29

Nonconservative Loading

TABLE OF CONTENTS

§29.1. Introduction

In Chapter 5 a mechanical system was defined to be *conservative* when both external and internal forces are derivable from a potential. In this course we consider only elastic systems; consequently the internal forces are derivable from an strain (internal) energy potential U. Thus the conservative/nonconservative character depends on whether the *external loads* are conservative or nonconservative.

Conservative applied forces $\mathbf{f}$ may be derived from the external loads potential V by differentiating with respect to the state variables:

$$\mathbf{f} = \frac{\partial V}{\partial \mathbf{u}}. \tag{29.1}$$

Nonconservative forces, on the other hand, are not expressable as (29.1). They have to be worked out directly at the force level.

In the present Chapter we will give examples of both force types in conjunction with the TL-formulated two-node bar element The main result is that consideration of nonconservative loads contributes an unsymmetric component, called *load stiffness*, to the tangent stiffness matrix.

Remark 29.1. The chief sources of nonconservative forces in various branches of engineering are:

1. Aerodynamic forces (aerospace, civil), hydrodynamic forces (mechanical, marine, chemical), aircraft and rocket propulsion forces (aerospace).
2. Gyroscopic forces (aerospace, electrical).
3. Active control systems (aerospace, electrical, mechanical).

§29.2. Potential Force Example: Gravity

Consider the two-node, three-dimensional bar element immersed in a gravity field of constant strength g acting along the global $-Z$ axis, as illustrated in Figure 29.1. The bar has reference length L_0, reference area A_0 and mass density ρ. The element coordinate systems are labeled as follows:

$$\begin{aligned} &\bar{x}_0, \bar{y}_0, \bar{z}_0 && \text{in the reference configuration } \mathcal{C}_0 \\ &\bar{x}, \bar{y}, \bar{z} && \text{in the current configuration } \mathcal{C} \end{aligned}$$

This distinction between local coordinate systems is introduced here as it becomes necessary in later Sections.

Take a differential element of bar of length $d\bar{x}_0$ in $\mathcal{C}_0$. This moves to a corresponsing position in $\mathcal{C}$, with a vertical displacement of u_z with respect to $\mathcal{C}_0$. See Figure 29.2. The work potential gained by this displacement is

$$dV = -\rho g A_0 u_z(\bar{x}_0)\, d\bar{x}_0 \tag{29.2}$$

The external potential of the bar element is obtained by linearly interpolating $u_z = (1-\zeta)u_{z1} + \zeta u_{z2}$, $\zeta = \bar{x}/L_0$ and integrating over the bar length:

$$\begin{aligned} V &= -\int_0^L \rho g A_0 u_z d\bar{x}_0 = -\int_0^1 A_0 g \,[\,1-\zeta \quad \zeta\,] \begin{bmatrix} u_{z1} \\ u_{z2} \end{bmatrix} L_0 d\zeta \\ &= -\rho g A_0 L_0\, \tfrac{1}{2}(u_{z1} + u_{z2}). \end{aligned} \tag{29.3}$$

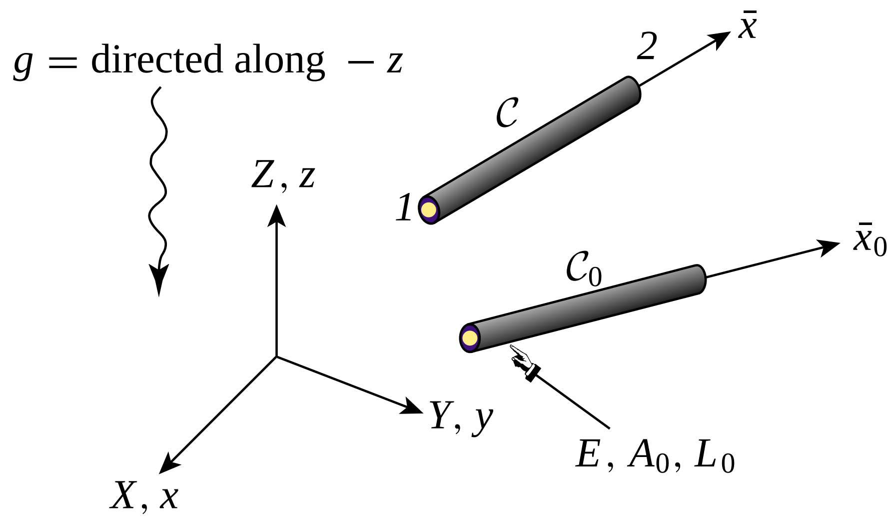

Figure 29.1. TL bar element displacing in a gravity field g.

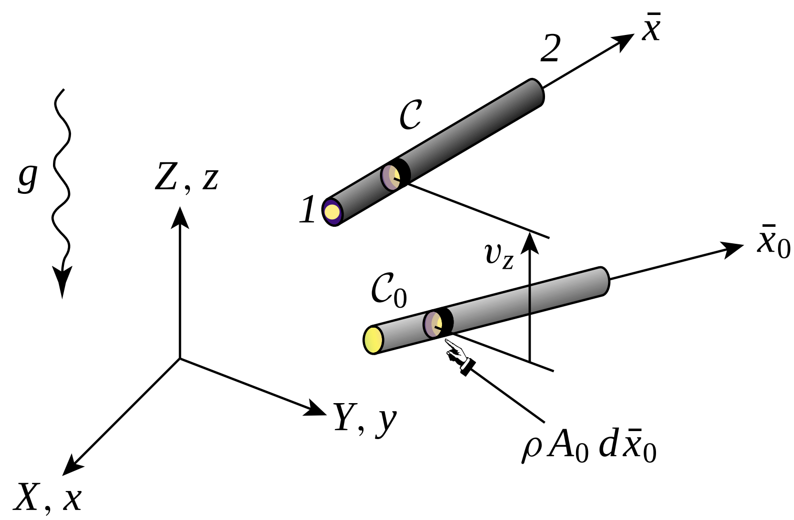

Figure 29.2. Calculation of external potential.

(As usual in the TL kinematic description, all quantities are referred to $\mathcal{C}_0$.) It follows that the external force vector for the element is

$$\mathbf{f}_g = \frac{\partial V}{\partial \mathbf{u}} = \begin{bmatrix} \partial V/\partial u_{x1} \\ \partial V/\partial u_{y1} \\ \partial V/\partial u_{z1} \\ \partial V/\partial u_{x2} \\ \partial V/\partial u_{y2} \\ \partial V/\partial u_{z2} \end{bmatrix} = -\tfrac{1}{2}\rho A_0 L_0 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{29.4}$$

This can also be derived through basic principles of statics. Note that this vector is *independent of*

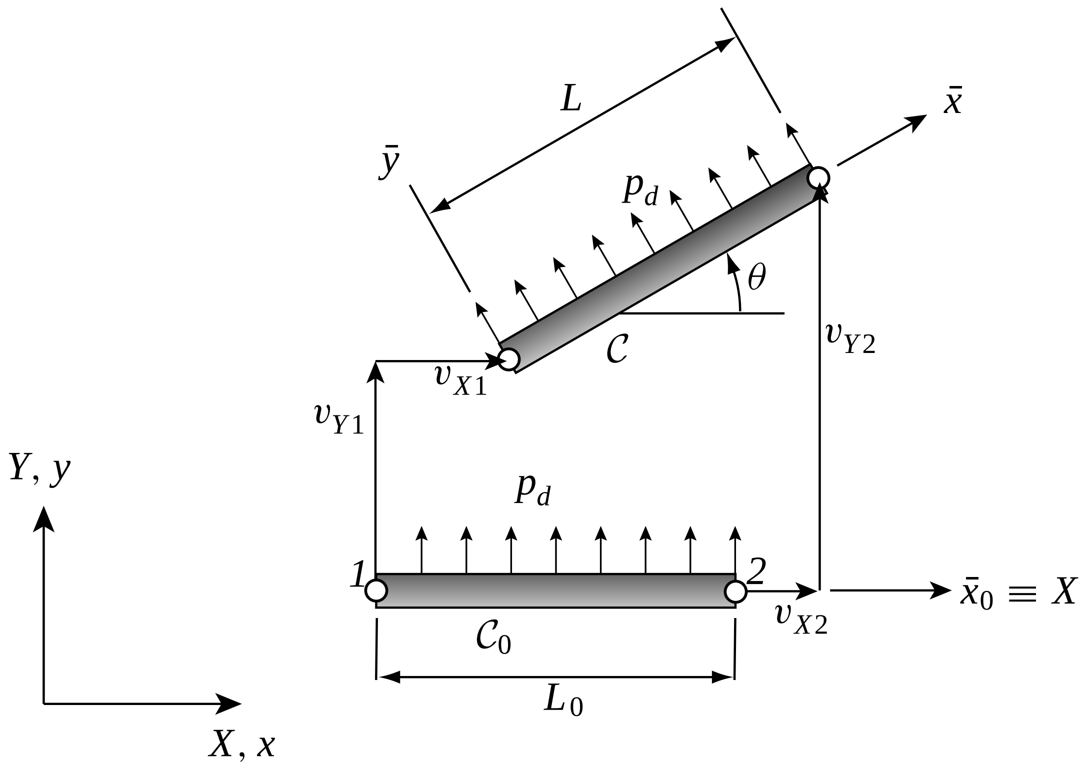

Figure 29.3. 2D bar under constant “follower” pressure p_d.

the current configuration. This is a distinguishing feature of external work potentials that *depend linearly on the displacements*, such as (29.3).

§29.3. Follower Load and Associated Load Stiffness

To illustrate the concept of *load stiffness* with a minimum of mathematics, let us consider a two-dimensional specialization. The bar element originally lies along the x axis in the reference configuration $\mathcal{C}_0$ and moves in the (x, y) plane to $\mathcal{C}$, which forms an angle θ with x. The bar is under a a *constant* pressure p_d that is always normal to the element as it displaces, as shown in Figure 29.3. This kind of applied force is called a *follower* load in the literature.[1]

From statics the external force vector is obviously

$$\mathbf{f} = \tfrac{1}{2} p_d L \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \\ -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \tag{29.5}$$

From geometry

$$\cos\theta = \frac{L_0 + u_{X21}}{L}, \quad \sin\theta = \frac{u_{Y21}}{L}, \quad \text{with} \quad u_{X21} = u_{X2} - u_{X1}, \quad u_{Y21} = u_{Y2} - u_{Y1}, \tag{29.6}$$

[1] Such loads are often applied by fluids at rest or in motion. The latter case is studied in Sections 29.4-5.

Consequently

$$\mathbf{f} = \tfrac{1}{2} p_d \begin{bmatrix} -u_{Y21} \\ L_0 + u_{X21} \\ 0 \\ -u_{Y21} \\ L_0 + u_{X21} \\ 0 \end{bmatrix}. \tag{29.7}$$

Take now the partial of the negative of this external load vector with respect to **u**. The result is a matrix with dimensions of stiffness, denoted by $\mathbf{K}_L$:

$$\mathbf{K}_L = -\frac{\partial \mathbf{f}}{\partial \mathbf{u}} = \tfrac{1}{2} p_d \begin{bmatrix} 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{29.8}$$

$\mathbf{K}_L$ is called a *load stiffness matrix*. It arises from *displacement-dependent loads*.[2] We can see from this example that $\mathbf{K}_L$ is *unsymmetric*. A consequence of this fact is that (29.2) does not have a potential V that is a function of the node displacements.[3]

§29.4. General Characterization of the Load Stiffness

Suppose that we have a one-parameter *conservative* system with displacement dependent forces. Then

$$\Pi = U(\mathbf{u}) - V(\mathbf{u}, \lambda), \tag{29.9}$$

where the external potential $V = V(\mathbf{u}, \lambda)$ depends on the displacements **u** in a general fashion. Then

$$\mathbf{r} = \frac{\partial \Pi}{\partial \mathbf{u}} = \frac{\partial U}{\partial \mathbf{u}} - \frac{\partial V}{\partial \mathbf{u}} = \mathbf{p} - \mathbf{f}, \tag{29.10}$$

$$\mathbf{K} = \frac{\partial \mathbf{r}}{\partial \mathbf{u}} = \frac{\partial \mathbf{p}}{\partial \mathbf{u}} - \frac{\partial \mathbf{f}}{\partial \mathbf{u}}. \tag{29.11}$$

The partial $\partial\mathbf{p}/\partial\mathbf{u}$ gives $\mathbf{K}_M + \mathbf{K}_G$, the material plus geometric stiffness, as discussed in previous Chapters. The last term gives $\mathbf{K}_L$, the *conservative load stiffness*

$$\mathbf{K}_L = -\frac{\partial \mathbf{f}}{\partial \mathbf{u}} = -\frac{\partial^2 V}{\partial \mathbf{u}^2} \tag{29.12}$$

which is called the *conservative load stiffness*. This matrix is obviously symmetric because it is the negated Hessian of $V(\mathbf{u}, \lambda)$ with respect to **u**. Consequently

$$\boxed{\mathbf{K} = \mathbf{K}_M + \mathbf{K}_G + \mathbf{K}_L.} \tag{29.13}$$

[2] This source of nonlinearity was called force B.C. nonlinearity in Chapter 2.

[3] If $\mathbf{K}_L$ were symmetric we could work backwards and integrate (29.5), expressed in terms of the node displacements, to find the potential function V.

These three components of $\mathbf{K}$ are symmetric, and so is $\mathbf{K}$.

Now consider a more general structural system subject to both conservative and *non-conservative* loads:

$$\boxed{\mathbf{r} = \mathbf{p} - \mathbf{f}_c - \mathbf{f}_n,} \tag{29.14}$$

Here $\mathbf{f}_c = \partial V/\partial \mathbf{u}$ whereas $\mathbf{f}_n$ collects external forces not derivable from a potential. Then

$$\boxed{\mathbf{K} = \frac{\partial \mathbf{r}}{\partial \mathbf{u}} = \mathbf{K}_M + \mathbf{K}_G + \mathbf{K}_{Lc} + \mathbf{K}_{Ln}.} \tag{29.15}$$

The nonconservative load stiffness matrix, $\mathbf{K}_{Ln}$, is *unsymmetric*.

Remark 29.2. In practice one derives the total force $\mathbf{f}$ from statics, as in the example of §29.3, and obtains $\mathbf{K}_L$ by taking the partials with respect to the displacements in $\mathbf{u}$. If the resulting stiffness is unsymmetric the load is nonconservative. The splitting of $\mathbf{K}_L$ into a symmetric matrix $\mathbf{K}_{Lc}$ and unsymmetric part $\mathbf{K}_{Ln}$ can be done in a variety of ways. (If the unsymmetric part is required to be antisymmetric, however, the splitting is unique.)

§29.5. Forces Produced by Fluid Motion

To study in more detail a frequent source of non-conservative follower loads, suppose that the bar element is submerged in a moving fluid whose flow is independent of time — *i.e.*, a steady flow. See Figure 29.4. We neglect "feedback" effects on the flow due to the presence and motion of the bar. The steady notion can be described by the fluid-particle velocity field[4]

$$\mathbf{u}_f(X, Y, Z) = \begin{bmatrix} u_{fX}(X, Y, Z) \\ u_{fY}(X, Y, Z) \\ u_{fZ}(X, Y, Z) \end{bmatrix}, \tag{29.16}$$

For simplicity in the formulation below, we further assume that the velocity field is *uniform*, *i.e.*, does not depend upon (X, Y, Z), and that it is directed along the x axis:

$$\mathbf{u}_f = \begin{bmatrix} u_{fX} \\ 0 \\ 0 \end{bmatrix}, \tag{29.17}$$

where u_{fX} is independent of position.

By virtue of drag effects the fluid motion exerts a normal *drag force* p_d (force per unit length) upon the bar in the current configuration $\mathcal{C}$. The drag force is normal to the bar longitudinal axis $\bar{x}$ and it is a function of the magnitude of the velocity component *normal* to that axis. Furthermore if the bar cross section is circular or annular, the force is coaxial with the normal velocity vector. For additional simplicity we shall assume that the cross section satisfies such a geometric constraint[5]

[4] The symbol u and its vector counterparts $\mathbf{u}$ and $\vec{u}$ are commonly used in fluid mechanics to denote velocities rather than displacements as in structural and solid mechanics. In fact displacements are rarely used in fluids. Subscript f is introduced here to lessen the risk of confusion with structural displacements.

[5] For arbitrary cross sections, the fluid motion exerts drag and lift forces, the latter being normal to the bar axis and to the normal velocity vector. Lift forces are what makes airplanes fly. This more general situation is dealt with in treatises on aerodynamics, wind forces and hydraulics.

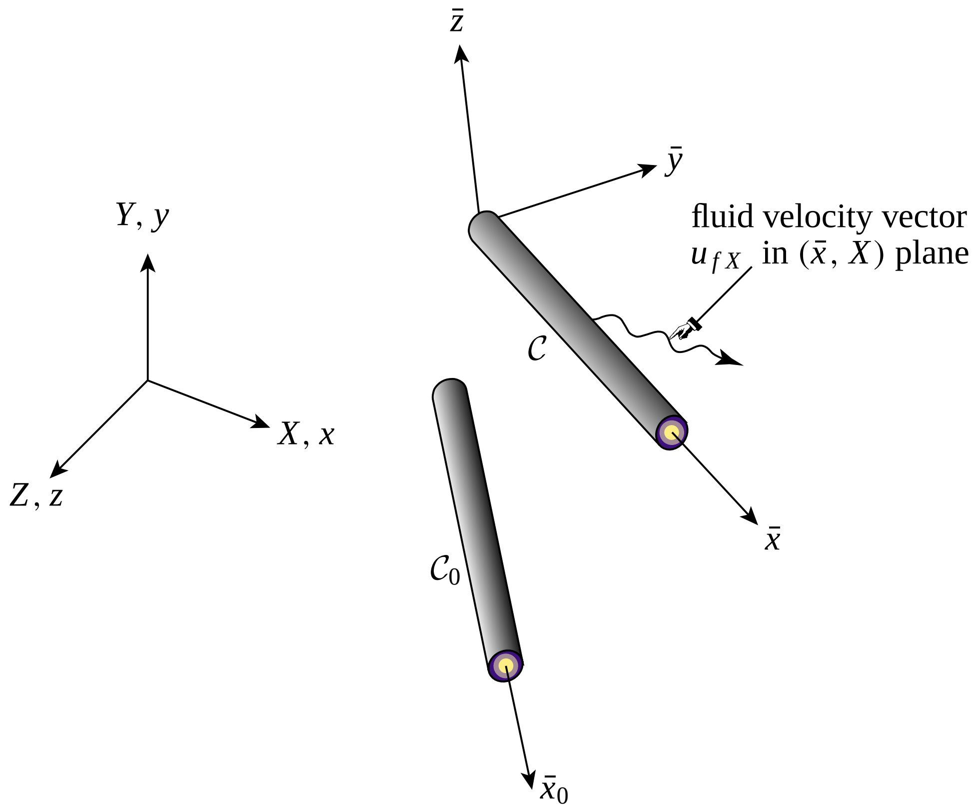

Figure 29.4 Bar element in steady fluid flow.

For slow (laminar) flow the drag force is proportional to the magnitude of the normal velocity component whereas if the motion is fully turbulent it is proportional to the square of that velocity. We assume here the latter case. Other drag-velocity dependencies can be similarly treated.

Consider the bar in the $(\bar{x}, X)$ plane as illustrated in Figure 29.5, and let $\bar{y}$ be defined as the normal to the element axis $\bar{x}$ that is located in this plane and forms an acute angle θ with x. The drag force on the element per unit length is directed along $\bar{y}$ and has the value

$$p_d = \tfrac{1}{2} C_d \, \rho_f \, d \, u_{fn}^2 \tag{29.18}$$

where C_d is the drag coefficient,[6] ρ_f the fluid mass density, d the "exposed width" (for a bar of circular cross-section, its external diameter), and u_{fn} the fluid-normal velocity $u_{fX} \cos\theta$ (see Figure). The total force on the element is $p_d L$, where L is the current length, and this force "lumps" into $\frac{1}{2} p_d L$ at each node.

In order to refer these forces to the global X, Y, Z axes, we need to know the direction cosines t_{21}, t_{22} and t_{23} of $\bar{y}$ with respect to x, y, z. Then the hydrodynamic node force vector in the (X, Y, Z)

[6] C_D is a dimensionless number tabulated in fluid dynamic handbooks

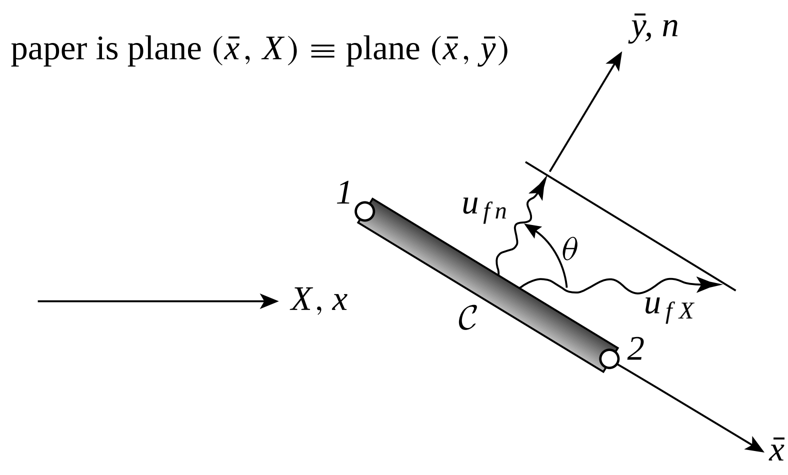

Figure 29.5. Normal fluid velocity component in the current bar configuration

system is

$$\mathbf{f} = \tfrac{1}{2} p_d L \begin{bmatrix} t_{21} \\ t_{22} \\ t_{23} \\ t_{21} \\ t_{22} \\ t_{23} \end{bmatrix} \tag{29.19}$$

To compute these direction cosines, one proceeds as follows:

(1) Compute the direction $\bar{z}$ by taking the cross product of $\bar{x}$ and X.

(2) Compute the direction $\bar{y}$ by taking the cross product of $\bar{z}$ and $\bar{x}$.

If $\bar{x}$ and X are parallel, step (1) does not define z but then the fluid flow occurs along the element axis and the pressure p_d vanishes.

Remark 29.3. If the fluid flow is uniform with speed u_{fj} along a general direction $\mathbf{j} \equiv \vec{j}$, the preceding derivation must be modified by taking $\vec{\bar{z}} = \vec{\bar{x}} \times \vec{j}$, $\vec{\bar{y}} = \vec{\bar{z}} \times \vec{\bar{x}}$, $\theta = \text{angle}(\vec{\bar{y}}, \vec{j})$. Observe that it would be incorrect to decompose u_{fj} onto its components in the X, Y and Z directions and superpose associated forces, because the drag force is nonlinear in the velocity.

Remark 29.4. If the flow is steady but nonuniform, numerical integration over elements is generally required. For this simple elkement integration with the flow velocity evaluated at the element center is often sufficient.

§29.6. Load Stiffness For 2D Fluid Motion

To show what kind of load stiffness is produced by fluid drag forces, consider again the case of Figure 29.4 but now make p_d depend on the "tilt" θ as explained in §29.3; see Figure 29.6. Since a turbulent-motion-induced drag force is proportional to the square of $u_{fn} = u_{fX} \cos\theta$, it may be expressed as

$$p_d = p_{d0} \cos^2\theta \tag{29.20}$$

where p_{d0} is p_d for $\theta = 0$ (bar normal to fluid motion).

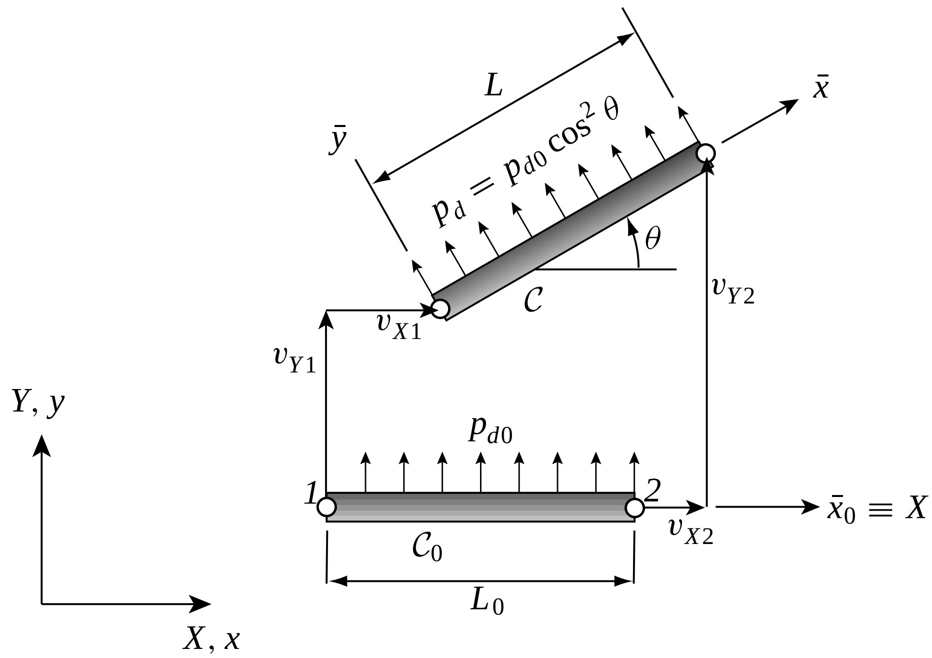

Figure 29.6. Follower pressure p_d on a 2D bar that depends on the "tilt angle" θ.

The external load vector is

$$\mathbf{f} = \tfrac{1}{2} p_{d0} L \begin{bmatrix} -\sin\theta\cos^2\theta \\ \cos^3\theta \\ 0 \\ -\sin\theta\cos^2\theta \\ \cos^3\theta \\ 0 \end{bmatrix} \tag{29.21}$$

To differentiate this expression under the assumption that p_{d0} does not depend on the node displacements, and that L is constant, we need partial derivative expressions such as

$$\begin{aligned} \frac{\partial(-\sin\theta\cos^2\theta)}{\partial u_{X21}} &= -2\sin\theta\cos\theta\,\frac{\partial\cos\theta}{\partial u_{X21}} - \cos^2\theta\,\frac{\partial\sin\theta}{\partial u_{X21}} \\ &= \frac{1}{L} sc(c^2 - 2s^2) = \frac{1}{L} sc(1 - 3s^2), \end{aligned} \tag{29.22}$$

etc. The resulting load stiffness $\mathbf{K}_L = -\partial\mathbf{f}/\partial\mathbf{u}$ is more complicated than (29.8), but still can be obtained in closed form.

If L is let to vary, then one can substitute $\cos\theta = (L_0 + u_{X21})/L$ and $\sin\theta = u_{Y21}/L$ to put $\mathbf{f}$ in terms of u_{X21} and u_{Y21}, and the differentiation to get $\mathbf{K}_L$ becomes straightforward. Thus the exact expression is in fact easier to work out than the approximate one. The details of the derivation are worked out in Exercise 29.5.

Homework Exercise for Chapter 29

Nonconservative Loading

EXERCISE 29.1 (A:20) Work out $\mathbf{f}_d$ for the case of a uniform flow of speed u_{fj} in a general direction $\vec{j}$ as described in Remark 29.2.

EXERCISE 29.2 (A:15) Specialize the result of Exercise 29.1 to the two dimensional case (bar and flow in the x, y plane). Differentiate to obtain $\mathbf{K}_L$, comparing with (29.20).

EXERCISE 29.3 (A:20) In the previous exercise take into account the effect of *friction* forces exerted on the bar by the flow. Use the linear model: the tangential friction force p_t per unit length of the bar is directed along $\bar{x}$ and has the value $C_f a u_{ft}$, where C_f is a friction coefficient, a is the "exposed perimeter" of the bar (for a circular cross section, $a = 2\pi d$), and $u_{ft} = u_{fj} \sin\theta$ is the tangential velocity (fluid velocity projected on the current bar direction, with proper sign).

EXERCISE 29.4 (A:20) Prove the formulas (29.10).

EXERCISE 29.5 (A:20) Complete the derivation of $\mathbf{K}_L$ in §29.6.

EXERCISE 29.6 (A:30) A simple example of a gyroscopic force is a *torsional moment* $\bar{M}_x$ directed along the longitudinal axis $\bar{x}$ of a beam-column element, which keeps pointing in that direction as the element moves and rotates. Obtain the gyroscopic force vector $\mathbf{f}_n$ and associated load stiffness $\mathbf{K}_{Ln}$ for a three-dimensional beam column of length L currently directed along the global x axis. The element degrees of freedom are

$$\mathbf{u}^T = [\, u_{x1} \quad u_{y1} \quad u_{z1} \quad \theta_{x1} \quad \theta_{y1} \quad \theta_{z1} \quad u_{x2} \quad u_{y2} \quad u_{z2} \quad \theta_{x2} \quad \theta_{y2} \quad \theta_{z2} \,]. \tag{E29.1}$$

For this "moment tilting" analysis it is sufficient to assume that: (a) node 1 stays fixed, (b) the element remain straight, and (c) any deviations from the current x direction are infinitesimal.