30

Dynamics & Vibration Overview

TABLE OF CONTENTS

§30.1. Introduction

The development in previous chapters pertain to static analysis, in which all quantities are independent of time. This kind of analysis applies also to *quasi-static* scenarios, where the state varies with time but does so slowly that inertial and damping effects can be ignored. For example one may imagine situations such as a roof progressively burdened by falling snow before collapse, the filling of a dam, or the construction of a tunnel. Or foundation settlements: think of the Pisa tower before leaning stopped. The quasi-static assumption is commonly used in design even for loads that vary in a faster time scale. For example, vehicles travelling over a bridge or wind effects on buildings.[1]

By contrast *dynamic analysis* is appropriate when the variation of displacements with time is so rapid that inertial effects cannot be ignored. There are numerous practical examples: earthquakes, rocket launches, vehicle crashes, explosive forming, air blasts, underground explosions, rotating machinery, airplane flutter. The structural accelerations, which are second derivatives with respect to time, must be kept in the governing equations. Damping effects, which are associated with velocities (the first temporal derivatives of displacements), may be also included. However, *passive damping* effects are often neglected as they tend to take energy out of a system and thus reduce the response amplitude.

Dynamic analysis may be performed in the time domain or the frequency domain. The latter is restricted in scope in that it applies to *linear* structural models, or to linearized fluctuations about an equilibrium state. The frequency domain embodies naturally the analysis of free vibrations, which is the focus of the present Chapter.

§30.2. Semidiscrete Equations of Motion

The essence of structural analysis is mastering forces. In the development of FEM, this was understood by the pioneers of the first generation, as narrated in §1.7.1. With the victory of the Direct Stiffness Method (DSM) by 1970, displacements came to the foreground as primary computational variables because they scale well into complicated systems.

To understand *dynamic* analysis, that dual role must be kept in mind. Displacements become even more important as computational variables. After all, velocities and accelerations are temporal derivatives of displacements. There is no easy way to do the job with forces only, since dynamics is about *motion*. On the other hand, the fundamental governing equations of structural dynamics are *force balance* statements. They are elaborate versions of Newtonian mechanics.

This Newtonian viewpoint is illustrated in Table 30.1 for several modeling scenarios that span statics, dynamics and vibrations. For notational simplicity it is assumed that the structure has been discretized in space, for example by the FEM. The right column shows the vector form of the governing equations as *force balance* statements. The table defines nomenclature.

When the model is time dependent, the relations shown in the right column of Table 30.1 are called *semidiscrete equations of motion*. The qualifier "semidiscrete" says that the time dimension has

[1] The quasi-static assumption can be done during design if dynamic effects can be accounted for through appropiate safety factors. For many types of structures (e.g., buildings, bridges, offshore towers) these are specified in building codes. This saves time when dynamic effects are inherently nondeterministic, as in traffic, winds or wave effects.

Table 30.1. Discrete Structural Mechanics Expressed as Force Balance Statements

Case	*Problem type*	*Governing force balance equations*
I	General nonlinear dynamics	$\underbrace{\mathbf{p}(\mathbf{u},\dot{\mathbf{u}},\ddot{\mathbf{u}},t)}_{internal} = \underbrace{\mathbf{f}(\mathbf{u},\dot{\mathbf{u}},t)}_{external}$
II	General nonlinear statics	$\underbrace{\mathbf{p}(\mathbf{u})}_{internal} = \underbrace{\mathbf{f}(\mathbf{u})}_{external}$
III	Flexible structure nonlinear dynamics	$\underbrace{\mathbf{p}_i(\mathbf{u},\dot{\mathbf{u}},\ddot{\mathbf{u}},t)}_{inertial} + \underbrace{\mathbf{p}_d(\mathbf{u},\dot{\mathbf{u}},t)}_{damping} + \underbrace{\mathbf{p}_e(\mathbf{u},t)}_{elastic} = \underbrace{\mathbf{f}(\mathbf{u},t)}_{external}$
IV	Flexible structure linear dynamics	$\underbrace{\mathbf{M}\,\ddot{\mathbf{u}}(t)}_{inertial} + \underbrace{\mathbf{C}\,\dot{\mathbf{u}}(t)}_{damping} + \underbrace{\mathbf{K}\,\mathbf{u}(t)}_{elastic} = \underbrace{\mathbf{f}(t)}_{external}$
V	Linear elastostatics	$\underbrace{\mathbf{K}\,\mathbf{u}}_{elastic} = \underbrace{\mathbf{f}}_{external}$
VI	Dynamic perturbations	$\underbrace{\mathbf{M}(\mathbf{u})\,\ddot{\mathbf{d}}(t)}_{inertial} + \underbrace{\mathbf{C}(\mathbf{u})\,\dot{\mathbf{d}}(t)}_{damping} + \underbrace{\mathbf{K}(\mathbf{u})\,\mathbf{d}(t)}_{elastic} + \underbrace{\mathbf{p}(\mathbf{u}) = \mathbf{f}(\mathbf{u})}_{static\ equilibrium}$
VII	Damped forced vibrations	$\underbrace{\mathbf{M}\,\ddot{\mathbf{u}}(t)}_{inertial} + \underbrace{\mathbf{C}\,\dot{\mathbf{u}}(t)}_{damping} + \underbrace{\mathbf{K}\,\mathbf{u}(t)}_{elastic} = \underbrace{\mathbf{f}_p(t)}_{periodic}$
VIII	Damped free vibrations	$\underbrace{\mathbf{M}\,\ddot{\mathbf{u}}(t)}_{inertial} + \underbrace{\mathbf{C}\,\dot{\mathbf{u}}(t)}_{damping} + \underbrace{\mathbf{K}\,\mathbf{u}(t)}_{elastic} = \mathbf{0}$
IX	Undamped free vibrations	$\underbrace{\mathbf{M}\,\ddot{\mathbf{u}}(t)}_{inertial} + \underbrace{\mathbf{K}\,\mathbf{u}(t)}_{elastic} = \mathbf{0}$

Symbol $\mathbf{u}$ is array of total displacement DOFs; $\mathbf{d}$ in case VI is a linearized perturbation of $\mathbf{u}$.
Symbol t denotes time. Superposed dots abbreviate time derivatives: $\dot{\mathbf{u}} = d\mathbf{u}/dt$, $\ddot{\mathbf{u}} = d^2\mathbf{u}/dt^2$, etc.
The history $\mathbf{u} = \mathbf{u}(t)$ is called the *response* of the system. This term is extendible to nonlinear statics.
Initial force effects $\mathbf{f}_I$ may be accommodated in forced cases by taking $\mathbf{f} = \mathbf{f}_I$ when $\mathbf{u} = \mathbf{0}$.

not been discretized: t is still a continuous variable. This legalizes the use of time differentiation, abbreviated by superposed dots, to bring in velocities and accelerations.

This table may be scanned "top down" by starting with the most general case I: nonlinear structural dynamics, branching down to more restricted but specific forms. Along the way one finds in case V an old friend: the DSM master equations $\mathbf{K}\,\mathbf{u} = \mathbf{f}$ for linear elastostatics, treated in previous Chapters. The last case IX: undamped free vibrations, is that treated in this and next two Chapters. Some brief comments are made as regards damped and forced vibrations.

§30.2.1. Vibrations as Equilibrium Disturbance

An elastic structure is placed in motion through some short-term disturbance, for example an impulse. Remove the disturbance. If wave propagation effects are ignored and the structure remains elastic, it will keep on oscillating in a combination of time-periodic patterns called *vibration modes*. Associated with each vibration mode is a characteristic time called *vibration period*. The inverse of

a period, normalized by appropriate scaling factors, is called a *vibration frequency*. The structure is said to be vibrating, or more precisely *undergoing free vibrations*. In the absence of damping mechanisms an elastic structure will vibrate forever. The presence of even minute amounts of viscous damping, however, will cause a gradual decrease in the amplitude of the oscillations. These will eventually cease.[2]

If the disturbances are sufficiently small to warrant linearization, this scenario fits case VI of Table 30.1, therein labeled "dynamic perturbations." Its main application is the investigation of *dynamic stability* of equilibrium configurations. If the perturbation $\mathbf{d}(t)$ is unbounded under some initial conditions, that equilibrium configuration[3] is said to be dynamically unstable.

The analysis of case VI does not belong to an introductory course because it requires advanced mathematical tools. Moreover it often involve nondeterministic (stochastic) effects. Cases VII through IX are more tractable in an introductory course. In these, fluctuations are linearized about an undeformed and unstressed state defined by $\mathbf{u} = \mathbf{0}$. Thus $\mathbf{d}$ (the perturbed displacement) becomes simply $\mathbf{u}$ (the total displacement). Matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are called the mass, damping and stiffness matrices, respectively. These matrices are independent of $\mathbf{u}$ since they are evaluated at the undeformed state $\mathbf{u} = \mathbf{0}$. Two scenarios are of interest in practice:

1. *Forced Vibrations*. The system is subjected to a time dependent force $\mathbf{f}(t)$. The response $\mathbf{u}(t)$ is determined from the linear dynamics equation: $\mathbf{M}\,\ddot{\mathbf{u}}(t) + \mathbf{C}\,\dot{\mathbf{u}}(t) + \mathbf{K}\,\mathbf{u}(t) = \mathbf{f}(t)$ of case IV. Of particular interest in resonance studies in when $\mathbf{f}(t)$ is periodic in time, which is case VII.
2. *Free Vibrations*. The external force is zero for $t > 0$. The response $\mathbf{u}(t)$ is determined from initial conditions. If damping is viscous and light, the undamped model gives conservative answers and is much easier to handle numerically. Consequently the model of case IX is that generally adopted during design studies.

§30.2.2. Undamped Free Vibrations

From the foregoing discussion it follows that case IX: undamped free vibrations is of paramount importance in design. The governing equation is

$$\mathbf{M}\,\ddot{\mathbf{u}}(t) + \mathbf{K}\,\mathbf{u}(t) = \mathbf{0}. \tag{30.1}$$

This expresses a force balance[4] in the following sense: in the absence of external loads the internal elastic forces $\mathbf{K}\,\mathbf{u}$ balance the negative of the inertial forces $\mathbf{M}\ddot{\mathbf{u}}$. The only ingredient beyond the by now familiar $\mathbf{K}$ is the mass matrix $\mathbf{M}$. The size of these matrices will be denoted by n_f, the number of degrees of freedom upon application of support conditions.

Equation (30.1) is linear and homogeneous. Its general solution is a linear combination of exponentials. Under matrix definiteness conditions discussed later the exponentials can be expressed as

---

[2] Mathematically a damped oscillation also goes on forever. Eventually, however, the motion amplitude reaches a molecular scale level at which a macroscopic idealization does not apply. At such point the oscillations in the physical structure can be considered to have ceased.

[3] Usually obtained through a nonlinear static analysis. This kind of study, called dynamic stability analysis, is covered under Nonlinear Finite Element Methods.

[4] Where is $f = ma$? To pass to internal forces change the sign of f: $f_{int} + ma = ku + ma = 0$. Replace by matrices and vectors and you have (30.1).

a combination of trigonometric functions: sines and cosines of argument ωt. A compact representation of such functions is obtained by using the exponential form $e^{j\omega t}$, where $j = \sqrt{-1}$:

$$\mathbf{u}(t) = \sum_i \mathbf{v}_i \, e^{j\omega_i t}. \tag{30.2}$$

Here ω_i is the i^{th} *circular frequency*, expressed in radians per second, and $\mathbf{v}_i \neq \mathbf{0}$ the corresponding vibration mode shape, which is independent of t.

§30.2.3. The Vibration Eigenproblem

Replacing $\mathbf{u}(t) = \mathbf{v}\, e^{j\omega t}$ in (30.1) segregates the time dependence to the exponential: $(-\omega^2\mathbf{M} + \mathbf{K})\,\mathbf{v}\, e^{j\omega t} = \mathbf{0}$. Since $e^{j\omega t}$ is not identically zero, it can be dropped leaving the algebraic condition:

$$(-\omega^2\mathbf{M} + \mathbf{K})\,\mathbf{v} = \mathbf{0}. \tag{30.3}$$

Because $\mathbf{v}$ cannot be the null vector, this equation is an algebraic eigenvalue problem in ω^2. The eigenvalues $\lambda_i = \omega_i^2$ are the roots of the characteristic polynomial be indexed by i:

$$\det(\mathbf{K} - \omega_i^2\mathbf{M}) = 0. \tag{30.4}$$

Dropping the index i this eigenproblem is usually written as

$$\mathbf{K}\,\mathbf{v} = \omega^2\mathbf{M}\mathbf{v}. \tag{30.5}$$

If $\mathbf{M}$ and $\mathbf{K}$ satisfy some mild conditions, solutions of (30.5) are denoted by ω_i and $\mathbf{v}_i$. This are called the *vibration frequencies* or *eigenfrequencies*, and the it vibration modes or *eigenmodes*, respectively. The set of all ω_i is called the *frequency spectrum* or simply *spectrum*. .

§30.2.4. Eigensystem Properties

Both stiffness $\mathbf{K}$ and mass $\mathbf{M}$ are symmetric matrices. In addition $\mathbf{M}$ is nonnegative. Nothing more can be assumed in general. For example, if $\mathbf{K}$ incorporates Lagrangian multipliers from the treatment of a MFC, as explained in Chapter 10, it will be indefinite.

If $\mathbf{M}$ is positive definite, the following properties hold.

1. There are n_f squared vibration frequencies ω_i^2, which are roots of the characteristic polynomial (30.4). These are not necessarily distinct. A root of (30.6) that appears m times is said to have multiplicity m.[5]
2. All roots ω_i^2 of (30.6) are real. The corresponding eigenmodes $\mathbf{v}_i$ have real entries.
3. If $\mathbf{K}$ is nonnegative, $\omega_i^2 \geq 0$ and the frequencies $\omega_i = +\sqrt{\omega_i^2}$ are also real and nonnegative. Furthermore, if $\mathbf{K}$ is positive definite, all $\omega_i^2 > 0$ and consequently $+\sqrt{\omega_i} > 0$.

If $\mathbf{M}$ is nonnegative, care must be exercised; this case is discussed in an Exercise. If $\mathbf{M}$ is indefinite (which should never happen in structures) all of the foregoing properties are lost.

[5] For example, a free-free (fully unsupported) structure has n_R zero frequencies, where n_R is the number of rigid body modes.

Example 30.1. This illustrates the weird things that can happen if $\mathbf{M}$ is indefinite. Consider

$$\mathbf{K} = \begin{bmatrix} \alpha & 1 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} 0 & 1 \\ 1 & 1+\beta \end{bmatrix}, \tag{30.6}$$

where α and β vary from 1 to -1. Then

$$\mathbf{M}^{-1}\mathbf{K} = \frac{1}{2\alpha - 1}\begin{bmatrix} 1 & (1-\beta) \\ \alpha & (-1+\alpha+\alpha\beta) \end{bmatrix}. \tag{30.7}$$

The eigenvalues are

$$\omega_{1,2}^2 = \frac{-2+\alpha+\alpha\beta \pm \sqrt{\alpha[4-4\beta+\alpha(1+\beta)^2]}}{4\alpha - 2}. \tag{30.8}$$

These are complex if the radicand is negative. But that is not all. If $\alpha \to 0$ one eigenvalue goes to ∞. If $\alpha = 0$, $\mathbf{A} = \mathbf{M}^{-1}\mathbf{K}$ is a 2×2 Jordan block and one eigenvector is lost.

§30.3. Solving The Vibration Eigenproblem

In what follows we often denote $\lambda_i = \omega_i^2$ to agree more closely with the conventional notation for the algebraic eigenproblem.

§30.3.1. Determinant Roots

Mathematically the ω_i^2 are the roots of the characteristic equation (30.4). The simple minded approach is to expand the determinant to get the characteristic polynomial $P(\omega_i^2)$ and get their roots:

$$\det(\mathbf{K} - \omega_i^2\mathbf{M}) = P(\omega_i^2) = 0. \tag{30.9}$$

This approach is deprecated by numerical analysts. It seems as welcome as anthrax. Indeed for numerical floating point computations of large systems it risks numerical overflow; moreover the roots of the characteristic polynomial can be very ill-conditioned with respect to coefficients.

For small systems and using either exact or symbolic computation there is nothing wrong with this if the roots can be expressed exactly in terms of the coefficients, as in the above example.

§30.3.2. Reduction to the Standard Eigenproblem

The standard algebraic eigenproblem has the form

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}. \tag{30.10}$$

Most library routines included in packages such as *Matlab* and *Mathematica* are designed to solve this eigenproblem. If $\mathbf{A}$ is symmetric the eigenvalues λ_i are real; moreover there exist a complete system of eigenvectors $\mathbf{x}_i$. If these are normalied to length one: $||\mathbf{x}_i||_2 = 1$ they satisfy the orthonormality conditions

$$\mathbf{x}_i^T\mathbf{x}_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}, \qquad \mathbf{x}_i^T\mathbf{A}\mathbf{x}_j = \lambda_i, \tag{30.11}$$

where δ_{ij} is the Kronecker delta. If the $\mathbf{x}_i$ are collected as columns of a matrix $\mathbf{X}$, the foregoing conditions can be expressed as $\mathbf{X}^T\mathbf{X} = \mathbf{I}$ and $\mathbf{X}^T\mathbf{K}\mathbf{X} = \mathbf{\Lambda} = \mathbf{diag}\lambda_i$.

§30.3.3. Unsymmetric Reduction

If $\mathbf{M}$ is nonsingular, a simple way to reduce $\mathbf{Kv} = \omega^2\mathbf{Mv}$ to standard form is to premultiply both sides by $\mathbf{M}^{-1}$ whence

$$\mathbf{M}^{-1}\mathbf{Kv} = \omega^2\mathbf{v} \quad \Rightarrow \mathbf{Ax} = \lambda\mathbf{x}, \quad \text{with} \quad \mathbf{A} = \mathbf{M}^{-1}\mathbf{K}, \quad \lambda = \omega^2, \quad \mathbf{x} = \mathbf{v}. \tag{30.12}$$

The fastest way to form $\mathbf{A}$ is by solving $\mathbf{MA} = \mathbf{K}$ for $\mathbf{A}$. One nice feature of (30.12) is that the eigenvectors need not be backtransformed, as happens in symmetry-preserving methods.

As in the case of the characteristic polynomial, this is deprecated by numerical analysts, also not so vehemently. Their objection is that $\mathbf{A}$ is not generally symmetric even if $\mathbf{K}$ and $\mathbf{M}$ are. So $\mathbf{Ax} = \lambda\mathbf{x}$ has to be submitted to an unsymmetric eigensolver. Thus risks contaminating the spectrum with complex numbers. Plus, it is slower.

The writer's experience is that (30.12) works perfectly fine for small systems. If tiny imaginary components appear, they are set to zero and life goes on.

§30.3.4. Symmetry Preserving Reduction

It is possible to retain symmetry by proceeding as follows. Decompose the mass matrix as

$$\mathbf{M} = \mathbf{L}\mathbf{L}^T \tag{30.13}$$

This is the Cholesky decomposition, which can be carried out to completion if $\mathbf{M}$ is positive definite. Then

$$\mathbf{A} = \mathbf{L}^{-1}\mathbf{K}\mathbf{L}^{-T}. \tag{30.14}$$

The demonstraion is in one of the Exercises. The symmetric eigenproblem can be handled by standard library routines, which give back all the eigenvalues and eigenvectors. The square root of the eigenvalues give the vibration frequencies and the vibration modes are recovered from the relation $\mathbf{Lv}_i = \mathbf{x}_i$, which can be handled by standard library routines.

Notes and Bibliography

The literature on dynamics and vibrations of structures is quite large. It is sufficient to cite here titles that incorporate modern analysis methods: Clough and Penzien [25], Geradin and Rixen [62], Meirovich [95,96] and Wilson [151].

Several books in matrix methods and FEM books contain at least an introductory treatment of dynamics. Citable textbooks include Bathe [9], Cook, Malkus and Plesha [28], Hughes [78]. Despite their age, Przemieniecki [116] remains a useful source of mass matrices, and Pestel and Leckie [106] contains a catalog of transfer matrices (an early 1960 method suitable for small computers).

As regards books on linear algebra matrix theory and matrix calculus see the Bibliography cited in Appendix A. The most elegant coverage is that of Strang [130]. Two comprehensive references on matrix computations in general are Golub and VanLoan [65] and Stewart [126]. The former is more up to date as regard recent literature. Bellman [14] contains more advanced material. Stewart and Sun [127] cover the sensitivity analysis of standard and generalized eigenproblems.

There are comprehensive books that treat the algebraic eigenproblem. Wilkinson's masterpiece [152] is dated in several subjects, particularly the generalized eigenproblem and the treatment of large eigenproblems. But it is still unsurpassed as the "bible" of backward error analysis. More up to date in methods is Parlett [105], which is however restricted to the symmetric eigenproblem.

As regards source code for matrix computations, the Handbook compilation of Algol 60 procedures by Wilkinson and Reisch [153] is elegant and still useful as template for other languages. Half of the handbook deals with eigenvalue problems. By contrast, the description of Fortran EISPACK code [60] suffers from the inherent ugliness and unreadability of Fortran IV.

References

Referenced items moved to Appendix R.

Homework Exercises for Chapter 30 - Dynamics & Vibration Overview

EXERCISE 30.1 [A:15]. A 3-element model of a bar in 1D gives

$$\mathbf{M} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}. \tag{E30.1}$$

Solve the vibration eigenproblem and show the natural frequencies and associated vibration modes. Normalize the latter so that $\mathbf{V}^T\mathbf{M}\mathbf{V} = \mathbf{I}$ ("mass normalized eigenvectors").

EXERCISE 30.2 [A:25]. In (E30.1) replace the (4,4) mass entry by $2 - \alpha$ and the (4,4) stiffness entry by $1 - \alpha/2$. Using *Matlab* or *Mathematica*, solve the eigenproblem for α varing from $0 to 4$ in 0.5 increments. Discuss what happens to the frequencies and vibration modes as α goes to 2 and beyond. Explain.

EXERCISE 30.3 [D:20]. Eigenvectors can be scaled by arbitrary nonzero factors. Discuss 4 ways in which the eigenvectors $\mathbf{v}_i$ of $\mathbf{K}\mathbf{v}_i = \omega_i^2\mathbf{M}\mathbf{v}_i$ can be normalized, and what assumptions are necessary in each case.