H

The Small Strain TL C1 Plane Beam

§H.1 SUMMARY

This Appendix derives the discrete equations of a geometrically nonlinear, C^1 (Hermitian), prismatic, plane beam-column in the framework of the Total Lagrangian (TL) description. The formulation is restricted to the three deformational degrees of freedom: d, θ_1 and θ_2 shown in Figure H.1. The element rigid body motions have been removed by forcing the transverse deflections at the end nodes to vanish. The strains are assumed to be *small* while the cross section rotations θ are small but finite.

Given the foregoing kinematic limitations, this element is evidently of no use *per se* in geometrically nonlinear analysis. Its value is in providing the local equations for a TL/CR formulation

§H.2 FORMULATION OF GOVERING EQUATIONS

§H.2.1 Kinematics

We consider a geometrically nonlinear, prismatic, homogenous, isotropic elastic, plane beam element that deforms in the x, y plane as shown in Figure H.1. The element has cross section area A_0 and moment of inertia I_0 in the reference configuration, and elastic modulus E.

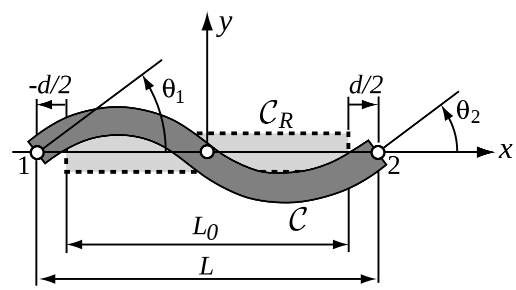

Figure H.1 Kinematics of TL Hermitian beam element

The plane motion of the beam is described by the two dimensional displacement field $\{u_x(x,y), u_y(x,y)\}$ where u_x and u_y are the axial and transverse displacement components, respectively, of arbitrary points within the element. The rotation of the cross section is $\theta(x)$, which is assumed small. The following kinematic assumptions of thin beam theory are used

$$\begin{bmatrix} u_x(x,y) \\ u_y(x,y) \end{bmatrix} = \begin{bmatrix} u_x^a(x) - y\dfrac{\partial u_y^a(x)}{\partial x} \\ u_y^a(x) \end{bmatrix} = \begin{bmatrix} u_x^a(x) - y\theta(x) \\ u_y^a(x) \end{bmatrix} \tag{H.1}$$

where u_x^a and u_y^a denote the displacements of the neutral axes, and $\theta(x) = \partial u_y^a/\partial x$ is the rotation of the cross section. The three degrees of freedom of the beam element are

$$\mathbf{u}^e = \begin{bmatrix} d \\ \theta_1 \\ \theta_2 \end{bmatrix} \tag{H.2}$$

§H.2.2 Strains

We introduce the notation

$$\epsilon = \frac{\partial u_x}{\partial x}, \qquad \kappa = \frac{\partial \theta}{\partial x} = \frac{\partial^2 u_y^a}{\partial\, x^2}. \tag{H.3}$$

for engineering axial strain and beam curvature, respectively. The exact Green-Lagrange measure of axial strain is

$$e = \frac{\partial u_x}{\partial x} + \tfrac{1}{2}\left(\frac{\partial u_x}{\partial x}\right)^2 + \tfrac{1}{2}\left(\frac{\partial u_y}{\partial x}\right)^2 = \epsilon - y\kappa + \tfrac{1}{2}(\epsilon - y\kappa)^2 + \tfrac{1}{2}\theta^2 \tag{H.4}$$

This can be expressed in terms of the displacement gradients as follows:

$$e = \mathbf{h}^T\mathbf{g} + \tfrac{1}{2}\mathbf{g}^T\mathbf{H}\mathbf{g} = \mathbf{c}^T\mathbf{g} \tag{H.5}$$

where

$$\mathbf{g} = \begin{bmatrix} \partial u_x^a/\partial x \\ \partial u_y^a/\partial x \\ \partial^2 u_y^a/\partial x^2 \end{bmatrix} = \begin{bmatrix} \epsilon \\ \theta \\ \kappa \end{bmatrix}, \qquad \mathbf{h} = \begin{bmatrix} 1 \\ 0 \\ -y \end{bmatrix}, \qquad \mathbf{H} = \begin{bmatrix} 1 & 0 & -y \\ 0 & 1 & 0 \\ -y & 0 & y^2 \end{bmatrix} \tag{H.6}$$

We simplify this expression by dropping all y dependent terms form the $\mathbf{H}$ matrix:

$$\hat{\mathbf{H}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{H.7}$$

The simplified axial strain is

$$e = \mathbf{h}^T\mathbf{g} + \tfrac{1}{2}\mathbf{g}^T\hat{\mathbf{H}}\mathbf{g} = \epsilon - y\kappa + \tfrac{1}{2}\epsilon^2 + \tfrac{1}{2}\theta^2 \tag{H.8}$$

The rational for this selective simplification is that $e_a = \epsilon + \frac{1}{2}\epsilon^2$ is the GL mean axial strain. If the $\frac{1}{2}\epsilon^2$ term is retained, a simpler geometric stiffness is obtained. The term $\frac{1}{2}\theta^2$ is the main nonlinear effect contributed by the section rotations.

The vectors that appear in the CCF formulation of TL finite elements discussed in Chapters 10-11 are

$$\mathbf{b} = \mathbf{h} + \mathbf{H}\mathbf{g} = \begin{bmatrix} 1+\epsilon \\ \theta \\ -y \end{bmatrix}, \quad \mathbf{c} = \mathbf{h} + \tfrac{1}{2}\mathbf{H}\mathbf{g} = \begin{bmatrix} 1+\frac{1}{2}\epsilon \\ \frac{1}{2}\theta \\ -y \end{bmatrix}, \tag{H.9}$$

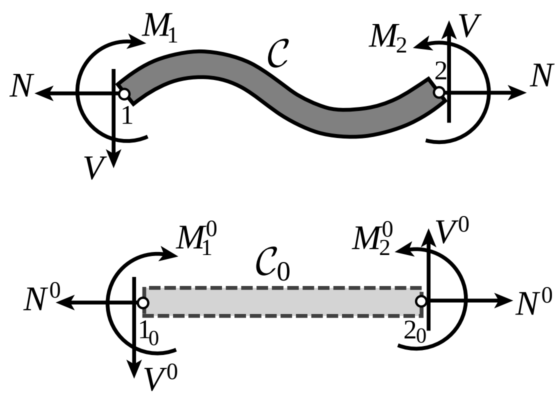

Figure H.2. Stress resultants in reference and current configurations. Configurations shown offset for clarity.

§H.2.3 Stresses and Stress Resultants

The stress resultants in the reference configuration are N^0, M_1^0 and M_2^0. The initial shear force is $V^0 = (M_1^0 - M_2^0)/L_0$. The axial force N^0 and transvese shear force V^0 are constant along the element, whereas the bending moment $M^0(x)$ is linearly interpolated from $M^0 = M_1^0(1-x/L_0) + M_2^0 x/L_0$. See Figure H.2 for sign conventions. The initial PK2 axial stress is computed using beam theory:

$$s^0 = \frac{N^0}{A_0} - \frac{M^0 y}{I_0} \tag{H.10}$$

Denote by N, V and M the stress resultants in the current configuration. Whereas N and V are constant along the element, $M = M(x)$ varies linearly along the length because this is a Hermitian model, which relies on cubic transverse displacements. Consequently we will define its variation by the two node values M_1 and M_2. The shear V is recovered from equilibrium as $V = (M_1 - M_2)/L$, which is also constant. The PK2 axial stress in the current state is $s = s^0 + Ee = s^0 + E\mathbf{c}^T\mathbf{g}$, or inserting (H.9):

$$s = s^0 + E\left(\epsilon + \tfrac{1}{2}\epsilon^2 + \tfrac{1}{2}\theta^2 - y\kappa\right) \tag{H.11}$$

§H.2.4 Constitutive Equations

Integrating (H.11) over the cross section one gets the constitutive equations in terms of resultants:

$$\begin{aligned} N &= \int_{A_0} s\,dA = s^0 A_0 + EA_0(\epsilon + \tfrac{1}{2}\epsilon^2 + \tfrac{1}{2}\theta^2) = N^0 + EA_0(e_a + \tfrac{1}{2}\theta^2),\\ M &= -\int_{A_0} ys\,dA = M^0 + EI_0\kappa \end{aligned} \tag{H.12}$$

§H.2.5 Strain Energy Density

We shall use the CCF formulation presented in Chapter 10 to derive the stiffness equations. Using $\alpha = \beta = 1$ (not a spectral form) one obtains the core energy of a beam particle as

$$\mathcal{U} = \\ = \tfrac{1}{2}\mathbf{g}^T \left(E \begin{bmatrix} (1+\frac{1}{2}\epsilon)^2 + \frac{1}{4}\theta^2 - \frac{1}{3}y\kappa & \frac{1}{3}\theta & -y(1+\frac{1}{3}\epsilon) \\ \frac{1}{3}\theta & \frac{1}{4}(\epsilon^2+\theta^2) + \frac{1}{3}(\epsilon - y\kappa) & -\frac{1}{3}y\theta \\ -y(1+\frac{1}{3}\epsilon) & -\frac{1}{3}y\theta & y^2 \end{bmatrix} + s^0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \mathbf{g} \tag{H.13}$$

Integration over this cross section yields the strain energy per unit of beam length:

$$\begin{aligned} U_A &= \tfrac{1}{2}\mathbf{g}^T \int_{A_0} (E\,\mathbf{c}\,\mathbf{c}^T + s^0\mathbf{H})\,dA\ \mathbf{g} \\ &= \tfrac{1}{2}\mathbf{g}^T \left(E \begin{bmatrix} (1+\frac{1}{2}\epsilon)^2 A_0 & \frac{1}{2}(1+\frac{1}{2}\epsilon)\theta A_0 & 0 \\ \frac{1}{2}(1+\frac{1}{2}\epsilon)\theta A_0 & \frac{1}{4}\theta^2 A_0 & 0 \\ 0 & 0 & I_0 \end{bmatrix} + N^0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \mathbf{g} \end{aligned} \tag{H.14}$$

To obtain the element energy it is necessary to specify the variation of ϵ, θ and κ along the beam. At this point shape functions have to be introduced.

§H.2.6 Shape Functions

Define the isoparametric coordinate $\xi = 2x/L_0$. The displacement interpolation is taken to be the same used for the linear beam element:

$$\begin{bmatrix} u_x^a \\ u_y^a \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\xi & 0 & 0 \\ 0 & \frac{1}{8}L_0(1-\xi)^2(1+\xi) & \frac{1}{8}L_0(1+\xi)^2(1-\xi) \end{bmatrix} \begin{bmatrix} d \\ \theta_1 \\ \theta_2 \end{bmatrix}. \tag{H.15}$$

From this the displacement gradients are

$$\mathbf{g} = \begin{bmatrix} \epsilon \\ \theta \\ \kappa \end{bmatrix} = \frac{1}{L_0} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4}L_0(\xi-1)(3\xi+1) & \frac{1}{4}L_0(1+\xi)(3\xi-1) \\ 0 & 3\xi - 1 & 3\xi + 1 \end{bmatrix} \begin{bmatrix} d \\ \theta_1 \\ \theta_2 \end{bmatrix} = \mathbf{G}\mathbf{u}^e. \tag{H.16}$$

The rotation θ varies quadritically and the curvature θ linearly. The node values are obtained on setting $\xi = \pm 1$:

$$\mathbf{g}_1 = \begin{bmatrix} \epsilon_1 \\ \theta_1 \\ \kappa_1 \end{bmatrix} = \frac{1}{L_0} \begin{bmatrix} 1 & 0 & 0 \\ 0 & L_0 & 0 \\ 0 & -4 & -2 \end{bmatrix} \mathbf{u}^e, \qquad \mathbf{g}_2 = \begin{bmatrix} \epsilon_2 \\ \theta_2 \\ \kappa_2 \end{bmatrix} = \frac{1}{L_0} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & L_0 \\ 0 & 2 & 4 \end{bmatrix} \mathbf{u}^e \tag{H.17}$$

§H.2.7 Element Energy

The strain energy of the element can be now obtained by expressing the gradients $\mathbf{g} = \mathbf{G}\mathbf{u}^e$ and integrating over the length. the result can be expressed as

$$U^e = \int_{-\frac{1}{2}L_0}^{\frac{1}{2}L_0} U_A\,dA = \int_{-1}^{1} U_A\,\tfrac{1}{2}L_0\,d\xi = \tfrac{1}{2}(\mathbf{u}^e)^T\mathbf{K}^U\mathbf{u}^e \tag{H.18}$$

where the energy stiffness is the sum of three contributions: $\mathbf{K}^U = \mathbf{K}^U_a + \mathbf{K}^U_b + \mathbf{K}^U_N$. These come from the axial deformations, bending deformations and initial stress, respectively:

$$\mathbf{K}^U_a = \frac{EA_0}{L_0}\begin{bmatrix} (1+\frac{1}{2}\epsilon)^2 & \frac{(1+\frac{1}{2}\epsilon)(4\theta_1-\theta_2)L_0}{60} & \frac{(1+\frac{1}{2}\epsilon)(-\theta_1+4\theta_2)L_0}{60} \\ \frac{(1+\frac{1}{2}\epsilon)(4\theta_1-\theta_2)L_0}{60} & \frac{(12\theta_1^2-3\theta_1\theta_2+\theta_2^2)L_0^2}{840} & \frac{(-3\theta_1^2+4\theta_1\theta_2-3\theta_2^2)L_0^2}{1680} \\ \frac{(1+\frac{1}{2}\epsilon)(-\theta_1+4\theta_2)L_0}{60} & \frac{(-3\theta_1^2+4\theta_1\theta_2-3\theta_2^2)L_0^2}{1680} & \frac{(\theta_1^2-3\theta_1\theta_2+12\theta_2^2)L_0^2}{840} \end{bmatrix},$$
$$\mathbf{K}^U_b = \frac{EI_0}{L_0}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 4 \end{bmatrix}, \quad \mathbf{K}^U_N = \frac{N_0}{L_0}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2L_0^2/15 & L_0^2/30 \\ 0 & L_0^2/30 & 2L_0^2/15 \end{bmatrix}. \tag{H.19}$$

§H.3 INTERNAL FORCE

The internal force $\mathbf{p}$ is obtained as the derivative

$$\mathbf{p} = \frac{\partial U^e}{\partial \mathbf{u}^e} = \left(\mathbf{K}^U + \tfrac{1}{2}(\mathbf{u}^e)^T \frac{\partial \mathbf{K}^U}{\partial \mathbf{u}^e}\right)\mathbf{u}^e = \mathbf{K}^p \mathbf{u}^e \tag{H.20}$$

The internal force stiffness is again the sum of three contributions: $\mathbf{K}^p = \mathbf{K}^p_a + \mathbf{K}^p_b + \mathbf{K}^p_N$. These come from the axial deformations, bending deformations and initial stress, respectively:

$$\mathbf{K}^p_a = \frac{EA_0}{L_0}\begin{bmatrix} 1+\frac{3}{2}\epsilon+\frac{1}{2}\epsilon^2 & \frac{(3+2\epsilon)(4\theta_1-\theta_2)L_0}{120} & \frac{(3+2\epsilon)(-\theta_1+4\theta_2)L_0}{120} \\ \frac{(3+2\epsilon)(4\theta_1-\theta_2)L_0}{120} & \frac{(12\theta_1^2-3\theta_1\theta_2+\theta_2^2)L_0^2}{420} & \frac{(-3\theta_1^2+4\theta_1\theta_2-3\theta_2^2)L_0^2}{840} \\ \frac{(3+2\epsilon)(-\theta_1+4\theta_2)L_0}{120} & \frac{(-3\theta_1^2+4\theta_1\theta_2-3\theta_2^2)L_0^2}{840} & \frac{(\theta_1^2-3\theta_1\theta_2+12\theta_2^2)L_0^2}{420} \end{bmatrix},$$
$$\mathbf{K}^p_b = \frac{EI_0}{L_0}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 4 \end{bmatrix}, \quad \mathbf{K}^p_N = \frac{N_0}{L_0}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2L_0^2/15 & L_0^2/30 \\ 0 & L_0^2/30 & 2L_0^2/15 \end{bmatrix}. \tag{H.21}$$

§H.4 TANGENT STIFFNESS

The tangent stiffness $\mathbf{K}$ is obtained as the derivative

$$\mathbf{K} = \frac{\partial \mathbf{p}}{\partial \mathbf{u}^e} = \left(\mathbf{K}^r + (\mathbf{u}^e)^T \frac{\partial \mathbf{K}^r}{\partial \mathbf{u}^e}\right)\mathbf{u}^e \tag{H.22}$$

This is again the sum of three contributions: $\mathbf{K} = \mathbf{K}_a + \mathbf{K}_b + \mathbf{K}_N$, which come from the axial deformations, bending deformations and current stress, respectively:

$$
\mathbf{K}_a = \frac{E A_0}{L_0}\begin{bmatrix} (1+\epsilon)^2 & \dfrac{(1+\epsilon)(4\theta_1-\theta_2)L_0}{30} & \dfrac{(1+\epsilon)(-\theta_1+4\theta_2)L_0}{30} \\ \dfrac{(1+\epsilon)(4\theta_1-\theta_2)L_0}{30} & \dfrac{(12\theta_1^2-3\theta_1\theta_2+\theta_2^2)L_0^2}{210} & \dfrac{(-3\theta_1^2+4\theta_1\theta_2-3\theta_2^2)L_0^2}{420} \\ \dfrac{(1+\epsilon)(-\theta_1+4\theta_2)L_0}{30} & \dfrac{(-3\theta_1^2+4\theta_1\theta_2-3\theta_2^2)L_0^2}{420} & \dfrac{(\theta_1^2-3\theta_1\theta_2+12\theta_2^2)L_0^2}{210} \end{bmatrix},
$$

$$
\mathbf{K}_b = \frac{E I_0}{L_0}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 4 \end{bmatrix}, \quad \mathbf{K}_N = \frac{N}{30 L_0}\begin{bmatrix} 30 & 0 & 0 \\ 0 & 4L_0^2 & -L_0^2 \\ 0 & -L_0^2 & 4L_0^2 \end{bmatrix}. \tag{H.23}
$$

The material stiffness is $\mathbf{K}_M = \mathbf{K}_a + \mathbf{K}_b$ and the geometric stiffness is $\mathbf{K}_G = \mathbf{K}_N$.