10

The Core-Congruential Formulation: Core Equations

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In Chapter 8 it was noted that two methods for developing geometrically nonlinear elements based on the Total Lagrangian description exist: the Standard Formulation (SF) and the Core-Congruential Formulation or CCF. Chapters 10 and 11 cover the second method. The present exposition is taken largely from a recent survey article [1].

§10.1. Introduction

There is an elegant Total Lagrangian (TL) formulation of geometrically nonlinear mechanical finite elements that has received little attention in the literature. This will be referred to as the *Core-Congruential Formulation*, or CCF, in the sequel. The key concepts, presented by Rajasekaran and Murray [2] in 1973, evolved from the analysis and reinterpretation of the pioneer work of Mallet and Marcal [3] as well as Murray's previous work in geometrically nonlinear finite element analysis [4]. The discussion of Reference [2] by Felippa [5] provided parametric expressions for the stiffness matrices that appear at various levels of the discrete governing equations. This work originated what is called here the Direct Core Congruential Formulation, or DCCF.

In 1987 this course presented the derivation of several elements using the DCCF. Preparation of homework assignments and feedback from students in this and follow-up offerings helped to streamline the material. Subsequently Crivelli's doctoral thesis [6] used the CCF in the systematic development of a three-dimensional nonlinear Timoshenko beam element capable of undergoing arbitrarily large rotations. Challenges posed by this application pushed this formulation beyond frontiers hitherto deemed impassable by a TL element with rotational degrees of freedom. This development was summarily reported in a survey article by Felippa and Crivelli [7] and explained in more detail in a subsequent paper by Crivelli and Felippa [8].

A lesson gained from this research is that, when dealing with 3D finite rotations, the CCF should be applied in a staged fashion that allows the systematic examination of additional terms arising in the transformations to physical degrees of freedom. That transformation methodology gave rise to what is here called the Generalized CCF, or GCCF.

Both DCCF and GCCF share the same "divide and conquer" philosophy. However, the core equations as well as subsequent steps that transform those equations to physical freedoms vary in complexity. To simplify the exposition while focusing on the essential aspects, Sections 10.3 through 10.7 focus on the DCCF. Examples of application to elements amenable to the direct treatment are presented. The GCCF is discussed in Chapter 11, and illustrated with applications to 2D and 3D beam elements.

Remark 10.1. Several authors have expressed the belief that the approximation performance of TL-based elements degrades beyond moderate rotations, and an updated Lagrangian or corotational description is necessary for handling truly large motions. For example, in 1986 Mathiasson, Bengtsson and Samuelsson [9] concluded that "The TL formulation can only be used in problems with small or moderate displacements." More recently Bergan and Mathisen [10] voice a similar opinion: "it is commonly known that in a step by step TL formulation artificial strains easily arise in beam elements due to nonhomogeneities in the displacement expansions in transverse and longitudinal directions." Our experience shows that such limitations are not inherent in the TL description but instead emerge when *a priori* kinematic approximations are made to simplify element derivations. The 3D beam element just cited exhibits computational and approximation performance for very large rotations comparable to those based on the co-rotational and Updated Lagrangian descriptions while retaining certain advantages listed in the Conclusions.

§10.2. Overview

§10.2.1. Basic Concepts

The original development of the CCF was concerned with the construction of TL stiffness matrices for geometrically nonlinear analysis through the congruential-transformation pattern

$$\mathbf{K}^{level} = \int_{V_0} \mathbf{G}^T \mathbf{S}^{level} \mathbf{G}\, dV, \tag{10.1}$$

where $\mathbf{S}$ is the *core* stiffness matrix, $\mathbf{K}$ the physical stiffness in terms of the nodal degrees of freedom $\mathbf{v}$, $\mathbf{G}$ a core-to-physical-freedom transformation matrix assumed to be independent of $\mathbf{v}$, V_0 the appropriate reference integration volume, and in which "level" identifies the *governing equation level at which the stiffness matrix is used*.

The three variational levels of interest in practice are: energy (level 0), force equilibrium (level 1), and first-order incremental equilibrium (level 2). Qualifiers "residual-force" and "secant-stiffness" are also used for level 1, and "tangent-stiffness" used for level 2.

The core stiffness matrix is expressed in terms of the *displacement gradients* at each material point. Displacement gradients $\mathbf{g}$ make a better choice of core variables than finite strains because for elements with translational degrees of freedom (DOFs) they can be expressed linearly in terms of node displacements $\mathbf{v}$ as $\mathbf{g} = \mathbf{G}\mathbf{v}$, a property that validates (10.1) for all levels. As discussed below, such elements fall under the purview of the Direct CCF.

The qualifier "core" emphasizes the goal of *independence of* $\mathbf{S}^{level}$ *with respect to discretization decisions* such as element geometry, shape functions, and choice of nodal degrees of freedom. Such a dependence is introduced by the congruential transformation indicated in (10.1) and the integration over the element volume.

§10.2.2. Direct and Generalized CCF

The basic schematics of the CCF, mathematically expressed through (10.1), may be diagrammed as

Core Stiffness Equations	$\Longrightarrow$	**Congruential Transformation Equations**	$\Longrightarrow$	**Physical-DOF Stiffness Equations**

But this panoramic view needs to be rendered more precise. If the relation between core DOFs (the displacement gradients $\mathbf{g}$) and the physical DOFs (the node displacements $\mathbf{v}$ of a finite element model) is *linear*, these transformations do not depend on level:

(0) **Core Energy Stiffness** *(1)* **Core Secant Stiffness** *(1)* **Core Internal Force** *(2)* **Core Tangent Stiffness**	$\Longrightarrow$	**Congruential Transformation Equations**	$\Longrightarrow$	**Physical Energy Stiffness** **Physical Secant Stiffness** **Physical Internal Force** **Physical Tangent Stiffness**

In this diagram, numbers annotated within the "core box" denote the variational level of the governing equation in use. Internal force and secant stiffness are two alternative governing-equation expressions at level 1. The energy level (level 0) may also be expressed in several ways, but this is not shown in the diagrams to reduce clutter. Under the aforementioned assumption we obtain the Direct Core Congruential Formulation, or DCCF.

If the relation between displacement gradients **g** and node displacements **v** is *nonlinear*, the transformations sketched above are not only more complex but depend on variational level and possibly the expression form used within a level. This complication arises when elements with rotational degrees of freedom such as beams, plates and shells are considered. It gives rise to the Generalized Core Congruential Formulation, or GCCF.

Two variants of the GCCF may be distinguished. If the relation between **g** and **v** is nonlinear but *algebraic*, the transformation equations do vary with level but in principle are still possible as illustrated in the following diagram.

(0) **Core Energy Stiffness** *(1)* **Core Secant Stiffness**	$\Longrightarrow$	**S-Congruential Transformation Equations**	$\Longrightarrow$	**Physical Energy Stiffness** **Physical Secant Stiffness**
(1) **Core Internal Force** *(2)* **Core Tangent Stiffness**	$\Longrightarrow$	**T-Congruential Transformation Equations**	$\Longrightarrow$	**Physical Internal Force** **Physical Tangent Stiffness**

Here "T-Congruential" and "S-Congruential" are abbreviations for "Tangent Congruential" and "Secant-Congruential," respectively. Such a distinction is elaborated upon in Chapter 11.

If the relation between **g** and **v** is nonlinear and can be expressed only in *non-integrable differential* form, the "Secant Transformation Equations" of the preceding diagram do not generally exist, and the diagram must be truncated:

(0) **Core Energy Stiffness** *(1)* **Core Secant Stiffness**				
(1) **Core Internal Force** *(2)* **Core Tangent Stiffness**	$\Longrightarrow$	**T-Congruential Transformation Equations**	$\Longrightarrow$	**Physical Internal Force** **Physical Tangent Stiffness**

These two variants of the GCCF are called Algebraic GCCF and Differential GCCF and denoted by acronyms AGCCF and DGCCF, respectively, in the sequel. The main distinction between AGCCF and DGCCF is that it makes no sense to talk about missing quantities, such as the physical secant stiffness, with the latter.

The original development of the CCF outlined in the Introduction focused on elements with translational-degree-of-freedom configurations. For such elements the Direct form of the CCF, or DCCF, is sufficient. Sections §10.3 through §10.7 focus on that form, leaving the development and application of the GCCF to the next Chapter.

§10.2.3. The CCF Philosophy: Divide and Conquer

The CCF derivation of the finite element equations naturally reflects the outlined framework. It proceeds through two phases: a core phase followed by a transformation phase. In the initial phase *core* energy, secant and tangent stiffness matrices as well as internal force vectors are obtained. These matrices and vectors pertain to individual *particles*. For the stiffness matrices they are collectively represented by the term $\mathbf{S}^{level}$ in (10.1).

The key goal is to try to make such core equations as independent as possible with respect to finite-element discretization decisions such as element geometry, shape functions, selection of nodal degrees of freedom and (in the case of rotational DOFs) rotational parametrizations. To emphasize this independence, the term *core* was coined. Complete independence is in fact achievable if the relation between displacement gradients $\mathbf{g}$ and $\mathbf{v}$ is linear, which characterizes the DCCF. The goal has to be tempered if the relation is nonlinear because dependencies may arise at the tangent stiffness level. Such dependencies create the so-called *complementary geometric stiffness* terms, which are characteristic of elements that fall under purview of the GCCF.

In the transformation phase, these core forms are transformed to physical DOFs, *i.e.* element node displacements. The transformation may be done directly for simple elements and in multistage fashion for complex ones. In particular, multistage transformations are recommended for elements that require the Differential GCCF such as 3D beam and shell elements. In this case the transformation phase is decomposed into transformation stages that progressively "bind" particles into lines, areas or volumes through kinematic constraints, and eventually link the element domain to the nodal degrees of freedom. Decisions such as the choice of specific parametrizations for finite rotations may be deferred to final stages.

What are the differences between the CCF and the more conventional Total Lagrangian formulation of nonlinear finite elements? If kinematic exactness is maintained throughout, the final discrete equations are identical. This is shown in Appendix 1 for the DCCF applied to continuum elements. But in geometrically nonlinear analysis approximations of various kinds are common, especially in structural elements with rotational degrees of freedom such as beams, plates and shells. In the conventional formulation it is quite difficult to assess *a priori* the effect of seemingly innocuous approximations "thrown into the pot," and *a posteriori* exhaustive testing of complex situations becomes virtually impossible. Sample: how does the neglect of higher order terms in the axial deformation of a spinning 3D beam affects torsional buckling?

The staged approach recommended for the GCCF permits a better control over such assumptions. The core equations are physically transparent, clearly displaying the effect of material behavior, displacement gradients and prestresses. In the ensuing transformation sequence the origin of each term can be accurately traced, and on that basis informed decisions on retention or dropping made. This process can be aided by computer by testing subproblems that isolate the physics modeled by specific terms.

From this discussion it follows that, from the standpoint of element development, evaluation and testing, the most significant advantage that can be claimed for the CCF is the clean separation of physical effects. The importance of this factor should not be underestimated, because physical transparency is the key to success in nonlinear analysis.

§10.3. Historical Background

In 1968 Mallet and Marcal [3] attempted to establish a standard nomenclature for geometrically nonlinear finite element structural analysis based on the Total Lagrangian (TL) kinematic description. Consider a discrete, finite element model of a static structural system under dead loading with nodal displacement degrees of freedom collected in array $\mathbf{v}$. Displacements are measured from a fixed reference configuration $\mathcal{C}_0$ to a current configuration $\mathcal{C}$. The virtual-work conjugate forces, independent of $\mathbf{v}$, are collected in array $\mathbf{p}$. The system has a total potential energy function $\Pi = U - P$ that is the difference between the strain energy U and the loads potential $P = \mathbf{p}^T\mathbf{v}$. The residual node forces are $\mathbf{r} = \partial\Pi/\partial\mathbf{v}$, and the symbol Δ denotes increment associated with the variation of the current configuration. (In keeping up with the spirit of Reference [3] actual variations are used below rather than virtual ones; the latter are identified by the usual δ prefix.)

Mallet and Marcal expressed the total potential energy, the residual (force-balance) equilibrium equations, and the incremental equilibrium equations as follows:

$$\Pi = U - P = \tfrac{1}{2}\mathbf{v}^T\left[\mathbf{K}_0 + \tfrac{1}{3}\mathbf{N}_1 + \tfrac{1}{6}\mathbf{N}_2\right]\mathbf{v} - \mathbf{p}^T\mathbf{v}, \tag{10.2}$$

$$\mathbf{r} = \frac{\partial\Pi}{\partial\mathbf{v}} = \left[\mathbf{K}_0 + \tfrac{1}{2}\mathbf{N}_1 + \tfrac{1}{3}\mathbf{N}_2\right]\mathbf{v} - \mathbf{p} = \mathbf{0}, \tag{10.3}$$

$$\Delta\mathbf{r} = \left[\mathbf{K}_0 + \mathbf{N}_1 + \mathbf{N}_2\right]\,\Delta\mathbf{v} - \Delta\mathbf{p} = \mathbf{0}. \tag{10.4}$$

Here $\mathbf{K}_0$ is the *linear stiffness matrix* evaluated at the reference configuration, whereas $\mathbf{N}_1$ and $\mathbf{N}_2$ are *nonlinear stiffness matrices*, also evaluated at the reference configuration, that depend linearly and quadratically, respectively, on the node displacements $\mathbf{v}$. The $\mathbf{N}$ matrices were said "to repeat" in the foregoing expressions. (This old notation has not survived; presently symbol $\mathbf{N}$ is most commonly used to identify matrices of element shape functions.)

Five years later Rajasekaran and Murray [2] examined more critically the structure of the matrices that appear in the above equations. In that investigation they chose to start from the "core" stiffness matrices corresponding to $\mathbf{K}$, $\mathbf{N}_1$ and $\mathbf{N}_2$ expressed in terms of displacement gradients, and in doing so laid down the main idea of the CCF. Working with specific elements they showed that the nonlinear stiffness matrices $\mathbf{N}_1$ and $\mathbf{N}_2$ are *not uniquely determined*. Indeed (10.2)-(10.4) as written are unique only for a single degree of freedom. They did not present, however, a general expression valid for arbitrary elements. This was partly done by Felippa [5], who in the discussion of Reference [2] considered again those equations, rewritten here in a more general and compact form:

$$\Pi = \tfrac{1}{2}\mathbf{v}^T\mathbf{K}^U\mathbf{v} + \left(\mathbf{p}^0 - \mathbf{p}\right)^T\mathbf{v}, \tag{10.5}$$

$$\mathbf{r} = \mathbf{K}^r\,\mathbf{v} + \mathbf{p}^0 - \mathbf{p} = \mathbf{f} - \mathbf{p} = \mathbf{0}, \tag{10.6}$$

$$\Delta\mathbf{r} = \mathbf{K}\,\Delta\mathbf{v} - \Delta\mathbf{p} = \mathbf{0}, \tag{10.7}$$

in which the notation of this paper — rather than that of Reference [5] — is used. Here $\mathbf{K}^U$, $\mathbf{K}^r$ and $\mathbf{K}$ denote the *energy*, *secant* and *tangent* stiffness matrices, respectively. (Energy and secant stiffnesses are not denoted by $\mathbf{K}^e$ and $\mathbf{K}^s$ because such symbols are used for other purposes in the finite element course noted in the Introduction.) In addition, $\mathbf{p}^0$ is the *prestress force vector*, which vanishes if the reference configuration is stress free and was omitted in that discussion, [5] and $\mathbf{f} = \mathbf{K}^r\mathbf{v} + \mathbf{p}^0$ is the internal force vector. The tangent stiffness is of course fundamental in incremental-iterative solution methods and stability analysis, while the secant stiffness (by itself or in the internal-force form $\mathbf{K}^r\mathbf{v}+\mathbf{p}^0$) is important in pseudo-force methods. The energy stiffness enjoys limited application *per se* but has theoretical importance as source for the other two.

In linear problems $\mathbf{K}^U=\mathbf{K}^r=\mathbf{K}=\mathbf{K}_0$ and the three stiffness matrices coalesce. But in nonlinear problems not only do the matrices differ but, as shown in the next section, $\mathbf{K}^U$ and $\mathbf{K}^r$ may involve arbitrary scalar coefficients. Such parametrized expressions were given by Felippa [5] under the following restrictions:

(R1) $\mathbf{K}^r$ is symmetric.
(R2) The reference configuration is stress free.
(R3) The finite strain measure is quadratic in the displacement gradients.
(R4) The transformation between core and physical freedoms is linear.

The following treatment eliminates restrictions (R1) and (R2) altogether, and the other two selectively. It should be noted that restriction (R4) is the condition that, with present terminology, characterizes the DCCF.

§10.4. Core Stiffness Equations

§10.4.1. TL Description of Particle Motion

A conservative, geometrically nonlinear structure under dead loading is viewed as a continuum undergoing finite displacements $\mathbf{u}$. These displacements are measured from a fixed *reference* configuration $\mathcal{C}_0$ to a variable *current* configuration $\mathcal{C}$. *No discretization into finite elements is implied at this stage*. We confine our attention to the case in which the material behavior stays within the linear elastic range, thus implying *small deformational strains but arbitrarily large rotations*. Corresponding points or *particles* in the reference and current configuration are referred to a fixed Cartesian coordinate system and have the coordinates X_i and x_i ($i=1,\ldots n_d$), respectively, where n_d is the number of space dimensions. The displacement field components are $u_i=x_i-X_i$.

Let the state of strain at a particle in the current configuration be characterized by n_s strains e_i ($i=1,2,\ldots n_s$) collected in an array $\mathbf{e}$, and let the corresponding conjugate stresses be s_i ($i=1,2,\ldots n_s$), collected in an array $\mathbf{s}$. Using the summation convention the elastic stress-strain relations are written

$$s_i=s_i^0+E_{ij}e_j,\quad \text{with}\quad E_{ij}=E_{ji},\qquad \text{or}\quad \mathbf{s}=\mathbf{s}^0+\mathbf{E}\mathbf{e},\tag{10.8}$$

where s_i^0 are stresses in the reference configuration (stresses that remain if $e_i=0$, also called prestresses) and E_{ij} are elastic moduli arranged as a $n_s\times n_s$ square array in the usual manner.

Let $\pi,\mathcal{U},\mathcal{P},\mathbf{\Psi},\mathbf{\Phi}$ and $\mathbf{\Upsilon}$ denote the analogues of Π, U, P, $\mathbf{p}$, $\mathbf{f}$ and $\mathbf{r}$, respectively, at the particle level. (The first three acquire the meaning of energy densities, whereas $\mathbf{\Psi}$ is a dead-loading body force density independent of $\mathbf{u}$.) The strain energy density can be expressed as

$$\mathcal{U}=e_is_i^0+\tfrac{1}{2}e_iE_{ij}e_j=\mathbf{e}^T\mathbf{s}^0+\tfrac{1}{2}\mathbf{e}^T\mathbf{E}\mathbf{e}.\tag{10.9}$$

The total strain energy U is obtained by integrating (10.9) over the structure volume: $U=\int_{V_0}\mathcal{U}\,dV$; the integration taking place — as can be expected in a TL description — over the reference configuration geometry.

Next, introduce the n_g displacement gradients $g_{mn} = \partial u_m / \partial X_n$. These are subsequently identified as g_i ($i = 1, 2, \ldots n_g$) so they can be conveniently arranged in a one-dimensional array $\mathbf{g}$. Following Rajasekaran and Murray [2] and Felippa [5] assume that the strains e_i are linked to the displacement gradients through matrix relations of the form

$$\mathbf{e}_i = \mathbf{h}_i^T \mathbf{g} + \tfrac{1}{2} \mathbf{g}^T \mathbf{H}_i \, \mathbf{g}, \qquad i = 1, 2, \ldots n_s \tag{10.10}$$

where $\mathbf{h}_i$ and $\mathbf{H}_i$ are arrays of dimension $n_g \times 1$ and $n_g \times n_g$, respectively, with $\mathbf{H}_i$ symmetric. In the original References [2,5] it was assumed that $\mathbf{H}_i$ is independent of $\mathbf{g}$, which is the case for the Green-Lagrange strain measure. This restriction, labeled (R3) in §10.3, will be enforced below except in §10.4.5.

§10.4.2. Energy Variations

As noted previously, for deriving core equations we regard the displacement gradients $\mathbf{g}$ as degrees of freedom. On substituting (10.8) and (10.10) into (10.9) we obtain the "core counterparts" of (10.5)–(10.7), in which $\mathbf{v}$ has become $\mathbf{g}$:

$$\pi = \mathcal{U} - \mathcal{P} = \tfrac{1}{2} \mathbf{g}^T \mathbf{S}^U \mathbf{g} + (\boldsymbol{\Psi}^0 - \boldsymbol{\Psi})^T \mathbf{g}, \tag{10.11}$$

$$\boldsymbol{\Upsilon} = \frac{\partial \mathcal{H}}{\partial \mathbf{g}} = \mathbf{S}^r \mathbf{g} + \boldsymbol{\Psi}^0 - \boldsymbol{\Psi} = \boldsymbol{\Phi} - \boldsymbol{\Psi} = \mathbf{0}, \tag{10.12}$$

$$\Delta \boldsymbol{\Upsilon} = \mathbf{S} \, \Delta \mathbf{g} - \Delta \boldsymbol{\Psi} = \mathbf{0}. \tag{10.13}$$

Here $\mathbf{S}^U$, $\mathbf{S}^r$ and $\mathbf{S}$ denote the energy, secant and tangent core stiffness matrices, and $\boldsymbol{\Psi}^0$, which is independent of $\mathbf{g}$, is the core counterpart of $\mathbf{p}^0$.

With this notation the first and second variations of the strain energy density can be expressed as

$$\delta \mathcal{U} = \delta \mathbf{g}^T (\mathbf{S}^U \mathbf{g} + \boldsymbol{\Psi}^0) + \tfrac{1}{2} \mathbf{g}^T \delta \mathbf{S}^U \mathbf{g} = \delta \mathbf{g}^T \left(\mathbf{S}^r \mathbf{g} + \boldsymbol{\Psi}^0 \right) = \delta \mathbf{g}^T \boldsymbol{\Phi}, \tag{10.14}$$

$$\delta^2 \mathcal{U} = \delta \mathbf{g}^T \mathbf{S}^r \, \delta \mathbf{g} + \delta \mathbf{g}^T \delta \mathbf{S}^r \mathbf{g} + (\delta^2 \mathbf{g})^T \boldsymbol{\Phi} = \delta \mathbf{g}^T \mathbf{S} \, \delta \mathbf{g} + (\delta^2 \mathbf{g})^T \boldsymbol{\Phi}. \tag{10.15}$$

These variational equations implicitly determine $\mathbf{S}^r$, $\boldsymbol{\Phi}$ and $\mathbf{S}$ from $\mathbf{S}^U$ and $\boldsymbol{\Psi}^0$. If the linearity restriction (R4) holds, the term in $\delta^2 \mathbf{g}$ drops out as explained in the Remark below, and

$$\delta^2 \mathcal{U} = \delta \mathbf{g}^T \mathbf{S} \, \delta \mathbf{g}. \tag{10.16}$$

Remark 10.2. If $\mathbf{g} = \mathbf{G} \mathbf{v}$ with $\mathbf{G}$ independent of $\mathbf{v}$, $\delta^2 \, \mathbf{g} = \mathbf{G} \, \delta^2 \mathbf{v} = \mathbf{0}$ because $\mathbf{v}$ are independent variables. On the other hand, if displacement gradients are nonlinear functions of node displacements expressable as $g_i = g_i(v_j)$, then

$$\delta g_i = \frac{\partial g_i}{\partial v_j} \delta v_j = G_{ij} \, \delta v_j, \qquad \delta^2 g_i = \frac{\partial^2 g_i}{\partial v_j \partial v_k} \delta v_j \delta v_k + \frac{\partial g_i}{\partial v_j} \cancelto{0}{\delta^2} v_j = F_{ijk} \, \delta v_j \, \delta v_k. \tag{10.17}$$

Thus $\delta \mathbf{g}$ is still $\mathbf{G} \, \delta \mathbf{v}$ but $\delta^2 \mathbf{g} = (\mathbf{F} \, \delta \mathbf{v}) \, \delta \mathbf{v}$, where $\mathbf{F}$ is a cubic array. The presence of the term $\delta^2 \mathbf{g}$ is taken into account in the GCCF discussed in Chapter 11.

§10.4.3. Parametrized Forms

For convenience introduce the following $n_g \times n_g$ matrices (with summation convention on $i, j = 1, \dots n_g$ implied):

$$\begin{aligned} &\mathbf{S}_0 = E_{ij}\,\mathbf{h}_i\mathbf{h}_j, \quad \mathbf{S}_1 = E_{ij}\,\mathbf{h}_i\mathbf{g}^T\mathbf{H}_j, \quad \mathbf{S}_1^* = E_{ij}\,(\mathbf{h}_i^T\mathbf{g})\,\mathbf{H}_j, \\ &\mathbf{S}_2 = E_{ij}\,\mathbf{H}_i\,\mathbf{g}\mathbf{g}^T\mathbf{H}_j, \qquad \mathbf{S}_2^* = E_{ij}\,(\mathbf{g}^T\mathbf{H}_i\mathbf{g})\,\mathbf{H}_j, \end{aligned} \tag{10.18}$$

in which parentheses are used to emphasize the grouping of *scalar* quantities such as $\mathbf{g}^T\mathbf{H}_i\mathbf{g}$. It may be then verified that, if assumptions (R3)-(R4) of §10.3 hold, the core stiffnesses and prestress vector in (10.13)–(10.15) possess the general form:

$$\begin{aligned} \mathbf{S}^U(\alpha,\beta) &= \mathbf{S}_0 + \tfrac{1}{2}\alpha(\mathbf{S}_1+\mathbf{S}_1^T) + (1-\alpha)\mathbf{S}_1^* + \tfrac{1}{4}\beta\mathbf{S}_2 + \tfrac{1}{4}(1-\beta)\mathbf{S}_2^* + s_i^0\mathbf{H}_i \\ &= \mathbf{S}_0 + \tfrac{1}{2}\alpha(\mathbf{S}_1+\mathbf{S}_1^T) + (\tfrac{1}{2}-\alpha)\mathbf{S}_1^* + \tfrac{1}{4}\beta(\mathbf{S}_2-\mathbf{S}_2^*) + \tfrac{1}{2}(s_i^0+s_i)\mathbf{H}_i, \\ &= \mathbf{S}_0 + \tfrac{1}{2}\alpha(\mathbf{S}_1+\mathbf{S}_1^T) - \alpha\mathbf{S}_1^* + \tfrac{1}{4}\beta\mathbf{S}_2 - \tfrac{1}{4}(1+\beta)\mathbf{S}_2^* + s_i\mathbf{H}_i, \\ \mathbf{S}^r(\phi,\psi) &= \mathbf{S}_0 + \tfrac{1}{2}\mathbf{S}_1 + \phi\mathbf{S}_1^T + (1-\phi)\mathbf{S}_1^* + \tfrac{1}{4}(2-\psi)\mathbf{S}_2 + \tfrac{1}{4}\psi\mathbf{S}_2^* + s_i^0\mathbf{H}_i \\ &= \mathbf{S}_0 + \tfrac{1}{2}\mathbf{S}_1 + \phi\mathbf{S}_1^T + (\tfrac{1}{2}-\phi)\mathbf{S}_1^* + \tfrac{1}{4}(2-\psi)\mathbf{S}_2 + \tfrac{1}{4}(\psi-1)\mathbf{S}_2^* + \tfrac{1}{2}(s_i^0+s_i)\mathbf{H}_i, \\ &= \mathbf{S}_0 + \tfrac{1}{2}\mathbf{S}_1 + \phi\mathbf{S}_1^T - \phi\mathbf{S}_1^* + \tfrac{1}{4}(2-\psi)(\mathbf{S}_2-\mathbf{S}^*) + s_i\mathbf{H}_i, \\ \mathbf{S} &= \mathbf{S}_0 + \mathbf{S}_1 + \mathbf{S}_1^T + \mathbf{S}_1^* + \mathbf{S}_2 + \tfrac{1}{2}\mathbf{S}_2^* + s_i^0\mathbf{H}_i = \mathbf{S}_0 + \mathbf{S}_1 + \mathbf{S}_1^T + \mathbf{S}_2 + s_i\mathbf{H}_i, \\ \mathbf{\Psi}^0 &= s_i^0\mathbf{h}_i. \end{aligned} \tag{10.19}$$

Here α, β, ϕ and ψ are arbitrary scalar coefficients in the sense that $\mathbf{g}^T\mathbf{S}^U\mathbf{g}$ and $\mathbf{S}^r\mathbf{g}$ are independent of them. In fact,

$$\mathbf{\Phi} = \mathbf{S}^r\mathbf{g} + \mathbf{\Psi}^0 = s_i\mathbf{b}_i, \tag{10.20}$$

where $\mathbf{b}_i$ is defined in (10.25) below. The expressions (10.19) are more general than those originally given by Felippa [5] because restrictions (R1)-(R2) noted in §10.3 are no longer enforced. Note that the secant core stiffness $\mathbf{S}^r$ becomes symmetric if $\phi = 1/2$.

The "repeatable forms" (10.2)–(10.4) of Mallet and Marcal are obtained if $\alpha = \beta = \psi = 2/3$ and $\phi = 1/2$, in which case the combinations $\mathbf{S}_1 + \mathbf{S}_1^T + \mathbf{S}_1^*$ and $\mathbf{S}_2 + \frac{1}{2}\mathbf{S}_2^*$ become the core counterparts of $\mathbf{N}_1$ and $\mathbf{N}_2$, respectively. But this observation has largely historical interest. More physically relevant are the following combinations:

$$\begin{aligned} &\mathbf{S}_D = \mathbf{S}_1 + \mathbf{S}_1^T + \mathbf{S}_2, \qquad \mathbf{S}_M = \mathbf{S}_0 + \mathbf{S}_D, \\ &\mathbf{S}_G = \mathbf{S}_1^* + \tfrac{1}{2}\mathbf{S}_2^* + s_i^0\mathbf{H}_i = s_i\mathbf{H}_i. \end{aligned} \tag{10.21}$$

These are the core versions of the *initial-displacement*, *material* and *geometric* stiffness, respectively. The core tangent stiffness is $\mathbf{S} = \mathbf{S}_0 + \mathbf{S}_D + \mathbf{S}_G = \mathbf{S}_M + \mathbf{S}_G$.

If the Generalized CCF is required for downstream element development as explained in Chapter 11, $\mathbf{S}_G = s_i\mathbf{H}_i$ is called the *principal core geometric stiffness* and is denoted by $\mathbf{S}_{GP}$. In this case the combination

$$\mathbf{S} = \mathbf{S}_M + \mathbf{S}_{GP}, \tag{10.22}$$

receives the name *principal core tangent stiffness*.

Remark 10.3. Finite element practicioners may be surprised at the nonuniqueness of $\mathbf{S}^U$ and $\mathbf{S}^r$. It appears to contradict the fact that, given two square matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ and an arbitrary nonzero test vector $\mathbf{x}$, $\mathbf{A}_1\mathbf{x} = \mathbf{A}_2\mathbf{x}$ for all $\mathbf{x}$ implies $\mathbf{A}_1 = \mathbf{A}_2$. But this is not necessarily true if $\mathbf{A}_1$ and $\mathbf{A}_2$ are functions of $\mathbf{x}$. More precisely, the energy core stiffness is not unique because

$$\mathbf{g}^T(\mathbf{S}_1 - \mathbf{S}_1^*)\mathbf{g} = 0, \qquad \mathbf{g}^T(\mathbf{S}_1^T - \mathbf{S}_1^*)\mathbf{g} = 0, \qquad \mathbf{g}^T(\mathbf{S}_2 - \mathbf{S}_2^*)\mathbf{g} = 0, \tag{10.23}$$

and the secant core stiffness is not unique because

$$(\mathbf{S}_1^T - \mathbf{S}_1^*)\mathbf{g} = \mathbf{0}, \qquad (\mathbf{S}_2 - \mathbf{S}_2^*)\mathbf{g} = \mathbf{0}. \tag{10.24}$$

Adding "gage terms" such as those of (10.24) multiplied by arbitrary coefficients does not change $\delta\mathcal{U}$ and consequently the secant stiffness acquires two free parameters. Uniqueness holds for the tangent stiffness because the test vectors are the virtual displacement gradient variations, and $\mathbf{S}$ is not a function of $\delta\mathbf{g}$.

Remark 10.4. Because of (10.23), an additional free parameter appears in $\mathbf{S}^U$ if unsymmetry is allowed. If symmetry is enforced the first two gage expressions must be combined to read $\mathbf{g}^T(\mathbf{S}_1 + \mathbf{S}_1^T - 2\mathbf{S}_1^*)\mathbf{g} = 0$.

§10.4.4. Spectral Forms

There is a more compact alternative expression of the core stiffnesses that offers theoretical as well as implementational advantages at the cost of some generality. Define vectors $\mathbf{b}_i$ and $\mathbf{c}_i$ as

$$e_i = \mathbf{c}_i^T\mathbf{g}, \quad \mathbf{c}_i = \mathbf{h}_i + \tfrac{1}{2}\mathbf{H}_i\mathbf{g}, \qquad \mathbf{b}_i = \frac{\partial e_i}{\partial \mathbf{g}} = \mathbf{h}_i + \mathbf{H}_i\mathbf{g}. \tag{10.25}$$

Then the spectral forms (so called because of the formal similarity of equations (10.26)–(10.28) with the spectral decomposition of a matrix as the sum of rank-one matrices) are

$$\mathbf{S}^U(1,1) = \mathbf{S}^U\big|_{\alpha=\beta=1} = E_{ij}\,\mathbf{c}_i\mathbf{c}_j^T + s_i^0\mathbf{H}_i, \tag{10.26}$$

$$\mathbf{S}^r(0,0) = \mathbf{S}^r\big|_{\phi=\psi=0} = E_{ij}\,\mathbf{b}_i\mathbf{c}_j^T + s_i^0\mathbf{H}_i, \tag{10.27}$$

$$\mathbf{S}^r(\tfrac{1}{2},1) = \mathbf{S}^r\big|_{\phi=\frac{1}{2},\psi=1} = E_{ij}\,\mathbf{c}_i\mathbf{c}_j^T + \tfrac{1}{2}(s_i + s_i^0)\mathbf{H}_i, \tag{10.28}$$

$$\mathbf{S} = E_{ij}\,\mathbf{b}_i\mathbf{b}_j^T + s_i\mathbf{H}_i = \mathbf{S}_M + \mathbf{S}_G. \tag{10.29}$$

Note that $\mathbf{S}^r(\frac{1}{2},1)$ is symmetric but $\mathbf{S}^r(0,0)$ is not. It is seen that for energy and secant stiffnesses, compactness is paid in terms of settling for specific coefficients.

Remark 10.5. The foregoing relations may be easily verified by noting that

$$\begin{aligned}
E_{ij}\,\mathbf{c}_i\mathbf{c}_j^T &= \mathbf{S}_0 + \tfrac{1}{2}(\mathbf{S}_1 + \mathbf{S}_1^T) + \tfrac{1}{4}\mathbf{S}_2,\\
E_{ij}\,\mathbf{b}_i\mathbf{c}_j^T &= \mathbf{S}_0 + \tfrac{1}{2}\mathbf{S}_1 + \mathbf{S}_1^T + \tfrac{1}{2}\mathbf{S}_2,\\
E_{ij}\,\mathbf{b}_i\mathbf{b}_j^T &= \mathbf{S}_0 + \mathbf{S}_1 + \mathbf{S}_1^T + \mathbf{S}_2,\\
E_{ij}\frac{\partial(\mathbf{c}_i\mathbf{c}_j^T)}{\partial\mathbf{g}} &= E_{ij}\left[\mathbf{c}_i\left(\frac{\partial\mathbf{c}_j}{\partial\mathbf{g}}\right)^T + \left(\frac{\partial\mathbf{c}_i}{\partial\mathbf{g}}\right)\mathbf{c}_j^T\right] = \mathbf{S}_1^* + \tfrac{1}{2}\mathbf{S}_2^* = E_{ij}e_j\mathbf{H}_i = (s_i - s_i^0)\mathbf{H}_i,\\
E_{ij}\frac{\partial^2(\mathbf{c}_i\mathbf{c}_j^T)}{\partial\mathbf{g}^2} &= 2E_{ij}\,\frac{\partial\mathbf{c}_i}{\partial\mathbf{g}}\left(\frac{\partial\mathbf{c}_j}{\partial\mathbf{g}}\right)^T = \tfrac{1}{2}\mathbf{S}_2^*,
\end{aligned} \tag{10.30}$$

and seeking these patterns in the general parametrized expressions (10.20).

§10.4.5. Generalization to $\mathbf{H}(\mathbf{g})$

If the $\mathbf{H}_i$ depend on $\mathbf{g}$, as it generally happens if strain measures other than Green-Lagrange's are used, the secant and tangent stiffness core equations become more complex because of the presence of first and second $\mathbf{g}$-derivatives of $\mathbf{H}_i$. The changes in the core variational equations (10.14)–(10.15) can be succintly expressed as

$$\delta\mathcal{U} = \delta\mathbf{g}^T \left((\mathbf{S}^r + \widehat{\mathbf{S}}^r)\mathbf{g} + \mathbf{\Psi}^0 + \widehat{\mathbf{\Psi}}^0 \right) = \delta\mathbf{g}^T (\mathbf{\Phi} + \widehat{\mathbf{\Phi}}), \tag{10.31}$$

$$\delta^2\mathcal{U} = \delta\mathbf{g}^T (\mathbf{S} + \widehat{\mathbf{S}})\, \delta\mathbf{g} + (\delta^2\mathbf{g})^T (\mathbf{\Phi} + \widehat{\mathbf{\Phi}}). \tag{10.32}$$

where $\widehat{\mathbf{S}}^r$, $\widehat{\mathbf{S}}$ and $\widehat{\mathbf{\Phi}}$ are additional core terms that arise on account of the dependence of the $\mathbf{H}_i$ on $\mathbf{g}$.

The parametrization and efficient characterization of such terms for several strain measures of interest in practice, notably logarithmic and midpoint strains, are presently open problems. Such topics would in fact be good candidates for term projects in advanced nonlinear finite element courses.

§10.5. Core Stiffness Derivation Examples

Because the core equations reflect the motion of an individual particle, their form is primarily determined by the choice of components of $\mathbf{s}$, $\mathbf{e}$ and $\mathbf{g}$ that are retained in the strain energy density. This choice is in turn a byproduct of the mathematical idealization of the actual structure or structural component.

Several cases are worked out below to illustrate the basic steps. The core expressions developed in these examples do not force commitment to specific elements, only to a mathematical model. For example the bar core equations may be subsequently used to develop 2-node straight elements or 3-node curved ones. Some specific elements based on these equations are derived in Chapter 11.

§10.5.1. Bar in 3D Space

The particle belongs to a bar moving in 3D space. The only energy contribution is due to the axial (longitudinal) stress. We have $n_d = 3$, $n_s = 1$ and $n_g = 3$. To simplify node subscripting, Cartesian systems and displacement components will be denoted by $\{X, Y, Z\}$, $\{x, y, z\}$ and $\{u_X, u_Y, u_Z\}$ rather than $\{X_1, X_2, X_3\}$, $\{x_1, x_2, x_3\}$ and $\{u_1, u_2, u_3\}$, respectively. In the reference configuration $\mathcal{C}_0$ the bar is referred to a *local* Cartesian system $\{\bar{X}, \bar{Y}, \bar{Z}\}$, with $\bar{X}$ located along the bar axis. See Figure 10.1.

With reference to this local system, the motion of a particle initially at $\bar{X}$ is defined by the displacement components $\bar{u}_X = \bar{u}_X(\bar{X})$, $\bar{u}_Y = \bar{u}_Y(\bar{X})$ and $\bar{u}_Z = \bar{u}_Z(\bar{X})$. The three displacement gradients that intervene in the definition of nonlinear strains are

$$\mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} \partial\bar{u}_X/\partial\bar{X} \\ \partial\bar{u}_Y/\partial\bar{X} \\ \partial\bar{u}_Z/\partial\bar{X} \end{bmatrix}. \tag{10.33}$$

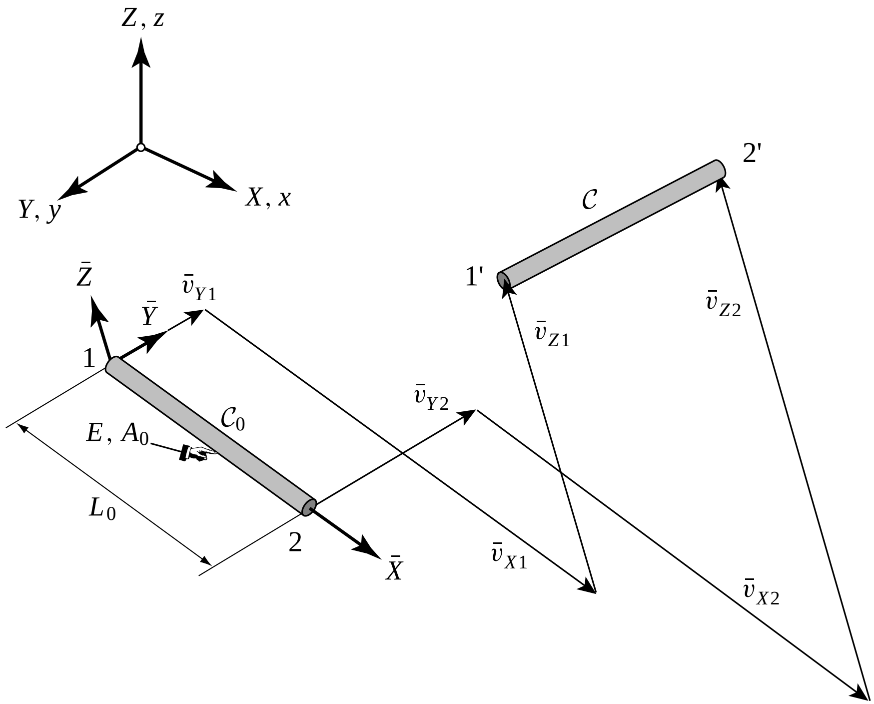

Figure 10.1. A 2-node bar element in 3D space.

As uniaxial strain measure we adopt the Green-Lagrange (GL) axial strain, defined as

$$
\begin{aligned}
e \equiv e_1 &= \frac{\partial \bar{u}_X}{\partial \bar{X}} + \tfrac{1}{2}\left[\left(\frac{\partial \bar{u}_X}{\partial \bar{X}}\right)^2 + \left(\frac{\partial \bar{u}_Y}{\partial \bar{X}}\right)^2 + \left(\frac{\partial \bar{u}_Z}{\partial \bar{X}}\right)^2\right] = g_1 + \tfrac{1}{2}(g_1^2 + g_2^2 + g_3^2) \\
&= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} + \tfrac{1}{2} \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \mathbf{h}^T \mathbf{g} + \tfrac{1}{2} \mathbf{g}^T \mathbf{H} \mathbf{g}.
\end{aligned}
\tag{10.34}
$$

Thus for this choice of strain, $\mathbf{h}_1^T \equiv \mathbf{h}^T = [\,1 \quad 0 \quad 0\,]$ and $\mathbf{H}_1 \equiv \mathbf{H}$ is the 3×3 identity matrix. The conjugate stress measure $s_1 \equiv s$ is the second Piola-Kirchhoff (PK2) axial stress. The stress-strain relation is $s = s^0 + Ee$, where s^0 and s are PK2 axial stresses in the reference and current configurations, respectively, and E is Young's modulus.

Because $\mathbf{H}$ is independent of $\mathbf{g}$, to form the core stiffnesses in local coordinates we can directly use the spectral expressions (10.26)–(10.29). First construct the vectors

$$
\mathbf{c} \equiv \mathbf{c}_1 = \begin{bmatrix} 1 + \tfrac{1}{2} g_1 \\ \tfrac{1}{2} g_2 \\ \tfrac{1}{2} g_3 \end{bmatrix}, \qquad \mathbf{b} \equiv \mathbf{b}_1 = \begin{bmatrix} 1 + g_1 \\ g_2 \\ g_3 \end{bmatrix}, \tag{10.35}
$$

which inserted into the spectral forms yield

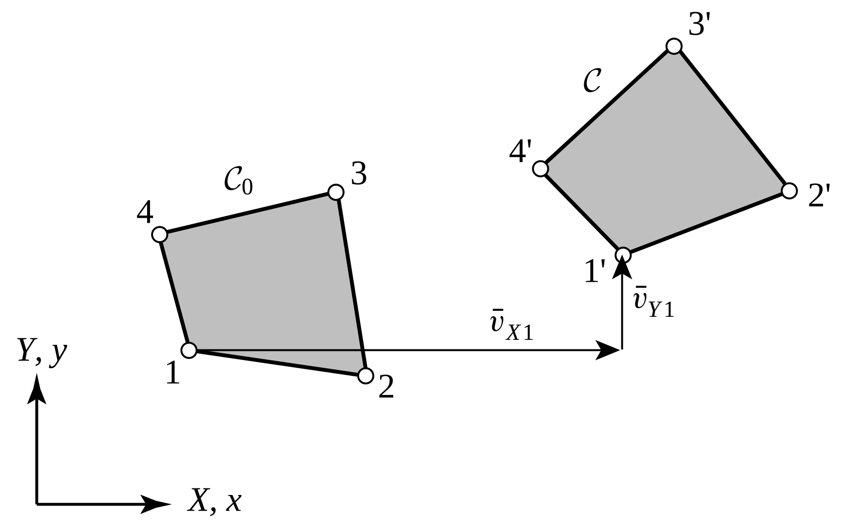

Figure 10.2. A 4-node plane stress element in 2D space.

$$\mathbf{S}^U(1,1) = E\,\mathbf{c}\mathbf{c}^T + s^0\mathbf{H} = E\begin{bmatrix} (1+\frac{1}{2}g_1)^2 & \frac{1}{2}g_2(1+\frac{1}{2}g_1) & \frac{1}{2}g_3(1+\frac{1}{2}g_1) \\ & \frac{1}{4}g_2^2 & \frac{1}{4}g_2g_3 \\ symm & & \frac{1}{4}g_3^2 \end{bmatrix} + s^0\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (10.36)$$

$$\mathbf{S}^r(\tfrac{1}{2},1) = E\,\mathbf{c}\mathbf{c}^T + s^m\mathbf{H} = E\begin{bmatrix} (1+\frac{1}{2}g_1)^2 & \frac{1}{2}g_2(1+\frac{1}{2}g_1) & \frac{1}{2}g_3(1+\frac{1}{2}g_1) \\ & \frac{1}{4}g_2^2 & \frac{1}{4}g_2g_3 \\ symm & & \frac{1}{4}g_3^2 \end{bmatrix} + s^m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (10.37)$$

$$\mathbf{S} = E\,\mathbf{b}\mathbf{b}^T + s\mathbf{H} = E\begin{bmatrix} (1+g_1)^2 & g_2(1+g_1) & g_3(1+g_1) \\ & g_2^2 & g_2g_3 \\ symm & & g_3^2 \end{bmatrix} + s\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (10.38)$$

In equation (10.37), $s^m = \frac{1}{2}(s^0 + s) = s^0 + \frac{1}{2}Ee$ is the average or "half-way" stress. The clean separation into material and geometric (initial-stress) stiffnesses should be noted.

§10.5.2. Plate in Plane Stress

As second example we consider a particle that pertains to a plate in plane stress (membrane), constrained to move in its plane. See Figure 10.2. As usual we consider only the motion of the midplane. The Cartesian reference system and displacement components will be denoted by $\{X, Y\}$, $\{x, y\}$ and $\{u_X, u_Y\}$ rather than $\{X_1, X_2\}$, $\{x_1, x_2\}$ and $\{u_1, u_2\}$, respectively. The element displacement field of a generic particle originally at (X, Y) is defined by the two components $u_X = u_X(X, Y)$ and $u_Y = u_Y(X, Y)$. Three in-plane PK2 stresses contribute to the strain energy and four displacement gradients appear in the corresponding GL strain. Consequently $n_d = 2$, $n_s = 3$ and $n_g = 4$. The four displacement gradients are arranged as

$$\mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{bmatrix} = \begin{bmatrix} \partial u_X/\partial X \\ \partial u_Y/\partial X \\ \partial u_X/\partial Y \\ \partial u_Y/\partial Y \end{bmatrix} \tag{10.39}$$

The strain measures chosen are the three components e_i $(i = 1, 2, 3)$ of the GL strains defined in the usual manner:

$$e_1 = e_{XX} = g_1 + \tfrac{1}{2}(g_1^2 + g_2^2) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \mathbf{g} + \tfrac{1}{2}\mathbf{g}^T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{g}, \tag{10.40}$$

$$e_2 = e_{YY} = g_4 + \tfrac{1}{2}(g_3^2 + g_4^2) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}^T \mathbf{g} + \tfrac{1}{2}\mathbf{g}^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{g}, \tag{10.41}$$

$$e_3 = e_{XY} + e_{YX} = g_2 + g_3 + g_1 g_3 + g_2 g_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}^T \mathbf{g} + \tfrac{1}{2}\mathbf{g}^T \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \mathbf{g}, \tag{10.42}$$

from which expressions for $\mathbf{h}_i$ and $\mathbf{H}_i$ $(i = 1, 2, 3)$ follow. For brevity, only the derivation of the tangent stiffness matrix will be described. Begin by forming the vectors

$$\mathbf{b}_1 = \begin{bmatrix} 1 + g_1 \\ g_2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 0 \\ 0 \\ g_3 \\ 1 + g_4 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} g_3 \\ 1 + g_4 \\ 1 + g_1 \\ g_2 \end{bmatrix}. \tag{10.43}$$

Then from (10.29) we get the core stiffness

$$\mathbf{S} = E_{ij}\mathbf{b}_i\mathbf{b}_i^T + s_i\mathbf{H}_i = \mathbf{S}_M + \mathbf{S}_G, \tag{10.44}$$

where $s_i = s_i^0 + E_{ij}e_j$, $(i, j = 1, 2, 3)$, are the PK2 stresses in the current configuration.

In full and using the abbreviations $a_1 = 1 + g_1$, $a_4 = 1 + g_4$ we get

$$\mathbf{S}_M = \begin{bmatrix} E_{11}a_1^2 + 2E_{13}a_1g_3 + E_{33}g_3^2 & E_{11}a_1g_2 + E_{13}(a_1a_4 + g_2g_3) + E_{33}a_4g_3 & E_{12}a_1g_3 + E_{13}a_1^2 + E_{23}g_3^2 + E_{33}a_1g_3 & E_{12}a_1a_4 + E_{13}a_1g_2 + E_{23}a_4g_3 + E_{33}g_2g_3 \\ & E_{11}g_2^2 + 2E_{13}a_4g_2 + E_{33}a_4^2 & E_{12}g_2g_3 + E_{13}a_1g_2 + E_{23}a_4g_3 + E_{33}a_1a_4 & E_{12}a_4g_2 + E_{13}g_2^2 + E_{23}a_4^2 + E_{33}a_4g_2 \\ & & E_{22}g_3^2 + 2E_{23}a_1g_3 + E_{33}a_1^2 & E_{22}a_4g_3 + E_{23}(a_1a_4 + g_2g_3) + E_{33}a_1g_2 \\ \mathit{symm} & & & E_{22}a_4^2 + 2E_{23}a_4g_2 + E_{33}g_2^2 \end{bmatrix} \tag{10.45}$$

$$\mathbf{S}_G = \begin{bmatrix} s_1 & 0 & s_3 & 0 \\ & s_1 & 0 & s_3 \\ & & s_2 & 0 \\ symm & & & s_2 \end{bmatrix}. \tag{10.46}$$

§10.5.3. Plate Bending

This is similar to the previous example in that the structure is a flat thin plate but now motion in 3D space $\{X, Y, Z\}$ is allowed. With this increased freedom the plate is capable of membrane stretching and bending. For the latter a Kirchhoff mathematical model is assumed. The three energy-contributing GL strains are now functions of six gradients. Consequently $n_d = 3$, $n_s = 3$ and $n_g = 6$. The contributing gradients are arranged as

$$\mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \end{bmatrix} = \begin{bmatrix} \partial u_X/\partial X \\ \partial u_Y/\partial X \\ \partial u_Z/\partial Z \\ \partial u_X/\partial Y \\ \partial u_Y/\partial Y \\ \partial u_Z/\partial Y \end{bmatrix} \tag{10.47}$$

The three GL strains are defined as

$$e_1 = e_{XX} = g_1 + \tfrac{1}{2}(g_1^2 + g_2^2 + g_3^2) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \mathbf{g} + \tfrac{1}{2}\mathbf{g}^T \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{g}, \tag{10.48}$$

$$e_2 = e_{YY} = g_5 + \tfrac{1}{2}(g_4^2 + g_5^2 + g_6^2) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}^T \mathbf{g} + \tfrac{1}{2}\mathbf{g}^T \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{g}, \tag{10.49}$$

$$e_3 = e_{XY} + e_{YX} = g_2 + g_4 + g_1 g_4 + g_2 g_5 + g_3 g_6 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}^T \mathbf{g} + \tfrac{1}{2}\mathbf{g}^T \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \mathbf{g}, \tag{10.50}$$

which define $\mathbf{h}_i$ and $\mathbf{H}_i$, $i = 1, 2, 3$. When one reaches this level of bookkeeping it is more expedient and less error-prone to obtain the core matrices through symbolic manipulation. For example, the following *Macsyma* program forms $\mathbf{S}_M$ and $\mathbf{S}_G$ in matrices `SM` and `SG`, respectively:

```
 h1: matrix([1],[0],[0],[0],[0],[0])$
 h2: matrix([0],[0],[0],[0],[1],[0])$
 h3: matrix([0],[1],[0],[1],[0],[0])$
 g:  matrix([g1],[g2],[g3],[g4],[g5],[g6])$
 HH1:matrix([1,0,0,0,0,0],[0,1,0,0,0,0],[0,0,1,0,0,0],
            [0,0,0,0,0,0],[0,0,0,0,0,0],[0,0,0,0,0,0])$
 HH2:matrix([0,0,0,0,0,0],[0,0,0,0,0,0],[0,0,0,0,0,0],
            [0,0,0,1,0,0],[0,0,0,0,1,0],[0,0,0,0,0,1])$
 HH3:matrix([0,0,0,1,0,0],[0,0,0,0,1,0],[0,0,0,0,0,1],
            [1,0,0,0,0,0],[0,1,0,0,0,0],[0,0,1,0,0,0])$
 b1:h1+HH1.g$ b2:h2+HH2.g$ b3:h3+HH3.g$
SM:E11* b1.transpose(b1)+E22*b2.transpose(b2)+E33*b3.transpose(b3)
 + E12*(b1.transpose(b2)+b2.transpose(b1))
 + E13*(b1.transpose(b3)+b3.transpose(b1))
 + E23*(b2.transpose(b3)+b3.transpose(b2))$
ratvars(g6,g5,g4,g3,g2,g1,a5,a1,E11,E12,E13,E22,E23,E33)$
SM:ratsimp(SM)$
SG:ratsimp(s1*HH1+s2*HH2+s3*HH3)$
```

These matrices may be automatically converted to TEX by appropriate *Macsyma* statements (not shown above). That output was reformatted by hand for inclusion here. For the core tangent stiffness this semi-automated process yields

$$
\begin{aligned}
S_M(1,1) &= E_{33}g_4^2 + 2E_{13}(1+g_1)g_4 + E_{11}(1+g_1)^2\\
S_M(1,2) &= E_{13}((1+g_1)(1+g_5)+g_2g_4) + E_{33}g_4(1+g_5) + E_{11}(1+g_1)g_2\\
S_M(1,3) &= E_{13}((1+g_1)g_6+g_3g_4) + E_{33}g_4g_6 + E_{11}(1+g_1)g_3\\
S_M(1,4) &= E_{23}g_4^2 + E_{33}(1+g_1)g_4 + E_{12}(1+g_1)g_4 + E_{13}(1+g_1)^2\\
S_M(1,5) &= E_{12}((1+g_1)(1+g_5)) + E_{23}g_4(1+g_5) + E_{33}g_2g_4 + E_{13}(1+g_1)g_2\\
S_M(1,6) &= E_{12}(1+g_1)g_6 + E_{23}g_4g_6 + E_{33}g_3g_4 + E_{13}(1+g_1)g_3\\
S_M(2,2) &= E_{33}(1+g_5)^2 + 2E_{13}g_2(1+g_5) + E_{11}g_2^2\\
S_M(2,3) &= E_{33}(1+g_5)g_6 + E_{13}(g_2g_6+g_3(1+g_5)) + E_{11}g_2g_3\\
S_M(2,4) &= E_{33}((1+g_1)(1+g_5)) + E_{23}g_4(1+g_5) + E_{12}g_2g_4 + E_{13}(1+g_1)g_2\\
S_M(2,5) &= E_{23}(1+g_5)^2 + E_{33}g_2(1+g_5) + E_{12}g_2(1+g_5) + E_{13}g_2^2\\
S_M(2,6) &= E_{23}(1+g_5)g_6 + E_{12}g_2g_6 + E_{33}g_3(1+g_5) + E_{13}g_2g_3\\
S_M(3,3) &= E_{33}g_6^2 + 2E_{13}g_3g_6 + E_{11}g_3^2\\
S_M(3,4) &= E_{33}(1+g_1)g_6 + E_{23}g_4g_6 + E_{12}g_3g_4 + E_{13}(1+g_1)g_3\\
S_M(3,5) &= E_{23}(1+g_5)g_6 + E_{33}g_2g_6 + E_{12}g_3(1+g_5) + E_{13}g_2g_3\\
S_M(3,6) &= E_{23}g_6^2 + E_{33}g_3g_6 + E_{12}g_3g_6 + E_{13}g_3^2\\
S_M(4,4) &= E_{22}g_4^2 + 2E_{23}(1+g_1)g_4 + E_{33}(1+g_1)^2\\
S_M(4,5) &= E_{23}((1+g_1)(1+g_5)+g_2g_4) + E_{22}g_4(1+g_5) + E_{33}(1+g_1)g_2\\
S_M(4,6) &= E_{23}((1+g_1)g_6+g_3g_4) + E_{22}g_4g_6 + E_{33}(1+g_1)g_3\\
S_M(5,5) &= E_{22}(1+g_5)^2 + 2E_{23}g_2(1+g_5) + E_{33}g_2^2\\
S_M(5,6) &= E_{22}(1+g_5)g_6 + E_{23}(g_2g_6+g_3(1+g_5)) + E_{33}g_2g_3\\
S_M(6,6) &= E_{22}g_6^2 + 2E_{23}g_3g_6 + E_{33}g_3^2
\end{aligned}
\tag{10.51}
$$

(which can be further compacted by introducing the auxiliary symbols $a_1 = 1+g_1$ and $a_5 = 1+g_5$ as done in §10.5.3) and

$$
\mathbf{S}_G = \mathbf{S}_{GP} = \begin{bmatrix} s_1 & 0 & 0 & s_3 & 0 & 0\\ 0 & s_1 & 0 & 0 & s_3 & 0\\ 0 & 0 & s_1 & 0 & 0 & s_3\\ s_3 & 0 & 0 & s_2 & 0 & 0\\ 0 & s_3 & 0 & 0 & s_2 & 0\\ 0 & 0 & s_3 & 0 & 0 & s_2 \end{bmatrix}. \tag{10.52}
$$

Remark 10.6. If the plate element to which the particle belong has (as usual) rotational freedoms, an additional geometric stiffness (the complementary geometric stiffness) appears in the transformation phase. Because of this, the core geometric stiffness (10.52) has been relabeled as $\mathbf{S}_{GP}$, where subscript P means "principal."

Remark 10.7. The core stiffness matrices may also be used for part of the formulation of thin-shell facet elements, with the proviso that global reference axes $\{X, Y, Z\}$ are to be replaced by a local coordinate system $\{\bar{X}, \bar{Y}, \bar{Z}\}$ with $\bar{Z}$ normal to the element midplane.

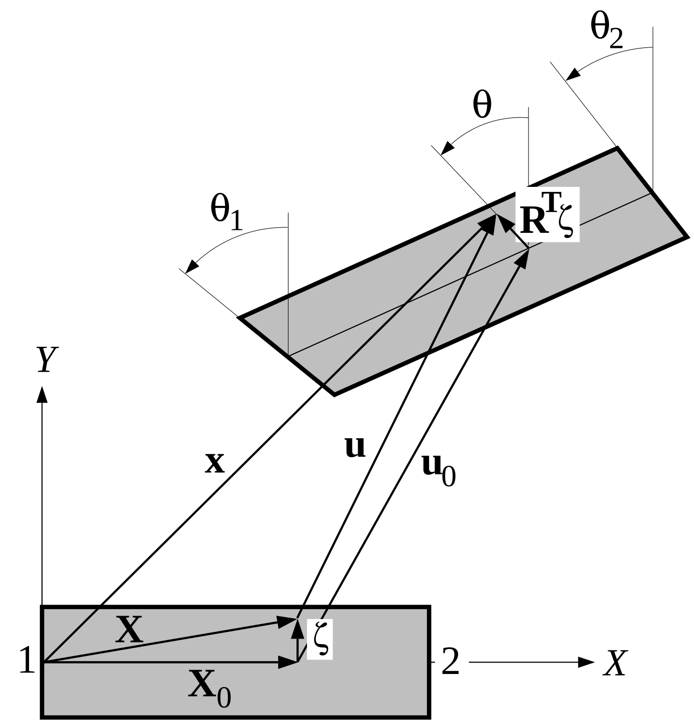

Figure 10.3. Kinematics of 2D Timoshenko beam element

§10.5.4. 2D Timoshenko Beam

Consider next an isotropic Timoshenko plane beam that moves in the (X,Y) plane. For notational simplicity it is assumed that the longitudinal axis of the beam is aligned with X. The only PK2 stresses that contribute to the strain energy are the axial stress $s_1 \equiv s_{XX}$ and the mean shear stress $s_2 \equiv s_{XY}$. The corresponding GL strains are the axial strain $e_1 \equiv e_{XX}$ and the section-averaged shear strain $e_2 \equiv \gamma_{XY} = e_{XY}+e_{YX}$. The constitutive equations are $s_1 = s_1^0+Ee_1$ and $s_2 = s_2^0+G\,e_2$, where E and G are the Young's modulus and shear modulus, respectively, of the material. The treatment outlined below is slightly modified from that of a course term project by Alexander, de la Fuente and Haugen. [11]

The finite displacements are described in a local coordinate system that is attached to the initial position of the beam, as illustrated in Figure 1. Under the usual kinematic assumptions of the Timoshenko beam model (plane sections remain plane but not necessarily normal to the deformed centroidal axis) the coordinates of a particle in the underformed and deformed configurations may be written

$$\mathbf{X} = \mathbf{X}_0 + \boldsymbol{\zeta}, \qquad \mathbf{X}_0 = \begin{bmatrix} X \\ 0 \end{bmatrix}, \qquad \boldsymbol{\zeta} = \begin{bmatrix} 0 \\ Y \end{bmatrix}, \tag{10.53}$$

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{R}^T\boldsymbol{\zeta}, \qquad \mathbf{x}_0 = \begin{bmatrix} X + u_{0X} \\ u_{0Y} \end{bmatrix}, \quad \mathbf{R}^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \tag{10.54}$$

where u_{0X} and u_{0Y} are the components of the centroidal displacement vector $\mathbf{u}_0$. Subtracting (10.53) from (10.54) gives the element displacement field

$$\mathbf{u} = \mathbf{x} - \mathbf{X} = \begin{bmatrix} u_X \\ u_Y \end{bmatrix} = \begin{bmatrix} u_{0X} - Y\sin\theta \\ u_{0Y} + Y(\cos\theta - 1) \end{bmatrix}. \tag{10.55}$$

Four displacement gradients contribute to the GL strains. Thus for this case we have $n_d = 2$, $n_s = 2$ and $n_g = 4$. The four contributing displacement gradients are arranged in the usual pattern:

$$\mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{bmatrix} = \begin{bmatrix} \partial u_X/\partial X \\ \partial u_Y/\partial X \\ \partial u_X/\partial Y \\ \partial u_Y/\partial Y \end{bmatrix}. \tag{10.56}$$

For future use in Chapter 11 we note that the gradients can be written in terms of generalized section freedoms as

$$\mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{bmatrix} = \begin{bmatrix} \epsilon - Y\kappa\cos\theta \\ \gamma - Y\kappa\sin\theta \\ -\sin\theta \\ \cos\theta - 1 \end{bmatrix}, \tag{10.57}$$

in which $\epsilon = \partial u_{0X}/\partial X$ is a generalized axial strain, $\gamma = \partial u_{0Y}/\partial X$ a generalized shear strain, and $\kappa = \partial\theta/\partial X$ is the beam curvature.

The matrix form of the GL strains is

$$e_1 = g_1 + \tfrac{1}{2}(g_1^2 + g_2)^2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \mathbf{g} + \tfrac{1}{2}\mathbf{g}^T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{g}, \tag{10.58}$$

$$e_2 = g_2 + g_3 + g_1 g_3 + g_2 g_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}^T \mathbf{g} + \tfrac{1}{2}\mathbf{g}^T \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \mathbf{g}, \tag{10.59}$$

which define $\mathbf{h}_1$, $\mathbf{h}_2$, $\mathbf{H}_1$ and $\mathbf{H}_2$. On introducing the auxiliary vectors

$$\mathbf{b}_1 = \begin{bmatrix} 1 + g_1 \\ g_2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} g_3 \\ 1 + g_4 \\ 1 + g_1 \\ g_2 \end{bmatrix}, \quad \mathbf{c}_1 = \begin{bmatrix} 1 + \frac{1}{2}g_1 \\ \frac{1}{2}g_2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} \frac{1}{2}g_3 \\ 1 + \frac{1}{2}g_4 \\ 1 + \frac{1}{2}g_1 \\ \frac{1}{2}g_2 \end{bmatrix}, \tag{10.60}$$

the spectral core stiffness matrices and internal force vector can be written

$$\mathbf{S}^U = E\,\mathbf{c}_1\mathbf{c}_1^T + G\,\mathbf{c}_2\mathbf{c}_2^T + s_1^0\mathbf{H}_1 + s_2^0\mathbf{H}_2\,, \tag{10.61}$$

$$\mathbf{S}^r = E\,\mathbf{c}_1\mathbf{c}_1^T + G\,\mathbf{c}_2\mathbf{c}_2^T + \tfrac{1}{2}(s_1^0 + s_1)\mathbf{H}_1 + \tfrac{1}{2}(s_2^0 + s_2)\mathbf{H}_2\,, \tag{10.62}$$

$$\mathbf{S} = \mathbf{S}_M + \mathbf{S}_{GP}, \qquad \mathbf{S}_M = E\mathbf{b}_1\mathbf{b}_1^T + G\mathbf{b}_2\mathbf{b}_2^T, \quad \mathbf{S}_{GP} = s_1\mathbf{H}_1 + s_2\mathbf{H}_2\,, \tag{10.63}$$

$$\mathbf{\Phi} = s_1\mathbf{b}_1 + s_2\mathbf{b}_2. \tag{10.64}$$

Because beam elements have rotational freedoms, a complementary geometric stiffness matrix appears when carrying out the transformation phase. This term is considered in the subsequent GCCF treatment of this element in Chapter 11.

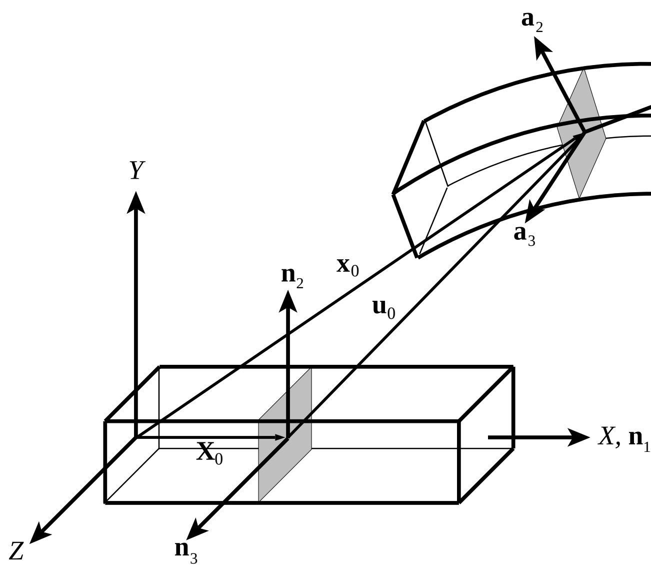

Figure 10.4. Kinematics of 3D Timoshenko beam element.

§10.5.5. 3D Timoshenko Beam: Kinematics

The last example of derivation of core equations involve a TL 3D Timoshenko beam capable of arbitrarily large rotations. The following material is largely extracted from a recent paper by Crivelli and Felippa [8] as well as Crivelli's thesis [6] and is continued with the DGCCF transformation phase in Chapter 11. The notation used in those references has been slightly edited to fit that of the present article.

As in the 2D case, the beam is isotropically elastic with Young's modulus E and shear modulus G. The reference configuration of the beam is straight and prismatic although not necessarily stress free. A *local reference frame* $\mathbf{n}_i$ is attached to it, with $\mathbf{n}_1$ directed along the longitudinal axis (the locus of cross section centroids). Axes $\mathbf{n}_2$ and $\mathbf{n}_3$ are in the plane of the left-end cross section; these will be eventually aligned with the principal inertia axes to simplify some algebraic expressions. Along these axes we attach the coordinate system $\{X, Y, Z\}$. This description is schematically shown in Figure 2. We further define a set of *moving* frames, denoted by $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$, parametrized by the longitudinal coordinate X. Initially these frames coincide with $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$, and displace rigidly attached to the cross-sections of the moving current configuration.

A beam particle originally at (X, Y, Z) displaces to

$$\mathbf{x}(\mathbf{X}) = \mathbf{x}_0(X) + \mathbf{R}^T(X)\boldsymbol{\zeta}(Y, Z), \qquad \zeta^T = [\,0 \quad Y \quad Z\,], \tag{10.65}$$

where $\mathbf{x}_0$ describes the position of the centroid of the given cross-section, $\mathbf{R}$ is a 3-by-3 orthogonal matrix function that orients the displaced cross section, and $\boldsymbol{\zeta}$ is a cross-section position vector. The displacement field is

$$\mathbf{u} = \mathbf{x} - \mathbf{X} = \mathbf{u}_0 + (\mathbf{R}^T - \mathbf{I})\zeta. \tag{10.66}$$

where $\mathbf{u}_0(X) = \mathbf{x}_0(X) - \mathbf{X}_0(X)$ is the centroidal displacement (see Figure 2).

In the sequel 3×3 skew-symmetric matrices are consistently denoted by placing a tilde over their axial 3-vector symbol; for example

$$\tilde{\mathbf{a}} = \textbf{spin}\,(\mathbf{a}) = \begin{bmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{bmatrix}, \qquad \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \textbf{axial}\,(\tilde{\mathbf{a}}). \tag{10.67}$$

The skew-symmetric *curvature matrix* $\tilde{\boldsymbol{\kappa}}$ is defined by $\tilde{\boldsymbol{\kappa}} = \mathbf{R}(d\mathbf{R}^T/dX)$, which is the rate of change of the orthogonal rotation matrix $\mathbf{R}$ with respect to the longitudinal coordinate. The curvature vector is $\boldsymbol{\kappa} = \textbf{axial}\,(\tilde{\boldsymbol{\kappa}})$. We shall also require later the variation of angular orientation $\delta\boldsymbol{\Theta}$, defined as the axial vector of the skew matrix $\mathbf{R}\,\delta\mathbf{R}^T$:

$$\widetilde{\delta\boldsymbol{\Theta}} = \mathbf{R}\,\delta\mathbf{R}^T = -\delta\mathbf{R}\,\mathbf{R}^T, \qquad \delta\boldsymbol{\Theta} = \textbf{axial}\,(\widetilde{\delta\boldsymbol{\Theta}}), \tag{10.68}$$

All displacement gradients g_{ij} appear in the GL strain measures. To maintain compactness the nine gradients are partitioned into three 3-vectors:

$$\mathbf{g}_1 = \begin{bmatrix} \partial u_X/\partial X \\ \partial u_Y/\partial X \\ \partial u_Z/\partial X \end{bmatrix}, \qquad \mathbf{g}_2 = \begin{bmatrix} \partial u_X/\partial Y \\ \partial u_Y/\partial Y \\ \partial u_Z/\partial Y \end{bmatrix}, \qquad \mathbf{g}_3 = \begin{bmatrix} \partial u_X/\partial Z \\ \partial u_Y/\partial Z \\ \partial u_Z/\partial Z \end{bmatrix}, \tag{10.69}$$

The 9-component gradient vector is $\mathbf{g}^T = [\mathbf{g}_1^T \quad \mathbf{g}_2^T \quad \mathbf{g}_3^T]$, but this symbol is not used directly here. Also introduce the 3-vectors

$$\mathbf{h}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{h}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{h}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{10.70}$$

With the help of these quantities, explicit expressions for the displacement gradient vectors $\mathbf{g}$ can be given as

$$\begin{aligned} \mathbf{g}_1 &= \frac{d\mathbf{u}_0}{dX} + \mathbf{R}^T\tilde{\boldsymbol{\kappa}}\boldsymbol{\zeta} = \frac{d\mathbf{u}_0}{dX} + \mathbf{R}^T\tilde{\boldsymbol{\zeta}}^T\boldsymbol{\kappa}, \\ \mathbf{g}_2 &= (\mathbf{R}^T - \mathbf{I})\mathbf{h}_2, \qquad \mathbf{g}_3 = (\mathbf{R}^T - \mathbf{I})\mathbf{h}_3. \end{aligned} \tag{10.71}$$

The only nonzero components of the GL strain tensor can be written

$$\begin{aligned} e_1 \equiv e_{11} = {}& \qquad\qquad \mathbf{h}_1^T\mathbf{g}_1 + \tfrac{1}{2}\mathbf{g}_1^T\mathbf{H}\mathbf{g}_1, \\ e_2 \equiv \gamma_{12} = 2e_{12} ={}& \mathbf{h}_2^T\mathbf{g}_1 + \mathbf{h}_1^T\mathbf{g}_2 + \tfrac{1}{2}(\mathbf{g}_1^T\mathbf{H}\mathbf{g}_2 + \mathbf{g}_2^T\mathbf{H}\mathbf{g}_1), \\ e_3 \equiv \gamma_{13} = 2e_{13} ={}& \mathbf{h}_3^T\mathbf{g}_1 + \mathbf{h}_1^T\mathbf{g}_3 + \tfrac{1}{2}(\mathbf{g}_1^T\mathbf{H}\mathbf{g}_3 + \mathbf{g}_3^T\mathbf{H}\mathbf{g}_1), \end{aligned} \tag{10.72}$$

where $\mathbf{H}$ is here the 3×3 identity matrix. Note that from the orthogonality of the rotation matrix $\mathbf{R}$ we find

$$\begin{aligned} e_{22} &= \mathbf{h}_2^T\mathbf{g}_2 + \tfrac{1}{2}\mathbf{g}_2^T\mathbf{g}_2 \\ &= R_{22} - 1 + \tfrac{1}{2}\big(R_{21}^2 + (R_{22}-1)^2 + R_{23}^2\big) = R_{22} - 1 + \tfrac{1}{2}(2 - 2R_{22}) = 0, \\ 2e_{23} &= \mathbf{h}_2^T\mathbf{g}_3 + \mathbf{h}_3^T\mathbf{g}_2 + \mathbf{g}_2^T\mathbf{g}_3 \\ &= R_{32} + R_{23} + R_{21}R_{31} + R_{22}R_{32} - R_{32} + R_{23}R_{33} - R_{23} = 0, \end{aligned} \tag{10.73}$$

and similarly $e_{33} = 0$. This confirms that the only nonzero strains are (10.72).

The strains (10.72) may be rewritten in a more physically suggestive form:

$$\begin{aligned} e_1 &= e_{11} = e_b + e_f, & \gamma &= \gamma_{12} + \gamma_{13}, \\ e_b &= \left(\frac{d\mathbf{u}_0}{dX}\right)^T \left(\mathbf{h}_1 + \tfrac{1}{2}\frac{d\mathbf{u}_0}{dX}\right), & e_2 &= \gamma_{12} = \gamma_2 + \overline{\gamma}_2 = \mathbf{h}_2^T \boldsymbol{\phi} + \mathbf{h}_2^T \tilde{\boldsymbol{\zeta}}^T \boldsymbol{\kappa}, \\ e_f &= \boldsymbol{\zeta}^T \boldsymbol{\mathcal{K}}_e + \tfrac{1}{2}\boldsymbol{\kappa}^T \tilde{\boldsymbol{\zeta}}\tilde{\boldsymbol{\zeta}}^T \boldsymbol{\kappa} \simeq \boldsymbol{\zeta}^T \boldsymbol{\mathcal{K}}_e, & e_3 &= \gamma_{13} = \gamma_3 + \overline{\gamma}_3 = \mathbf{h}_3^T \boldsymbol{\phi} + \mathbf{h}_3^T \tilde{\boldsymbol{\zeta}}^T \boldsymbol{\kappa}. \end{aligned} \tag{10.74}$$

Here e_b, e_f are stretching and flexural normal strains, γ_2 and γ_3 represent bending-induced shear strains, and $\overline{\gamma}_2$, $\overline{\gamma}_3$ are torsion-induced shear strains. The last term in e_f represents a squared-curvature contribution to flexure, which can usually be neglected (cf. Remark 9.2). The strain energy stored in the current configuration is

$$U = \int_{L_0}\int_{A_0} \mathcal{U}\, dA\, dX, \quad \text{with} \quad \mathcal{U} = \tfrac{1}{2}E e_1^2 + \tfrac{1}{2}G\left(e_2^2 + e_3^2\right) + s_1^0 e_1 + s_2^0 e_{12} + s_3^0 e_3. \tag{10.75}$$

§10.5.6. 3D Timoshenko Beam: Core equations

The PK2 stresses associated with the GL strains (10.72) are $s_1 \equiv s_{11} = s_{XX}$, $s_2 \equiv s_{12} = s_{XY}$ and $s_3 \equiv s_{13} = s_{XZ}$. The constitutive equations are $s_1 = s_1^0 + Ee_1$, $s_2 = s_2^0 + Ge_2$ and $s_3 = s_3^0 + Ge_3$. The spectral core stiffnesses can be compactly expressed in terms of the vectors $\mathbf{c}_i = \mathbf{h}_i + \frac{1}{2}\mathbf{H}\mathbf{g}_i$ and $\mathbf{b}_i = \mathbf{h}_i + \mathbf{H}\mathbf{g}_i$ for $i = 1, 2, 3$, where no subscript is needed in $\mathbf{H} \equiv \mathbf{I}$. Applying the spectral formulas of §10.4.4 we obtain for the 9×9 core energy stiffness

$$\mathbf{S}^U = \begin{bmatrix} E\mathbf{S}_1^U + G(\mathbf{S}_2^U + \mathbf{S}_3^U) & G\mathbf{S}_4^U & G\mathbf{S}_5^U \\ G\mathbf{S}_4^{U\,T} & G\mathbf{S}_1^U & 0 \\ G\mathbf{S}_5^{U\,T} & 0 & G\mathbf{S}_1^U \end{bmatrix} + \begin{bmatrix} s_1^0\mathbf{H} & s_2^0\mathbf{H} & s_3^0\mathbf{H} \\ s_2^0\mathbf{H} & \mathbf{0} & \mathbf{0} \\ s_3^0\mathbf{H} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{10.76}$$

where $\mathbf{S}_1^U = \mathbf{c}_1\mathbf{c}_1^T$, $\mathbf{S}_2^U = \mathbf{c}_2\mathbf{c}_2^T$, $\mathbf{S}_3^U = \mathbf{c}_3\mathbf{c}_3^T$, $\mathbf{S}_4^U = \mathbf{c}_2\mathbf{c}_1^T$ and $\mathbf{S}_5^U = \mathbf{c}_3\mathbf{c}_1^T$. At the residual level we obtain for $\mathbf{S}^r$ a form similar to (10.76) except that the prestresses s_i^0, $i = 1, 2, 3$ have to be replaced by the midpoint stresses $\frac{1}{2}(s_i^0 + s_i)$. The internal force vector conjugate to $\delta\mathbf{g}$ is $\boldsymbol{\Phi} = \mathbf{S}^r\mathbf{g} + \boldsymbol{\Phi}^0 = \boldsymbol{\Phi}_\sigma + \boldsymbol{\Phi}_\tau$, in which

$$\boldsymbol{\Phi}_\sigma = \begin{bmatrix} s_1\mathbf{b}_1 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \quad \boldsymbol{\Phi}_\tau = \begin{bmatrix} s_2\mathbf{b}_2 + s_3\mathbf{b}_3 \\ s_2\mathbf{b}_1 \\ s_3\mathbf{b}_1 \end{bmatrix}, \tag{10.77}$$

represent the contribution of the normal and shear stresses, respectively.

The principal core tangent stiffness matrix $\mathbf{S} = \mathbf{S}_M + \mathbf{S}_{GP}$ is obtained from (10.29). The material stiffness is

$$\mathbf{S}_M = \begin{bmatrix} E\mathbf{S}_1 + G(\mathbf{S}_2 + \mathbf{S}_3) & G\mathbf{S}_4 & G\mathbf{S}_5 \\ G\mathbf{S}_4^T & G\mathbf{S}_1 & 0 \\ G\mathbf{S}_5^T & 0 & G\mathbf{S}_1 \end{bmatrix}, \tag{10.78}$$

where $\mathbf{S}_1 = \mathbf{b}_1\mathbf{b}_1^T$, $\mathbf{S}_2 = \mathbf{b}_2\mathbf{b}_2^T$, $\mathbf{S}_3 = \mathbf{b}_3\mathbf{b}_3^T$, $\mathbf{S}_4 = \mathbf{b}_2\mathbf{b}_1^T$ and $\mathbf{S}_5 = \mathbf{b}_3\mathbf{b}_1^T$. The principal geometric stiffness is

$$\mathbf{S}_{GP} = \begin{bmatrix} s_1\mathbf{H} & s_2\mathbf{H} & s_3\mathbf{H} \\ s_2\mathbf{H} & \mathbf{0} & \mathbf{0} \\ s_3\mathbf{H} & \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} (s_1^0 + Ee_1)\mathbf{H} & (s_2^0 + Ge_2)\mathbf{H} & (s_2^0 + Ge_3)\mathbf{H} \\ (s_2^0 + Ge_2)\mathbf{H} & \mathbf{0} & \mathbf{0} \\ (s_2^0 + Ge_3)\mathbf{H} & \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{10.79}$$

The contribution of $\left(\delta^2\mathbf{g}\right)^T \mathbf{\Phi}$ to the complementary geometric stiffness depends on the target variables in the ensuing transformation phase. Because this transformation requires the DGCCF, it is taken up in Chapter 11.

§10.6. References

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Homework Exercises for Chapter 10

Not assigned

EXERCISE 10.1

Show that the "core" forms $\mathbf{g}^T\mathbf{S}^U\mathbf{g}$ and $\mathbf{S}^r\,\mathbf{g}$, with $\mathbf{S}^U$ and $\mathbf{S}^r$ given by (10.19), are independent of the coefficients α, β, ϕ and ψ. It is sufficient to work this exercise for for $i,\ j = 1$, that is, only one stress, strain and modulus is considered. Keep, however, $\mathbf{g}$, $\mathbf{h}$ and $\mathbf{H}$ generic.

Note: This is a good exercise in matrix gymnastics. If you are "rusty" in matrix magic, reading the Addendum below is strongly recommended.

Addendum: Matrix Product Properties

In carrying out the manipulations required by Exercise 10.1, the following properties of matrix products ought to be kept in mind.

(1) Let $\mathbf{x}$ and $\mathbf{y}$ be two conforming vectors. Since their inner product is a scalar, obviously

$$\mathbf{x}^T\mathbf{y} = \mathbf{y}^T\mathbf{x}.$$

(2) If $\mathbf{x}$ and $\mathbf{y}$ are two conforming vectors and $\mathbf{A}$ a conforming matrix, the quadratic form $\mathbf{x}^T\mathbf{A}\mathbf{y}$ is also a scalar; consequently

$$\mathbf{x}^T\mathbf{A}\mathbf{y} = \mathbf{y}^T\mathbf{A}^T\mathbf{x}.$$

Furthermore, if $\mathbf{A}$ is square and symmetric, $\mathbf{x}^T\mathbf{A}\mathbf{y} = \mathbf{y}^T\mathbf{A}\mathbf{x}$.

(3) If $\mathbf{x}$ and $\mathbf{y}$ are conforming vectors, $\mathbf{x}\mathbf{y}^T$ and $\mathbf{y}\mathbf{x}^T$ are rank-one square matrices (the transpose of each other). Furthermore, $\mathbf{x}\mathbf{x}^T$ is a symmetric matrix.

(4) Scalars can be moved to any position within a matrix product. If the scalar is in itself the result of a vector or matrix product, the components may be transposed as per rules (1) and (2). For example, if $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are conforming matrices, and $\mathbf{x}$ and $\mathbf{y}$ are conforming vectors,

$$\mathbf{x}^T\mathbf{y}\mathbf{A}\mathbf{B} = \mathbf{A}\mathbf{x}^T\mathbf{y}\mathbf{B} = \mathbf{A}\mathbf{B}\mathbf{x}^T\mathbf{y} = \mathbf{A}\mathbf{B}\mathbf{y}^T\mathbf{x}, \quad \text{etc.}$$

$$\mathbf{x}^T\mathbf{A}\mathbf{y}\mathbf{B}\mathbf{C} = \mathbf{y}^T\mathbf{A}^T\mathbf{x}\mathbf{B}\mathbf{C} = \mathbf{B}\mathbf{x}^T\mathbf{A}\mathbf{y}\mathbf{C} = \mathbf{B}\mathbf{y}^T\mathbf{A}^T\mathbf{x}\mathbf{C}, \quad \text{etc.}$$