6

Conservative Systems

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§6.1. Introduction

It was noted in previous Chapters that if a structure *and* the forces acting upon it collectively form a *conservative* mechanical system, the residual force vector $\mathbf{r}$ may be expressed as the gradient of the total potential energy Π with respect to the state vector:

$$\boxed{\mathbf{r} = \frac{\partial \Pi}{\partial \mathbf{u}}.} \tag{6.1}$$

Furthermore, the decompositions $\Pi = U - P$ and $\mathbf{r} = \mathbf{p} - \mathbf{f}$ are related in the sense that

$$\mathbf{p} = \frac{\partial U}{\partial \mathbf{u}}, \qquad \mathbf{f} = \frac{\partial P}{\partial \mathbf{u}}. \tag{6.2}$$

where $\mathbf{p}$ and $\mathbf{f}$ are the internal and external forces, respectively, U is the internal energy — which reduces to the strain energy in the problems considered in this course — and P is the potential of the applied loads, the negative of which is called the external work function W.

The force equilibrium equations $\mathbf{r} = \mathbf{0}$ or $\mathbf{f} = \mathbf{p}$ express the fact that the *total potential energy is stationary with respect to variations of the state vector when the structure is in static equilibrium*. Mathematically:

$$\delta\Pi = \mathbf{r}^T \delta\mathbf{u} = \left(\frac{\partial \Pi}{\partial \mathbf{u}}\right)^T \delta\mathbf{u} = 0. \tag{6.3}$$

where $\delta\mathbf{u}$ denotes a virtual displacement, δ being the variation symbol. Since $\delta\mathbf{u}$ is arbitrary, (6.3) implies that $\mathbf{r} = \mathbf{0}$.

If the structural system is conservative there are substantial advantages in taking advantage of that property:

(1) If discrete force equilibrium equations are worked out by hand (either for complete structures or finite elements) derivation from a potential is usually simpler than direct use of equilibrium, because differentiation is a straightforward and less error prone operation, especially as regards signs. Exercise 6.3 gives an example of this.

(2) The transformation of residual equations to different coordinate systems is simplified because of the invariance properties of energy functions.

(3) The conventional finite element discretization method relies on the availability of an *internal* energy functional.

(4) The tangent stiffness matrix is symmetric. Consequently equation solvers (and eigensolvers) can take advantage of this property.

(5) Loss of stability can be assessed by the singular stiffness criterion, which is static in nature. If the system is nonconservative, loss of stability may have to be tested by a dynamic criterion, which is always more difficult and computationally expensive.

This Chapter introduces the concepts of internal and external potential for systems with finite degrees of freedom. The presentation is not general in nature but relies on a few simple examples complemented with exercises. The material is intended to serve as a "bridge" to the formulation of geometrically nonlinear finite elements, which starts in the next Chapter.

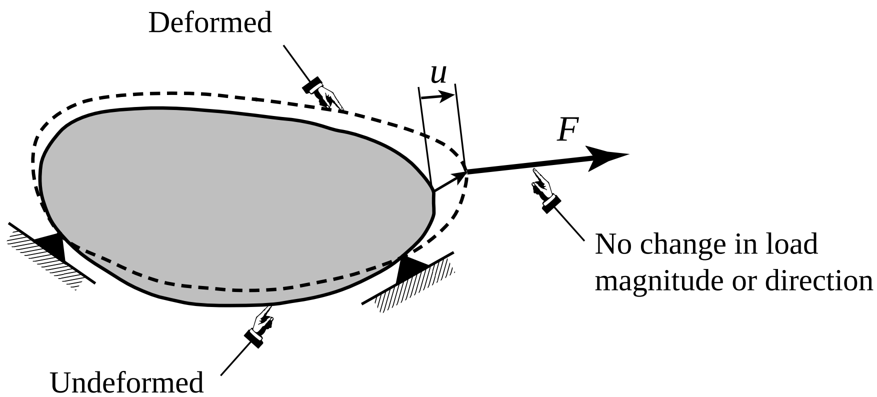

Figure 6.1. Structure under concentrated dead load F

§6.2. The Load Potential

The concept of *load potential* is the easiest to understand. This function, called P, is the potential of the work done by the applied or prescribed forces working on the displacements of the points on which those forces act. The negative of this potential $W = -P$ is called the work function, but this function will not be used in the present course.

§6.2.1. Concentrated Dead Loads

For a concrete example, consider a structure loaded by a single concentrated force F that *does not change in magnitude or direction* as the structure displaces; see Figure 6.1). A force with these properties is called a *dead load*.

If u is the deflection of the point of application of F *in the direction of the force*, then the work performed is obviously Fu. Consequently,

$$P = Fu. \tag{6.4}$$

If the structure is subjected to n loads F_k ($k = 1, \ldots n$) and the corresponding deflections in the direction of the forces are called u_k, then

$$P = \sum_{i=1}^{n} F_k u_k. \tag{6.5}$$

In general these forces will be defined by their three components along the axes x, y, z and are more properly represented by vectors $\mathbf{f}_k$. For example, if at location $k = 3$ we have a force F_3 acting in the y-direction,

$$\mathbf{f}_3 = \begin{bmatrix} 0 \\ F_3 \\ 0 \end{bmatrix}. \tag{6.6}$$

Likewise, the displacement of points of application of $\mathbf{f}_k$ is denoted by vector $\mathbf{u}_k$. The vector generalization of (6.5) is then the sum of n inner products:

$$P = \sum_{k=1}^{n} \mathbf{f}_k^T \mathbf{u}_k. \tag{6.7}$$

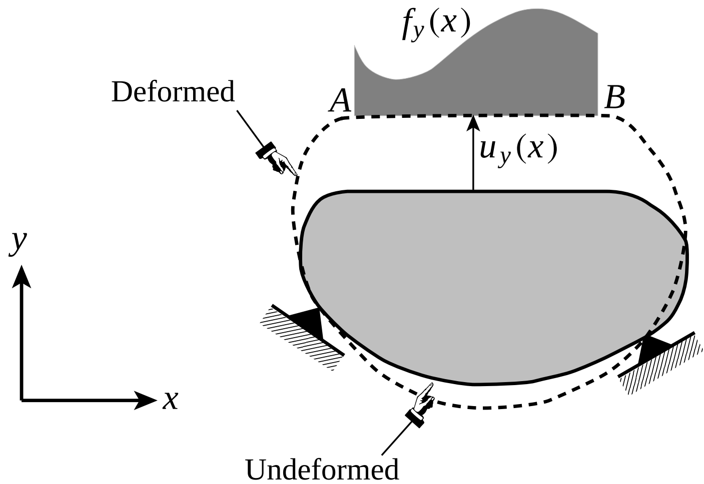

Figure 6.2. Structure under under line load $f_y(x)$ (directed upward) over segment AB.

Finally, if all applied force components are collected in the external force vector **f** (augmented with zero entries as necessary to be in one-to-one correspondence with the state vector **u**) then we have the compact inner-product expression

$$P = \mathbf{f}^T \mathbf{u}. \tag{6.8}$$

§6.2.2. Distributed Dead Loads

For distributed forces invariant in magnitude and direction, a spatial integration process is necessary to obtain P. These forces may include line loads, surface loads or volume loads (body forces).

For example, consider the structure of Figure 6.2, on which a *dead line load* $f_y(x)$ acts in the y direction along segment AB of the x axis. Then

$$P = \int_{x_A}^{x_B} f_y(x)\, u_y(x)\, dx, \tag{6.9}$$

where $u_y(x)$ is the y-displacement component of points on segment (A,B). A similar technique can be used for volume (body) forces as illustrated in Exercise 6.1.

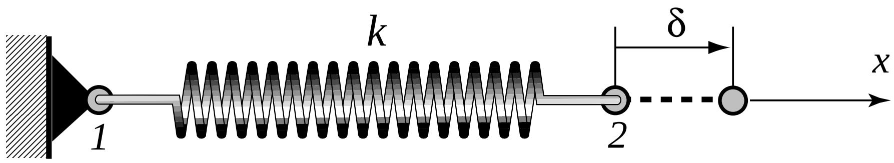

Figure 6.3. Linear spring of stiffness k deforming along its axis.

Remark 6.1. Substantial mathematical complications arise if some forces are functions of the displacements. For example, in slender structures under aerodynamic pressure loads the change of direction of the forces as the structure deflects may have to be considered in the stability analysis. These so-called "follower" forces, which introduce force B.C. nonlinearities, are considered later in the course. Suffices to say here that no loads potential P generally exist in such cases and the system is nonconservative.

§6.3. The Internal Energy: A Linear Spring

The internal energy, called U, is the recoverable mechanical work "stored" in the material of the structure by virtue of its elastic deformation. When this work is expressed in terms of strains and stresses, as in following Chapters, it is called the *strain energy*. Note that only flexible bodies can store strain energy; a rigid body cannot.

We shall illustrate the internal energy concept here by considering the simplest of all structural elements already encountered in linear finite element analysis: a *linear spring* of stiffness k, illustrated in Figure 6.3. Generalization to more complicated structures and structural components will be made in subsequent Chapters.

If the spring is undeformed, its internal energy U can be conventionally taken as zero (because an energy function can be adjusted by an arbitrary constant without changing its gradients). Now let the spring deform slowly (to avoid inertial effects) such that its two ends separate by a distance δ called the *elongation*. The internal spring force $\bar{f}$ for an intermediate elongation $0 \le \bar{\delta} \le \delta$ is $\bar{f} = k\bar{\delta}$. An elementary result of mechanics is that the strain energy taken up by the spring in its deformed state is

$$U = \int_0^\delta \text{spring-force} \times \, d(\text{elongation}) = \int_0^\delta (k\bar{\delta})\, d\bar{\delta} = \tfrac{1}{2}k\delta^2. \tag{6.10}$$

Suppose that the spring is fixed at end 1 and that end 2 can move only along the x axis, as in Figure 6.3. Call u the x displacement of end 2. Then $\delta = u - 0 = u$ and the strain energy is $U = \frac{1}{2}ku^2$. According to (6.2) the internal force, which in this case is just the spring axial force p, is the derivative of U with respect to u:

$$p = \frac{\partial U}{\partial u} = ku. \tag{6.11}$$

This is *linear* in the displacement u so nothing has changed so far with respect to linear finite element analysis.

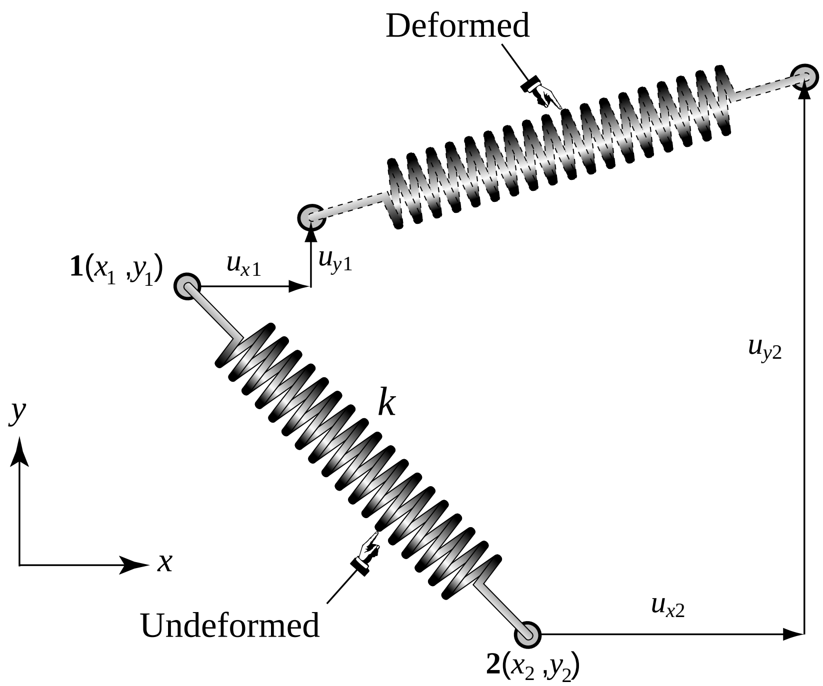

Figure 6.4. Linear spring of stiffness k displacing on the x, y plane.

§6.4. The Internal Energy: How Geometric Nonlinearities Arise

Now suppose that the spring can *move arbitrarily* on the plane x, y, as depicted in Figure 6.4. The position of the deformed spring is completely defined by the four displacement components u_{x1}, u_{y1}, u_{x2} and u_{y2}, which we collect in the state vector

$$\mathbf{u} = \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \end{bmatrix}. \tag{6.12}$$

Let ℓ and ℓ_d denote the spring lengths in the undeformed and deformed configurations, respectively. The elongation δ is given by

$$\delta = \ell_d - \ell = \sqrt{(\ell_x + \Delta_x)^2 + (\ell_y + \Delta_y)^2} - \sqrt{\ell_x^2 + \ell_y^2}, \tag{6.13}$$

where $\Delta_x = u_{x2} - u_{x1}$, $\Delta_y = u_{y2} - u_{y1}$, $\ell_x = x_2 - x_1$, $\ell_y = y_2 - y_1$, in which x_1, y_1, x_2 and y_2 denote the x, y coordinates of the end nodes of the undeformed spring. Consequently

$$\begin{aligned} U &= \tfrac{1}{2}k\delta^2 = \tfrac{1}{2}k(\ell^2 + \ell_d^2 - 2\ell\ell_d) \\ &= \tfrac{1}{2}k\left(2\ell^2 + 2\ell_x\Delta_x + \Delta_x^2 + 2\ell_y\Delta_y + \Delta_y^2 - 2\ell\sqrt{(\ell_x + \Delta_x)^2 + (\ell_y + \Delta_y)^2}\right). \end{aligned} \tag{6.14}$$

The components of the internal forces are

$$\mathbf{p} = \frac{\partial U}{\partial \mathbf{u}} = \begin{bmatrix} \dfrac{\partial U}{\partial u_{x1}} \\ \dfrac{\partial U}{\partial u_{y1}} \\ \dfrac{\partial U}{\partial u_{x2}} \\ \dfrac{\partial U}{\partial u_{y2}} \end{bmatrix}. \tag{6.15}$$

The actual expressions of the components in (6.15), which are nonlinear functions of the displacements, are worked out in Exercise 6.2.

The important points that emerge from this example are:

1. The internal forces are *nonlinear* functions of the displacements, although the spring itself remains constitutively linear. This nonlinearity comes in as a result of geometric effects, and is thus properly called *geometric nonlinearity*.

2. The effect of geometric nonlinearities can be traced to the *change in direction* of the spring. Because if the spring stretches along its original axis the internal force remains linear in the displacements. This change of direction is measured by *rotations*.

Even for this simple case the exact nonlinear equations are quite nasty, involving irrational functions of the displacements. The second property, however, shows that approximations to the exact nonlinear equations may be made when the change in direction is "small" in some sense. This feature is illustrated in Exercise 6.3.

§6.5. Internal Energy: Additivity Property

If the structure consists of m linear springs, each of which absorbs an internal energy U_k, the total internal energy is the sum of the individual spring energies:

$$U = U_1 + U_2 + \ldots + U_m. \tag{6.16}$$

This additivity property is of course general because energies are *scalar* quantities. It applies to arbitrary structures decomposed into structural components such as finite elements. Furthermore, (6.16) is not affected by whether the structure is linear or nonlinear.

The last property explains why finite element equations should be derived from energy functions if such functions exist. That is not, however, always possible.

Homework Exercises for Chapter 6

Conservative Systems

Note: the use of a symbolic algebra package, such as `Mathematica` *or MathCad, is recommended for Exercises 6.3 and 6.4 to avoid tedious algebra and generate plots quickly. (There could be a gain from hours to minutes).*

EXERCISE 6.1 [A:15] A body of volume V and density ρ is in an uniform gravity field g acting along the $-z$ axis. The body displaces to another position defined by the small-displacement field $u(x, y, z)$. Find the expression of the load potential P as an integral over the body if the change in shape of the body is negligible.

EXERCISE 6.2 [A:20] Work out the expression of the internal forces for (6.15). Then extend this relation to the three-dimensional case in which the ends of the spring move by u_{x1}, u_{y1}, u_{z1}, u_{x2}, u_{y2}, u_{z2} in the x, y, z space.

EXERCISE 6.3 [A+N/C:30] Consider the shallow arch model shown in Figure E6.1. This consists of two identical linear springs of axial stiffness k pinned to each other and to unmoving pinned supports as shown. The springs are assumed able to resist both tensile and compressive forces. The distance between the supports is $2L$. The undeformed springs form an angle α with the horizontal axis.

The central pin in loaded by a dead vertical force of magnitude f, positive downwards, which is parametrized as $f = \lambda k L$. Only *symmetrical* deformations of the arch are to be considered for this Exercise. Consequently the system has just *one* degree of freedom which we take to be the displacement u under the load, also positive downwards. The response of this system exhibits the snap-through behavior sketched in Figure E6.2.

(a) Show that the internal energy U and load potential P of the two-spring system are given by

$$U = kL^2\left(\frac{1}{\cos\alpha} - \frac{1}{\cos\theta}\right)^2, \qquad P = f u, \tag{E6.1}$$

where θ is the angle shown in Figure E6.1, which is linked to u by the relation $\tan\theta + u/L = \tan\alpha$.

(b) Derive the exact equilibrium equation

$$r(u, \lambda) = \frac{\partial \Pi}{\partial u} = 0, \tag{E6.2}$$

in which $\Pi = U - P$ is the total potential energy, and $\lambda = f/(kL)$ is the dimensionless state parameter. For convenience rewrite this as

$$r(\mu, \lambda) = 0, \tag{E6.3}$$

in terms of the dimensionless state parameter

$$\mu = \frac{u}{L\tan\alpha}. \tag{E6.4}$$

(c) Derive the exact equation for the limit load parameters

$$\left.\frac{\partial\lambda(\mu)}{\partial\mu}\right|_{\mu=\mu_L,\lambda=\lambda_L} = 0. \tag{E6.5}$$

(Hint: the exact equation in terms of the angular coordinate θ is $\cos^3\theta_L = \cos\alpha$). Solve this trigonometric equation[1] for the limit-load parameters λ_{L1} and λ_{L2} and the dimensionless displacements μ_{L1} and μ_{L2} at those points assuming that $\alpha = 30^\circ$.

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[1] Equation (E6.5) is equivalent to $\det K = 0$ because for a one-DOF system $\det K = K = \partial\lambda/\partial\mu$.

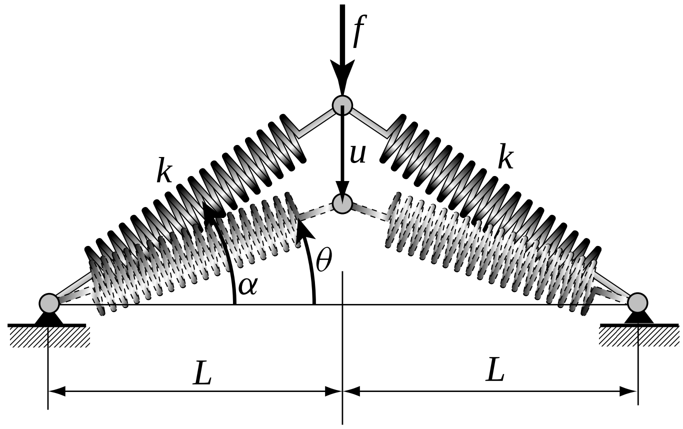

Figure E6.1. Two-spring model of shallow arch.

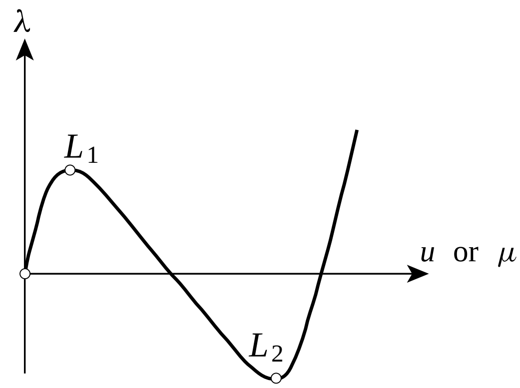

Figure E6.2. Snap-through response of shallow arch (sketch).

(d) If the arch initially is and remains sufficiently "shallow" throughout its snap-through behavior, we may make the small-angle approximations,

$$\cos\alpha \approx 1 - \tfrac{1}{2}\alpha^2, \quad \cos\theta \approx 1 - \tfrac{1}{2}\theta^2, \quad \sin\alpha \approx \tan\alpha \approx \alpha, \quad \sin\theta \approx \tan\theta \approx \theta. \tag{E6.6}$$

Recast the energy, equilibrium equations, and limit load equations in terms of these approximations, obtaining U as a quartic polynomial in θ, r as a cubic polynomial in θ, etc, then replace in terms of μ. As a check, the residual equation in terms of λ and μ should be given by (4.16). Calculate the limit load parameters λ_{L1} and λ_{L2}, and the dimensionless displacements μ_{L1} and μ_{L2} at those loads. Verify that these displacements correspond to the angles $\theta_L = \pm\alpha/\sqrt{3}$.

(e) Draw the control-state response curves $r(\mu, \lambda) = 0$, derived using the exact nonlinear equations and those from the small-angle approximations on the λ, μ plane (as in the sketch of Figure E6.2, going up to $\mu \approx 2.5$) for $\alpha = 30^\circ$.

EXERCISE 6.4 [A+N:15] Derive the current stiffness parameter κ defined in Equation (5.8) for the approximate (small-angle) model of the two-spring arch of Exercise 6.3. Plot the variation of $\kappa(\mu)$ as μ varies from 0 to μ_{L1} at the first limit point, with μ along the horizontal axis. Does κ vanish at the limit point?