Eigenvalue problem in structural dynamics

$$\mathbf{k}\phi = \lambda \mathbf{m}\phi$$

$\lambda_n = \omega_n^2$ are the roots of the characteristic equation

$$p(\lambda) = \det[\mathbf{k} - \lambda \mathbf{m}] = 0$$

In this section Rayleigh's quotient and its properties are presented because it is needed in vector iteration methods. If Eq. (10.11.1) is premultiplied by ϕ^T, the following scalar equation is obtained:

$$\phi^T \mathbf{k} \phi = \lambda \phi^T \mathbf{m} \phi$$

The positive definiteness of $\mathbf{m}$ guarantees that $\phi^T \mathbf{m} \phi$ is nonzero, so that it is permissible to solve for λ:

$$\lambda = \frac{\phi^T \mathbf{k} \phi}{\phi^T \mathbf{m} \phi} \tag{10.12.1}$$

Rayleigh's quotient has the following properties:

1. When ϕ is an eigenvector ϕ_n of Eq. (10.11.1), Rayleigh's quotient is equal to the corresponding eigenvalue λ_n.
2. If ϕ is an approximation to ϕ_n with an error that is a first-order infinitesimal, Rayleigh's quotient is an approximation to λ_n with an error which is a second-order infinitesimal (i.e., Rayleigh's quotient is *stationary* in the neighborhoods of the true eigenvectors).
3. Rayleigh's quotient is bounded between $\lambda_1 \equiv \omega_1^2$ and $\lambda_N \equiv \omega_N^2$, the smallest and largest eigenvalues.

INVERSE VECTOR ITERATION METHOD

1. Determine $\bar{\mathbf{x}}_{j+1}$ by solving the algebraic equations:

$$\mathbf{k}\bar{\mathbf{x}}_{j+1} = \mathbf{m}\mathbf{x}_j \tag{10.13.3}$$

2. Obtain an estimate of the eigenvalue by evaluating Rayleigh's quotient:

$$\lambda^{(j+1)} = \frac{\bar{\mathbf{x}}_{j+1}^T \mathbf{k}\bar{\mathbf{x}}_{j+1}}{\bar{\mathbf{x}}_{j+1}^T \mathbf{m}\bar{\mathbf{x}}_{j+1}} = \frac{\bar{\mathbf{x}}_{j+1}^T \mathbf{m}\mathbf{x}_j}{\bar{\mathbf{x}}_{j+1}^T \mathbf{m}\bar{\mathbf{x}}_{j+1}} \tag{10.13.4}$$

3. Check convergence by comparing two successive values of λ:

$$\frac{|\lambda^{(j+1)} - \lambda^{(j)}|}{\lambda^{(j+1)}} \le \text{tolerance} \tag{10.13.5}$$

4. If the convergence criterion is not satisfied, normalize $\bar{\mathbf{x}}_{j+1}$:

$$\mathbf{x}_{j+1} = \frac{\bar{\mathbf{x}}_{j+1}}{(\bar{\mathbf{x}}_{j+1}^T \mathbf{m}\bar{\mathbf{x}}_{j+1})^{1/2}} \tag{10.13.6}$$

   and go back to the first step and carry out another iteration using the next j.

5. Let l be the last iteration [i.e., the iteration that satisfies Eq. (10.13.5)]. Then

$$\lambda_1 \doteq \lambda^{(l+1)} \qquad \phi_1 \doteq \frac{\bar{\mathbf{x}}_{l+1}}{(\bar{\mathbf{x}}_{l+1}^T \mathbf{m}\bar{\mathbf{x}}_{l+1})^{1/2}} \tag{10.13.7}$$

Example 10.14

The floor masses and story stiffnesses of the three-story frame, idealized as a shear frame, are shown in Fig. E10.14, where $m = 100$ kips/g, and $k = 168$ kips/in. Determine the fundamental frequency ω_1 and mode shape ϕ_1 by inverse vector iteration.

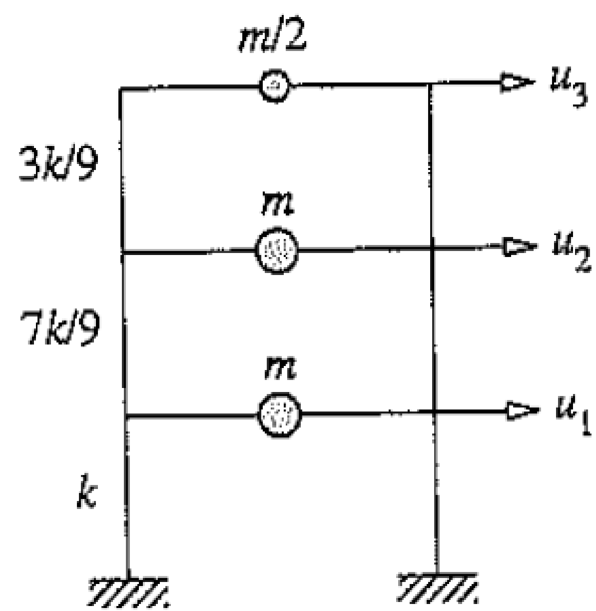

Figure E10.14

Solution The mass and stiffness matrices for the system are

$$\mathbf{m} = m\begin{bmatrix} 1 & & \\ & 1 & \\ & & \frac{1}{2} \end{bmatrix} \qquad \mathbf{k} = \frac{k}{9}\begin{bmatrix} 16 & -7 & 0 \\ -7 & 10 & -3 \\ 0 & -3 & 3 \end{bmatrix}$$

where $m = 0.259$ kip-sec^2/in. and $k = 168$ kips/in.

The inverse iteration algorithm of Eqs. (10.13.3) to (10.13.7) is implemented starting with an initial vector $\mathbf{x}_1 = \langle 1 \quad 1 \quad 1\rangle^T$ leading to Table E10.14. The final result is $\omega_1 \doteq \sqrt{144.14} = 12.006$ and $\phi_1 \doteq \langle 0.6377 \quad 1.2752 \quad 1.9122\rangle^T$.

Iteration	$\mathbf{x}_j$	$\bar{\mathbf{x}}_{j+1}$	$\lambda^{(j+1)}$	$\mathbf{x}_{j+1}$
1	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0.0039 \\ 0.0068 \\ 0.0091 \end{bmatrix}$	147.73	$\begin{bmatrix} 0.7454 \\ 1.3203 \\ 1.7676 \end{bmatrix}$
2	$\begin{bmatrix} 0.7454 \\ 1.3203 \\ 1.7676 \end{bmatrix}$	$\begin{bmatrix} 0.0045 \\ 0.0089 \\ 0.0130 \end{bmatrix}$	144.29	$\begin{bmatrix} 0.6574 \\ 1.2890 \\ 1.8800 \end{bmatrix}$
3	$\begin{bmatrix} 0.6574 \\ 1.2890 \\ 1.8800 \end{bmatrix}$	$\begin{bmatrix} 0.0044 \\ 0.0089 \\ 0.0132 \end{bmatrix}$	144.15	$\begin{bmatrix} 0.6415 \\ 1.2785 \\ 1.9052 \end{bmatrix}$
4	$\begin{bmatrix} 0.6415 \\ 1.2785 \\ 1.9052 \end{bmatrix}$	$\begin{bmatrix} 0.0044 \\ 0.0089 \\ 0.0133 \end{bmatrix}$	144.14	$\begin{bmatrix} 0.6384 \\ 1.2758 \\ 1.9109 \end{bmatrix}$
5	$\begin{bmatrix} 0.6384 \\ 1.2758 \\ 1.9109 \end{bmatrix}$	$\begin{bmatrix} 0.0044 \\ 0.0088 \\ 0.0133 \end{bmatrix}$	144.14	$\begin{bmatrix} 0.6377 \\ 1.2752 \\ 1.9122 \end{bmatrix}$

Similarly, the vector $\bar{\mathbf{x}}_{j+1}$ after j iteration cycles can be expressed as

$$\bar{\mathbf{x}}_{j+1} = \frac{1}{\lambda_1^j} \sum_{n=1}^{N} \left(\frac{\lambda_1}{\lambda_n}\right)^j \phi_n q_n \tag{10.13.13}$$

Since $\lambda_1 < \lambda_n$ for $n > 1$, $(\lambda_1/\lambda_n)^j \to 0$ as $j \to \infty$, and only the $n = 1$ term in Eq. (10.13.13) remains significant, indicating that

$$\bar{\mathbf{x}}_{j+1} \to \frac{1}{\lambda_1^j} \phi_1 q_1 \quad \text{as} \quad j \to \infty \tag{10.13.14}$$

Thus $\bar{\mathbf{x}}_{j+1}$ converges to a vector proportional to ϕ_1. Furthermore, the normalized vector $\mathbf{x}_{j+1}$ of Eq. (10.13.6) converges to ϕ_1, which is mass orthonormal.

The rate of convergence depends on λ_1/λ_2, the ratio that appears in the second term in the summation of Eq. (10.13.13). The smaller this ratio is, the faster is the convergence; this implies that convergence is very slow when λ_2 is nearly equal to λ_1. For such situations the convergence rate can be improved by the procedures of Section 10.14.

If only the first natural mode ϕ_1 and the associated natural frequency ω_1 are required, there is no need to proceed further. This is an advantage of the iteration method. It is unnecessary to solve the complete eigenvalue problem to obtain one or two of the modes.

VECTOR ITERATION WITH SHIFTS:

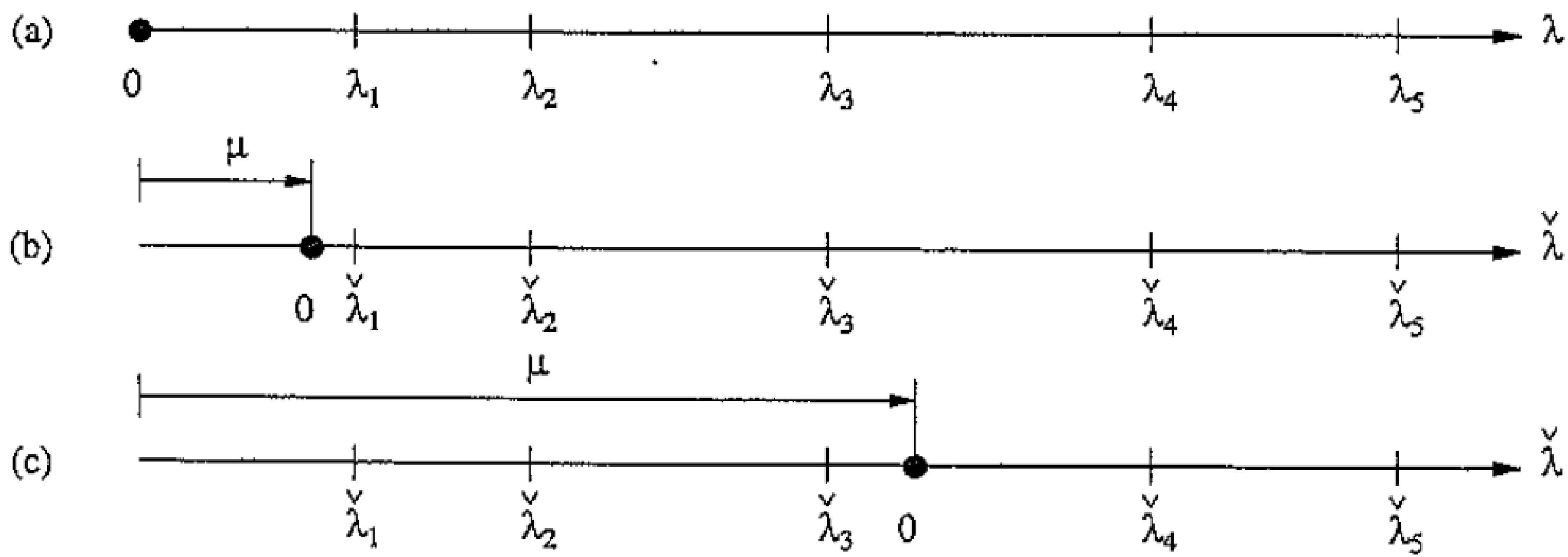

Figure 10.14.1 (a) Eigenvalue spectrum; (b) eigenvalue measured from a shifted origin; (c) location of shift point for convergence to λ_3.

$$\breve{\mathbf{k}}\phi = \breve{\lambda}\mathbf{m}\phi$$

$$\breve{\mathbf{k}} = \mathbf{k} - \mu\mathbf{m} \qquad \breve{\lambda} = \lambda - \mu$$

Example 10.15

Determine the natural frequencies and modes of vibration of the system of Example 10.14 by inverse vector iteration with shifting.

Solution Equation (10.14.1) with a selected shift μ is solved by inverse vector iteration. Selecting the shift $\mu_1 = 100$, $\check{\mathbf{k}}$ is calculated from Eq. (10.14.2) and the inverse vector iteration

TABLE E10.15a VECTOR ITERATION WITH SHIFT: FIRST EIGENPAIR

Iteration	$\mathbf{x}_j$	μ	$\bar{\mathbf{x}}_{j+1}$	$\lambda^{(j+1)}$	$\mathbf{x}_{j+1}$
1	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	100	$\begin{bmatrix} 0.0114 \\ 0.0218 \\ 0.0313 \end{bmatrix}$	144.60	$\begin{bmatrix} 0.6759 \\ 1.2933 \\ 1.8610 \end{bmatrix}$
2	$\begin{bmatrix} 0.6759 \\ 1.2933 \\ 1.8610 \end{bmatrix}$	100	$\begin{bmatrix} 0.0145 \\ 0.0289 \\ 0.0432 \end{bmatrix}$	144.15	$\begin{bmatrix} 0.6401 \\ 1.2769 \\ 1.9083 \end{bmatrix}$
3	$\begin{bmatrix} 0.6401 \\ 1.2769 \\ 1.9083 \end{bmatrix}$	100	$\begin{bmatrix} 0.0144 \\ 0.0289 \\ 0.0433 \end{bmatrix}$	144.14	$\begin{bmatrix} 0.6377 \\ 1.2752 \\ 1.9122 \end{bmatrix}$
4	$\begin{bmatrix} 0.6377 \\ 1.2752 \\ 1.9122 \end{bmatrix}$	100	$\begin{bmatrix} 0.0144 \\ 0.0289 \\ 0.0433 \end{bmatrix}$	144.14	$\begin{bmatrix} 0.6375 \\ 1.2750 \\ 1.9125 \end{bmatrix}$

TABLE E10.15b VECTOR ITERATION WITH SHIFT: SECOND EIGENPAIR

Iteration	$\mathbf{x}_j$	μ	$\bar{\mathbf{x}}_{j+1}$	$\lambda^{(j+1)}$	$\mathbf{x}_{j+1}$
1	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	600	$\begin{bmatrix} 0.0044 \\ 0.0028 \\ -0.0133 \end{bmatrix}$	605.11	$\begin{bmatrix} 0.8030 \\ 0.5189 \\ -2.4277 \end{bmatrix}$
2	$\begin{bmatrix} 0.8030 \\ 0.5189 \\ -2.4277 \end{bmatrix}$	600	$\begin{bmatrix} 0.0197 \\ 0.0201 \\ -0.0373 \end{bmatrix}$	648.10	$\begin{bmatrix} 1.0062 \\ 1.0221 \\ -1.8994 \end{bmatrix}$
3	$\begin{bmatrix} 1.0062 \\ 1.0221 \\ -1.8994 \end{bmatrix}$	600	$\begin{bmatrix} 0.0201 \\ 0.0201 \\ -0.0405 \end{bmatrix}$	648.64	$\begin{bmatrix} 0.9804 \\ 0.9778 \\ -1.9717 \end{bmatrix}$
4	$\begin{bmatrix} 0.9804 \\ 0.9778 \\ -1.9717 \end{bmatrix}$	600	$\begin{bmatrix} 0.0202 \\ 0.0202 \\ -0.0404 \end{bmatrix}$	648.65	$\begin{bmatrix} 0.9827 \\ 0.9829 \\ -1.9642 \end{bmatrix}$

Starting with the shift $\mu_1 = 600$ and the same $\mathbf{x}_1$, the inverse iteration algorithm leads to Table E10.15b. The final result is $\omega_2 = \sqrt{648.65} = 25.468$ and $\phi_2 = \langle 0.9827 \quad 0.9829 \quad -1.9642 \rangle^T$. Convergence is attained in four iteration cycles.

TABLE E10.15c VECTOR ITERATION WITH SHIFT: THIRD EIGENPAIR

Iteration	$\mathbf{x}_j$	μ	$\bar{\mathbf{x}}_{j+1}$	$\lambda^{(j+1)}$	$\mathbf{x}_{j+1}$
1	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	1500	$\begin{bmatrix} 0.0198 \\ -0.0156 \\ 0.0054 \end{bmatrix}$	1510.6	$\begin{bmatrix} 1.5264 \\ -1.2022 \\ 0.4148 \end{bmatrix}$
2	$\begin{bmatrix} 1.5264 \\ -1.2022 \\ 0.4148 \end{bmatrix}$	1500	$\begin{bmatrix} 0.1167 \\ -0.0832 \\ 0.0333 \end{bmatrix}$	1513.5	$\begin{bmatrix} 1.5784 \\ -1.1261 \\ 0.4509 \end{bmatrix}$
3	$\begin{bmatrix} 1.5784 \\ -1.1261 \\ 0.4509 \end{bmatrix}$	1500	$\begin{bmatrix} 0.1168 \\ -0.0834 \\ 0.0334 \end{bmatrix}$	1513.5	$\begin{bmatrix} 1.5778 \\ -1.1270 \\ 0.4508 \end{bmatrix}$

Starting with the shift $\mu_1 = 1500$ and the same $\mathbf{x}_1$, the inverse iteration algorithm leads to Table E10.15c. The final result is $\omega_3 = \sqrt{1513.5} = 38.904$ and $\phi_3 = \langle 1.5778 \;\; -1.1270 \;\; 0.4508 \rangle^T$. Convergence is attained in three cycles.