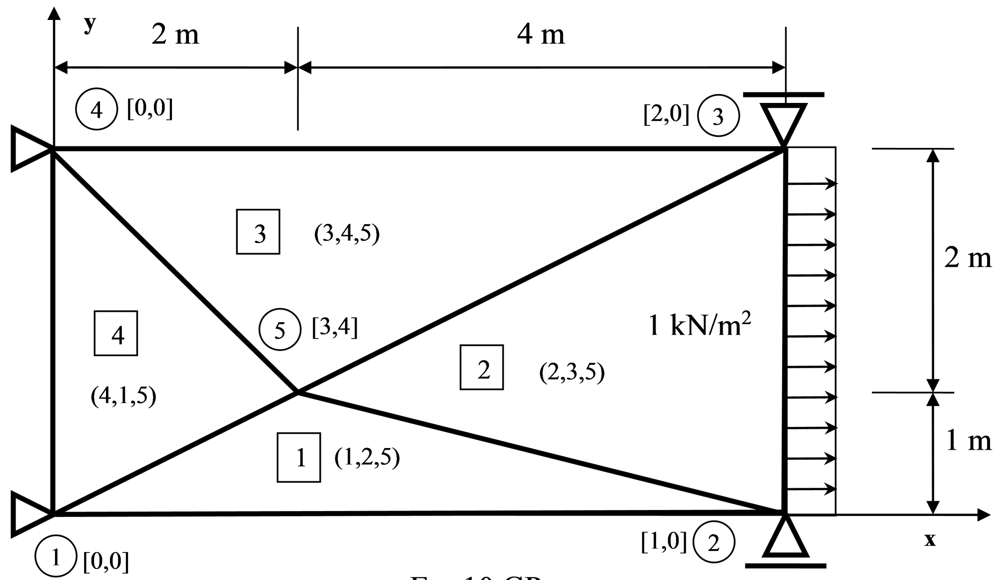
# Cvičení č. 9 - Rovinná napjatost

## Příklad č. 1 - Tahový patch test




E = 10 GPa
ν = 0.1
t = 1.0 m

**Slabé řešení:**

$$\delta \boldsymbol{r}^{\mathsf{T}} \left\{ \sum_{e=1}^{n} \boldsymbol{L}^{eT} \left[ \overbrace{\int_{\Omega} \boldsymbol{B}^e(\boldsymbol{x})^{\mathsf{T}} \boldsymbol{D}^e(\boldsymbol{x}) \boldsymbol{B}^e(\boldsymbol{x})\, d\Omega}^{\boldsymbol{K^e}} \boldsymbol{L}^e \boldsymbol{r} - \overbrace{\int_{\Omega} \boldsymbol{N}^e(\boldsymbol{x})^{\mathsf{T}} \overline{\boldsymbol{X}}(\boldsymbol{x}) d\Omega}^{\boldsymbol{f_\Omega^e}} - \underbrace{\int_{\Gamma_p} \boldsymbol{N}^e(\boldsymbol{x})^{\mathsf{T}} \overline{\boldsymbol{p}}(\boldsymbol{x})\, d\Gamma_p}_{\boldsymbol{f_\Gamma^e}} \right] \right\} = 0$$

$$\varepsilon^e(x) \approx B^e(x) r^e$$
$$\sigma^e(x) \approx D^e(x) B^e(x) r^e$$

**Derivace interpolačních funkcí:**

$$B^e = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix}$$

$$B^e = \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

**Matice tuhosti pro rovinnou napjatost:**

$$D = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

## Matice tuhosti prvku:

$$K^e = \int_{\Omega_e} B^{eT} D^e B^e d\Omega = B^{eT} D^e B^e \int_{\Omega_e} d\Omega = AtB^{eT} D^e B^e$$

## Okrajové podmínky:

### Předepsané zatížení na hranici:

$$f_{\Gamma_p} = \int_{\Gamma_p} N^e(x)^T \bar{p}(x) d\Gamma_p$$

In [16]:
```
% funkce pocita plochu trojuhelnikoveho prvku
% In:
%   xe - vektor x-ovych souradnic uzlu prvku (x1,x2,x3)
%   ye - vektor y-ovych souradnic uzlu prvku (y1,y2,y3)
%
% Out:
%   Ae - plocha prvku
%
function Ae = area_triangle (xe,ye)

Ae=(1/2)*((xe(2)*ye(3)-xe(3)*ye(2))-(xe(1)*ye(3)-xe(3)*ye(1))+(xe(1)*ye(2)-xe(2)*ye(1))) ;

end

% funkce pocita delky stran trojuhelnikoveho prvku
% In:
%   xe - vektor x-ovych souradnic uzlu prvku (x1,x2,x3)
%   ye - vektor y-ovych souradnic uzlu prvku (y1,y2,y3)
%
% Out:
%   le - delky hran prvku (3,1)
%
function le = length_triangle (xe,ye)

le = zeros(3,1) ;
le(1) = sqrt((xe(2)-xe(1))*(xe(2)-xe(1))+(ye(2)-ye(1))*(ye(2)-ye(1)));
le(2) = sqrt((xe(3)-xe(2))*(xe(3)-xe(2))+(ye(3)-ye(2))*(ye(3)-ye(2)));
le(3) = sqrt((xe(1)-xe(3))*(xe(1)-xe(3))+(ye(1)-ye(3))*(ye(1)-ye(3)));
end

% funkce pocita matici tuhosti 2d trojuhelnikoveho stenoveho prvku s linearni aproximaci
% In:
%   xe - vektor x-ovych souradnic uzlu prvku (x1,x2,x3)
%   ye - vektor y-ovych souradnic uzlu prvku (y1,y2,y3)
%   E - Younguv modul pruznosti
%   nu - Poissonuv soucinitel
%
% Out:
%   ke - matice tuhosti prvku (6,6)
%   dbe - matice na vypocet napeti (3,6)
%   de - materialova matice (3,3)
%   be - mative derivaci bazovych funkci (3,6)
%
function [ke,dbe,de,be] = plane_stress (xe,ye,E,nu)

Ae = area_triangle(xe,ye) ;

y23 = ye(2)-ye(3); y31 = ye(3)-ye(1); y12 = ye(1)-ye(2);
x32 = xe(3)-xe(2); x13 = xe(1)-xe(3); x21 = xe(2)-xe(1);

be = (1/(2*Ae))*[ y23, 0, y31, 0, y12, 0 ;
                  0, x32, 0, x13, 0, x21 ;
                  x32,y23,x13,y31,x21,y12] ;

de = (E/(1-nu*nu))*[ 1, nu, 0;
                    nu, 1, 0;
                    0, 0, (1-nu)/2 ];

dbe = de*be ;

ke = Ae*be'*de*be;

end

% funkce pro lokalizaci matice tuhosti prvku
% do matice tuhosti konstrukce
% vyzaduje existenci globalni matice kodovych cisel - lm
%
%
% Vstup:
%   k - matice tuhosti konstrukce
%   ke- matice tuhosti prvku
%   lm- matice kodovych cisel
```

```
%   e - cislo prvku
%
% Vystup:
%   k - matice tuhosti konstrukce
%
function k = assembly (k,ke,lm,e)

ndof = size(ke,1);
for i=1:ndof
        ia = lm(e,i);
        if ia ~=0
            for j=1:ndof
                ja=lm(e,j);
                if ja ~=0
                  k(ia,ja)=k(ia,ja)+ke(i,j);
                end
            end
        end
    end
end

% funkce pro lokalizaci matice prave strany
% vyzaduje existenci globalni matice kodovych cisel - lm
%
%
% Vstup:
%   f - matice prave strany
%   fe- matice prave strany prvku
%   lm- matice kodovych cisel
%   e - cislo prvku
%
% Vystup:
%   f - matice prave strany
%
function f = assemblyf (f,fe,lm,e)

ndof = size(fe,1);
for i=1:ndof
        ia = lm(e,i);
        if ia ~=0
           f(ia,1)=f(ia,1)+fe(i,1);
        end
    end
end
```

In [17]:

```
%pocet prvku
nelem=4;
nnodes=5;
E=10;
nu = 0.1;

% pole kodovych cisel uzlu
id = [ 5 6; 1 7; 2 8; 9 10; 3 4 ];

%pole uzlovych cisel prvku
ide = [1 2 5; 2 3 5; 3 4 5; 4 1 5];

% pocet neznamych
neq = 10;

% pole souradnic
x =[0.0, 6.0, 6.0, 0.0, 2.0];
y =[0.0, 0.0, 3.0, 3.0, 1.0];

ke = zeros(6,6);
dbe = zeros(3,6);
be = zeros(3,6);
de = zeros(3,3);

k = zeros(neq,neq);
f = zeros(neq,1);

for i = 1:nelem
    xe = [x(ide(i,:))] ;
    ye = [y(ide(i,:))] ;
    [ke,dbe,de,be] = plane_stress(xe,ye,E,nu) ;
    lm = [id(ide(i,1),:), id(ide(i,2),:), id(ide(i,3),:)];
    k=assembly(k,ke,lm,1);
end

f = [ 1.5, 1.5, 0.0, 0.0] ;

%reseni posunu
kuu=k(1:4,1:4)
fuu=f(:)
u=kuu\fuu;
u

ug = zeros(neq,1);
ug(1:4)=u(:)

nodal_u = [];
% Vypis posunu podle uzlu
for i=1:size(id,1)
    nodal_u = [nodal_u;ug(id(i,1)),ug(id(i,2))];
end
nodal_u
```

```
kuu =

    7.07071   -2.18855   -7.07071    2.77778
   -2.18855    5.89226   -3.53535   -2.77778
   -7.07071   -3.53535   31.81818    0.00000
    2.77778   -2.77778    0.00000   50.56818

fuu =

   1.50000
   1.50000
   0.00000
   0.00000

u =

   5.9400e-01
   5.9400e-01
   1.9800e-01
   5.6938e-19

ug =

   0.59400
   0.59400
   0.19800
   0.00000
   0.00000
   0.00000
   0.00000
   0.00000
   0.00000
   0.00000

nodal_u =

   0.00000   0.00000
   0.59400   0.00000
   0.59400   0.00000
   0.00000   0.00000
   0.19800   0.00000
```

## Vyhodnocení výsledků:

**Vnitřní síly na hranách:**

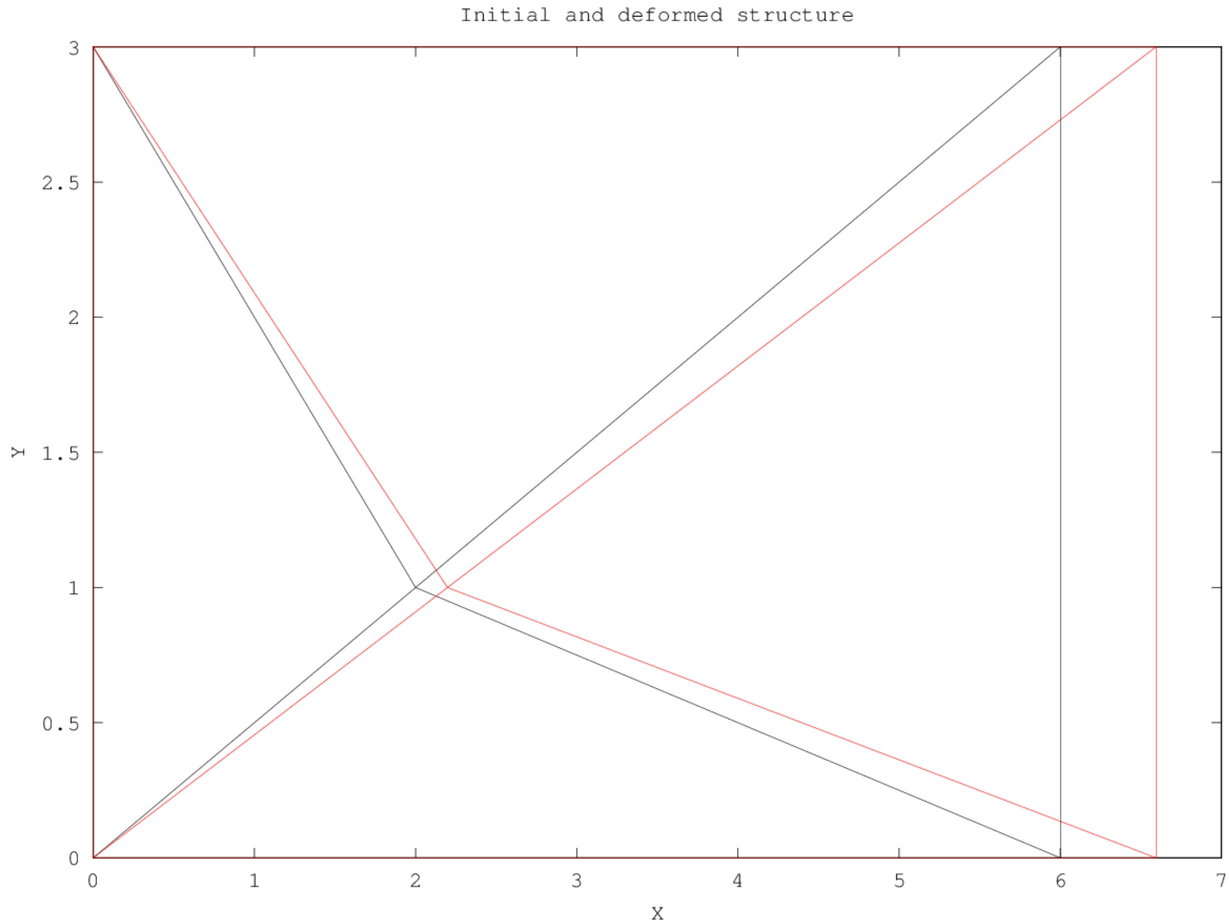
$$g^e = \begin{Bmatrix} \sigma_n \\ \tau \end{Bmatrix} = \begin{bmatrix} cos(\alpha) & sin(\alpha) \\ -sin(\alpha) & cos(\alpha) \end{bmatrix} \begin{Bmatrix} p_x \\ p_y \end{Bmatrix} = \begin{bmatrix} cos(\alpha) & sin(\alpha) \\ -sin(\alpha) & cos(\alpha) \end{bmatrix} \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix} \begin{Bmatrix} cos(\alpha) \\ sin(\alpha) \end{Bmatrix}$$

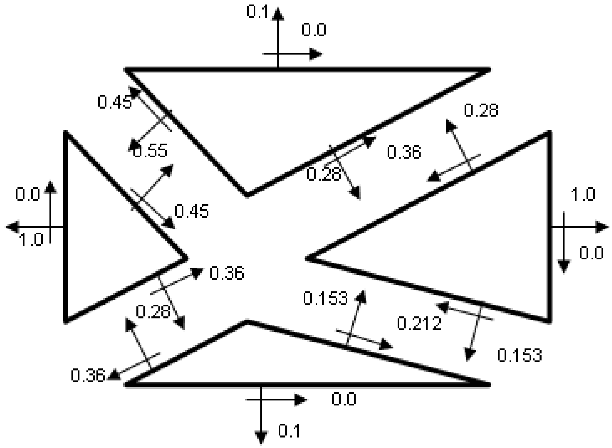
In [18]:

```
% funkce pocita napeti na hrnach 2d trojuhelnikoveho stenoveho prvku s linearni aproximaci
% In:
%   xe - vektor x-ovych souradnic uzlu prvku (x1,x2,x3)
%   ye - vektor y-ovych souradnic uzlu prvku (y1,y2,y3)
%   sig - matice napeti v tezisti (3,1)
%
% Out:
%   pse - napeti na hranach (6,1)
%
function pse = boundary_stress (xe,ye,sig)

pse = zeros(6,1) ;

A = [ sig(1), sig(3);
      sig(3), sig(2)] ;

y23 = ye(2)-ye(3); y31 = ye(3)-ye(1); y12 = ye(1)-ye(2);
x32 = xe(3)-xe(2); x13 = xe(1)-xe(3); x21 = xe(2)-xe(1);

le = length_triangle (xe,ye) ;

c1 = -y12/le(1) ;
s1 = -x21/le(1) ;

T = [ c1, s1; -s1, c1 ];

temp = T*A*[c1;s1];

pse(1) = temp(1,1) ;
pse(2) = temp(2,1) ;

c1 = -y23/le(2) ;
s1 = -x32/le(2) ;

T = [ c1, s1; -s1, c1 ];

temp = T*A*[c1;s1];

pse(3) = temp(1,1) ;
pse(4) = temp(2,1) ;

c1 = -y31/le(3) ;
s1 = -x13/le(3) ;

T = [ c1, s1; -s1, c1 ];

temp = T*A*[c1;s1];

pse(5) = temp(1,1) ;
pse(6) = temp(2,1) ;

end

%reseni relativnich deformaci a napeti

for i = 1:nelem
    xe = [x(ide(i,:))] ;
    ye = [y(ide(i,:))] ;
    [ke,dbe,de,be] = plane_stress(xe,ye,E,nu) ;
    lm = [id(ide(i,1),:), id(ide(i,2),:), id(ide(i,3),:)] ;
    ul = [ug(lm)] ;
    eps = be*ul;
    sig = dbe*ul;
    pse = boundary_stress (xe,ye,sig);
    eps'
    sig'
    pse'
end

% Plot deformed configuration over the initial configuration
for i = 1:nnodes
    xnew(i) = x(i) + ug(id(i,1));
    ynew(i) = y(i) + ug(id(i,2));
```

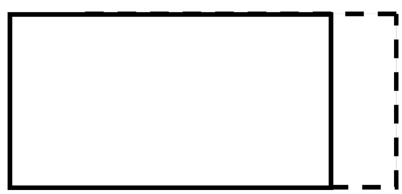
```
end

figure;
for i = 1:nelem
    xe = [x(ide(i,:))] ;
    ye = [y(ide(i,:))] ;
    plot(xe,ye,'k');hold on;
    xe = [xnew(ide(i,:))] ;
    ye = [ynew(ide(i,:))] ;
    plot(xe,ye,'r');hold on;
end
title('Initial and deformed structure'); xlabel('X'); ylabel('Y');
```

```
ans =

   0.099000   0.000000   0.000000

ans =

   1.0000e+00   1.0000e-01   1.1102e-16

ans =

   1.0000e-01  -1.1102e-16   1.5294e-01  -2.1176e-01   2.8000e-01   3.6000e-01

ans =

   0.099000   0.000000  -0.000000

ans =

   1.0000e+00   1.0000e-01  -6.4702e-19

ans =

   1.0000e+00  -6.4702e-19   2.8000e-01   3.6000e-01   1.5294e-01  -2.1176e-01

ans =

   0.099000  -0.000000   0.000000

ans =

   1.0000e+00   1.0000e-01  -5.5511e-17

ans =

   1.0000e-01   5.5511e-17   5.5000e-01  -4.5000e-01   2.8000e-01   3.6000e-01

ans =

   0.099000   0.000000   0.000000

ans =

   1.0000e+00   1.0000e-01   1.2940e-18

ans =

   1.0000e+00   1.2940e-18   2.8000e-01   3.6000e-01   5.5000e-01  -4.5000e-01
```


Initial and deformed structure
3
2.5
2
1.5
1
0.5
0
Y
0
1
2
3
4
5
6
7
X




**Analytické řešení:**



Předpokládáme, že je aproximace posunutí na prvku lineární (implikuje konstantní relativní deformace a konstantní napětí):

$$u = ax, v = 0$$

Z geometrických rovnic:

$$\varepsilon_x = \frac{\partial u}{\partial x} = a$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = 0$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

Z konstitutivních rovnic: $$

# \left{\begin{array}{c} \sigma_x \ \sigma*y* \ *tau*{xy} \end{array}\right}

\frac{E}{1-\nu^2}\left[

$$1 \quad \nu \quad 0 \nu \quad 1 \quad 0 0 \quad 0 \quad \frac{1-\nu}{2}$$

\right] \left{\begin{array}{c} \varepsilon_x \ \varepsilon*y* \ *gamma*{xy} \end{array}\right}

$$Zokrajovýchpodmínek:$$

\sigma_x = 1.0

$$Celkem:$$

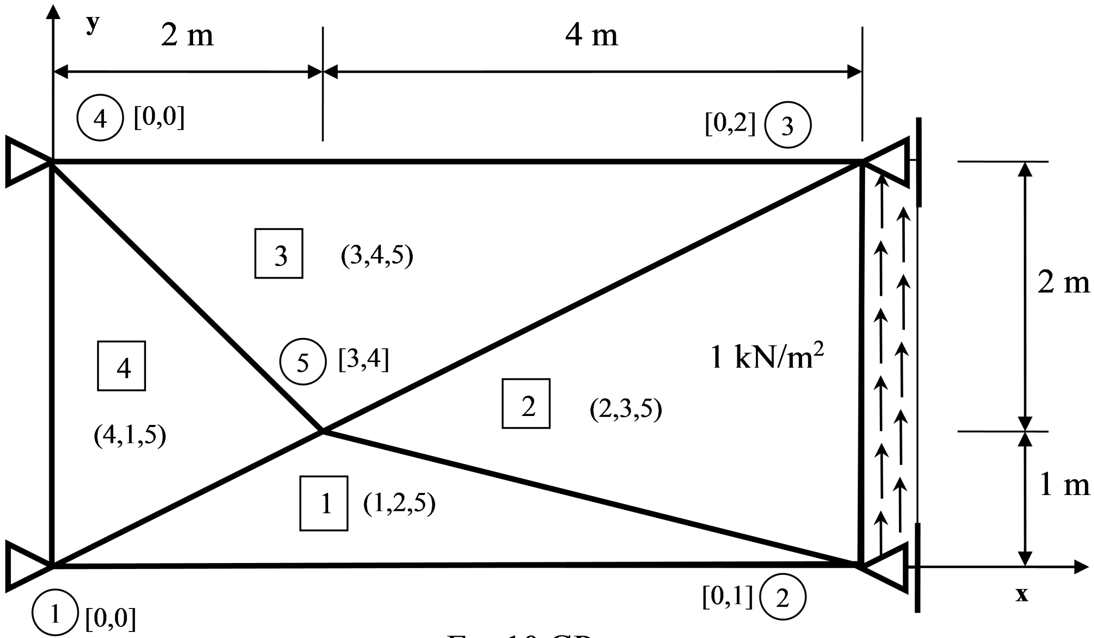
\sigma_x = \frac{E}{1-\nu^2}a=1.0 \implies a=0.099=\varepsilon_x

\sigma*y = \frac{E}{1-\nu^2}\nu a=0.1*

*tau*{xy} = 0

u(6)=ax=0.099 \cdot 6=0.594 $$

## Příklad č. 2 - Smykový patch test




E = 10 GPa
ν = 0.1
t = 1.0 m

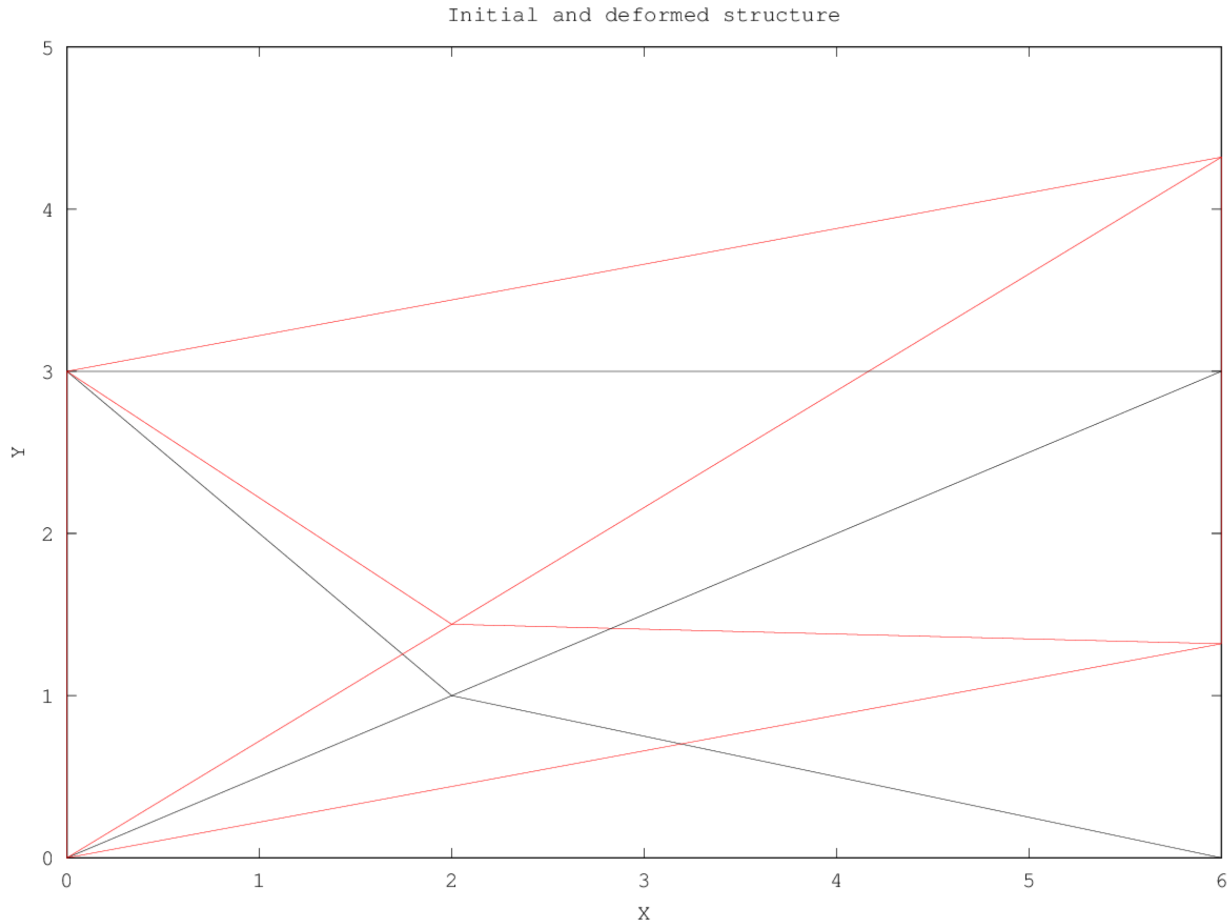
In [19]:

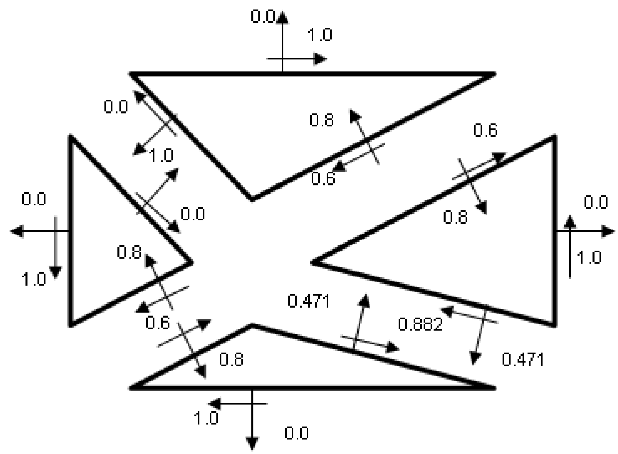
```
%pocet prvku
nelem=4;
nnodes=5;
E=10;
nu = 0.1;

% pole kodovych cisel uzlu
id = [ 5 6; 7 1; 8 2; 9 10; 3 4 ];

%pole uzlovych cisel prvku
ide = [1 2 5; 2 3 5; 3 4 5; 4 1 5];

% pocet neznamych
neq = 10;

% pole souradnic
x =[0.0, 6.0, 6.0, 0.0, 2.0];
y =[0.0, 0.0, 3.0, 3.0, 1.0];

ke = zeros(6,6);
dbe = zeros(3,6);
be = zeros(3,6);
de = zeros(3,3);

k = zeros(neq,neq);
f = zeros(neq,1);

for i = 1:nelem
    xe = [x(ide(i,:))] ;
    ye = [y(ide(i,:))] ;
    [ke,dbe,de,be] = plane_stress(xe,ye,E,nu) ;
    lm = [id(ide(i,1),:), id(ide(i,2),:), id(ide(i,3),:)];
    k=assembly(k,ke,lm,1);
end

f = [ 1.5, 1.5, 0.0, 0.0] ;

%reseni posunu
kuu=k(1:4,1:4)
fuu=f(:)
u=kuu\fuu;
u

ug = zeros(neq,1);
ug(1:4)=u(:) ;

nodal_u = [];
% Vypis posunu podle uzlu
for i=1:size(id,1)
    nodal_u = [nodal_u;ug(id(i,1)),ug(id(i,2))];
end
nodal_u
```

```
kuu =

   11.23737   -6.35522    2.77778  -11.23737
   -6.35522    9.36448   -2.77778   -5.61869
    2.77778   -2.77778   31.81818    0.00000
  -11.23737   -5.61869    0.00000   50.56818

fuu =

   1.50000
   1.50000
   0.00000
   0.00000

u =

   1.3200e+00
   1.3200e+00
   1.9979e-17
   4.4000e-01

nodal_u =

   0.00000   0.00000
   0.00000   1.32000
   0.00000   1.32000
   0.00000   0.00000
   0.00000   0.44000
```

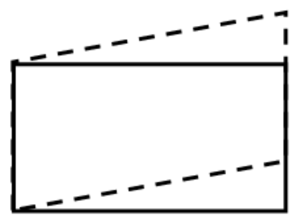
In [20]:
```
%reseni relativnich deformaci a napeti

for i = 1:nelem
    xe = [x(ide(i,:))] ;
    ye = [y(ide(i,:))] ;
    [ke,dbe,de,be] = plane_stress(xe,ye,E,nu) ;
    lm = [id(ide(i,1),:), id(ide(i,2),:), id(ide(i,3),:)] ;
    ul = [ug(lm)] ;
    eps = be*ul;
    sig = dbe*ul;
    pse = boundary_stress (xe,ye,sig);
    eps'
    sig'
    pse'
end

% Plot deformed configuration over the initial configuration
for i = 1:nnodes
    xnew(i) = x(i) + ug(id(i,1));
    ynew(i) = y(i) + ug(id(i,2));
end

figure;
for i = 1:nelem
    xe = [x(ide(i,:))] ;
    ye = [y(ide(i,:))] ;
    plot(xe,ye,'k');hold on;
    xe = [xnew(ide(i,:))] ;
    ye = [ynew(ide(i,:))] ;
    plot(xe,ye,'r');hold on;
end
title('Initial and deformed structure'); xlabel('X'); ylabel('Y');
```

```
ans =

   0.00000   0.00000   0.22000

ans =

   2.2204e-16   1.7764e-15   1.0000e+00

ans =

   1.7764e-15  -1.0000e+00   4.7059e-01  -8.8235e-01  -8.0000e-01  -6.0000e-01

ans =

  -4.9948e-18   5.5511e-17   2.2000e-01

ans =

   6.0570e-17   8.8313e-16   1.0000e+00

ans =

   6.0570e-17   1.0000e+00  -8.0000e-01  -6.0000e-01   4.7059e-01  -8.8235e-01

ans =

   0.00000  -0.00000   0.22000

ans =

  -5.5511e-17  -4.4409e-16   1.0000e+00

ans =

  -4.4409e-16  -1.0000e+00   1.0000e+00  -2.2204e-16  -8.0000e-01  -6.0000e-01

ans =

   0.00000   0.00000   0.22000

ans =

   1.0090e-16   1.0090e-17   1.0000e+00

ans =

   1.0090e-16   1.0000e+00  -8.0000e-01  -6.0000e-01   1.0000e+00  -5.5511e-17
```


Initial and deformed structure
5
4
3
2
1
0
Y
0
1
2
3
4
5
6
X




**Analytické řešení:**



Předpokládáme, že je aproximace posunutí na prvku lineární (implikuje konstantní relativní deformace a konstantní napětí):

$$u = 0, v = ax$$

Z geometrických rovnic:

$$\varepsilon_x = \frac{\partial u}{\partial x} = 0$$
$$\varepsilon_y = \frac{\partial v}{\partial y} = 0$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = a$$

Z konstitutivních rovnic: $$

# \left{\begin{array}{c} \sigma_x \ \sigma*y* \ *tau*{xy} \end{array}\right}

\frac{E}{1-\nu^2}\left[

$$1 \quad \nu \quad 0 \nu \quad 1 \quad 0 0 \quad 0 \quad \frac{1-\nu}{2}$$

\right] \left{\begin{array}{c} \varepsilon_x \ \varepsilon*y* \ *gamma*{xy} \end{array}\right}

$$Zokrajovýchpodmínek:$$

\tau*{xy} = 1.0*

$$Celkem:$$

*tau*{xy} = G\gamma*{xy}=\frac{E}{1-\nu^2}\frac{1-\nu}{2}a=1.0 \implies a=0.22=\gamma*{xy}

v(6)=ax=0.22 \cdot 6 = 1.32 $$