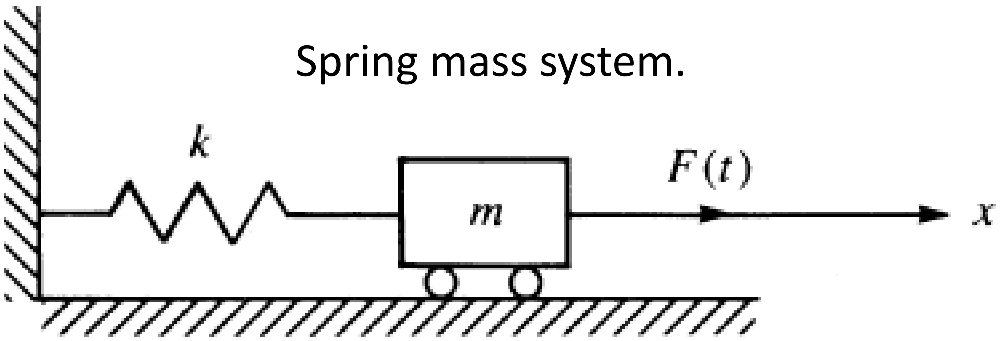
Structural Dynamics



Spring mass system.

The spring force is given by $T = kx$ and $F(t)$ is the driving force.
Start by applying Newton's second law ($F=ma$).

$$F(t) - kx = m\ddot{x}$$

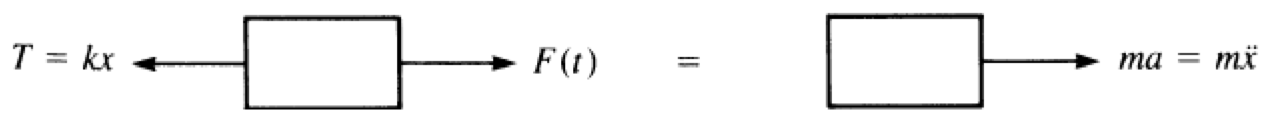
We will now look at free vibrations.

Considering the free vibration of the mass—that is, when $F(t) = 0$.

$$\omega^2 = \frac{k}{m}$$

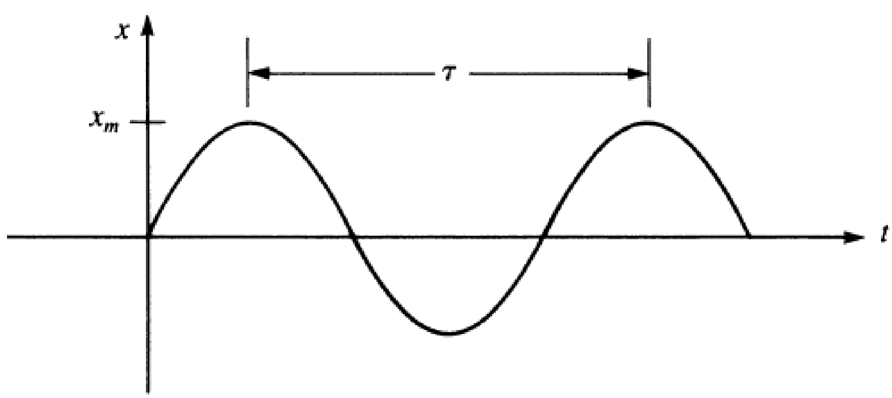
Setting $m\ddot{x} = 0$.

We obtain. $$\ddot{x} + \omega^2 x = 0$$



Free body diagrams of spring mass system.

The free vibration of the system will take the form of simple harmonic motion below.



Time/displacement curve.

$$\ddot{x} + \omega^2 x = 0$$

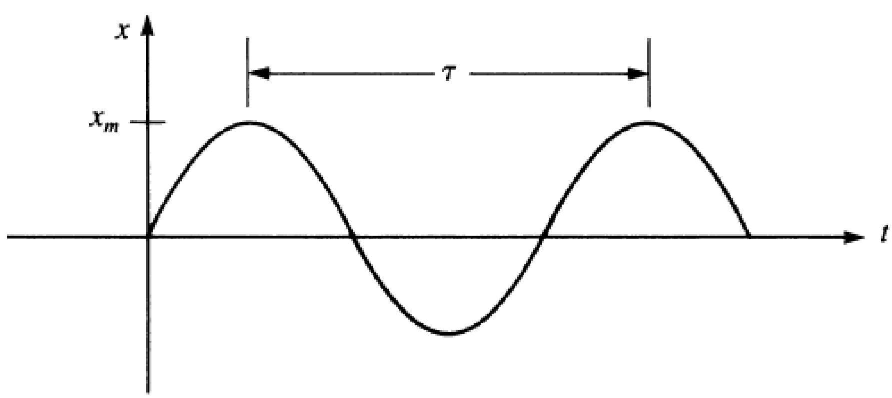
The motion defined above is called simple harmonic motion. The displacement and acceleration are proportional but of opposite directions..

x_m is the maximum displacement or amplitude of the vibration.

The period τ is the time necessary to make a full cycle.

$$\tau = \frac{2\pi}{\omega}$$

where τ is measured in seconds. Also the frequency in hertz ($\mathrm{Hz} = 1/\mathrm{s}$) is $f = 1/\tau = \omega/(2\pi)$.

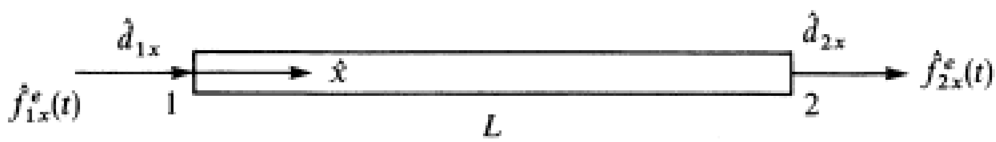


Time/displacement curve.

All vibrations are damped to some degree by friction forces. These forces may be caused by dry or Coulomb friction between rigid bodies, by internal friction between molecules within a deformable body, or by fluid friction when a body moves in a fluid. These result in natural circular vibration frequencies that are less than those calculated using free vibration.

Lets start with a 1D bar to look at vibrations.

Figure 16–4 shows the typical bar element of length L, cross-sectional area A, and mass density ρ (with typical units of lb-s^2/in^4), with nodes 1 and 2 subjected to external time-dependent loads $\hat{f}_x^e(t)$.



We will assume a displacement function taking the form of. $\hat{u} = a_1 + a_2\hat{x}$

So the shape functions will be. $\hat{u} = N_1\hat{d}_{1x} + N_2\hat{d}_{2x}$

$$N_1 = 1 - \frac{\hat{x}}{L} \qquad N_2 = \frac{\hat{x}}{L}$$

The strain/displacement relationship is given by.

$$\{\varepsilon_x\} = \frac{\partial \hat{u}}{\partial \hat{x}} = [B]\{\hat{d}\}$$

$$[B] = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \qquad \{\hat{d}\} = \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

The stress/strain relationship is given by.

$$\{\sigma_x\} = [D]\{\varepsilon_x\} = [D][B]\{\hat{d}\} = [D][B][U]$$

Time dependence: The bar is not in equilibrium under a time-dependent force so $f_{1x} \neq f_{2x}$.

At each node 'the external (applied) force f_x^e minus the internal force is equal to the nodal mass times acceleration.'

We add this internal force $F = ma$ to each nodal force to obtain.

$$\hat{f}_{1x}^e = \hat{f}_{1x} + m_1 \frac{\partial^2 \hat{d}_{1x}}{\partial t^2} \qquad \hat{f}_{2x}^e = \hat{f}_{2x} + m_2 \frac{\partial^2 \hat{d}_{2x}}{\partial t^2}$$

The masses m_1 and m_2 are lumped to each of the nodes to obtain.

$$\begin{Bmatrix} \hat{f}_{1x}^e \\ \hat{f}_{2x}^e \end{Bmatrix} = \begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} + \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial^2 \hat{d}_{1x}}{\partial t^2} \\ \dfrac{\partial^2 \hat{d}_{2x}}{\partial t^2} \end{Bmatrix}$$

The external nodal force becomes. $\{\hat{f}^e(t)\} = [\hat{k}]\{\hat{d}\} + [\hat{m}]\{\ddot{\hat{d}}\}$

The element stiffness matrix is. $[\hat{k}] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

The lumped mass matrix is given by. $[\hat{m}] = \frac{\rho AL}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The acceleration term is. $\{\ddot{\hat{d}}\} = \frac{\partial^2\{\hat{d}\}}{\partial t^2}$

Another way to handle the mass is through the development of the consistent mass matrix where use the shape function to model the mass along the bar.

We will use D'Alembert's principle and introduce an effective body force X^e as.

$$\{X^e\} = -\rho\{\ddot{\hat{u}}\}$$

the minus sign indicates that the acceleration produces D'Alembert's body forces opposite in the direction as the acceleration.

We will use $\{f_b\} = \iiint\limits_V [N]^T \{X\}\, dV$ and substitute

$\{X^e\} = -\rho\{\ddot{u}\}$ for $\{X\}$ gives us.

$$\{f_b\} = -\iiint\limits_V \rho [N]^T \{\ddot{u}\}\, dV$$

Using $\{\hat{u}\} = [N]\{\hat{d}\}$, and completing the derivatives.

$$\{\dot{\hat{u}}\} = [N]\{\dot{\hat{d}}\} \qquad \{\ddot{\hat{u}}\} = [N]\{\ddot{\hat{d}}\}$$

where $\{\dot{\hat{d}}\}$ and $\{\ddot{\hat{d}}\}$ are the nodal velocities and accelerations

$$\{f_b\} = -\iiint\limits_V \rho [N]^T [N]\, dV \{\ddot{\hat{d}}\} = -[\hat{m}]\{\ddot{\hat{d}}\}$$

The element mass matrix is given by. $[\hat{m}] = \iiint\limits_V \rho [N]^T [N] \, dV$

Substituting for interpolation functions: $$[\hat{m}] = \iiint\limits_V \rho \begin{Bmatrix} 1 - \dfrac{\hat{x}}{L} \\ \dfrac{\hat{x}}{L} \end{Bmatrix} \begin{bmatrix} 1 - \dfrac{\hat{x}}{L} & \dfrac{\hat{x}}{L} \end{bmatrix} dV$$

For a 1D bar one gets: $$[\hat{m}] = \rho A \int_0^L \begin{Bmatrix} 1 - \dfrac{\hat{x}}{L} \\ \dfrac{\hat{x}}{L} \end{Bmatrix} \begin{bmatrix} 1 - \dfrac{\hat{x}}{L} & \dfrac{\hat{x}}{L} \end{bmatrix} d\hat{x}$$

$$[\hat{m}] = \rho A \int_0^L \begin{Bmatrix} 1 - \dfrac{\hat{x}}{L} \\ \dfrac{\hat{x}}{L} \end{Bmatrix} \begin{bmatrix} 1 - \dfrac{\hat{x}}{L} & \dfrac{\hat{x}}{L} \end{bmatrix} d\hat{x}$$

Complete the matrix multiplications.

$$[\hat{m}] = \rho A \int_0^L \begin{bmatrix} \left(1 - \dfrac{\hat{x}}{L}\right)^2 & \left(1 - \dfrac{\hat{x}}{L}\right)\dfrac{\hat{x}}{L} \\ \left(1 - \dfrac{\hat{x}}{L}\right)\dfrac{\hat{x}}{L} & \left(\dfrac{\hat{x}}{L}\right)^2 \end{bmatrix} d\hat{x}$$

Upon integration

$$[\hat{m}] = \frac{\rho A L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

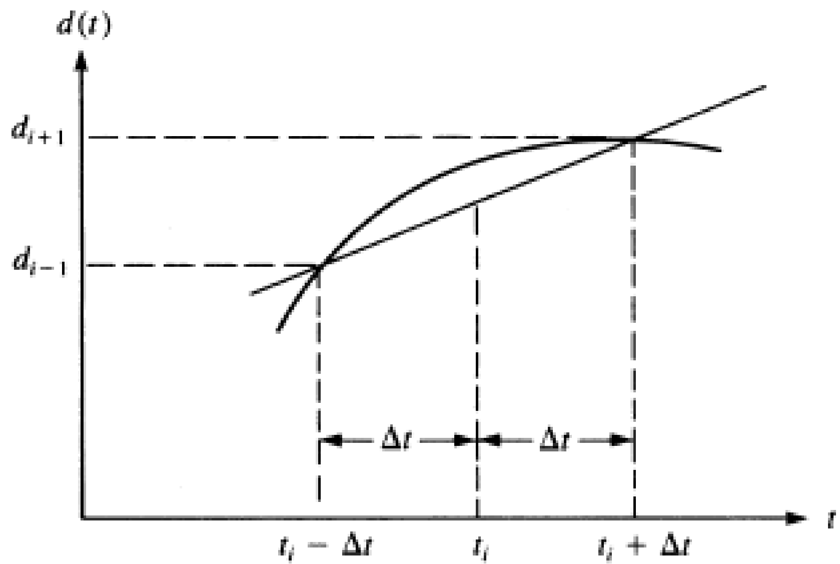
Assembly

$$\{F(t)\} = [K]\{d\} + [M]\{\ddot{d}\}$$

where

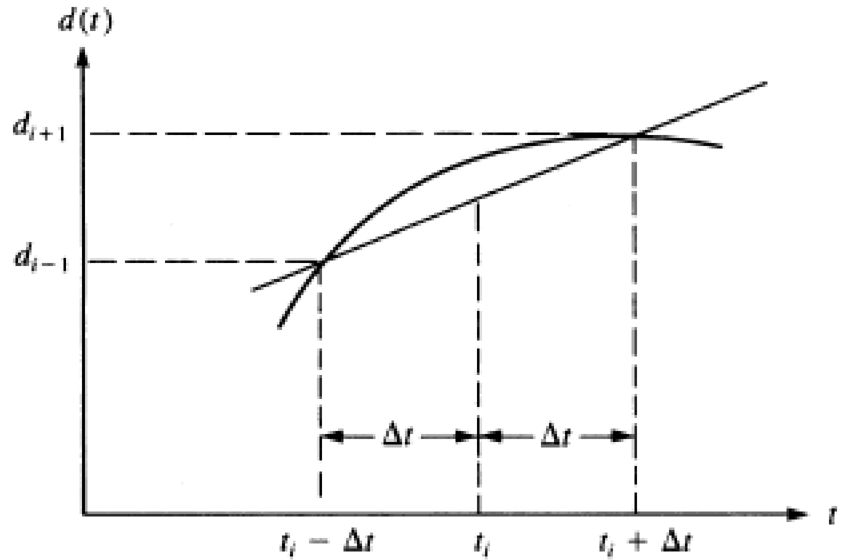
$$[K] = \sum_{e=1}^{N}[k^{(e)}] \qquad [M] = \sum_{e=1}^{N}[m^{(e)}] \qquad \{F\} = \sum_{e=1}^{N}\{f^{(e)}\}$$

Numerical Integration in Time



Several different methods are available to do the numerical integration.

We will cover central-difference and the Newmark-Beta.



The central difference method is based on finite difference expressions in time for velocity and acceleration at time t given by the following:

$$\dot{\underline{d}}_i = \frac{\underline{d}_{i+1} - \underline{d}_{i-1}}{2(\Delta t)}$$

$$\ddot{\underline{d}}_i = \frac{\dot{\underline{d}}_{i+1} - \dot{\underline{d}}_{i-1}}{2(\Delta t)}$$

With a Taylor expansion acceleration can be defined in terms of displacements.

$$\ddot{\underline{d}}_i = \frac{\underline{d}_{i+1} - 2\underline{d}_i + \underline{d}_{i-1}}{(\Delta t)^2}$$

$$\underline{\ddot{d}}_i = \frac{\underline{d}_{i+1} - 2\underline{d}_i + \underline{d}_{i-1}}{(\Delta t)^2}$$

Rearranging gives us. $\underline{d}_{i+1} = 2\underline{d}_i - \underline{d}_{i-1} + \underline{\ddot{d}}_i(\Delta t)^2$

This will be used to determine the nodal displacements in the next time step $i+1$ knowing the displacements at time steps i and $i-1$ and the acceleration at time i.

From $\{F(t)\} = [K]\{d\} + [M]\{\ddot{d}\}$

we obtain $\underline{\ddot{d}}_i = \underline{M}^{-1}(\underline{F}_i - \underline{K}\underline{d}_i)$

$$\underline{d}_{i+1} = 2\underline{d}_i - \underline{d}_{i-1} + \ddot{\underline{d}}_i(\Delta t)^2$$

To obtain an expression for $\underline{d}_{i+1}$, we first multiply above by the mass matrix M and then substitute

$$\ddot{\underline{d}}_i = \underline{M}^{-1}(\underline{F}_i - \underline{K}\underline{d}_i)$$

for $\ddot{\underline{d}}_i$ into the equation at the top of the page to obtain:

$$\underline{M}\underline{d}_{i+1} = 2\underline{M}\underline{d}_i - \underline{M}\underline{d}_{i-1} + (\underline{F}_i - \underline{K}\underline{d}_i)(\Delta t)^2$$

and rearranging.

$$\underline{M}\underline{d}_{i+1} = (\Delta t)^2\underline{F}_i + [2\underline{M} - (\Delta t)^2\underline{K}]\underline{d}_i - \underline{M}\underline{d}_{i-1}$$

We start the computations by determining the displacement at $\underline{d}_{i-1}$

$$\underline{d}_{i-1} = \underline{d}_i - (\Delta t)\dot{\underline{d}}_i + \frac{(\Delta t)^2}{2}\ddot{\underline{d}}_i \qquad \text{Eqn 1.}$$

1. Given: $\underline{d}_0, \dot{\underline{d}}_0$, and $\underline{F}_i(t)$.
2. If $\ddot{\underline{d}}_0$ is not initially given, solve $\ddot{\underline{d}}_i = \underline{M}^{-1}(\underline{F}_i - \underline{K}\underline{d}_i)$ at t = 0 then $\ddot{\underline{d}}_0$ is

$$\ddot{\underline{d}}_0 = \underline{M}^{-1}(\underline{F}_0 - \underline{K}\underline{d}_0)$$

3. Solve Eqn. 1 $t = -\Delta t$ for $\underline{d}_{-1}$; that is,

$$\underline{d}_{-1} = \underline{d}_0 - (\Delta t)\dot{\underline{d}}_0 + \frac{(\Delta t)^2}{2}\ddot{\underline{d}}_0$$

4. Having solved for $\underline{d}_{-1}$ in step 3, now solve for $\underline{d}_1$ using $\underline{M}\underline{d}_{i+1} = (\Delta t)^2\underline{F}_i + [2\underline{M} - (\Delta t)^2\underline{K}]\underline{d}_i - \underline{M}\underline{d}_{i-1}$

$$\underline{d}_1 = \underline{M}^{-1}\{(\Delta t)^2\underline{F}_0 + [2\underline{M} - (\Delta t)^2\underline{K}]\underline{d}_0 - \underline{M}\underline{d}_{-1}\}$$

5. With $\underline{d}_0$ initially given, and $\underline{d}_1$ determined from step 4, use $\underline{M}\underline{d}_{i+1} = (\Delta t)^2\underline{F}_i + [2\underline{M} - (\Delta t)^2\underline{K}]\underline{d}_i - \underline{M}\underline{d}_{i-1}$ to obtain

$$\underline{d}_2 = \underline{M}^{-1}\{(\Delta t)^2\underline{F}_1 + [2\underline{M} - (\Delta t)^2\underline{K}]\underline{d}_1 - \underline{M}\underline{d}_0\}$$

6. Using $\ddot{\underline{d}}_i = \underline{M}^{-1}(\underline{F}_i - \underline{K}\underline{d}_i)$ $\ddot{\underline{d}}_1$ =

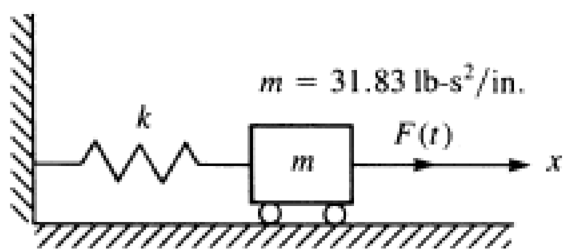
$$\ddot{\underline{d}}_1 = \underline{M}^{-1}(\underline{F}_1 - \underline{K}\underline{d}_1)$$

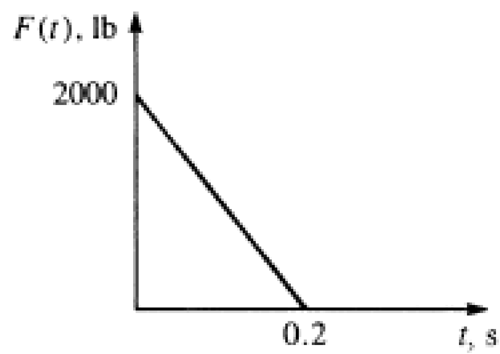
7. Using the result of step 5 and the boundary condition for $\underline{d}_0$ given in step 1, determine the velocity at the first time step by $\dot{\underline{d}}_i = \dfrac{\underline{d}_{i+1} - \underline{d}_{i-1}}{2(\Delta t)}$

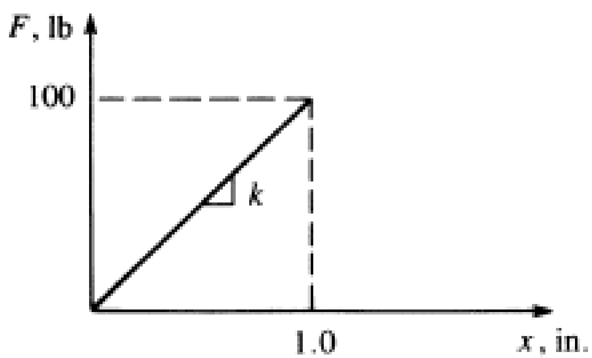
$$\dot{\underline{d}}_1 = \frac{\underline{d}_2 - \underline{d}_0}{2(\Delta t)}$$

8. Use steps 5–7 repeatedly to obtain the displacement, acceleration, and velocity for all other time steps.

Determine the displacement, velocity, and acceleration at 0.05-s time intervals up to 0.2 s for the one-dimensional spring-mass oscillator subjected to the time-dependent forcing function shown in the figure below. This forcing function is a typical one assumed for blast loads. The restoring spring force versus displacement curve is also provided. [Note that the bar in the figure represents a one-element bar with its left end fixed and right node subjected to $F(t)$ when a lumped mass is used.







Step 1

At time = 0, the displacement and velocity are $d_0 = 0 \qquad \dot{d}_0 = 0$

m = 31.83 lb-s/in^2

The initial acceleration at $t = 0$ is obtained as

$$\ddot{\underline{d}}_i = \underline{M}^{-1}(\underline{F}_i - \underline{K}\underline{d}_i) \qquad \ddot{d}_0 = \frac{2000 - 100(0)}{31.83} = 62.83 \text{ in./s}^2$$

where we have used $\underline{F}(0) = 2000$ lb and $\underline{K} = 100$ lb/in.

Step 3

The displacement d_{-1} is obtained as

$$\underline{d}_{-1} = \underline{d}_0 - (\Delta t)\underline{\dot{d}}_0 + \frac{(\Delta t)^2}{2}\underline{\ddot{d}}_0 \qquad d_{-1} = 0 - 0 + \frac{(0.05)^2}{2}(62.83) = 0.0785 \text{ in.}$$

Step 4

The displacement at time $t = 0.05$ s is $\underline{d}_1 = \underline{M}^{-1}\{(\Delta t)^2\underline{F}_0 + [2\underline{M} - (\Delta t)^2\underline{K}]\underline{d}_0 - \underline{M}\,\underline{d}_{-1}\}$

$$\begin{aligned} d_1 &= \frac{1}{31.83}\{(0.05)^2(2000) + [2(31.83) - (0.05)^2(100)]0 - (31.83)(0.0785)\} \\ &= 0.0785 \text{ in.} \end{aligned}$$

Step 5 Use $\underline{d}_2 = \underline{M}^{-1}\{(\Delta t)^2\underline{F}_1 + [2\underline{M} - (\Delta t)^2\underline{K}]\underline{d}_1 - \underline{M}\underline{d}_0\}$

Having obtained d_1, we now determine the displacement at time $t = 0.10$ s as

$$\begin{aligned} d_2 &= \frac{1}{31.83}\{(0.05)^2(1500) + [2(31.83) - (0.05)^2(100)](0.0785) - (31.83)(0)\} \\ &= 0.274 \text{ in.} \end{aligned}$$

Step 6

The acceleration at time $t = 0.05$ s is

$$\ddot{\underline{d}}_1 = \underline{M}^{-1}(\underline{F}_1 - \underline{K}\underline{d}_1) \qquad \ddot{d}_1 = \frac{1}{31.83}[1500 - 100(0.0785)] = 46.88 \text{ in./s}^2$$

Step 7

The velocity at time $t = 0.05$ s is

$$\dot{\underline{d}}_1 = \frac{\underline{d}_2 - \underline{d}_0}{2(\Delta t)} \qquad \dot{d}_1 = \frac{0.274 - 0}{2(0.05)} = 2.74 \text{ in./s}$$

Step 8

Repeated use of steps 5–7 will result in the displacement, acceleration, and velocity for additional time steps as desired. We will now perform one more time-step iteration of the procedure.

Repeating step 5 for the next time step, we have

$$d_3 = \frac{1}{31.83}\{(0.05)^2(1000) + [2(31.83) - (0.05)^2(100)](0.274) - (31.83)(0.0785)\} = 0.546 \text{ in.}$$

Repeating step 6 for the next time step, we have

$$\ddot{d}_2 = \frac{1}{31.83}[1000 - 100(0.274)] = 30.56 \text{ in./s}^2$$

t (s)	$F(t)$ (lb)	d_i (in.)	Q (lb)	$\ddot{d}_i$ (in./s^2)	$\dot{d}_i$ (in./s)	d_i (exact)
0	2000	0	0	62.83	0	0
0.05	1500	0.0785	7.85	46.88	2.74	0.0718
0.10	1000	0.274	27.40	30.56	4.68	0.2603
0.15	500	0.546	54.64	13.99	5.79	0.5252
0.20	0	0.854	85.35	−2.68	6.07	0.8250
0.25	0	1.154	115.4	−3.63	5.91	1.132

Finally, repeating step 7 for the next time step, we obtain

$$\dot{d}_2 = \frac{0.546 - 0.0785}{2(0.05)} = 4.68 \text{ in./s}$$

Table 16–1 summarizes the results obtained through time $t = 0.25$ s. In Table 1, $Q = kd_i$ is the restoring spring force. Also, the exact analytical solution for displacement based on the

$$y = \frac{F_0}{k}(1 - \cos\omega t) + \frac{F_0}{kt_d}\left(\frac{\sin\omega t}{\omega} - t\right)$$

where $F_0 = 2000$ lb, $k = 100$ lb/in., $t_d = 0.2$ s, and

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{100}{31.83}} = 1.77 \text{ rad/s} \qquad \blacksquare$$

Newmark-Beta Method

Newmark-Beta equations

$$\underline{\dot{d}}_{i+1} = \underline{\dot{d}}_i + (\Delta t)[(1-\gamma)\underline{\ddot{d}}_i + \gamma\underline{\ddot{d}}_{i+1}]$$
$$\underline{d}_{i+1} = \underline{d}_i + (\Delta t)\underline{\dot{d}}_i + (\Delta t)^2[(\tfrac{1}{2}-\beta)\underline{\ddot{d}}_i + \beta\underline{\ddot{d}}_{i+1}]$$

where β and γ are parameters chosen by the user. The parameter β is generally chosen between 0 and 1/4, and γ is often taken to be 1/2.

γ = 1/2 and β = 1/4 , gives stable analysis results.

To find $\underline{d}_{i+1}$, we first multiply

$$\underline{d}_{i+1} = \underline{d}_i + (\Delta t)\underline{\dot{d}}_i + (\Delta t)^2[(\tfrac{1}{2} - \beta)\underline{\ddot{d}}_i + \beta\underline{\ddot{d}}_{i+1}]$$

by the mass matrix $\underline{M}$:

$$\underline{M}\,\underline{d}_{i+1} = \underline{M}\,\underline{d}_i + (\Delta t)\underline{M}\,\underline{\dot{d}}_i + (\Delta t)^2\underline{M}(\tfrac{1}{2} - \beta)\underline{\ddot{d}}_i] + \beta(\Delta t)^2[\underline{F}_{i+1} - \underline{K}\,\underline{d}_{i+1}]$$

and then substitute $\underline{\ddot{d}}_i = \underline{M}^{-1}(\underline{F}_i - \underline{K}\,\underline{d}_i)$

$$\underline{M}\,\underline{d}_{i+1} = 2\underline{M}\,\underline{d}_i - \underline{M}\,\underline{d}_{i-1} + (\underline{F}_i - \underline{K}\,\underline{d}_i)(\Delta t)^2$$

Combining like terms

$$\underline{M}\,\underline{d}_{i+1} = (\Delta t)^2\underline{F}_i + [2\underline{M} - (\Delta t)^2\underline{K}]\underline{d}_i - \underline{M}\,\underline{d}_{i-1}$$

Finally, dividing $\underline{M}\,\underline{d}_{i+1} = (\Delta t)^2 \underline{F}_i + [2\underline{M} - (\Delta t)^2 \underline{K}]\underline{d}_i - \underline{M}\,\underline{d}_{i-1}$

by $\beta(\Delta t)^2$:

$$\underline{K}'\underline{d}_{i+1} = \underline{F}'_{i+1}$$

where

$$\underline{K}' = \underline{K} + \frac{1}{\beta(\Delta t)^2}\underline{M}$$

$$\underline{F}'_{i+1} = \underline{F}_{i+1} + \frac{\underline{M}}{\beta(\Delta t)^2}\left[\underline{d}_i + (\Delta t)\dot{\underline{d}}_i + \left(\frac{1}{2} - \beta\right)(\Delta t)^2 \ddot{\underline{d}}_i\right]$$

1. Starting at time $t = 0$, $\underline{d}_0$ is known from the given boundary conditions on displacement, and $\dot{\underline{d}}_0$ is known from the initial velocity conditions.

2. Solve $\ddot{\underline{d}}_0 = \underline{M}^{-1}(\underline{F}_0 - \underline{K}\underline{d}_0)$ unless $\ddot{\underline{d}}_0$ is known from an initial

3. Solve $\underline{K}'\underline{d}_{i+1} = \underline{F}'_{i+1}$ for $\underline{d}_1$, because $\underline{F}_{i+1}$ is known for all time steps and $\underline{d}_0, \dot{\underline{d}}_0$, and $\ddot{\underline{d}}_0$ are now known from steps 1 and 2.

4. Solve $\ddot{\underline{d}}_1 = \dfrac{1}{\beta(\Delta t)^2}\left[\underline{d}_1 - \underline{d}_0 - (\Delta t)\dot{\underline{d}}_0 - (\Delta t)^2\left(\dfrac{1}{2} - \beta\right)\ddot{\underline{d}}_0\right]$

5. Solve $\dot{\underline{d}}_{i+1} = \dot{\underline{d}}_i + (\Delta t)[(1-\gamma)\ddot{\underline{d}}_i + \gamma\ddot{\underline{d}}_{i+1}]$ directly for $\dot{\underline{d}}_1$.

6. Using the results of steps 4 and 5, go back to step 3 to solve for $\underline{d}_2$ and then to steps 4 and 5 to solve for $\ddot{\underline{d}}_2$ and $\dot{\underline{d}}_2$. Use steps 3–5 repeatedly to solve for $\underline{d}_{i+1}, \dot{\underline{d}}_{i+1}$, and $\ddot{\underline{d}}_{i+1}$.

Determine the displacement, velocity, and acceleration at 0.1-s time increments up to a time of 0.5 s for the one-dimensional spring-mass oscillator subjected to the time-dependent forcing function shown in Figure , along with the restoring spring force versus displacement curve. Assume the oscillator is initially at rest. Let $\beta = \frac{1}{6}$ and $\gamma = \frac{1}{2}$, which corresponds to an assumption of linear acceleration within each time step.

Because we are again considering the single degree of freedom associated with the mass, the general matrix equations describing the motion reduce to single scalar equations. Again, we represent this single degree of freedom by d.

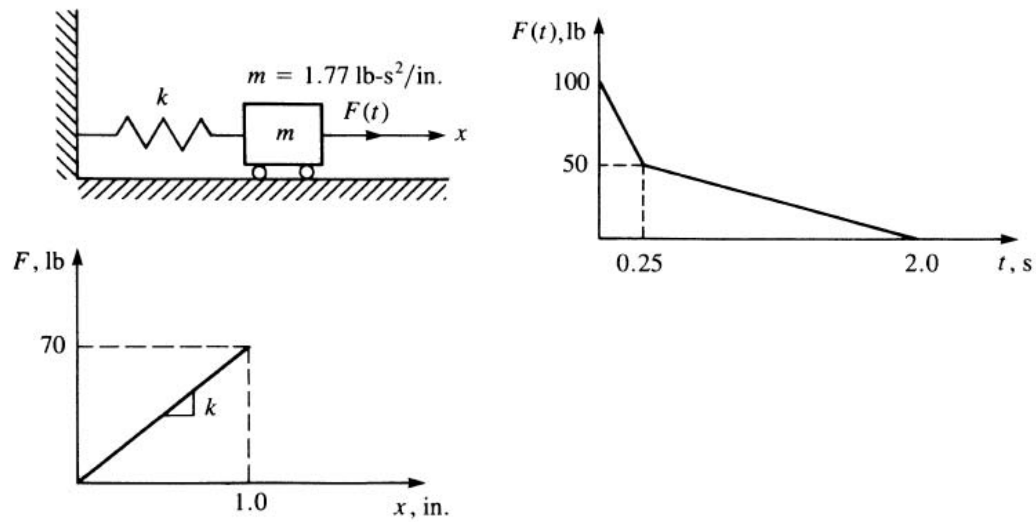


Figure Spring-mass oscillator subjected to a time-dependent force

$$\underline{K}' = \underline{K} + \frac{1}{\beta(\Delta t)^2}\underline{M}$$

$$\underline{F}'_{i+1} = \underline{F}_{i+1} + \frac{\underline{M}}{\beta(\Delta t)^2}\left[\underline{d}_i + (\Delta t)\dot{\underline{d}}_i + \left(\frac{1}{2} - \beta\right)(\Delta t)^2\ddot{\underline{d}}_i\right]$$

Step 1

At time $t = 0$, the initial displacement and velocity are zero; therefore,

$$d_0 = 0 \qquad \dot{d}_0 = 0$$

Step 2

The initial acceleration at $t = 0$ is obtained as

$$\ddot{d}_0 = \frac{100 - 70(0)}{1.77} = 56.5 \text{ in./s}^2$$

where we have used $\underline{F}_0 = 100$ lb and $\underline{K} = 70$ lb/in.

Step 3

We now solve for the displacement at time $t = 0.1$ s as

$$K' = 70 + \frac{1}{(\frac{1}{6})(0.1)^2}(1.77) = 1132 \text{ lb/in.}$$

$$F'_1 = 80 + \frac{1.77}{(\frac{1}{6})(0.1)^2}\left[0 + (0.1)(0) + \left(\frac{1}{2} - \frac{1}{6}\right)(0.1)^2(56.5)\right] = 280 \text{ lb}$$

$$d_1 = \frac{280}{1132} = 0.248 \text{ in.}$$

Step 4

Solve for the acceleration at time $t = 0.1$ s as

$$\ddot{d}_1 = \frac{1}{(\frac{1}{6})(0.1)^2}\left[0.248 - 0 - (0.1)(0) - (0.1)^2\left(\frac{1}{2} - \frac{1}{6}\right)(56.5)\right]$$

$$\ddot{d}_1 = 35.4 \text{ in./s}^2$$

Step 5

Solve for the velocity at time $t = 0.1$ s as

$$\dot{d}_1 = 0 + (0.1)[(1 - \tfrac{1}{2})(56.5) + (\tfrac{1}{2})(35.4)]$$

Step 6

Repeated use of steps 3–5 will result in the displacement, acceleration, and velocity for additional time steps as desired. We will now perform one more time-step iteration.

Repeating step 3 for the next time step ($t = 0.2$ s), we have

$$F_2' = 60 + \frac{1.77}{(\frac{1}{6})(0.1)^2}\left[0.248 + (0.1)(4.59) + \left(\frac{1}{2} - \frac{1}{6}\right)(0.1)^2(35.4)\right]$$

$$F_2' = 934 \text{ lb}$$

$$d_2 = \frac{934}{1132} = 0.825 \text{ in.}$$

Repeating step 4 for time step $t = 0.2$ s, we obtain

$$\ddot{d}_2 = \frac{1}{(\frac{1}{6})(0.1)^2}\left[0.825 - 0.248 - (0.1)(4.59) - (0.1)^2\left(\frac{1}{2} - \frac{1}{6}\right)(35.4)\right]$$

$$\ddot{d}_2 = 1.27 \text{ in./s}^2$$

Finally, repeating step 5 for time step $t = 0.2$ s, we have

$$\dot{d}_2 = 4.59 + (0.1)[(1 - \tfrac{1}{2})(35.4) + \tfrac{1}{2}(1.27)]$$

$$\dot{d}_2 = 6.42 \text{ in./s}$$

t (s)	$F(t)$ (lb)	d_i (in.)	Q (lb)	$\ddot{d}_i$ (in./s^2)	$\dot{d}_i$ (in./s)
0.	100	0	0	56.5	0
0.1	80	0.248	17.3	35.4	4.59
0.2	60	0.825	57.8	1.27	6.42
0.3	48.6	1.36	95.2	−26.2	5.17
0.4	45.7	1.72	120.4	−42.2	1.75
0.5	42.9	1.68	117.6	−42.2	−2.45

$Q = kd_i$ is the restoring spring force. ■

Given $\{F(t)\} = [K]\{d\} + [M]\{\ddot{d}\}$ we wish to solve this equation given

$$\underline{M}\ddot{\underline{d}} + \underline{K}\underline{d} = 0$$

The standard solution for $\underline{d}(t)$ is given by the harmonic equation in time

$$\underline{d}(t) = \underline{d}'e^{i\omega t}$$

where $\underline{d}'$ is the part of the nodal displacement matrix called *natural modes* that is assumed to be independent of time, i is the standard imaginary number given by $i = \sqrt{-1}$, and ω is a natural frequency.

Differentiating $\underline{d}(t) = \underline{d}'e^{i\omega t}$ twice with respect to time, we obtain

$$\ddot{\underline{d}}(t) = \underline{d}'(-\omega^2)e^{i\omega t}$$

Substitution of $\underline{d}(t) = \underline{d}'e^{i\omega t}$ and $\ddot{\underline{d}}(t) = \underline{d}'(-\omega^2)e^{i\omega t}$ into $\underline{M}\ddot{\underline{d}} + \underline{K}\underline{d} = 0$ gives

$$-\underline{M}\omega^2\underline{d}'e^{i\omega t} + \underline{K}\underline{d}'e^{i\omega t} = 0$$

Rearranging and combining terms gives us.

$$e^{i\omega t}(\underline{K} - \omega^2\underline{M})\underline{d}' = 0$$

Because $e^{i\omega t}$ is not zero, we get.

$$(\underline{K} - \omega^2\underline{M})\underline{d}' = 0$$

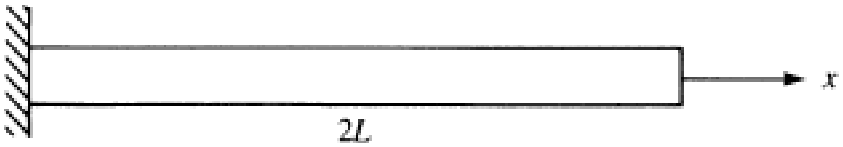
This equation has a nontrivial solution if and only if the determinant of the coefficient matrix of $\underline{d}'$ is zero, so we must have.

This equation has a nontrivial solution if and only if the determinant of the coefficient matrix of $\underline{d'}$ is zero, so we must solve.

$$|\underline{K} - \omega^2 \underline{M}| = 0$$

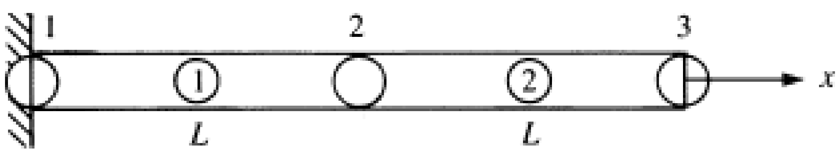
For the bar shown below has a length $2L$, modulus of elasticity E, mass density ρ, and cross-sectional area A, determine the first two natural frequencies.

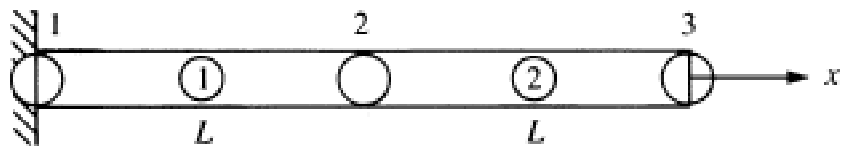
For simplicity, the bar is discretized into two elements each of length L as shown in below. To solve $|\underline{K} - \omega^2 \underline{M}| = 0$, we must develop the total stiffness matrix for the bar by using Eqn. 1 and either the lumped mass matrix or the consistent-mass matrix.



$$[\hat{k}] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The lumped-mass matrix will be used in this analysis with 2 elements.





$$\begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ [\hat{k}^{(1)}] = \dfrac{AE}{L} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \qquad \begin{array}{cc} & \begin{array}{cc} 2 & 3 \end{array} \\ [\hat{k}^{(2)}] = \dfrac{AE}{L} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

Assemble the global stiffness matrix.

$$[K] = \frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

The mass matrices are.

$$\begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ [\hat{m}^{(1)}] = \dfrac{\rho AL}{2} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \qquad \begin{array}{cc} & \begin{array}{cc} 2 & 3 \end{array} \\ [\hat{m}^{(2)}] = \dfrac{\rho AL}{2} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

Assembled $$[M] = \frac{\rho AL}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Assemble all the components and apply the boundary conditions
$\hat{d}_{1x} = 0$ (which is the same as $d_1' = 0$)

$$[K] = \frac{AE}{L}\begin{bmatrix} \cancel{1} & \cancel{-1} & \cancel{0} \\ \cancel{-1} & 2 & -1 \\ \cancel{0} & -1 & 1 \end{bmatrix} \qquad [M] = \frac{\rho AL}{2}\begin{bmatrix} \cancel{1} & \cancel{0} & \cancel{0} \\ \cancel{0} & 2 & 0 \\ \cancel{0} & 0 & 1 \end{bmatrix}$$

$$\left(\frac{AE}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \omega^2 \frac{\rho AL}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{Bmatrix} d_2' \\ d_3' \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\lambda = \omega^2$$

$$\left|\frac{AE}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \lambda \frac{\rho AL}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right| = 0$$

$$\left|\frac{AE}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \lambda\frac{\rho AL}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right| = 0$$

Dividing above by ρAL and letting $\mu = E/(\rho L^2)$, we obtain:

$$\begin{vmatrix} 2\mu - \lambda & -\mu \\ -\mu & \mu - \dfrac{\lambda}{2} \end{vmatrix} = 0$$

Take the determinant of this

gives. $$\lambda = 2\mu \pm \mu\sqrt{2}$$

so $$\lambda_1 = 0.60\mu \qquad \lambda_2 = 3.41\mu$$

For comparison, the exact solution is given by $\lambda = 0.616\mu$, whereas the consistent-mass approach yields $\lambda = 0.648\mu$.

The first and second natural frequencies are given by:

$$\omega_1 = \sqrt{\lambda_1} = 0.77\sqrt{\mu} \qquad \omega_2 = \sqrt{\lambda_2} = 1.85\sqrt{\mu}$$

Letting $E = 30 \times 10^6$ psi, $\rho = 0.00073$ lb-s^2/in^4, and $L = 100$ in., we obtain

$$\mu = E/(\rho L^2) = (30 \times 10^6)/[(0.00073)(100)^2] = 4.12 \times 10^6 \text{ s}^{-2}$$

$$\omega_1 = 1.56 \times 10^3 \text{ rad/s} \qquad \omega_2 = 3.76 \times 10^3 \text{ rad/s}$$

The natural frequencies are given by:

$$f_1 := \frac{\omega_1}{2\pi} \qquad f_2 := \frac{\omega_2}{2\pi}$$

$$f_1 = 248.282 \qquad f_2 = 598.423 \quad \text{Hz.}$$

$$\lambda = 2\mu \pm \mu\sqrt{2}$$

$$\lambda_1 = 0.60\mu \qquad \lambda_2 = 3.41\mu$$

Letting $E = 30 \times 10^6$ psi, $\rho = 0.00073$ lb-s^2/in^4, and $L = 100$ in., we obtain

$$\mu = E/(\rho L^2) = (30 \times 10^6)/[(0.00073)(100)^2] = 4.12 \times 10^6 \text{ s}^{-2}$$

We can obtain the displacements from here.

$$\begin{bmatrix} 2\mu - \lambda & -\mu \\ -\mu & \mu - \lambda/2 \end{bmatrix} \begin{Bmatrix} d2 \\ d3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \begin{aligned} 1.4\mu d_2'^{(1)} - \mu d_3'^{(1)} &= 0 \\ -\mu d_2'^{(1)} + 0.7\mu d_3'^{(1)} &= 0 \end{aligned}$$

It is customary to specify the value of one of the natural modes for a given ω_i or λ_i. With $\acute{d}_3^{(1)} = 1$ so from above equation $\acute{d}_2^{(1)} = 0.7$

$$d_3 := 1$$

$$d_2 := \frac{(d_3)}{(1.4)} \qquad d_2 = 0.714$$

Substituting λ_2 top equation. $$\begin{bmatrix} 2\mu - \lambda & -\mu \\ -\mu & \mu - \lambda/2 \end{bmatrix} \begin{Bmatrix} d2 \\ d3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

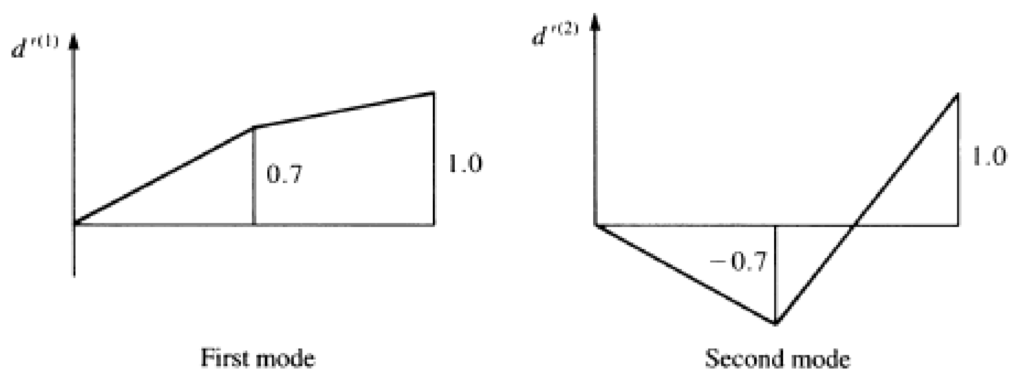
Again assuming ${d_3}^{\prime(1)} = 1$

$$\lambda_2 := \mu \cdot 3.41$$

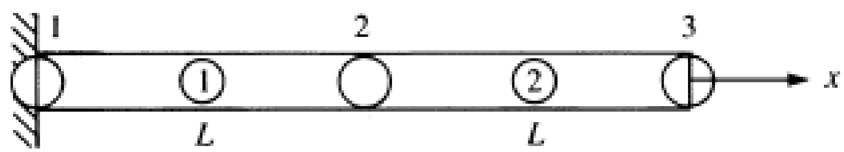
$$d3 := 1$$

$$\left(2\mu - \lambda_2\right) \cdot d2 - \mu \cdot d3$$

$$\frac{\mu}{2\mu - \lambda_2} = -0.709$$

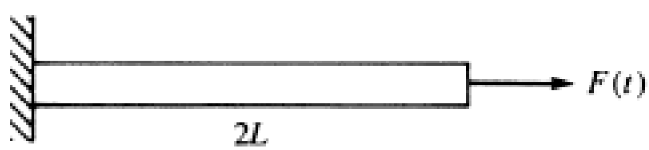


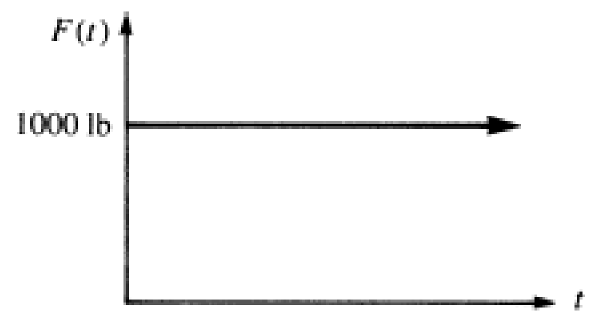
Using the previous results we shall now solve a time dependent problem using the same problem as the natural frequency work above.



$$E = 30 \times 10^6 \text{ psi}, \; \rho = 0.00073 \text{ lb-s}^2/\text{in}^4, \text{ and } L = 100 \text{ in.},$$

$$\mu = E/(\rho L^2) = (30 \times 10^6)/[(0.00073)(100)^2] = 4.12 \times 10^6 \text{ s}^{-2}$$





It has been shown that the time step must be less than or equal to 2 divided by the highest natural frequency when the central difference method is used. $\Delta t \leqslant 2/\omega_{\max}$.

In our case $\omega_1 = 1.56 \times 10^3$ rad/s $\omega_2 = 3.76 \times 10^3$ rad/s

Due to other practical consideration it is better to use.

$$\Delta t \leqslant \frac{3}{4}\left(\frac{2}{\omega_{\max}}\right)$$

$$\Delta t = \frac{3}{4}\left(\frac{2}{\omega_{\max}}\right) = \frac{1.5}{3.76 \times 10^3} = 0.40 \times 10^{-3} \text{ s}$$

An alternative guide (used only for a bar) for choosing the approximate time step is

$$\Delta t = \frac{L}{c_x}$$

C_x is called the longitudinal wave velocity and is given by: $c_x = \sqrt{E_x/\rho}$

$$\Delta t = \frac{L}{c_x} = \frac{100}{\sqrt{30 \times 10^6/0.00073}} = 0.48 \times 10^{-3} \text{ s}$$

For convenience we shall use a time step value of 0.25 second.

$$\frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \end{Bmatrix} + \frac{\rho A L}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{d}_{1x} \\ \ddot{d}_{2x} \\ \ddot{d}_{3x} \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 0 \\ F_3(t) \end{Bmatrix}$$

Step 1

Given: $d_{1x} = 0$ because of the fixed support at node 1, and all nodal displacements and velocities are zero at time $t = 0$; that is, $\underline{\dot{d}}_0 = 0$ and $\underline{d}_0 = 0$. Also, assume $\ddot{d}_{1x} = 0$ at all times.

Step 2 Thus eliminating the first row and columns of above and rearranging.

$$\underline{\ddot{d}}_0 = \begin{Bmatrix} \ddot{d}_{2x} \\ \ddot{d}_{3x} \end{Bmatrix}_{t=0} = \frac{2}{\rho A L}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\left[\begin{Bmatrix} 0 \\ 1000 \end{Bmatrix} - \frac{AE}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}\right]$$

$$\underline{M}^{-1} = \frac{2}{\rho A L}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{\ddot{d}}_i = \underline{M}^{-1}(\underline{F}_i - \underline{K}\underline{d}_i)$$

$$\underline{\ddot{d}}_0 = \frac{2000}{\rho A L}\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 27{,}400 \end{Bmatrix} \text{ in./s}^2$$

Step 3 we solve for $\underline{d}_{-1}$ as

$$\underline{d}_{-1} = \underline{d}_0 - (\Delta t)\underline{\dot{d}}_0 + \frac{(\Delta t)^2}{2}\underline{\ddot{d}}_0$$

Substituting the initial conditions on $\underline{\dot{d}}_0$ and $\underline{d}_0$ from step 1

$$\underline{\ddot{d}}_0 = \frac{2000}{\rho AL}\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 27{,}400 \end{Bmatrix} \text{ in./s}^2$$

$$\underline{d}_{-1} = 0 - (0.25 \times 10^{-3})(0) + \frac{(0.25 \times 10^{-3})^2}{2}(27{,}400)\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$$\begin{Bmatrix} d_{2x} \\ d_{3x} \end{Bmatrix}_{-1} = \begin{Bmatrix} 0 \\ 0.856 \times 10^{-3} \end{Bmatrix} \text{ in.}$$

Step 4

Solve for $\underline{d}_1$ using.

$$\underline{d}_1 = \underline{M}^{-1}\{(\Delta t)^2\underline{F}_0 + [2\underline{M} - (\Delta t)^2\underline{K}]\underline{d}_0 - \underline{M}\,\underline{d}_{-1}\}$$

$$\begin{Bmatrix} d_{2x} \\ d_{3x} \end{Bmatrix}_1 = \frac{2}{0.073}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\left\{(0.25 \times 10^{-3})^2\begin{Bmatrix} 0 \\ 1000 \end{Bmatrix} + \left[\frac{2(0.073)}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} - (0.25 \times 10^{-3})^2(30 \times 10^4)\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\right]\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \frac{0.073}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0.856 \times 10^{-3} \end{Bmatrix}\right\}$$

$$\begin{Bmatrix} d_{2x} \\ d_{3x} \end{Bmatrix}_1 = \frac{2}{0.073}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\left[\begin{Bmatrix} 0 \\ 0.0625 \times 10^{-3} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0.0312 \times 10^{-3} \end{Bmatrix}\right]$$

$$\begin{Bmatrix} d_{2x} \\ d_{3x} \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0.858 \times 10^{-3} \end{Bmatrix} \text{ in.} \qquad (\text{at } t = 0.25 \times 10^{-3} \text{ s})$$

Step 5

With $\underline{d}_0$ initially given and $\underline{d}_1$ determined from step 4,

$$\underline{d}_2 = \underline{M}^{-1}\{(\Delta t)^2\underline{F}_1 + [2\underline{M} - (\Delta t)^2\underline{K}]\underline{d}_1 - \underline{M}\,\underline{d}_0\}$$

$$= \frac{2}{0.073}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\left\{(0.25\times10^{-3})^2\begin{Bmatrix} 0 \\ 1000 \end{Bmatrix} + \left[\frac{2(0.073)}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} - (0.25\times10^{-3})^2(30\times10^4)\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\right]\right.$$
$$\left.\times\begin{Bmatrix} 0 \\ 0.858\times10^{-3} \end{Bmatrix} - \frac{0.073}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}\right\}$$

$$= \frac{2}{0.073}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\left[\begin{Bmatrix} 0 \\ 0.0625\times10^{-3} \end{Bmatrix} + \begin{Bmatrix} 0.0161\times10^{-3} \\ 0.0466\times10^{-3} \end{Bmatrix}\right]$$

$$\begin{Bmatrix} d_{2x} \\ d_{3x} \end{Bmatrix}_2 = \begin{Bmatrix} 0.221\times10^{-3} \\ 2.99\times10^{-3} \end{Bmatrix} \text{ in.} \qquad (\text{at } t = 0.50\times10^{-3}\text{ s})$$

Step 6

$$\underline{\ddot{d}}_1 = \frac{2}{0.073}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\left[\begin{Bmatrix} 0 \\ 1000 \end{Bmatrix} - (30 \times 10^4)\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0.858 \times 10^{-3} \end{Bmatrix}\right]$$

$$\begin{Bmatrix} \ddot{d}_{2x} \\ \ddot{d}_{3x} \end{Bmatrix}_1 = \begin{Bmatrix} 3526 \\ 20{,}345 \end{Bmatrix} \text{ in./s}^2 \qquad (\text{at } t = 0.25 \times 10^{-3} \text{ s})$$

Step 7

from step 5 and the boundary condition for $\underline{d}_0$ given in step 1, we

$$\underline{\dot{d}}_1 = \frac{\left[\begin{Bmatrix} 0.221 \times 10^{-3} \\ 2.99 \times 10^{-3} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}\right]}{2(0.25 \times 10^{-3})} \qquad \begin{Bmatrix} \dot{d}_{2x} \\ \dot{d}_{3x} \end{Bmatrix} = \begin{Bmatrix} 0.442 \\ 5.98 \end{Bmatrix} \text{ in./s} \qquad (\text{at } t = 0.25 \times 10^{-3} \text{ s})$$

Step 8

Repeating step 6 with $t = 0.50 \times 10^{-3}$ s, we obtain the nodal accelerations as

$$\underline{\ddot{d}}_2 = \frac{2}{0.073}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\left[\begin{Bmatrix} 0 \\ 1000 \end{Bmatrix} - 30 \times 10^4 \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0.221 \times 10^{-3} \\ 2.99 \times 10^{-3} \end{Bmatrix}\right]$$

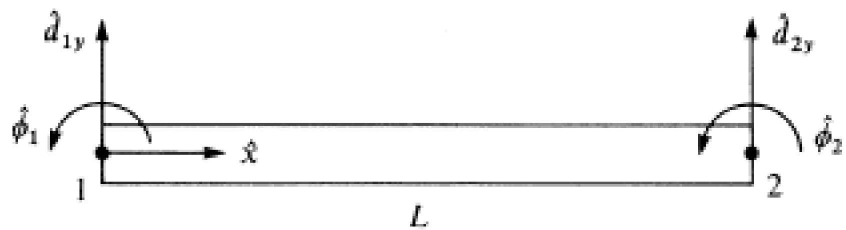
$$\begin{Bmatrix} \ddot{d}_{2x} \\ \ddot{d}_{3x} \end{Bmatrix}_2 = \begin{Bmatrix} 0 \\ 27{,}400 \end{Bmatrix} + \begin{Bmatrix} 10{,}500 \\ -22{,}800 \end{Bmatrix}$$

$$= \begin{Bmatrix} 10{,}500 \\ 4600 \end{Bmatrix} \text{in./s}^2 \qquad (\text{at } t = 0.5 \times 10^{-3} \text{ s})$$

Beam Element Dynamics

The lumped mass matrix for a beam element is given as follows for the beam element below.

$$
[\hat{m}] = \frac{\rho A L}{2} \begin{array}{c} \begin{matrix} \hat{d}_{1y} & \hat{\phi}_1 & \hat{d}_{2y} & \hat{\phi}_2 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}
$$



The mass moment of inertia of half of the beam segment about each end node using.

$$
I = \tfrac{1}{3}(\rho A L/2)(L/2)^2
$$

The consistent mass matrix is determined by using the shape functions:

$$[\hat{m}] = \iiint_V \rho [N]^T [N] \, dV$$

$$[\hat{m}] = \int_0^L \iint_A \rho \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{Bmatrix} [N_1 \quad N_2 \quad N_3 \quad N_4] \, dA \, d\hat{x}$$

$$N_1 = \frac{1}{L^3}(2\hat{x}^3 - 3\hat{x}^2 L + L^3)$$

$$N_2 = \frac{1}{L^3}(\hat{x}^3 L - 2\hat{x}^2 L^2 + \hat{x} L^3)$$

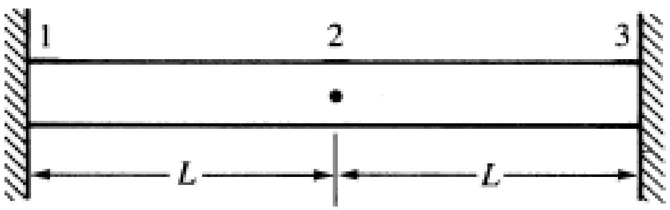
$$N_3 = \frac{1}{L^3}(-2\hat{x}^3 + 3\hat{x}^2 L)$$

$$N_4 = \frac{1}{L^3}(\hat{x}^3 L - \hat{x}^2 L^2)$$

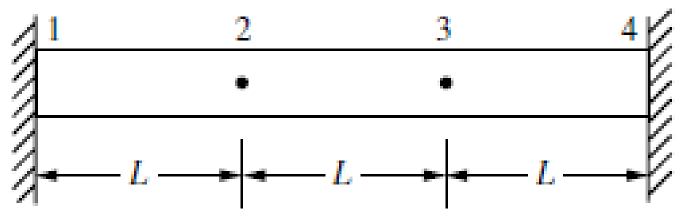
The mass matrix becomes.

$$[\hat{m}] = \frac{\rho A L}{420} \begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}$$

Consider the two beams below each has mass density ρ, modulus of elasticity E, cross-sectional area A, area moment of inertia I, and length 2L. he beam is discretized into (a) two beam elements of length L.



(a)



(b)

(a) Two-Element Solution

Using boundary conditions $d_{1y} = 0$, $\phi_1 = 0$, $d_{3y} = 0$, and $\phi_2 = 0$ to reduce the matrices) as:

$$\underline{K} = \frac{EI}{L^3} \begin{array}{c} \begin{matrix} d_{2y} & \phi_2 \end{matrix} \\ \begin{bmatrix} 24 & 0 \\ 0 & 8L^2 \end{bmatrix} \end{array} \qquad \underline{M} = \frac{\rho A L}{2} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\left|\frac{EI}{L^3}\begin{bmatrix} 24 & 0 \\ 0 & 8L^2 \end{bmatrix} - \omega^2 \rho A L \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right| = 0$$

Dividing above by ρAL we get.

$$\omega^2 = \frac{24EI}{\rho AL^4}$$

or

$$\omega = \frac{4.90}{L^2}\left(\frac{EI}{A\rho}\right)^{1/2}$$

The exact solution is

$$\omega = \frac{5.59}{L^2}\left(\frac{EI}{A\rho}\right)^{1/2}$$

(b) Three-Element Solution:

$$[\hat{m}^{(1)}] = \frac{\rho A L}{2} \begin{array}{c} \begin{array}{cccc} d_{1x} & \varphi_1 & d_{2x} & \varphi_2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \quad [\hat{m}^{(2)}] = \frac{\rho A L}{2} \begin{array}{c} \begin{array}{cccc} d_{2x} & \varphi_2 & d_{3x} & \varphi_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$[\hat{m}^{(3)}] = \frac{\rho A L}{2} \begin{array}{c} \begin{array}{cccc} d_{3x} & \varphi_3 & d_{4x} & \varphi_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Knowing that $d_{1y} = \varphi_1 = d_{4y} = \varphi_4$, we obtain the global mass matrix as

$$\underline{M} = \rho A L \begin{array}{c} \begin{array}{cccc} d_{2y} & \varphi_2 & d_{3y} & \varphi_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\underline{k}^{(1)} = \frac{EI}{L^3} \begin{array}{c} \begin{matrix} d_{1y} & \varphi_1 & d_{2y} & \varphi_2 \end{matrix} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array} \quad \underline{k}^{(2)} = \frac{EI}{L^3} \begin{array}{c} \begin{matrix} d_{2y} & \varphi_2 & d_{3y} & \varphi_3 \end{matrix} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array}$$

$$\underline{k}^{(3)} = \frac{EI}{L^3} \begin{array}{c} \begin{matrix} d_{3y} & \varphi_3 & d_{4y} & \varphi_4 \end{matrix} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array}$$

The global K matrix becomes.

$$\underline{K} = \frac{EI}{L^3} \begin{array}{c} \begin{matrix} d_{2y} & \varphi_2 & d_{3y} & \varphi_3 \end{matrix} \\ \begin{bmatrix} 12-12 & 6L+6L & -12 & 6L \\ 6L-6L & 4L^2+2L^2 & -6L & 2L^2 \\ -12 & -6L & 12+12 & -6L+6L \\ 6L & 2L^2 & -6L+6L & 4L^2+4L^2 \end{bmatrix} \end{array} = \frac{EI}{L^3} \begin{array}{c} \begin{matrix} d_{2y} & \varphi_2 & d_{3y} & \varphi_3 \end{matrix} \\ \begin{bmatrix} 0 & 12L & -12 & 6L \\ 0 & 6L^2 & -6L & 2L^2 \\ -12 & -6L & 24 & 0 \\ 6L & 2L^2 & 0 & 8L^2 \end{bmatrix} \end{array}$$

$$\left| \frac{EI}{L^3} \begin{bmatrix} 0 & 12L & -12 & 6L \\ 0 & 6L^2 & -6L & 2L^2 \\ -12 & -6L & 24 & 0 \\ 6L & 2L^2 & 0 & 8L^2 \end{bmatrix} - \omega^2 \rho A L \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right|$$

$$= \begin{vmatrix} \omega^2 \rho A L & 12EI/L^2 & -12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L & -6EI/L^2 & 2EI/L \\ -12EI/L^3 & -6EI/L^2 & 24EI/L^3 - \omega^2 \rho A L & 0 \\ 6EI/L^2 & 2EI/L & 0 & 8EI/L \end{vmatrix} = 0$$

$$\begin{vmatrix} -\omega^2 \beta & 12EI/L^2 & -12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L & -6EI/L^2 & 2EI/L \\ -12EI/L^3 & -6EI/L^2 & 24EI/L^3 - \omega^2 \beta & 0 \\ 6EI/L^2 & 2EI/L & 0 & 8EI/L \end{vmatrix} = 0$$

where $\beta = \rho A L$

Evaluating the determinant.

$$\frac{-1152\omega^2E^3I^3\beta}{L^5}+\frac{48\omega^4E^2I^2\beta^2}{L^2}+\frac{576E^4I^4}{L^8}-\frac{1296E^4I^4}{L^8}+\frac{96\omega^2E^3I^3\beta}{L^5}-\frac{4\omega^4\beta^2E^2I^2}{L^2}-\frac{6912E^4I^4}{L^8}=0$$

$$\frac{44\omega^4\beta^2E^2I^2}{L^2}-\frac{1056\omega^2\beta E^3I^3}{L^5}-\frac{7632E^4I^4}{L^8}=0$$

$$11\omega^4\beta^2-\frac{264\omega^2\beta EI}{L^3}-\frac{1908E^2I^2}{L^6}=0$$

$$11\omega^4\beta^2 - \frac{264\,\omega^2\beta EI}{L^3} - \frac{1908E^2\cdot I^2}{L^6} =0$$

$$\frac{11\omega^4\beta^2}{11} - \frac{264\,\omega^2\beta EI}{L^3\cdot 11} - \frac{1908E^2\cdot I^2}{L^6\cdot 11} =0$$

$$\omega^4\beta^2 - \frac{24\,\omega^2\beta EI}{L^3\cdot 11} - \frac{173.454E^2\cdot I^2}{L^6} =0$$

$$f(A) := (A)^2 - 24A - 173.454$$

$$C := f(A) \text{ coeffs}, A \rightarrow \begin{pmatrix} -173.454 \\ -24 \\ 1 \end{pmatrix}$$

$$r := \text{polyroots}(C)$$

$$r = \begin{pmatrix} -5.817 \\ 29.817 \end{pmatrix}$$

$$\omega_1{}^2\beta = \frac{-5.817254EI}{L^3} \quad \omega_1{}^2\beta = \frac{29.817254EI}{L^3}$$

Ignoring the negative root as it is not physically possible and solving explicitly for ω_1, we have

$$\omega_1{}^2 = \frac{29.817254EI}{\beta L^3}$$

or

$$\omega_1 = \sqrt{\frac{29.817254EI}{\beta L^3}} = \frac{5.46}{L^2}\sqrt{\frac{EI}{A\rho}}$$

Two Beam Elements: $\omega = \dfrac{4.90}{L^2}\sqrt{\dfrac{EI}{A\rho}}$

Three Beam Elements: $\omega = \dfrac{5.46}{L^2}\sqrt{\dfrac{EI}{A\rho}}$

Exact solution: $\omega = \dfrac{5.59}{L^2}\left(\dfrac{EI}{A\rho}\right)^{1/2}$