Cvičení č.6 - Stacionární vedení tepla v 1D

Diskretizovaná úloha

$$\left(\boldsymbol{K}^e_\Omega + \boldsymbol{K}^e_\Gamma\right) r^e = f^e_{\Gamma_c} + f^e_{\Gamma_p} + f^e_\Omega$$

- Levá strana
  - Matice vodivosti K^e_Ω

$$\boldsymbol{K}^e_\Omega = \int_\Omega \boldsymbol{B}^{eT} \lambda \, \boldsymbol{B}^e \, ds$$

  - Příspěvek do levé strany od přestupu tepla K^e_Γ

$$\boldsymbol{K}^e_\Gamma = \int_{\Gamma_c} \boldsymbol{N}^{eT} \alpha^e \boldsymbol{N}^e \, ds$$

- Pravá strana
  - Přestup tepla $f^e_{\Gamma_c}$

$$f^e_{\Gamma_c} = \int_{\Gamma_c} \boldsymbol{N}^{eT} \alpha^e T^e_0(x) \, ds$$

  - Předepsaný tok $f^e_{\Gamma_p}$

$$f^e_{\Gamma_p} = -\int_{\Gamma_p} \boldsymbol{N}^{eT} \bar{q}^e(x) \, ds$$

  - Vnitřní zdroj f^e_Ω

$$f^e_\Omega = \int_\Omega \boldsymbol{N}^{eT} Q^e(x) \, ds$$

Okrajové podmínky

- Dirichletova okrajová podmínka - předepsaná teplota na hranici

$$T(x) = \bar{T}(x), \qquad \text{pro } x \in \Gamma_T$$

- Neumannova okrajová podmínka - předepsaný tok na hranici

$$q_x(x) n(x) = \bar{q_x}(x), \qquad \text{pro } x \in \Gamma_{qp}$$

- Smíšená okrajová podmínka - přestup tepla

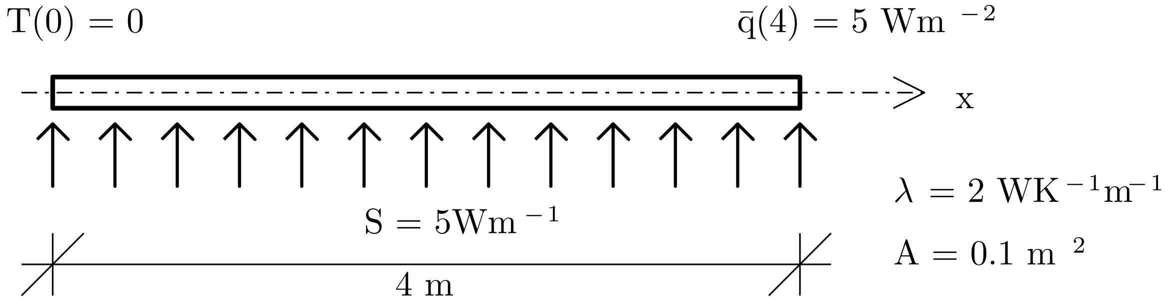
$$\bar{q_x}(x) = \alpha(x)(T(x) - T_0(x)), \qquad \text{pro } x \in \Gamma_{qc}$$

- Radiace - přestup tepla

$$\bar{q_x}(x) = \varepsilon(x)\sigma(x)(T^4(x) - T^4_\infty(x)), \qquad \text{pro } x \in \Gamma_{qc}$$

  (ε - povrchové vyzařování, σ - Stefan-Boltzmanova konstanta)

Příklad 1



Lineární aproximace

- Matice interpolačních funkcí

$$\boldsymbol{N}^e = \left[\frac{x_2 - x}{l^e}, \frac{x - x_1}{l^e} \right]$$

- Geometrická matice (derivace interpolačních funkcí)

$$\boldsymbol{B}^e = \left[\frac{-1}{l^e}, \frac{1}{l^e} \right]$$

- Matice vodivosti

$$\boldsymbol{K}^e = \int_\Omega \boldsymbol{B}^{eT} \lambda \, \boldsymbol{B}^e \, ds = \int_\Omega \begin{bmatrix} -1/l^e \\ 1/l^e \end{bmatrix} \lambda \begin{bmatrix} -1/l^e & 1/l^e \end{bmatrix} \, ds = \frac{\lambda}{l^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

- Vnitřní zdroj

$$f_\Omega^e = \int_\Omega \boldsymbol{N}^{eT} Q^e(x) \, ds = \int_{x_i}^{x^j} \begin{bmatrix} (x_j - x)/l^e \\ (x - x_i)/l^e \end{bmatrix} \frac{S}{A^e} dx = \begin{bmatrix} \left(x_j x - \frac{x^2}{2} \right) / l^e \\ \left(\frac{x^2}{2} - x_i x \right) / l^e \end{bmatrix}_{x_i}^{x^j} \frac{S}{A^e}$$

- Předepsaný tok $f_{\Gamma_p}^e$

$$f_{\Gamma_p}^e = -\int_{\Gamma_p} \boldsymbol{N}^{eT} \, \bar{q}^e(x) \, d\Gamma_p = -\int_{\Gamma_p} \begin{bmatrix} (x_j - x)/l^e \\ (x - x_i)/l^e \end{bmatrix} \bar{q}^e(x) \, d\Gamma_p = -\begin{bmatrix} (x_j - x)/l^e \\ (x - x_i)/l^e \end{bmatrix} \bar{q}(4) \bigg|_{x=4}$$

Řešení jedním prvkem

$l_1 = 4, \; x_1 = 0, \; x_2 = 4$

- Matice vodivosti

$$\boldsymbol{K}^1 = \frac{\lambda}{l_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

- Vnitřní zdroj

$$f_\Omega^1 = \begin{bmatrix} \left(x_2 x - \frac{x^2}{2} \right) / l_1 \\ \left(\frac{x^2}{2} - x_1 x \right) / l_1 \end{bmatrix}_{x_1}^{x^2} \frac{S}{A^e} = \begin{bmatrix} \left(4x - \frac{x^2}{2} \right) / 4 \\ \left(\frac{x^2}{2} \right) / 4 \end{bmatrix}_0^4 \frac{5}{0.1} = \begin{Bmatrix} 100.0 \\ 100.0 \end{Bmatrix}$$

- Předepsaný tok

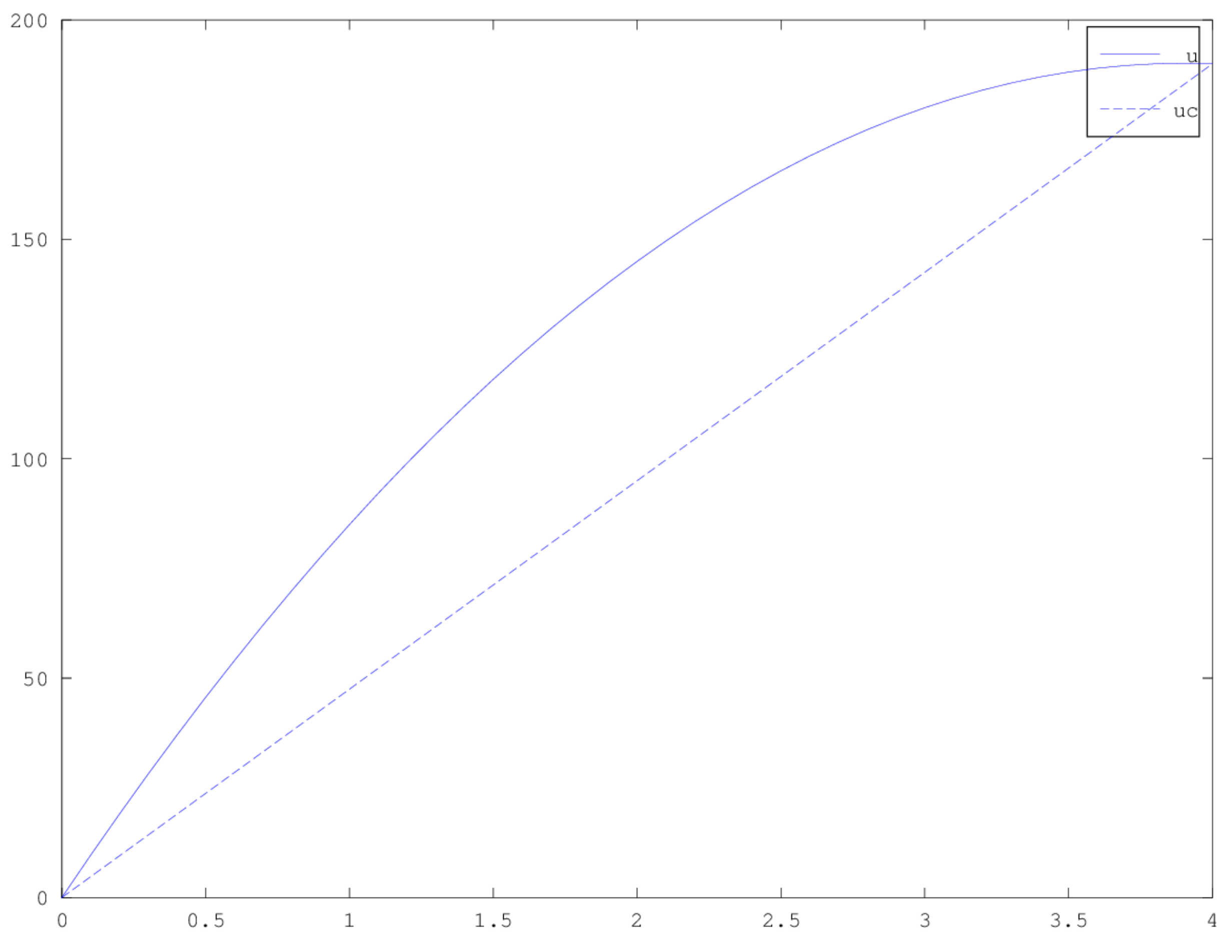
$$f_{\Gamma_p}^1 = -\begin{bmatrix} (x_2 - x)/l_1 \\ (x - x_1)/l_1 \end{bmatrix} \bar{q}(4) \bigg|_{x_2} = -\begin{bmatrix} (4 - x)/4 \\ x/4 \end{bmatrix} 5 \bigg|_{x_2} = -\begin{Bmatrix} 0.0 \\ 5.0 \end{Bmatrix}$$

Globální soustava rovnic

$$\begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{Bmatrix} 0 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} r_1 + 100 \\ 95 \end{Bmatrix}$$

In [2]:
```
l1 = 4;
A = 0.1;
lambda = 2;
x1 = 0; x2 = 4;

k1 = (lambda/l1)*[1 -1; -1 1];

K = k1;
fo = (5/0.1)*[(x2*x2-x2^2/2)/l1 (x2^2/2)/l1];
fgp = 5*[0; -1];
f1 = fo + fgp;
F = f1;

u = K(2,2)\F(2);
U = [0 u]

hold on;
x = 0:0.1:4;
plot (x, -12.5*x.^2+97.5*x, "b;u;")
plot ([0 l1], [U(1) U(2)], "b--;uc;")
```

```
U =

     0   190
```



Řešení dvěma prvky

Prvek 1: $l_1 = 2,\ x_1 = 0,\ x_2 = 2$

Prvek 2: $l_2 = 2,\ x_2 = 2,\ x_3 = 4$

- Matice vodivosti prvků 1, 2

$$\boldsymbol{K}^1 = \boldsymbol{K}^2 = \frac{\lambda}{l^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{2}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

- Matice vodivosti konstrukce (po lokalizaci)

$$\boldsymbol{K} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

- Vnitřní zdroj

$$f_\Omega^1 = \begin{bmatrix} \left(x_2 x - \frac{x^2}{2}\right)/l_1 \\ \left(\frac{x^2}{2} - x_1 x\right)/l_1 \end{bmatrix}_{x_1}^{x^2} \frac{S}{A^e} = \begin{bmatrix} \left(2x - \frac{x^2}{2}\right)/2 \\ \left(\frac{x^2}{2}\right)/2 \end{bmatrix}_0^2 \frac{5}{0.1} = \begin{Bmatrix} 50.0 \\ 50.0 \end{Bmatrix}$$

$$f_\Omega^2 = f_\Omega^1$$

- Předepsaný tok $f_{\Gamma_p}^e$

$$f_{\Gamma_p}^2 = -\begin{bmatrix} (x_3 - x)/l_2 \\ (x - x_2)/l_2 \end{bmatrix} \bar{q}(4)\bigg|_{x_3} = -\begin{bmatrix} (4 - x)/2 \\ (x - 2)/2 \end{bmatrix} 5\bigg|_{x_3} = -\begin{Bmatrix} 0.0 \\ 5.0 \end{Bmatrix}$$

Globální soustava rovnic

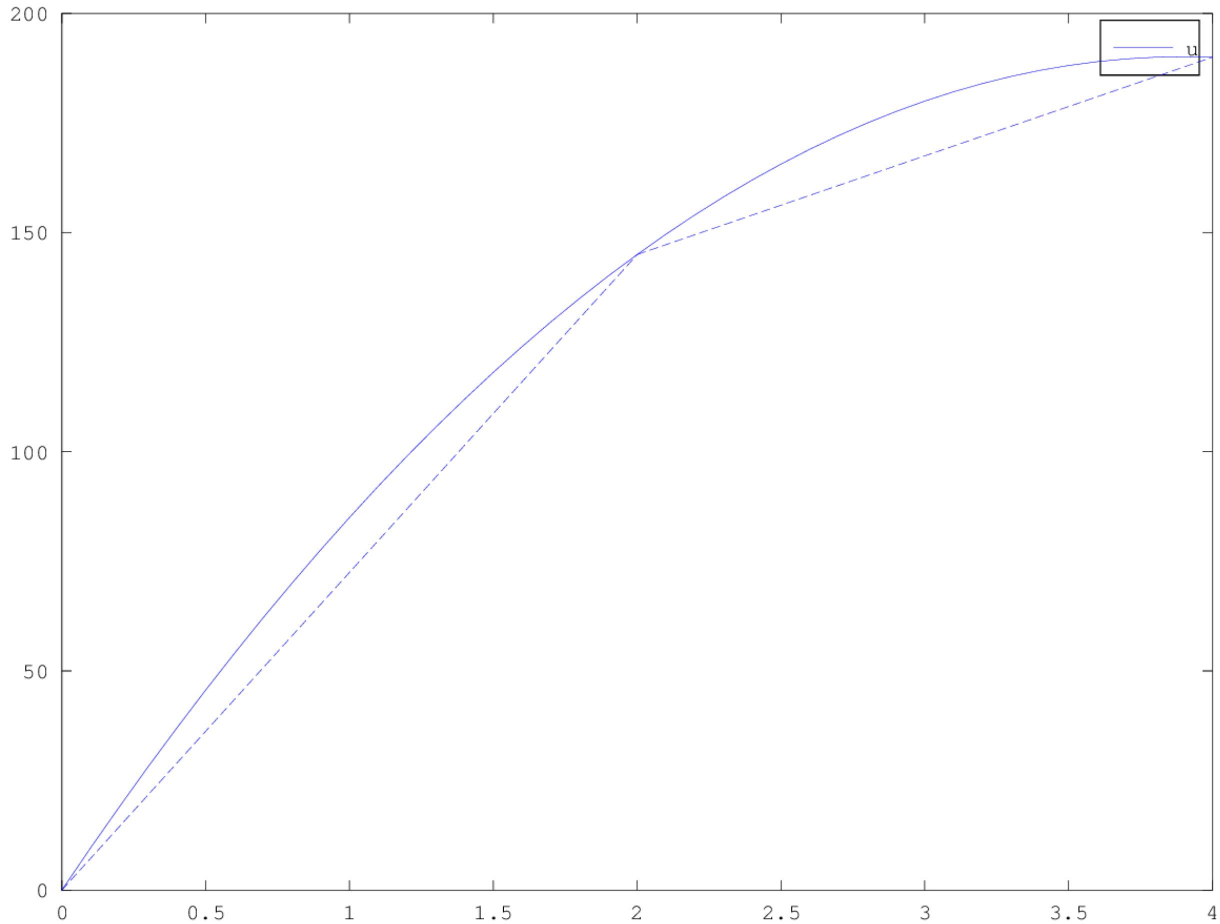
$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} r_1 + 50 \\ 100 \\ 45 \end{Bmatrix}$$

Řešení n prvky

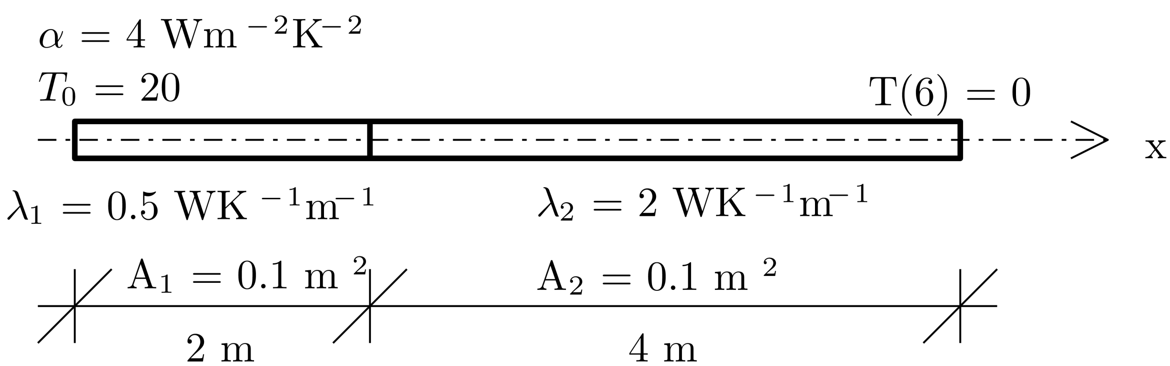
In [2]:

```
n = 2
l = 4/n;

A = 0.1;
lambda = 2;
x1 = 0; x2 = 4/n;

ki = (lambda/l)*[1 -1; -1 1];
fo = (5/0.1)*[(x2*x2-x2^2/2)/l (x2^2/2)/l];
fgp = 5*[0 -x2/l];

K = zeros (n+1);
F = zeros (n+1, 1);
for i=1:n
    loc = [i i+1];
    K(loc, loc) += ki;
    F(loc)+= fo';
end
F([n n+1]) += fgp';
u = K(2:n+1, 2:n+1)\(F(2:n+1,1));
U = [0 ; u]

#plot analytical solution
hold on;
x = 0:0.1:4;
plot (x, -12.5*x.^2+97.5*x, "b;u;")

#plot obtained solution
for i=1:n
    plot ([(i-1)*l i*l], [U(i) U(i+1)], "b--")
end
```

```
n =  2
U =

     0.00000
   145.00000
   190.00000
```



Příklad 2



Řešení dvěma prvky s lineární aproximací

- Matice vodivosti

$$\boldsymbol{K}^1 = \frac{\lambda_1}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{0.5}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\boldsymbol{K}^2 = \frac{\lambda_2}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{2}{4}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

- Příspěvek do matice soustavy od přestupu tepla

$$\boldsymbol{K}_\Gamma^1 = \int_{\Gamma_c} \boldsymbol{N}^{eT} \alpha^e \boldsymbol{N}^e \; ds = \begin{bmatrix} (x_2 - x)/l_1 \\ (x - x_1)/l_1 \end{bmatrix} \alpha \left[\frac{x_2 - x}{l_1}, \; \frac{x - x_1}{l_1}\right]\Bigg|_{x_1} = 4\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

- Přestup tepla

$$f_{\Gamma_c}^1 = \int_{\Gamma_c} \boldsymbol{N}^{eT} \alpha^e T_0(x) \; ds = \begin{bmatrix} (x_2 - x)/l_1 \\ (x - x_1)/l_1 \end{bmatrix} \alpha T_0 \Bigg|_{x_1} = 4 \cdot 20 \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 80 \\ 0 \end{Bmatrix}$$

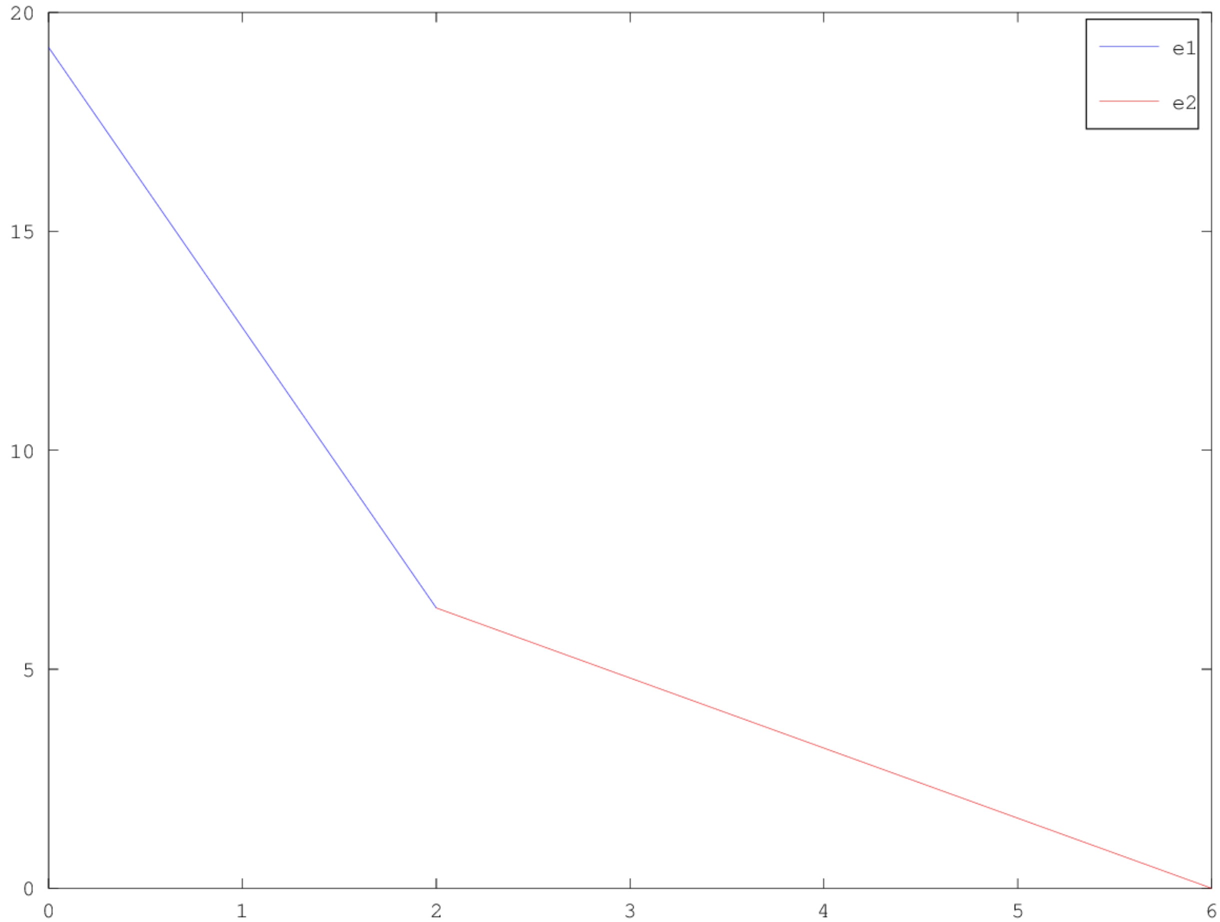
Globální soustava rovnic

$$\begin{bmatrix} 4.25 & -0.25 & 0 \\ -0.25 & 0.75 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 80 \\ 0 \\ r_1 \end{Bmatrix}$$

In [2]:

```
l1 = 2; l2 = 4;
A = 0.1;
lambda1 = 0.5; lambda2 = 2;
x1 = 0; x2 = 2; x3 = 6;
alpha = 4;
T0 = 20;

loc1 = [1 2];
loc2 = [2 3];

k1 = (lambda1/l1)*[1 -1; -1 1];
k1 += alpha*[1 0; 0 0];
k2 = (lambda2/l2)*[1 -1; -1 1];

fgc1 = alpha*T0*[1; 0];

K = zeros(3,3);
K(loc1,loc1) += k1;
K(loc2,loc2) += k2

F = zeros(3,1);
F(loc1) += fgc1

u = K(1:2,1:2)\F(1:2);
U = [u; 0]

hold on;
plot ([x1 x2], [U(1) U(2)], "b-;e1;")
plot ([x2 x3], [U(2) U(3)], "r-;e2;")
```

```
K =

   4.25000  -0.25000   0.00000
  -0.25000   0.75000  -0.50000
   0.00000  -0.50000   0.50000

F =

   80
    0
    0

U =

   19.20000
    6.40000
    0.00000
```



In [ ]: