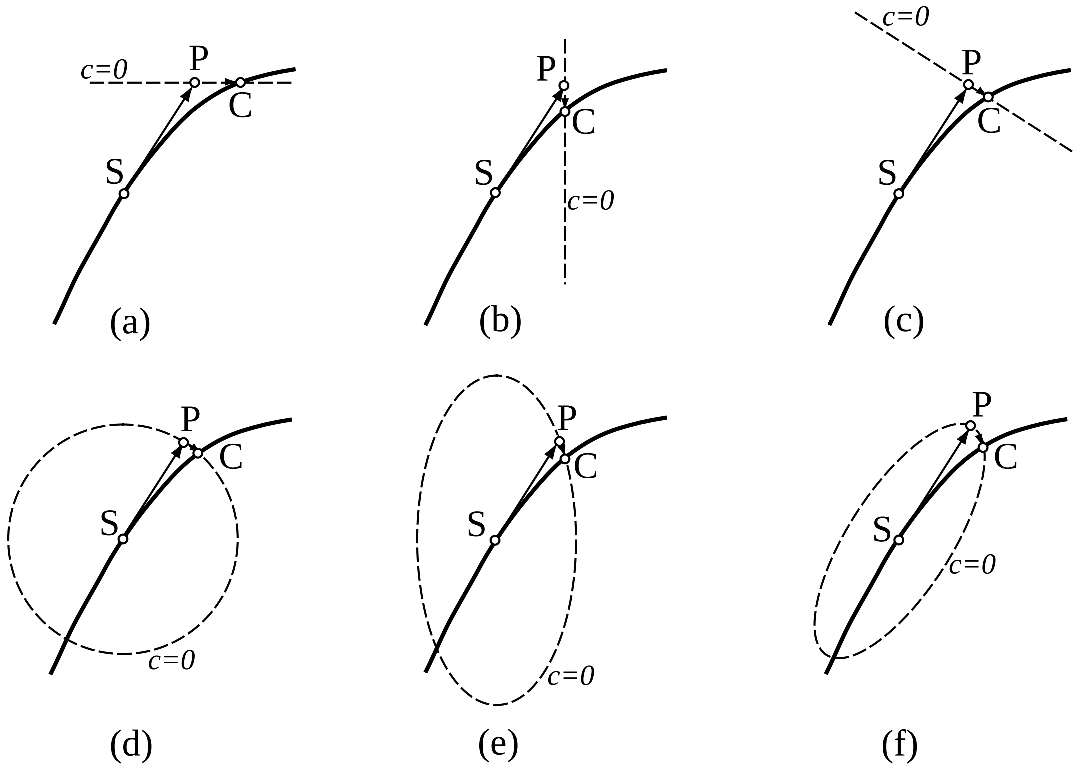

Figure 16.3. Geometric representation of constraint equations for a one-dof problem, with state u and control parameter λ plotted horizontally and vertically, respectively. (a) load control, (b) state control, (c) arclength control, (d) hyperspherical control, (e) global hyperelliptical control, and (f) local hyperelliptic control.

§16.6. Constraint Strategy

So far the form of the constraint equation, (16.2) or (16.3), has been left arbitrary. In the sequel we list, roughly in order of ascending complexity, instances that are either important in the applications or have historical interest. In what follows ℓ is always a *dimensionless* scalar that characterizes the size of the increment. Six constraints are pictured in Figure 16.3. In this figure, c is the constraint curve, S is the last solution point, P the predicted point and C the converged solution.

§16.6.1. λ Control

At each step $\Delta\lambda_n = \ell_n$, where ℓ is a dimensionless scalar. The constraint equation is (16.8) listed in Example 16.1. This is generally called λ-control. Often the parameter λ is associated with a loading amplitude, in which case this is called *load control*. The physical analogy would be a test machine in which the operator increases the load to specific values.

The differential form (16.3) has

$$\mathbf{a} = \mathbf{0}, \qquad g = 1. \tag{16.19}$$

As noted in Example 16.1, this constraint form fails as critical points are approached.

§16.6.2. State Control

This consists of specifying a norm of $\Delta\mathbf{u}_n$, for example the Euclidean norm:

$$c(\Delta\mathbf{u}_n) \equiv (\Delta\mathbf{u}_n^T\,\Delta\mathbf{u}_n)^2 - \ell_n\,u^2 = 0, \tag{16.20}$$

where u is a reference value with dimensions of displacement, which is introduce for scaling purposes. An alternative way of doing that consists of using the scaled increment of §4.6:

$$\Delta\widetilde{\mathbf{u}}_n^T\,\Delta\widetilde{\mathbf{u}}_n - \ell_n^2 = 0. \tag{16.21}$$

(See also Remark below.) The differential form (16.3) has

$$\mathbf{a}^T = 2\Delta\mathbf{u}_n, \qquad g = 0. \tag{16.22}$$

Remark 16.7. In the finite element literature the term *displacement control* has been traditionally associated with the case in which the magnitude of only one of the components of $\mathbf{u}$, say u_i, is specified, which is tantamount to choosing a special infinity norm of $\mathbf{u}$. This old technique was used in the mid-1960s by Argyris and Felippa.[4] There is a generalization of single displacement control in which several reference displacements are used. This multiple dimensional *hyperplane control* has been investigated by Powell, Bergan and others.[5]

§16.6.3. Arclength Control

Arclength control consists of specifying a distance $|\Delta s| = \ell$ along the path tangent. The constraint equation is (16.10) in Example 16.2. This form has scaling problems since it intermixes $\mathbf{u}$ and λ. It is generally preferable to work with the scaled quantities of §4.6 in which case the constraint becomes

$$\Delta\widetilde{s}_n - \ell_n = \frac{1}{\widetilde{f}_n}\,\left|\widetilde{\mathbf{v}}_n^T\,\Delta\widetilde{\mathbf{u}}_n + \Delta\lambda_n\right| - \ell_n = 0, \tag{16.23}$$

The differential form (16.3) for the unscaled form (16.9) is

$$\mathbf{a}^T = \mathbf{v}_n/f_n, \qquad g = 1/f_n. \tag{16.24}$$

and for (16.24)

$$\mathbf{a}^T = \mathbf{v}_n\mathbf{S}^2/\widetilde{f}_n, \qquad g = 1/\widetilde{f}_n. \tag{16.25}$$

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[4] J. H. Argyris, Continua and Discontinua, in *Proceedings Conference on Matrix Methods in Structural Engineering*, AFFDL-TR-66-80, Wright-Patterson AFB, Dayton, Ohio, 11–189 (1966).
C. A. Felippa, Refined Finite Element Analysis of Linear and Nonlinear Two-dimensional Structures, Ph.D. thesis, Dept. of Civil Engrg, University of California, Berkeley (1966).

[5] G. H. Powell and J. Simons, Improved Iteration Strategy for Nonlinear Structures, *Int. J. Num. Meth. Engrg.*, **17**, 1655–1667 (1981)
P. G. Bergan, G. Horrigmoe, B. Krakeland and T. H. Søreide, Solution Techniques for Nonlinear Finite Element Problems, *Int. J. Num. Meth. Engrg.*, **12**, 1677–1696 (1978)
P. G. Bergan, Solution Algorithms for Nonlinear Structural Problems, *Computers & Structures*, **12** 497–509 (1980)
P. G. Bergan and J. Simons, Hyperplane Displacement Control Methods in Nonlinear Analysis, in *Innovative Methods for Nonlinear Problems*, ed. by W. K. Liu, T. Belytschko and K. C. Park, Pineridge Press, Swansea, U.K., 345–364 (1984)

Without the scaling this becomes the constraint of Riks and Wempner,[6] also called *arclength control*. Geometrically the unscaled equation represents a hyperplane normal to $\mathbf{t}$, located a distance ℓ_n from the last solution point $S(\mathbf{u}_n, \lambda_n)$ in the state-control space. The scaled form admits a similar interpretation in the scaled state-control space space $(\mathbf{Su}, \lambda)$.

Remark 16.8. The "orthogonal trajectory" constraint discussed by Fried (see footnote 1) may be regarded as a generalization of the arclength constraint in which a traversal orthogonality condition is applied throughout the corrective phase. This differential constraint is interesting in that it does not fit the form (16.2) and may in fact be followed independently of the the predictor and past solution. But following the trajectory depends on $\mathbf{v} = \mathbf{K}^{-1}\mathbf{q}$ being frequently updated and is practical only with a true Newton corrector.

§16.6.4. (Global) Hyperelliptic Control

There is a wide family of constraints that combine the magnitude of $\Delta\lambda_n$ and a norm of $\Delta\mathbf{u}_n$. A frequently used combination is the hyperelliptic constraint

$$a_n^2 \Delta\mathbf{u}_n^T \Delta\mathbf{u}_n + b_n^2 \,(\Delta\lambda_n)^2 = \ell_n^2, \tag{16.26}$$

where scalar coefficients a and b may not be simultaneously zero.

More effective in practice is the scaled form of the above, namely

$$a_n^2 \Delta\tilde{\mathbf{u}}_n^T \Delta\tilde{\mathbf{u}}_n + b_n^2 \,(\Delta\lambda_n)^2 = \ell_n^2, \tag{16.27}$$

where all quantities are now dimensionless.

Geometrically these constraints corresponds to an hyperellipse that has the last solution as center, and includes other constraints as degenerate cases. The scaling parameters a and b were introduced by Padovan and Park.[7] The expression was rendered dimensionless by Felippa[8] who introduced scaling parameters and and discussed appropriate choices. If $a = b = 1$ in the unscaled form (16.27) we recover the hyperspherical constraint proposed (but not used) by Crisfield (*loc. cit.* in footnote 4).

The constraint gradients are

$$\mathbf{a} = 2a^2\Delta\mathbf{u}, \qquad g = 2b^2\Delta\lambda. \tag{16.28}$$

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[6] E. Riks, The Application of Newton's Method to the Problem of Elastic Stability, *Trans. ASME, J. Appl. Mech.*, **39**, 1060–1065 (1972)
G. A. Wempner, Discrete Approximations Related to Nonlinear Theories of Solids, *Int. J. Solids Structures*, **7**, 1581–1599 (1971).

[7] J. Padovan and S. Tovichakchaikul, Self-Adaptive Predictor-Corrector Algorithm for Static Nonlinear Structural Analysis, *Computers & Structures*, **15**, 365–377 (1982).
K. C. Park, A Family of Solution Algorithms for Nonlinear Structural Analysis Based on the Relaxation Equations, *Int. J. Num. Meth. Engrg.*, **18**, 1637–1647 (1982).

[8] C. A. Felippa, Dynamic Relaxation under General Increment Control, in *Innovative Methods for Nonlinear Problems*, ed. by W. K. Liu, T. Belytschko and K. C. Park, Pineridge Press, Swansea, U.K., 103–163 (1984).

§16.6.5. Local Hyperelliptic Control

This is a variation of the previous one in which we take a combination of $\Delta\overline{\lambda}$ and a norm of $\Delta\overline{\mathbf{u}}$, where $\Delta\overline{\lambda}$ and $\Delta\overline{\mathbf{u}}$ are to be determined according to a local coordinate system at $S(\mathbf{u}_n, \lambda_n)$:

$$c(\overline{\mathbf{u}}, \overline{\lambda}) = a^2(\overline{\mathbf{u}} - \overline{\mathbf{u}}_n)^T \mathbf{S}(\overline{\mathbf{u}} - \overline{\mathbf{u}}_n) + b^2(\overline{\lambda} - \overline{\lambda}_n)^2 - \ell_n^2 = 0, \tag{16.29}$$

where a and b are scalar coefficients and ℓ_n is prescribed. Geometrically this is a hyperellipse with principal axes in a coordinate system defined by $\Delta\overline{\lambda}$ and $\Delta\overline{\mathbf{u}}$. An attractive choice for the local system is provided by the path tangent vector $\mathbf{t}_n$ and the normal hyperplane at point $S(\mathbf{u}_n, \lambda_n)$. These are given by by (4.16) and (4.20) respectively, with $\mathbf{v} \equiv \mathbf{v}_n$.

Near critical points, $\mathbf{v} \to \infty$. In such a case we would like to recover the global system to avoid numerical difficulties. This is achieved by defining the new variables $\Delta\overline{\lambda}$ and $\Delta\overline{\mathbf{u}}$ according to

$$\Delta\overline{\lambda} = \mathbf{v}^T(\Delta\mathbf{u} - \mathbf{v}\Delta\lambda), \qquad \Delta\overline{\mathbf{u}} = \mathbf{v}(\mathbf{v}^T\Delta\mathbf{u} + \Delta\lambda). \tag{16.30}$$

Scaling of this constraint to achieve consistency is discussed by Skeie and Felippa (work cited in footnote 5), where additional computational details may be found. It turns out that this constraint can include all ones previously discussed as special regular or limit cases.

Remark 16.9. Another interesting strategy: the *work constraint* of Bathe and Dvorkin [9] limits the total external work spent during the corrective phase.

Remark 16.10. In path-independent problems that involve only geometric or conservative boundary-condition nonlinearities, it is generally best to maximize step lengths subject to stability and equilibrium accuracy constraints. Stability depends on the curvature of the response path, presence of critical points, and solution method used. Equilibrium accuracy depends chiefly on whether a corrective process is applied.

§16.7. Practical Solution Requirements

The remaining subsections describe various types of nonlinear structural analyses encountered in engineering practice, and the requirements they pose on solution procedures.

§16.7.1. Tracing the Response

"Tracing the response" is of interest for many nonlinear problems. For a typical stage, perform a sequence of incremental steps to find equilibrium states

$$\mathbf{u}_n,\ \lambda_n, \qquad n = 1, 2, \ldots$$

in sufficient number to ascertain the response $\mathbf{u} = \mathbf{u}(\lambda)$ of the structure within engineering requirements.

If the control parameter is associated with a fundamental load system, the response path is known as the *fundamental equilibrium path*, as it pertains to the service range in which the structure is supposed to operate.

One class of problems that fit this requirement is that in which structural deflections, rather than strength, are of primary importance in the design. For example, some large flexible space structures must meet rigorous "dimensional stability" tolerances while in service.

[9] K. J. Bathe and E. Dvorkin, On the Automatic Solution of Nonlinear Finite Element Equations, *Computers & Structures*, **17**, 871–879 (1983).

§16.7.2. Finding a Nonlinear Solution

A variant of the foregoing occurs if the primary objective of the analysis is to find a solution $\mathbf{u}$ corresponding to a given λ (for example, $\lambda = 1$), whereas tracing of the response path is in itself of little interest.

Very flexible structures that must operate in the nonlinear regime during service fit this problem class. The example of the suspension bridge under its own weight, discussed in §3.4, provides a good illustration. The undeflected "base" configuration $\mathbf{u} = \mathbf{0}$ is of little interest as it has no physical reality and the bridge never assumes it. It is merely a reference point for measuring deflections.

Under such circumstances, the chief consideration is that the accuracy with which the response path is traced is of little concern. Getting the final answer is the important thing. Once this reference configuration is obtained, "excursions" due to live loads, temperature variations, wind effects and the like may be the subject of further analysis staging.

§16.7.3. Stability Assessment

This is perhaps the most important application of nonlinear static analysis. The analyst is concerned with the value (or values) of λ closest to 0 at which the structure behavior is not uniquely determined by λ. These are the critical points discussed in Chapter 5. In physical terms, the system becomes uncontrollable and may "take off" dynamically.

Problem of this nature arise in stability design. The determination of limit points is called collapse or snapping analysis. The determination of bifurcation points is called buckling analysis.

§16.7.4. Post-buckling and Snap-through

Occassionally it is of interest to continue the nonlinear analysis beyond a limit or bifurcation point. Continuation past a limit point is post-collapse or snap-through analysis; continuation past a bifurcation point is post-buckling analysis.

Post-critical analyses are less commonly encountered in practice than the previous two types. They are of interest to ascertain imperfection sensitivity of primary structural components, or to assess strength reserve in fail-safe analysis under abnormal conditions such as construction, deployment or accidents.

Conventional load control is not generally sufficient to trace snap-through. This may be achieved, however, with the aid of the more general increment control strategies discussed above. Traversing bifurcation points is notoriously more difficult; a technique applicable to well isolated bifurcation points is discussed later in the context of augmented equations and auxiliary systems.

§16.7.5. Multiple Load Parameters

As discussed in Chapter 3, the case of multiple control parameters is reduced to a sequence of one-parameter analyses. The previous classification apply to individual stages, and not all stages necessarily fit the same type of analysis requirements.

The systematic determination of a complete equilibrium surface as the envelope of all response path is rarely pursued in practice aside from academic examples. For practical structures, an investigation of this type would put enormous demands on human and computer time and is doubtful whether the additional insight would justify such expenditures.

There is, however, a special case of multiparameter investigation that is gaining popularity for designing lightweight structures: stability interaction curves as envelopes of critical points.