11

The Core Congruential Formulation: Stiffness Equations

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The present Chapter completes the development of the Core Congruential Formulation (CCF) of Total Lagrangian (TL) elements. It present procedures to transform core equations to finite element stiffness equations.

§11.1. DCCF Transformation to Physical Freedoms

Core expressions for the internal-force vector and stiffness matrices of an individual TL element are given in (10.20)and (10.26)-(10.29), respectively. These expressions pertain to *material particles* of the structure. The behavior of each particle is expressed in terms of its displacement gradients collected in vector **g**. To create a discrete model the structure is subdivided into finite elements. Finite elements equations in terms of the physical DOFs collected in vector **v** are constructed through a combination of core-to-physical transformations and integration over element domains.

In this section we stay within the scope of the Direct CCF by assuming that the transformations between **g** and **v** are linear. Because all subsequent developments pertain to an individual element, no element identifiers are used to reduce indexing clutter.

Over an individual element the displacement field $\mathbf{u}^T = (u_1,\ u_2,\ u_3)$ is interpolated as

$$\mathbf{u} = \mathbf{N}\,\mathbf{v}, \tag{11.1}$$

where **v** now collects the element node-displacement degrees of freedom (DOFs) and $\mathbf{N} = \mathbf{N}(X_1, X_2, X_3)$ is a matrix of shape functions *independent* of **v**. Differentiating (11.1) with respect to the X_i and taking the first two **v** variations yields

$$\mathbf{g} = \mathbf{G}\mathbf{v}, \qquad \delta\mathbf{g} = \mathbf{G}\,\delta\mathbf{v}, \qquad \delta^2\mathbf{g} = \mathbf{0}, \tag{11.2}$$

(for the last one see Remark 10.2). Invariance of the strain energy variations δU and $\delta^2 U$ obtained by integrating (10.14)-(10.15) over the element reference volume yields

$$\mathbf{K}^U = \int_{V_0} \mathbf{G}^T\mathbf{S}^U\mathbf{G}\,dV, \qquad \mathbf{K}^r = \int_{V_0} \mathbf{G}^T\mathbf{S}^r\mathbf{G}\,dV, \qquad \mathbf{K} = \int_{V_0} \mathbf{G}^T\mathbf{S}\mathbf{G}\,dV, \tag{11.3}$$

$$\mathbf{f} = \int_{V_0} \mathbf{G}^T\mathbf{\Phi}\ dV, \qquad \mathbf{p} = \int_{V_0} \mathbf{G}^T\mathbf{\Psi}\ dV, \qquad \mathbf{p}^0 = \int_{V_0} \mathbf{G}^T\mathbf{\Psi}^0\ dV, \tag{11.4}$$

Although the dependency of $\mathbf{S}^{level}$ and $\mathbf{\Psi}$ on **g** is not made implicit in these equations, it must be remembered that the transformation $\mathbf{g} = \mathbf{G}\mathbf{v}$ also appears there. Because of the ensuing algebraic complexity, numerical integration is generally required unless the gradients are constant over the element.

Often **G** is expressed as a chain of transformations, some of which are position dependent and remain inside the element integral whereas others are not and may be taken outside. For example, in the bar element treated below, $\mathbf{G} = \mathbf{T}\,\bar{\mathbf{G}}$, where $\bar{\mathbf{G}}$ transforms **g** to local node displacements while **T** transforms local to global node displacements.

§11.2. DCCF Transformation Examples

§11.2.1. The Bar Element

The core equations for a geometrically nonlinear TL bar were derived in Section 5.1. These equations are now applied to the formulation of a two-node, linear-displacement, prismatic TL bar element. The element has constant reference area A_0 and initial length L_0. The two end nodes are located at (X_1, Y_1, Z_1) and (X_2, Y_2, Z_2), respectively. The node displacements are (v_{X1}, v_{Y1}, v_{Z1}) and (v_{X2}, v_{Y2}, v_{Z2}). The element displacement field in local coordinates $\{\bar{X}, \bar{Y}, \bar{Z}\}$ may be interpolated as

$$\bar{\mathbf{u}} = \begin{bmatrix} \bar{u}_X \\ \bar{u}_Y \\ \bar{u}_Z \end{bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 \end{bmatrix} \begin{bmatrix} \bar{v}_{X1} \\ \bar{v}_{Y1} \\ \bar{v}_{Z1} \\ \bar{v}_{X2} \\ \bar{v}_{Y2} \\ \bar{v}_{Z2} \end{bmatrix} = \mathbf{N}\,\bar{\mathbf{v}}, \tag{11.5}$$

where $N_1 = 1 - \bar{X}/L_0$ and $N_2 = \bar{X}/L$ are linear shape functions. Differentiating with respect to the reference coordinate we get

$$\mathbf{G} = \frac{1}{L_0} \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \bar{\mathbf{v}} = \frac{1}{L_0}\,\bar{\mathbf{G}}\bar{\mathbf{v}}, \tag{11.6}$$

This transformation may be applied to the core matrices and vectors derived in Chapter 10. For example, application to the core tangent stiffness (10.38) yields

$$\bar{\mathbf{K}} = \frac{1}{L_0^2} \int_{V_0} \bar{\mathbf{G}}^T \mathbf{S} \bar{\mathbf{G}}\, dV = \frac{A_0}{L_0}\, \bar{\mathbf{G}}^T (E\mathbf{b}\mathbf{b}^T + s\mathbf{H})\, \bar{\mathbf{G}}, \tag{11.7}$$

Finally, transformation to node displacements (v_{Xi}, v_{Yi}, v_{Zi}), $i = 1, 2$ is handled in the usual manner by writing the local-to-global transformation equation

$$\bar{\mathbf{u}} = \begin{bmatrix} \bar{u}_X \\ \bar{u}_Y \\ \bar{u}_Z \end{bmatrix} = \begin{bmatrix} T_{XX} & T_{XY} & T_{XZ} \\ T_{YX} & T_{YY} & T_{YZ} \\ T_{ZX} & T_{ZY} & T_{ZZ} \end{bmatrix} \begin{bmatrix} u_X \\ u_Y \\ u_Z \end{bmatrix} = \mathbf{T}\,\mathbf{u}, \tag{11.8}$$

which is valid for both end nodes giving $\bar{\mathbf{v}}_i = \mathbf{T}\mathbf{v}$, $i = 1, 2$. Consequently the element tangent stiffness matrix in local coordinates is given by

$$\mathbf{K} = \frac{A_0}{L_0} \begin{bmatrix} \mathbf{T}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{T}^T \end{bmatrix} \bar{\mathbf{G}}^T (E\mathbf{b}\mathbf{b}^T + s\mathbf{H})\, \bar{\mathbf{G}} \begin{bmatrix} \mathbf{T} & \mathbf{0} \\ \mathbf{0} & \mathbf{T} \end{bmatrix}. \tag{11.9}$$

For this simple element all entries may be obtained in closed form and no numerical integration is necessary.

§11.2.2. Iso-P Plane Stress Element

For the case of plane stress considered in §10.5.2, we shall assume that the associated finite elements are isoparametric displacement models with n nodes, and that (as usual for such models) the nodal freedoms are of translational type. The transformation to physical DOFs can then be handled within the purview of the DCCF.

As in §10.5.2 the reference system, current system and in-plane displacement components are denoted by $\{X, Y\}$, $\{x, y\}$ and $\{u_X, u_Y\}$, respectively. The element nodes are located at $\{X_i, Y_i\}$, $(i = 1, \ldots n)$ in the reference configuration $\mathcal{C}_0$ and move to $\{x_i = X_i + u_{Xi},\ y_i = Y_i + u_{Yi}\}$, $(i = 1, \ldots n)$ in the current configuration $\mathcal{C}$. The element displacement field may be expressed as

$$\begin{bmatrix} u_X \\ u_Y \end{bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & \ldots & 0 \\ 0 & N_1 & 0 & \ldots & N_n \end{bmatrix} \begin{bmatrix} v_{X1} \\ v_{Y1} \\ v_{X2} \\ \vdots \\ v_{Yn} \end{bmatrix} = \mathbf{N}\,\mathbf{v}, \tag{11.10}$$

in which N_i are appropriate isoparametric shape functions written in terms of natural coordinates such as ξ and η for quadrilaterals. The $\mathbf{G}$ matrix follows upon differentiation with respect to X and Y, and all core equations transformed as per (11.3)–(11.4). For example, the physical tangent stiffness is

$$\mathbf{K} = \int_{V_0} \mathbf{G}^T (\mathbf{S}_M + \mathbf{S}_G) \mathbf{G}\, dV, \tag{11.11}$$

where $\mathbf{S}_M$ and $\mathbf{S}_G$ are given by (10.45) and (10.46), respectively. As in the case of linear elements, (11.11) is most conveniently evaluated by numerical integration. Because several of the integrand matrices are sparse, in the interest of efficiency in the computer implementation the integrand may be symbolically evaluated through a computer algebra system such as *Macsyma*, *Maple* or *Mathematica*, and automatically converted to Fortran or C program statements before being encapsulated in the Gauss quadrature loop.

§11.3. The Generalized CCF

As discussed in §10.2, the Generalized Core Congruential Formulation or GCCF is required when the relation between displacement gradients $\mathbf{g}$ and finite element degrees of freedom $\mathbf{v}$ is nonlinear. This complication occurs in elements with rotational freedoms, such as beams, plates and shells, if *finite rotations are exactly treated*.

Recall the expression (10.15) of the second variation $\delta^2\mathcal{U}$ of the internal energy density. This expression has the core tangent stiffness $\mathbf{S}$ as kernel of the quadratic form in $\delta\mathbf{g}$. The core internal force $\mathbf{\Phi}$ also appears in the inner product $\left(\delta^2\mathbf{g}\right)^T \mathbf{\Phi}$. This second term may either survive or drop out depending on the relation of $\mathbf{g}$ with the target physical or generalized coordinates (the latter term is explained below) chosen in the CCF transformation phase. In the case of the DCCF, this term drops out and

$$\mathbf{S} = \mathbf{S}_M + \mathbf{S}_G \tag{11.12}$$

is *the* tangent core stiffness, which forward transforms as per (11.3). This is the situation considered so far. But if that term survives two things happen. First, (11.12) is relabeled as

$$\mathbf{S} = \mathbf{S}_M + \mathbf{S}_{GP}, \tag{11.13}$$

in which $\mathbf{S}$ and $\mathbf{S}_{GP}$ are called the principal core tangent stiffness and principal geometric stiffness, respectively. Second, transforming the term $\left(\delta^2\mathbf{g}\right)^T\,\mathbf{\Phi}$ to freedoms $\mathbf{v}$ produces a extra term in accordance with the schematics

$$\mathbf{K} = \mathbf{K}_M + \mathbf{K}_{GP} + \mathbf{K}_{GC}, \quad \mathbf{S}_M \rightarrow \mathbf{K}_M, \quad \mathbf{S}_{GP} \rightarrow \mathbf{K}_{GP}, \quad \left(\delta^2\mathbf{g}\right)^T\,\mathbf{\Phi} \mapsto \delta\mathbf{v}^T\mathbf{K}_{GC}\,\delta\mathbf{v}, \tag{11.14}$$

where $\rightarrow$ and $\mapsto$ symbolize DCCF-transformation and GCCF-transformation-styles, respectively. As can be seen the transformation phase produces a new term $\mathbf{K}_{GC}$ called the *complementary geometric stiffness*. That term *cannot be expressed in terms of the variation* $\delta\mathbf{g}$ *of the displacement gradients*. Consequently there is no "core complementary core geometric stiffness" $\mathbf{S}_{GC}$ that can be added to (11.13). Instead it appears as a "carry forward term" that materializes as a quadratic-form kernel upon transforming.

§11.3.1. Generalized Coordinates as Generic Target

For elements that require the GCCF treatment a one-shot transformation between $\mathbf{g}$ and $\mathbf{v}$ is often replaced by a multistage transformation. The degree of freedom sets used as intermediate targets of this process will be collectively referred to as "generalized coordinates" and identified as $\mathbf{q}$. Of course the final target: element node displacements $\mathbf{v}$, is a particular instance of such array of choices.

In §10.2 it was noted that two variants of the GGCF, qualified as algebraic and differential, should be distinguished in terms of consequences on the existence of physical stiffness equations at various variational levels. These variants are examined below. The ensuing development examines the transformation from displacement gradients $\mathbf{g}$ to a "generic target" set of generalized coordinates q_i collected in vector $\mathbf{q}$. These coordinates are assumed to be *independent*, a restriction removed later. Symbols $\mathbf{K}$ and $\mathbf{f}$ are used to denote tangent stiffness matrices and internal force vectors, respectively, in terms of $\mathbf{q}$.

§11.3.2. Algebraic Transformation

The Algebraic GCCF, or AGCCF, applies if the relation between $\mathbf{g}$ (source) and $\mathbf{q}$ (target) is nonlinear but algebraic. We have $\mathbf{g} = \mathbf{g}(\mathbf{q})$ or in index notation, $g_i = g_i(q_j)$. Differentiating with respect to the q_i variables yields

$$\begin{aligned}
\delta g_i &= \frac{\partial g_i}{\partial q_j}\delta q_j = G_{ij}\delta q_j, \quad \text{or } \delta\mathbf{g} = \mathbf{G}\,\delta\mathbf{q},\\
\delta^2 g_i &= \frac{\partial^2 g_i}{\partial q_j \partial q_k}\delta q_j \delta q_k + \frac{\partial g_i}{\partial q_j}\cancelto{0}{\delta^2 q_j} = F_{ijk}\delta q_j \delta q_k, \quad \text{or } \delta^2\mathbf{g} = (\mathbf{F}\,\delta\mathbf{q})\,\delta\mathbf{q},
\end{aligned} \tag{11.15}$$

Here $(\mathbf{F}\delta\mathbf{q})$ is the matrix $F_{ijk}\,\delta q_k = F_{kij}\,\delta q_k$; $\mathbf{F}$ being a cubic array. The array $\mathbf{G}$ receives the name *tangent transformation matrix*. The second term in the expansion of $\delta^2 g_i$ vanishes because the q_i are assumed to be independent target variables.

Enforcing invariance of $\delta^2 U$ yields the tangent stiffness transformation

$$\mathbf{K} = \int_{V_0} \left\{\mathbf{G}^T(\mathbf{S}_M + \mathbf{S}_{GP})\mathbf{G} + \mathbf{Q}\right\} dV = \mathbf{K}_M + \mathbf{K}_{GP} + \mathbf{K}_{GC} = \mathbf{K}_M + \mathbf{K}_G, \tag{11.16}$$

where the entries of $\mathbf{Q}$ are (cf. Remark 4.1) $Q_{ij} = Q_{ji} = F_{kij}\,\Phi_k$ with summation on $k = 1, \ldots n_g$. Note that $\mathbf{Q}$ is symmetric because $F_{kij} = F_{kji}$. Integration of $\mathbf{Q}$ over V_0 yields the complementary portion $\mathbf{K}_{GC}$ of the geometric stiffness $\mathbf{K}_G$.

The internal, applied and prestress force vectors transform according to the formulas in (11.4) with the $\mathbf{G}$ defined in (11.15):

$$\mathbf{f} = \int_{V_0} \mathbf{G}^T \mathbf{\Phi}\; dV, \qquad \mathbf{p} = \int_{V_0} \mathbf{G}^T \mathbf{\Psi}\; dV, \qquad \mathbf{p}^0 = \int_{V_0} \mathbf{G}^T \mathbf{\Psi}^0\; dV. \tag{11.17}$$

What happens to $\mathbf{K}^U$ and $\mathbf{K}^r$? They can be obtained, somewhat artificially, by constructing the matrix equation

$$\mathbf{g} = \mathbf{W}\mathbf{q}, \tag{11.18}$$

where $\mathbf{W}$ is called a *secant transformation matrix*. Generally this matrix is far from unique because its $n_g \times n_q$ entries must satisfy only n_g conditions. (Care has often to be given to the $q_j \to 0$ if $0/0$ limits appear in $\mathbf{W}$.) Using (11.18) we can proceed to form

$$\mathbf{K}^U = \int_{V_0} \mathbf{W}^T \mathbf{S}^U \mathbf{W}\, dV, \qquad \mathbf{K}^r = \int_{V_0} \mathbf{G}^T \mathbf{S}^r \mathbf{W}\, dV. \tag{11.19}$$

Because in general $\mathbf{W} \neq \mathbf{G}$, symmetry in the secant stiffness $\mathbf{K}^r$ cannot be expected even if $\mathbf{S}^r$ is symmetric.

Remark 11.1. The AGGCF is applicable to finite elements with degrees of freedoms that include *fixed-axis rotations*, because such rotations are integrable. Examples are provided by two-dimensional beams as well as plane stress (membrane) elements with drilling freedoms if only in-plane motions are allowed.

Remark 11.2. Why is $\mathbf{K}_{GC}$ called a geometric stiffness? Because it vanishes if the current configuration is stress free, in which case the core internal force $\mathbf{\Phi}$ vanishes and so does $\mathbf{Q}$.

§11.3.3. Differential Transformation

The Differential GCCF, or DGCCF, is required if the relation between $\mathbf{g}$ (source) and $\mathbf{q}$ (target) is only available as a non-integrable differential form between their variations:

$$\begin{aligned} \delta g_i &= G_{ij}\delta q_j, \quad \text{or} \quad \delta\mathbf{g} = \mathbf{G}\,\delta\mathbf{q}, \\ \delta^2 g_i &= \frac{\partial G_{ij}}{\partial q_k}\,\delta q_j\,\delta q_k = F_{ijk}\;\delta q_j\,\delta q_k, \quad \text{or} \quad \delta^2\mathbf{g} = (\mathbf{F}\delta\mathbf{q})\,\delta\mathbf{q}. \end{aligned} \tag{11.20}$$

The transformation equation (11.16) still applies for $\mathbf{K}$ whereas (11.17) holds for the force vectors. But no integral $\mathbf{g} = \mathbf{g}(\mathbf{q})$ as in the AGGCF exists. Consequently $\mathbf{K}^U$ and $\mathbf{K}^r$, which require a secant matrix relation of the form (11.18), cannot be constructed. Furthermore $\mathbf{Q}$ is not necessarily symmetric; a condition for that being $F_{kij} = F_{kji}$ or equivalently $\partial G_{ki}/\partial q_j = \partial G_{kj}/\partial q_i$.

Remark 11.3. For mechanical finite elements the DGCCF naturally arises when three-dimensional finite rotations are present as nodal degrees of freedom, because such rotations are non-integrable.

Remark 11.4. The relations (11.20) have points of resemblance with the case of non-holonomic constraints in analytical dynamics.

§11.3.4. Multistage Transformation

Up to this point the $\mathbf{q}$ have been assumed to be *independent* variables. But as previously noted, for complicated elements the GCCF transformations are more conveniently applied in *stages*. The target variables in one stage become the source variables for the next one.

What happens if the $\mathbf{q}$ are intermediate variables in a transformation chain? If the $\mathbf{q}$ are *linear* in the final independent degrees of freedom $\mathbf{v}$, all previous formulas hold because the DCCF applies for the remaining transformations, which are strictly congruential. But if the $\mathbf{q}$ are nonlinear in $\mathbf{v}$, or only a non-integrable differential relation exists, term $(\partial g_i/\partial q_j)\,\delta^2 q_j = G_{ij}\,\delta^2 q_j$ in the second of (11.15) survives. The net effect is that the geometric stiffness acquires a higher order component, implicitly defined as the kernel of

$$\int_{V_0} \Phi_i G_{ij}\,\delta^2 q_j\,dV, \tag{11.21}$$

This term cannot be resolved ("resolution" meaning explicit extraction of its stiffness kernel in the form of a complementary geometric stiffness) until the transformation chain reaches downstream variables that either are the final degrees of freedom (and thus independent), or depend linearly on such. It is difficult to state detailed rules that encompass all possible situations. Instead the treatment of the 2D and 3D beam element transformations in §11.4 and §11.11 illustrates the basic techniques for "carrying forward" terms such as (11.21).

§11.4. A 2-Node 2D Timoshenko Beam Element

We continue here with the derivation of a 2D, isotropic Timoshenko beam element started in §10.5.4. This example serves to illustrate the Algebraic GCCF. The specific element constructed here has two end nodes, six degrees of freedom, and reference length L_0. The cross section area $A \equiv A_0$ and moment of inertia $I = \int_A Y^2\,dA$ are constant along the element. Axis X is made to pass through the centroid so that $\int_A Y\,dA = 0$. Furthermore it is assumed that the cross section is doubly symmetric so that $\int_A Y^3\,dA = 0$.

The element displacement field, defined by $u_{0X}(X)$, $u_{0Y}(X)$ and $\theta(X)$, is interpolated with linear shape functions:

$$\begin{bmatrix} u_{0X} \\ u_{0Y} \\ \theta \end{bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 \end{bmatrix} \begin{bmatrix} v_{x1} \\ v_{y1} \\ \theta_1 \\ v_{x2} \\ v_{y2} \\ \theta_2 \end{bmatrix} = \mathbf{N}\mathbf{v}, \tag{11.22}$$

where $N_1 = 1 - (X/L_0)$ and $N_2 = 1 - N_1 = X/L_0$. Consequently

$$\epsilon = \frac{\partial u_{0X}}{\partial X} = \frac{v_{X2} - v_{X1}}{L_0}, \qquad \gamma = \frac{\partial u_{0Y}}{\partial X} = \frac{v_{X2} - v_{X1}}{L_0}, \qquad \kappa = \frac{\partial \theta}{\partial X} = \frac{\theta_2 - \theta_1}{L_0}, \tag{11.23}$$

are constant over the element.

§11.4.1. Generalized Coordinates and Stress Resultants

As intermediate set of generalized coordinates we take $\mathbf{q}^T = [\,\epsilon \quad \gamma \quad \kappa \quad \theta\,]$. These four quantities are constant over each cross section and may be viewed as cross-section orientation coordinates. Consequently when obtaining stiffness matrices and internal forces in terms of $\mathbf{q}$ it is convenient to integrate over the beam cross section. The resulting quantities appear naturally in terms of cross section stress-resultants as shown below. In terms of these generalized coordinates the auxiliary vectors $\mathbf{b}_i$ listed in (10.61) become

$$\mathbf{b}_1 = \begin{bmatrix} 1+g_1 \\ g_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+\epsilon - Y\kappa\cos\theta \\ \gamma - Y\kappa\sin\theta \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} g_3 \\ 1+g_4 \\ 1+g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 1+\epsilon - Y\kappa\cos\theta \\ \gamma - Y\kappa\sin\theta \end{bmatrix}, \tag{11.24}$$

The well known stress resultants of beam theory are the axial force N, transverse shear force V and bending moment M. They are obtained by integrating the PK2 stresses over the beam cross section:

$$\begin{aligned} N &= \int_{A_0} s_1\,dA = EA\big(\epsilon + \tfrac{1}{2}(\epsilon^2+\gamma^2)\big) + \tfrac{1}{2}EI\kappa^2 + N^0, \\ V &= \int_{A_0} s_2\,dA = GA_s\omega_\gamma + V^0, \\ M &= \int_{A_0} s_1 Y\,dA = -EI\kappa\omega_\epsilon + M^0, \end{aligned} \tag{11.25}$$

where $\omega_\epsilon = (1+\epsilon)\cos\theta + \gamma\sin\theta$ and $\omega_\gamma = \gamma\cos\theta - (1+\epsilon)\sin\theta$ can be viewed as generalized skew strains. In (11.25) N^0, V^0 and M^0 denote initial-stress resultants (stress resultants in $\mathcal{C}_0$, also called prestress forces), $A \equiv A_0$, $I = \int_{A_0} Y^2\,dA$, and $A_s = \mu A$, in which μ is the usual shear correction factor of Timoshenko beam theory. Because of the doubly-symmetric cross-section assumption, a term containing the third-section-moment $\int_{A_0} Y^3\,dA$ has been omitted from the expression for M.

In addition to N, V and M, the following higher order moment, which is absent from the linear theory, appears in the residual force and tangent stiffness:

$$C = \int_{A_0} s_1 Y^2\,dA = EI\big((\epsilon + \tfrac{1}{2}(\epsilon^2+\gamma^2)\big) + \tfrac{1}{2}E\mathcal{H}\kappa^2\big) + C^0, \tag{11.26}$$

in which $\mathcal{H} = \int_{A_0} Y^4\,dA$. If terms in κ^2 are neglected,

$$C - C^0 = (N - N^0)(I/A) = (N-N^0)r^2, \tag{11.27}$$

where $r = \sqrt{I/A}$ is the radius of gyration of the cross section. If such terms are retained this relation is only exact if $r^2 = \mathcal{H}/I$ and approximate otherwise.

Remark 11.5. One may verify that $\int_A s_2 Y\,dA$ vanishes identically. This serves as a check of the strain distribution equations.

§11.4.2. Transformation Matrices

The differential relations required to establish the tangent transformation are obtained from (10.57) as

$$\delta \mathbf{g} = \frac{\partial \mathbf{g}}{\partial \mathbf{q}} \delta \mathbf{q} = \begin{bmatrix} 1 & 0 & -Y\cos\theta & Y\kappa\sin\theta \\ 0 & 1 & -Y\sin\theta & -Y\kappa\cos\theta \\ 0 & 0 & 0 & -\cos\theta \\ 0 & 0 & 0 & -\sin\theta \end{bmatrix} \begin{bmatrix} \delta\epsilon \\ \delta\gamma \\ \delta\kappa \\ \delta\theta \end{bmatrix} = \mathbf{G}_1 \,\delta\mathbf{q}, \tag{11.28}$$

$$\delta \mathbf{q} = \frac{\partial \mathbf{q}}{\partial \mathbf{v}} \delta \mathbf{v} = \frac{1}{L_0} \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & L_0 - X & 0 & 0 & X \end{bmatrix} \begin{bmatrix} \delta v_{X1} \\ \delta v_{Y1} \\ \vdots \\ \delta\theta_2 \end{bmatrix} = \mathbf{G}_2 \,\delta\mathbf{v}, \tag{11.29}$$

The transformation relating $\delta\mathbf{g} = \mathbf{G}\,\delta\mathbf{v}$ may be obtained as the product

$$\mathbf{G} = \mathbf{G}_1\mathbf{G}_2 = \frac{1}{L_0} \begin{bmatrix} -1 & 0 & Y(\cos\theta + (L_0 - X)\kappa\sin\theta) & 1 & 0 & Y(-\cos\theta + X\kappa\sin\theta) \\ 0 & -1 & Y(\sin\theta - (L_0 - X)\kappa\cos\theta) & 0 & 1 & Y(-\sin\theta - X\kappa\cos\theta) \\ 0 & 0 & -(L_0 - X)\cos\theta & 0 & 0 & -X\cos\theta \\ 0 & 0 & -(L_0 - X)\sin\theta & 0 & 0 & -X\sin\theta \end{bmatrix} \tag{11.30}$$

but it is more instructive (as well as conducive to higher efficiency in the computer implementation) to perform the transformation phase in two stages.

Observe that the first transformation (from $\mathbf{g}$ to $\mathbf{q}$) is nonlinear and algebraic whereas the second one (from $\mathbf{q}$ to $\mathbf{v}$) is linear. Consequently we have to use the AGCCF for the first transformation but the second one can be done simply through the DCCF.

§11.4.3. Internal Force Vector

The internal force vector in terms of $\mathbf{q}$, denoted by $\mathbf{f}^q$, is obtained from the core expression (10.64) for $\mathbf{\Phi}$ and the matrix $\mathbf{G}_1$ given in (11.28):

$$\mathbf{f}^q = \int_{A_0} \mathbf{G}_1^T \,\mathbf{\Phi}\, dA_0 = \begin{bmatrix} f^q_\epsilon \\ f^q_\gamma \\ f^q_\kappa \\ f^q_\theta \end{bmatrix} = \begin{bmatrix} N(1+\epsilon) - M\kappa\cos\theta - V\sin\theta \\ N\gamma - M\kappa\sin\theta + V\cos\theta \\ -M\omega_\epsilon + C\kappa \\ -M\kappa\omega_\gamma - V\omega_\epsilon \end{bmatrix}. \tag{11.31}$$

Finally, application of (11.29) and integration over the element length yields

$$\mathbf{f} = \int_0^{L_0} \mathbf{G}_2^T \,\mathbf{f}^q \, dX = \begin{bmatrix} -f^q_\epsilon \\ -f^q_\gamma \\ -f^q_\kappa + \frac{1}{2} L_0 f^q_\theta \\ f^q_\epsilon \\ f^q_\gamma \\ f^q_\kappa + \frac{1}{2} L_0 f^q_\theta \end{bmatrix}. \tag{11.32}$$

This vector satisfies translational equilibrium.

§11.4.4. Tangent Stiffness Matrix

Transforming to generalized coordinates $\mathbf{q}$ produces three components of the tangent stiffness matrix:

$$\mathbf{K}^q = \int_{A_0} \big(\mathbf{G}_1^T(\mathbf{S}_M + \mathbf{S}_G)\mathbf{G}_1 + \mathbf{Q}\big)\, dA = \mathbf{K}^q_M + \mathbf{K}^q_{GP} + \mathbf{K}^q_{GC}. \tag{11.33}$$

The entries of $\mathbf{K}^q_M$, obtained through symbolic manipulation, are

$$\begin{aligned}
K^q_M(1,1) &= EA(1+\epsilon)^2 + GA_s \sin^2\theta + EI\kappa^2\cos^2\theta,\\
K^q_M(1,2) &= EA(1+\epsilon)\gamma - GA_s\sin\theta\cos\theta + EI\kappa^2\sin\theta\cos\theta,\\
K^q_M(1,3) &= EI\kappa\,\big((1+\epsilon)(1+\cos^2\theta) + \gamma\sin\theta\cos\theta\big),\\
K^q_M(1,4) &= EI\kappa^2\omega_\gamma\cos\theta + GA_s\omega_\epsilon\sin\theta,\\
K^q_M(2,2) &= EA\gamma^2 + GA_s\cos^2\theta + EI\kappa^2\sin^2\theta,\\
K^q_M(2,3) &= EI\kappa\,\big((1+\epsilon)\sin\theta\cos\theta + \gamma(1+\sin^2\theta)\big),\\
K^q_M(2,4) &= EI\kappa^2\omega_\gamma\sin\theta - GA_s\omega_\epsilon\cos\theta,\\
K^q_M(3,3) &= EI{\omega_\epsilon}^2 + E\mathcal{H}\kappa^2,\\
K^q_M(3,4) &= EI\kappa\,\big((1+\epsilon)\gamma(\cos^2\theta - \sin^2\theta) + \big(\gamma^2 - (1+\epsilon)^2\big)\sin\theta\cos\theta\big),\\
K^q_M(4,4) &= EI\kappa^2\phi_g^2 + GA_s{\omega_\epsilon}^2.
\end{aligned} \tag{11.34}$$

The principal geometric stiffness, which is readily worked out by hand, is

$$\mathbf{K}^q_{GP} = \begin{bmatrix} N & 0 & -M\cos\theta & M\kappa\sin\theta - V\cos\theta \\ & N & -M\sin\theta & -M\kappa\cos\theta - V\sin\theta \\ & & C & 0 \\ symm & & & C\kappa^2 \end{bmatrix} \tag{11.35}$$

The new term contributed by the AGCCF to $\mathbf{K}^q$ is the complementary geometric stiffness $\mathbf{K}^q_{GC}$. Its source is the matrix $\mathbf{Q}$ introduced in Section 8.2. The entries of $\mathbf{Q}$ are $Q_{ij} = (\partial^2 g_k/\partial q_i\,\partial q_j)\Phi_k$, where the components of $\mathbf{g}$ and $\mathbf{\Phi} = s_1\mathbf{b}_1 + s_2\mathbf{b}_2$ may be obtained from (10.57) and (11.24), respectively.

The entries of $\mathbf{Q}$ were symbolically generated by the following *Mathematica* module:

```
QmatrixOf2DTimoBeamElement[eps_,gamma_,kappa_,theta_,Em_,Gm_,Y_]:=
  Module[{g,h1,h2,H1,H2,e1,e2,s1,s2,b1,b2,phi,i,j,k},
    q={eps,gamma,kappa,theta}; phi={1,1,1,1};
    g={eps-Y*kappa*Cos[theta],gamma-Y*kappa*Sin[theta],
       -Sin[theta],Cos[theta]-1};
    gg={{g[[1]]},{g[[2]]},{g[[3]]},{g[[4]]}};
    h1={{1},{0},{0},{0}}; h2={{0},{1},{1},{0}};
    H1={{1,0,0,0},{0,1,0,0},{0,0,0,0},{0,0,0,0}};
    H2={{0,0,1,0},{0,0,0,1},{1,0,0,0},{0,1,0,0}};
    e1=(Transpose[h1].gg+(1/2)*Transpose[gg].H1.gg)[[1,1]];
    e2=(Transpose[h2].gg+(1/2)*Transpose[gg].H2.gg)[[1,1]];
    s1=Simplify[Em*e1]; s2=Simplify[Gm*e2];
```

```
b1={1+eps-Y*kappa*Cos[theta],gamma-Y*kappa*Sin[theta],0,0};
b2={-Sin[theta],Cos[theta],1+eps-Y*kappa*Cos[theta],
    gamma-Y*kappa*Sin[theta]};
phi=Simplify[s1*b1+s2*b2];
Q=Table[0,{4},{4}];
For[i=1,i<=4,i++, For[j=1,j<=4,j++, For[k=1,k<=4,k++,
  Q[[i,j]]=Q[[i,j]]+(D[D[g[[k]],q[[i]]],q[[j]]])*phi[[k]] ]]];
     Return[Q]
    ];
```

The output of this module was integrated over the cross section and pattern matched with the expression of the stress resultants (11.25)-(11.26) to produce

$$\mathbf{K}^q_{GC} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & -M\omega_\gamma \\ symm & & & -V\omega_\gamma + M\kappa\omega_\epsilon - C\kappa^2 \end{bmatrix}, \tag{11.36}$$

which added to (11.35) yields the geometric stiffness

$$\mathbf{K}^q_{G} = \begin{bmatrix} N & 0 & -M\cos\theta & M\kappa\sin\theta - V\cos\theta \\ & N & -M\sin\theta & -M\kappa\cos\theta - V\sin\theta \\ & & C & -M\omega_\gamma \\ symm & & & -V\omega_\gamma + M\kappa\omega_\epsilon \end{bmatrix}. \tag{11.37}$$

Finally, the tangent stiffness in terms of $\mathbf{q}$ is $\mathbf{K}^q = \mathbf{K}_M + \mathbf{K}_G$. Denoting the entries of $\mathbf{K}^q$ by K^q_{ij}, $i, j = 1, \ldots 4$ the tangent stiffness matrix $\mathbf{K}$ in terms of node displacements $\mathbf{v}$ is formed through the DCCF transformation

$$\mathbf{K} = \int_0^{L_0} \mathbf{G}_2^T \mathbf{K}^q \mathbf{G}_2 \, dX = \begin{bmatrix} K^q_{11} & K^q_{12} & K^q_{13} - \tfrac{1}{2}L_0K^q_{14} & -K^q_{11} & -K^q_{12} & -K^q_{13} - \tfrac{1}{2}L_0K^q_{14} \\ & K^q_{22} & K^q_{23} - \tfrac{1}{2}L_0K^q_{24} & -K^q_{12} & -K^q_{22} & -K^q_{23} - \tfrac{1}{2}L_0K^q_{24} \\ & & K^q_{33} - L_0K^q_{34} + \tfrac{1}{3}L_0^2K^q_{44} & -K^q_{13} + \tfrac{1}{2}L_0K^q_{14} & -K^q_{23} + \tfrac{1}{2}L_0K^q_{24} & -K^q_{33} + \tfrac{1}{6}L_0^2K^q_{44} \\ & & & K^q_{11} & K^q_{12} & K^q_{13} + \tfrac{1}{2}L_0K^q_{14} \\ & & & & K^q_{22} & K^q_{23} + \tfrac{1}{2}L_0K^q_{24} \\ symm & & & & & K^q_{33} + L_0K^q_{34} + \tfrac{1}{3}L_0^2K^q_{44} \end{bmatrix}. \tag{11.38}$$

The above rule can be applied to $\mathbf{K}_M$ and $\mathbf{K}_G$ should separate formation be desirable, as when setting up a stability eigenproblem.

If the reference configuration is not aligned with X, the preceding expressions apply to the local system $\{\bar{X}, \bar{Y}\}$. A final local-to-global transformation step, similar to that discussed for the 3D bar in §11.3, is then necessary. This step can be handled by a simple DCCF transformation, because the finite rotation θ remains the same in global coordinates.

Remark 11.6. The foregoing exact expressions contain curvature-squared terms typically in the combination $I\kappa^2$. This can be shown to be of order $(r/R)^2$ compared to other terms, where r is the radius of gyration of the cross section and $R = 1/\kappa$ the radius of curvature of the current configuration. For typical beams $(r/R)^2$ is 10^{-6} or less; consequently all such tiny terms may be dropped without visible loss of accuracy. For highly-bent extremely-thin beams, however, that ratio may go up to 0.01 in which case the κ^2 terms might have a noticeable though small effect if retained.

§11.4.5. Can a Secant Stiffness be Constructed?

To attempt the construction of a secant stiffness $\mathbf{K}^{rq}$ in terms of generalized coordinates $\mathbf{q}$ one should obtain a secant matrix form of the relationship $\mathbf{g} = \mathbf{g}(\mathbf{q})$. As noted previously such form is far from unique. One possible choice is

$$\mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ g_2 \\ g_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -Y\cos\theta & 0 \\ 0 & 1 & -Y\sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta/\theta \\ 0 & 0 & 0 & (\cos\theta - 1)/\theta \end{bmatrix} \begin{bmatrix} \epsilon \\ \gamma \\ \kappa \\ \theta \end{bmatrix} = \mathbf{W}_1\mathbf{q}, \tag{11.39}$$

which has the merit of not being too dissimilar from $\mathbf{G}_1$. Note that some care must be taken as regards some 0/0 limits. Then $\mathbf{K}^{rq} = \int_A \mathbf{G}_1^T \mathbf{S}^r \mathbf{W}_1 \, dA$, which may be easily worked out in closed form but is unsymmetric. Because $\mathbf{q}$ is linear in $\mathbf{v}$, the next transformation is simply $\mathbf{K}^r = \int_0^{L_0} \mathbf{G}_2^T \mathbf{K}^{rq} \mathbf{G}_2 \, dX$ which can be handled through a scheme similar to (11.38) but with an unsymmetric kernel matrix.

§11.5. A 2-Node 3D Timoshenko Beam Element

We continue here the development of a two-node 3D Timoshenko beam element started in §10.5.5. As can be surmised, the development is more complex and demanding than for its 2D counterpart. Only a summary taken from Crivelli's thesis [1] and Crivelli and Felippa [3] is presented here. The transformation phase to pass from the core equations to the element nodal degrees of freedom is carried out in three stages:

1. From particle displacement gradients $\mathbf{g}$ to generalized gradients $\mathbf{w}$ at each cross section. An integration over the cross section area is involved.
2. From generalized gradients $\mathbf{w}$ to cross-section orientation coordinates $\mathbf{q}$. The rotational parametrization is introduced at this stage.
3. From cross-section orientation to finite-element nodal degrees of freedom $\mathbf{v}$. An integration over the element length, as defined by the shape functions, is involved.

These transformation stages are summarized in Tables 11.1 and 11.2, which together also serve to define notation

§11.5.1. Transformation to Generalized Gradients

The first set of target variables are the *generalized gradients* $\mathbf{w}(X)$ at each reference cross section defined by the longitudinal coordinate X. The components of $\mathbf{w}$ are indirectly given through their first variation:

$$\delta\mathbf{w} = \begin{bmatrix} \dfrac{d\,\delta\mathbf{u}_0}{dX} & \dfrac{d\delta\mathbf{\Theta}}{dX} & \delta\mathbf{\Theta} \end{bmatrix}^T, \tag{11.40}$$

Table 11.1 Internal energy and its variations for 3D Timoshenko beam element

Core Particle $\mathbf{g}$	Section Gradients Cross-Section $\mathbf{w}$	Section Orientation Cross-Section $\mathbf{z}$	Physical DOF Whole Element $\mathbf{v}$
$\mathcal{U} = \frac{1}{2}\mathbf{g}^T\mathbf{S}^U\mathbf{g} + \mathbf{g}^T\mathbf{\Psi}^0$	—	—	—
$\delta\mathcal{U} = \delta\mathbf{g}^T(\mathbf{S}^r\mathbf{g} + \mathbf{\Psi}^0)$	$\delta U_G = \delta\mathbf{w}^T\boldsymbol{\mathcal{R}}$	$\delta U_z = \delta\mathbf{z}^T\mathbf{f}_z$	$\delta U = \delta\mathbf{v}^T\mathbf{f}$
$\delta^2\mathcal{U} = \delta\mathbf{g}^T\,\mathbf{S}\,\delta\mathbf{g} + \delta^2\mathbf{g}^T\mathbf{\Phi}$	$\delta^2 U_G = \delta\mathbf{w}^T\boldsymbol{\mathcal{S}}\,\delta\mathbf{w} + \mathcal{F}$	$\delta^2 U_z = \delta\mathbf{z}^T\mathbf{K}_z\,\delta\mathbf{z}$	$\delta^2 U = \delta\mathbf{v}^T\mathbf{K}\,\delta\mathbf{v}$

Table 11.2. Core-to-physical-DOFs transformations for 3D beam element

Core Level Particle $\mathbf{g}$	Section Gradients Cross-Section $\mathbf{w}$	Section Orientation Cross-Section $\mathbf{z}$	Physical DOF Whole Element $\mathbf{v}$
$\mathbf{\Phi}$	$\boldsymbol{\mathcal{R}} = \int_{A_0}\mathbf{W}^T\mathbf{\Phi}\,dA$	$\mathbf{f}_z = \mathbf{Z}^T\boldsymbol{\mathcal{R}}$	$\mathbf{f} = \int_0^{L_0}\mathbf{G}_z^T\mathbf{f}_z\,dX$
$\mathbf{S}$	$\boldsymbol{\mathcal{S}} = \int_{A_0}\mathbf{W}^T\mathbf{S}\mathbf{W}\,dA$	$\mathbf{K}_z = \mathbf{Z}^T\boldsymbol{\mathcal{S}}\mathbf{Z} + \boldsymbol{\mathcal{S}}_{GCz}$	$\mathbf{K} = \int_0^{L_0}\mathbf{G}_z^T\mathbf{K}_z\,\mathbf{G}_z\,dX$
$\delta\mathbf{g} = \mathbf{W}\,\delta\mathbf{w}$	$\delta\mathbf{w} = \mathbf{Z}\,\delta\mathbf{z}$	$\delta\mathbf{z} = \mathbf{G}_z\,\delta\mathbf{v}$	

where $\delta\mathbf{\Theta}$, defined in (10.68), measures the variation of angular orientation. Because this quantity is not generally integrable for three-dimensional motions, it is not possible to express $\mathbf{\Theta}$ as a unique function of the displacements. The variation of $\mathbf{g}_1$ is

$$\delta\mathbf{g}_1 = \frac{d\delta\mathbf{u}_0}{dX} + \mathbf{R}^T\tilde{\zeta}^T\frac{d\delta\mathbf{\Theta}}{dX} + \mathbf{R}^T\tilde{\zeta}^T\tilde{\boldsymbol{\kappa}}\delta\mathbf{\Theta} + \mathbf{R}^T\widetilde{\delta\mathbf{\Theta}}\tilde{\zeta}^T\boldsymbol{\kappa}, \tag{11.41}$$

where we used the relation [1] $\delta\boldsymbol{\kappa} = d\delta\mathbf{\Theta}/dX + \tilde{\boldsymbol{\kappa}}\delta\mathbf{\Theta}$. On using the commutative law $\tilde{\mathbf{a}}\mathbf{b} = \tilde{\mathbf{b}}^T\mathbf{a}$ and Jacobi's identity $\widetilde{\tilde{\mathbf{a}}\mathbf{b}} = \tilde{\mathbf{a}}\tilde{\mathbf{b}} - \tilde{\mathbf{b}}\tilde{\mathbf{a}}$ we may rewrite (11.41) as

$$\delta\mathbf{g}_1 = \frac{\partial\,\delta\mathbf{u}_0}{\partial X} + \mathbf{R}^T\tilde{\zeta}^T\frac{\partial\,\delta\mathbf{\Theta}}{\partial X} + \mathbf{R}^T\tilde{\boldsymbol{\kappa}}^T\tilde{\zeta}\,\delta\mathbf{\Theta} \tag{11.42}$$

For the other gradient vectors we have $\delta\mathbf{g}_2 = \delta\mathbf{R}^T\mathbf{h}_2 = \mathbf{R}^T\widetilde{\delta\mathbf{\Theta}}\mathbf{h}_2 = \mathbf{R}^T\tilde{\mathbf{h}}_2^T\delta\mathbf{\Theta}$ and $\delta\mathbf{g}_3 = \mathbf{R}^T\tilde{\mathbf{h}}_3^T\delta\mathbf{\Theta}$, which can be collected in matrix form as

$$\delta\mathbf{g} = \begin{bmatrix} \delta\mathbf{g}_1 \\ \delta\mathbf{g}_2 \\ \delta\mathbf{g}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{R}^T\tilde{\boldsymbol{\zeta}}^T & \mathbf{R}^T\tilde{\boldsymbol{\kappa}}^T\tilde{\boldsymbol{\zeta}} \\ 0 & 0 & \mathbf{R}^T\tilde{\mathbf{h}}_2^T \\ 0 & 0 & \mathbf{R}^T\tilde{\mathbf{h}}_3^T \end{bmatrix} \begin{bmatrix} \dfrac{d\,\delta\mathbf{u}_0}{dX} \\ \dfrac{d\,\delta\boldsymbol{\Theta}}{dX} \\ \delta\boldsymbol{\Theta} \end{bmatrix} = \begin{bmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \\ \mathbf{W}_3 \end{bmatrix} \delta\mathbf{w} = \mathbf{W}\,\delta\mathbf{w}, \tag{11.43}$$

where $\mathbf{I}$ is the 3-by-3 identity matrix and $\mathbf{W}_i$ are 3-by-9 matrices. The second variation of $\mathbf{g}$, which is required for the complementary geometric stiffness, is

$$\begin{aligned} \delta^2\mathbf{g}_1 &= \mathbf{R}^T\widetilde{\delta\boldsymbol{\Theta}}\tilde{\boldsymbol{\zeta}}^T\frac{d\,\delta\boldsymbol{\Theta}}{dX} + \mathbf{R}^T\widetilde{\delta\boldsymbol{\Theta}}\,\tilde{\boldsymbol{\zeta}}^T\tilde{\boldsymbol{\kappa}}\,\delta\boldsymbol{\Theta} + \mathbf{R}^T\widetilde{\delta\boldsymbol{\Theta}}\,\tilde{\boldsymbol{\zeta}}^T\frac{d\,\delta\boldsymbol{\Theta}}{dX} \\ &\quad + \mathbf{R}^T\widetilde{\delta\boldsymbol{\Theta}}\tilde{\boldsymbol{\zeta}}^T\tilde{\boldsymbol{\kappa}}\delta\boldsymbol{\Theta} + \delta^2\mathbf{R}^T\,\tilde{\boldsymbol{\zeta}}^T\boldsymbol{\kappa} + \mathbf{R}^T\tilde{\boldsymbol{\zeta}}^T\,\delta^2\boldsymbol{\kappa}, \\ \delta^2\mathbf{g}_2 &= \delta^2\mathbf{R}^T\,\mathbf{i}_2, \qquad \delta^2\mathbf{g}_3 = \delta^2\mathbf{R}^T\,\mathbf{i}_3 \end{aligned} \tag{11.44}$$

At this point it is appropriate to introduce the following section resultants:

$$\begin{aligned} &\mathcal{P} = A\sigma_b + \mathcal{P}^0, && s_b = E e_b, \\ &\boldsymbol{\mathcal{Q}} = \mu_s A + \boldsymbol{\mathcal{Q}}^0, && \boldsymbol{\tau} = \boldsymbol{\tau}_2 + \boldsymbol{\tau}_3, \quad \boldsymbol{\tau}_2 = G\gamma_2\mathbf{h}_2, \quad \boldsymbol{\tau}_3 = G\gamma_3\mathbf{h}_3, \\ &\boldsymbol{\mathcal{M}}_\sigma = E\,\mathbf{I}_S\mathcal{K}_e + \boldsymbol{\mathcal{M}}_\sigma^0, && \mathbf{I}_S = \int_{A_0}\zeta\zeta^T\,dA, \qquad \mathcal{K}_e = \tilde{\phi}\,\boldsymbol{\kappa}, \\ &\boldsymbol{\mathcal{M}}_\tau = \mu_t G\,\mathbf{I}_P\boldsymbol{\kappa} + \boldsymbol{\mathcal{M}}_\tau^0, && \mathbf{I}_P = \int_{A_0}\tilde{\boldsymbol{\zeta}}\,\tilde{\boldsymbol{\zeta}}^T\,dA. \end{aligned} \tag{11.45}$$

Here $\mathcal{P}$, $\boldsymbol{\mathcal{Q}}$, $\boldsymbol{\mathcal{M}}_\sigma$ and $\boldsymbol{\mathcal{M}}_\tau$ are axial forces, shear forces, bending moments and torsional moments, respectively, at the current configuration $\mathcal{C}$; $\mathcal{P}^0$, $\boldsymbol{\mathcal{Q}}^0$, $\boldsymbol{\mathcal{M}}_\sigma^0$ and $\boldsymbol{\mathcal{M}}_\tau^0$ are similar quantities at the reference configuration $\mathcal{C}_0$; μ_s and μ_t are transverse-shear and torsion coefficients that account for the actual shear stress distributions, respectively; and $\mathbf{I}_S$ and $\mathbf{I}_P$ are the cartesian and polar inertia tensors, respectively, of the cross section. Should the axes Y and Z be aligned with the principal inertia axes the latter simplified to

$$\mathbf{I}_S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix}, \qquad \mathbf{I}_P = \begin{bmatrix} I_{22}+I_{33} & 0 & 0 \\ 0 & I_{33} & 0 \\ 0 & 0 & I_{22} \end{bmatrix}. \tag{11.46}$$

Because the relation between $\mathbf{g}$ and $\mathbf{w}$ is of differential type the applicable transformation rules are those the DGCCF, and no energy or secant stiffness survives. Thus only the internal force vector $\boldsymbol{\mathcal{R}}$ and tangent stiffness $\boldsymbol{\mathcal{S}}$ associated with $\mathbf{w}$ are derived below.

Internal Force Vector. The generalized internal force vector is

$$\boldsymbol{\mathcal{R}} = \int_{A_0}\mathbf{W}^T\boldsymbol{\Phi}\,dA = \sum_i\int_{A_0}s_i\,\mathbf{W}^T\mathbf{b}_i\,dA = \boldsymbol{\mathcal{R}}_\sigma + \boldsymbol{\mathcal{R}}_\tau, \tag{11.47}$$

where $\boldsymbol{\mathcal{R}}_\sigma$ and $\boldsymbol{\mathcal{R}}_\tau$ are the contributions of the normal and shear stresses respectively. Detailed calculations result [1] in the following exact expressions:

$$\boldsymbol{\mathcal{R}}_\sigma = \begin{bmatrix} \mathbf{R}^T(\mathcal{P}\boldsymbol{\phi} + \tilde{\boldsymbol{\kappa}}\boldsymbol{\mathcal{M}}_\sigma) \\ \tilde{\boldsymbol{\phi}}^T\boldsymbol{\mathcal{M}}_\sigma \\ \widetilde{\mathcal{K}}_e^T\boldsymbol{\mathcal{M}}_\sigma \end{bmatrix}, \qquad \boldsymbol{\mathcal{R}}_\tau = \begin{bmatrix} \mathbf{R}^T\boldsymbol{\mathcal{Q}} \\ \boldsymbol{\mathcal{M}}_\tau \\ \tilde{\boldsymbol{\phi}}^T\boldsymbol{\mathcal{Q}} + \tilde{\boldsymbol{\kappa}}^T\boldsymbol{\mathcal{M}}_\tau \end{bmatrix}. \tag{11.48}$$

For small deformations in which the squared curvature may be neglected, $\mathbf{R} \approx \mathbf{I}$, $\tilde{\boldsymbol{\phi}} \approx \widetilde{\mathbf{h}}_1$, $\mathcal{K}_e \approx \boldsymbol{\kappa}$ and $\tilde{\boldsymbol{\kappa}}\boldsymbol{\mathcal{M}}_\sigma \approx \mathbf{0}$. If these approximations are made,

$$\boldsymbol{\mathcal{R}}_\sigma = \begin{bmatrix} \mathcal{P}\mathbf{h}_1 \\ \widetilde{\mathbf{h}}_1^T\boldsymbol{\mathcal{M}}_\sigma \\ \mathbf{0} \end{bmatrix}, \qquad \boldsymbol{\mathcal{R}}_\tau = \begin{bmatrix} \boldsymbol{\mathcal{Q}} \\ \boldsymbol{\mathcal{M}}_\tau \\ \widetilde{\mathbf{h}}_1^T\boldsymbol{\mathcal{Q}} \end{bmatrix}. \tag{11.49}$$

These resemble the classic linearized theory equations. Furthermore observe that the term $\mathcal{P}\mathbf{R}^T\boldsymbol{\phi}$ corresponds to the internal force of the TL 3D bar.

Tangent Stiffness. For the tangent stiffness we have the decomposition

$$\boldsymbol{\mathcal{S}} = \boldsymbol{\mathcal{S}}_M + \boldsymbol{\mathcal{S}}_{GP} + \boldsymbol{\mathcal{S}}_{GC}. \tag{11.50}$$

Furthermore, since $\mathbf{w}$ is nonlinear in downstream variables, the complementary geometric stiffness splits into two components:

$$\boldsymbol{\mathcal{S}}_{GC} = \boldsymbol{\mathcal{S}}_{GCw} + \boldsymbol{\mathcal{S}}_{GCq}, \tag{11.51}$$

where $\boldsymbol{\mathcal{S}}_{GCw}$ and $\boldsymbol{\mathcal{S}}_{GCq}$ contains terms that depend on the first and second variations, respectively, of $\mathbf{R}$ and $\boldsymbol{\kappa}$. The notation is suggested by the fact that $\boldsymbol{\mathcal{S}}_{GCw}$ can be *merged* into $\boldsymbol{\mathcal{S}}_{GP}$ to yield the geometric stiffness $\boldsymbol{\mathcal{S}}_{Gw} = \boldsymbol{\mathcal{S}}_{GP} + \boldsymbol{\mathcal{S}}_{GCw}$, which is associated with the generalized gradients $\mathbf{w}$ and *independent* of the rotational parametrization selected in the next set of target variables $\mathbf{q}$. On the other hand, the kernel $\boldsymbol{\mathcal{S}}_{GCq}$ cannot be extracted at the $\mathbf{w}$ level and must be *carried forward* to the $\mathbf{q}$ level because it is parametrization dependent. Each of the components in (11.50)-(11.51) may be expressed as the sum of two contributions, one from the normal stresses and one from the shear stresses:

$$\boldsymbol{\mathcal{S}}_M = \boldsymbol{\mathcal{S}}_{M\sigma} + \boldsymbol{\mathcal{S}}_{M\tau}, \quad \boldsymbol{\mathcal{S}}_{GP} = \boldsymbol{\mathcal{S}}_{GP\sigma} + \boldsymbol{\mathcal{S}}_{GP\tau}, \quad \boldsymbol{\mathcal{S}}_{GCx} = \boldsymbol{\mathcal{S}}_{GCx\sigma} + \boldsymbol{\mathcal{S}}_{GCx\tau}, \quad x = w, q. \tag{11.52}$$

Material Stiffness. The generalized core material stiffness is given by the congruential transformation

$$\boldsymbol{\mathcal{S}}_M = \int_{A_0} \mathbf{W}^T\mathbf{S}_M\mathbf{W}\,dA = \sum_i \int_{A_0} E_i\,\mathbf{W}^T\mathbf{b}_i\mathbf{b}_i^T\mathbf{W}\,dA = \boldsymbol{\mathcal{S}}_{M\sigma} + \boldsymbol{\mathcal{S}}_{M\tau}. \tag{11.53}$$

Carrying out the algebraic manipulations one obtains

$$\boldsymbol{\mathcal{S}}_{M\sigma} = E \begin{bmatrix} \mathbf{R}^T(\boldsymbol{\phi}\boldsymbol{\phi}^T + \tilde{\boldsymbol{\kappa}}^T\mathbf{I}_S\tilde{\boldsymbol{\kappa}})\mathbf{R} & \mathbf{R}^T\tilde{\boldsymbol{\kappa}}\mathbf{I}_S\tilde{\boldsymbol{\phi}} & \mathbf{R}^T\tilde{\boldsymbol{\kappa}}\mathbf{I}_S\widetilde{\mathcal{K}}_e \\ & \tilde{\boldsymbol{\phi}}^T\mathbf{I}_S\tilde{\boldsymbol{\phi}} & \tilde{\boldsymbol{\phi}}^T\mathbf{I}_S\widetilde{\mathcal{K}}_e \\ symm & & \widetilde{\mathcal{K}}_e^T\mathbf{I}_S\widetilde{\mathcal{K}}_e \end{bmatrix}, \tag{11.54}$$

$$\boldsymbol{\mathcal{S}}_{M\tau} = \mu G \begin{bmatrix} A\mathbf{R}^T\mathbf{I}_\perp\mathbf{R} & \mathbf{0} & A\mathbf{R}^T\mathbf{I}_\perp\tilde{\boldsymbol{\phi}} \\ & \mathbf{I}_P & \mathbf{I}_P\tilde{\boldsymbol{\kappa}} \\ symm & & A\tilde{\boldsymbol{\phi}}^T\mathbf{I}_\perp\tilde{\boldsymbol{\phi}} + \tilde{\boldsymbol{\kappa}}^T\mathbf{I}_P\tilde{\boldsymbol{\kappa}} \end{bmatrix}, \quad \text{in which } \mathbf{I}_\perp = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{11.55}$$

The contribution $\mathbf{R}^T\boldsymbol{\phi}\boldsymbol{\phi}^T\mathbf{R}$ is the core material stiffness of a TL 3D bar.

Geometric Stiffness due to Normal Stresses. It is convenient to work out together all geometric stiffness terms produced by the normal stresses, *i.e.*

$$\boldsymbol{\mathcal{S}}_{G\sigma} = \boldsymbol{\mathcal{S}}_{GP\sigma} + \boldsymbol{\mathcal{S}}_{GCw\sigma} + \boldsymbol{\mathcal{S}}_{GCq\sigma} = \boldsymbol{\mathcal{S}}_{Gw\sigma} + \boldsymbol{\mathcal{S}}_{GCq\sigma}. \tag{11.56}$$

The appropriate definitions are

$$\begin{aligned} \boldsymbol{\mathcal{S}}_{GP\sigma} &= \int_{A_0} s_{11}\mathbf{W}_1^T\mathbf{H}\mathbf{W}_1\,dA, \\ \boldsymbol{\mathcal{S}}_{GC\sigma} &= \int_{A_0} s_{11}\mathbf{b}_1\,\delta^2\mathbf{g}\;dA = \delta\mathbf{w}^T\boldsymbol{\mathcal{S}}_{GCw\sigma}\delta\mathbf{w} + \mathcal{F}(\delta^2\mathbf{R},\,\delta^2\boldsymbol{\kappa}), \end{aligned} \tag{11.57}$$

where $\mathcal{F}$ contains $\boldsymbol{\mathcal{S}}_{GCq}$ as $\mathbf{q}$ level kernel. Carrying out the algebraic manipulations one obtains

$$\boldsymbol{\mathcal{S}}_{Gw\sigma} = \boldsymbol{\mathcal{S}}_{GP\sigma} + \boldsymbol{\mathcal{S}}_{GCw\sigma} = \begin{bmatrix} \mathcal{P}\mathbf{I} & \mathbf{R}^T\widetilde{\boldsymbol{\mathcal{M}}}_\sigma^T & \mathbf{R}^T\tilde{\boldsymbol{\kappa}}^T\widetilde{\boldsymbol{\mathcal{M}}}_\sigma \\ & \mathbf{0} & \widetilde{\boldsymbol{\mathcal{M}}}_\sigma\tilde{\boldsymbol{\phi}} \\ symm & & \tilde{\boldsymbol{\phi}}\widetilde{\boldsymbol{\mathcal{M}}}_\sigma\tilde{\boldsymbol{\kappa}} + \tilde{\boldsymbol{\kappa}}^T\widetilde{\boldsymbol{\mathcal{M}}}_\sigma\tilde{\boldsymbol{\phi}} \end{bmatrix} \tag{11.58}$$

The term $\mathcal{P}\mathbf{I}$ corresponds to the core geometric stiffness of the 3D Tl bar.

The higher order term in (11.57) may be expressed as

$$\mathcal{F}_\sigma(\delta^2\mathbf{R},\,\delta^2\boldsymbol{\kappa}) = \boldsymbol{\mathcal{M}}_\sigma^T\tilde{\boldsymbol{\phi}}\delta^2\boldsymbol{\kappa} + \boldsymbol{\phi}^T\mathbf{R}\delta^2\mathbf{R}^T\tilde{\boldsymbol{\kappa}}\boldsymbol{\mathcal{M}}_\sigma\delta\mathbf{q}^T\left(\mathbf{V}(\tilde{\boldsymbol{\phi}}^T\boldsymbol{\mathcal{M}}_\sigma) + \mathbf{U}(\tilde{\boldsymbol{\kappa}}\boldsymbol{\mathcal{M}}_\sigma;\,\boldsymbol{\phi})\right)\delta\mathbf{q}, \tag{11.59}$$

Consequently

$$\boldsymbol{\mathcal{S}}_{GCq\sigma} = \mathbf{V}(\tilde{\boldsymbol{\phi}}^T\boldsymbol{\mathcal{M}}_\sigma) + \mathbf{U}(\tilde{\boldsymbol{\kappa}}\boldsymbol{\mathcal{M}}_\sigma;\,\boldsymbol{\phi}). \tag{11.60}$$

Because the next-level target variables $\mathbf{q}$ include the finite rotation parametrization, matrices $\mathbf{V}$ and $\mathbf{U}$ depend on that choice. They are the source of unsymmetries in the stiffness matrices when certain rotational parametrizations are adopted, such as the incremental rotation vector. If the rotational vector is chosen these matrices are symmetric.

Geometric Stiffness due to Shear Stresses. The contribution of the shear stresses to the geometric stiffness is

$$\boldsymbol{\mathcal{S}}_{G\tau} = \boldsymbol{\mathcal{S}}_{GP\tau} + \boldsymbol{\mathcal{S}}_{GCw\tau} + \boldsymbol{\mathcal{S}}_{GCq\tau} = \boldsymbol{\mathcal{S}}_{Gw\tau} + \boldsymbol{\mathcal{S}}_{GCq\tau}. \tag{11.61}$$

The appropriate definitions are

$$\begin{aligned} \boldsymbol{\mathcal{S}}_{GP\tau} &= \int_{A_0} s_{12}(\mathbf{W}_1^T\mathbf{H}\mathbf{W}_2 + \mathbf{W}_2\mathbf{H}\mathbf{W}_1) + s_{13}(\mathbf{W}_1^T\mathbf{H}\mathbf{W}_3 + \mathbf{W}_3\mathbf{H}\mathbf{W}_1)\,dA \\ \boldsymbol{\mathcal{S}}_{GC\tau} &= \int_{A_0} (s_{12}\mathbf{b}_2 + s_{13}\mathbf{b}_3)\,\delta^2\mathbf{g}\;dA = \delta\mathbf{w}^T\boldsymbol{\mathcal{S}}_{GCw\tau}\delta\mathbf{w} + \mathcal{F}_\tau(\delta^2\mathbf{R},\,\delta^2\boldsymbol{\kappa}). \end{aligned} \tag{11.62}$$

Carrying out manipulations one obtains the surprisingly simple form for $\boldsymbol{\mathcal{S}}_{Gw\tau}$

$$\boldsymbol{\mathcal{S}}_{Gw\tau} = \boldsymbol{\mathcal{S}}_{GP\tau} + \boldsymbol{\mathcal{S}}_{GCw\tau} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{R}^T \widetilde{\boldsymbol{\mathcal{Q}}}^T \\ & \mathbf{0} & \mathbf{0} \\ symm & & \mathbf{0} \end{bmatrix}. \tag{11.63}$$

The terms due to the second variation of $\mathbf{g}$ become

$$\mathcal{F}_\tau = \boldsymbol{\mathcal{Q}}^T \delta^2 \mathbf{R} \boldsymbol{\Phi} + \boldsymbol{\mathcal{M}}_\tau^T \delta^2 \boldsymbol{\kappa}. \tag{11.64}$$

The kernel carried forward to the $\mathbf{q}$ level is

$$\boldsymbol{\mathcal{S}}_{GCq\tau} = \mathbf{V}(\boldsymbol{\mathcal{M}}_\tau) + \mathbf{U}(\boldsymbol{\mathcal{Q}}\,;\boldsymbol{\Phi}). \tag{11.65}$$

§11.5.2. Transformation to the Rotational Vector

The second transformation stage passes from $\mathbf{w}$ to $\mathbf{z}$, which is a vector of generalized displacements, also associated with a beam section, which embodies the parametrization of the cross section rotation:

$$\mathbf{z} = \left\{ \frac{d\mathbf{u}_0}{dX} \quad \frac{d\boldsymbol{\alpha}}{dX} \quad \boldsymbol{\alpha} \right\}^T, \qquad \delta\mathbf{z} = \left\{ \frac{d\,\delta\mathbf{u}_0}{dX} \quad \frac{d\,\delta\boldsymbol{\alpha}}{dX} \quad \delta\boldsymbol{\alpha} \right\}^T. \tag{11.66}$$

Here $\boldsymbol{\alpha}$ denotes the rotational vector parametrization defined by the standard formulas

$$\boldsymbol{\alpha} = \ \textbf{axial}\,(\tilde{\alpha}), \qquad \mathbf{R} = \exp(\tilde{\boldsymbol{\alpha}}^T), \tag{11.67}$$

and which may be extracted from $\mathbf{R}$ by

$$\tilde{\alpha} = \log \mathbf{R} = \frac{\arcsin(\tau)}{2\tau}\,\textbf{axial}\,(\mathbf{R}^T - \mathbf{R}), \qquad \tau = \tfrac{1}{2} ||\,\textbf{axial}\,(\mathbf{R}^T - \mathbf{R})||. \tag{11.68}$$

Because only the variations of $\mathbf{w}$ are known the relation between $\mathbf{w}$ and $\mathbf{z}$ is also of differential type:

$$\delta\mathbf{w} = \mathbf{Z}\,\delta\mathbf{z}, \qquad \text{or} \qquad \delta\mathbf{w} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y}(\mathbf{z}) & \dfrac{d\mathbf{Y}(\mathbf{z})}{dX} \\ \mathbf{0} & \mathbf{0} & \mathbf{Y}(\mathbf{z}) \end{bmatrix} \begin{bmatrix} \dfrac{d\,\delta\mathbf{u}_0}{dX} \\ \dfrac{d\,\delta\boldsymbol{\alpha}}{dX} \\ \delta\boldsymbol{\alpha} \end{bmatrix}, \tag{11.69}$$

in which

$$\mathbf{Y}(\alpha) = \frac{\sin|\boldsymbol{\alpha}|}{|\boldsymbol{\alpha}|}\mathbf{I} + \left(1 - \frac{\sin|\boldsymbol{\alpha}|}{|\boldsymbol{\alpha}|}\right)\frac{\boldsymbol{\alpha}\boldsymbol{\alpha}^T}{|\boldsymbol{\alpha}|^2} - \frac{1-\cos|\boldsymbol{\alpha}|}{|\boldsymbol{\alpha}|^2}\tilde{\boldsymbol{\alpha}}. \tag{11.70}$$

On applying the transformations (11.69) we find for the internal force and the material and principal-geometric components of the tangent stiffness matrix:

$$\mathbf{f}_q = \mathbf{Z}^T(\boldsymbol{\mathcal{R}}\sigma + \boldsymbol{\mathcal{R}}\tau), \qquad \mathbf{K}_{Mq} = \mathbf{Z}^T(\boldsymbol{\mathcal{S}}_M)\mathbf{Z}, \qquad \mathbf{K}_{GPq} = \mathbf{Z}^T(\boldsymbol{\mathcal{S}}_{Gw}\mathbf{Z}. \tag{11.71}$$

The materialization of the geometric stiffness terms $\boldsymbol{\mathcal{S}}_{GCq\sigma}$ and $\boldsymbol{\mathcal{S}}_{GCq\tau}$ for the rotational vector needs additional work. We state here only the final result:

$$\mathbf{U}(\boldsymbol{\tau};\boldsymbol{\Phi}) = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ & \mathbf{0} & \mathbf{0} \\ symm & & \mathbf{U}^\tau \end{bmatrix}, \qquad \mathbf{T}(\boldsymbol{\mathcal{M}}_\tau) = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ & \mathbf{0} & \mathbf{T}_1^\tau \\ symm & & \mathbf{T}_2^\tau \end{bmatrix}. \tag{11.72}$$

where

$$
\begin{aligned}
\mathbf{U}^{\tau} &= c_1 \boldsymbol{\tau}^T \boldsymbol{\Phi} \mathbf{I} + c_2 \left(\boldsymbol{\tau} \boldsymbol{\Phi}^T + \boldsymbol{\Phi}^T \boldsymbol{\tau} \right) + c_3 \left(\boldsymbol{\tau}^T \boldsymbol{\Phi} \boldsymbol{\alpha} \boldsymbol{\alpha}^T + \boldsymbol{\tau}^T \tilde{\boldsymbol{\alpha}} \boldsymbol{\Phi} \mathbf{I} + \tilde{\boldsymbol{\tau}}^T \boldsymbol{\Phi} \boldsymbol{\alpha}^T + \boldsymbol{\alpha} \boldsymbol{\Phi}^T \tilde{\boldsymbol{\tau}} \right) \\
&\quad + c_5 \left(\boldsymbol{\alpha}^T \boldsymbol{\Phi} \left(\boldsymbol{\tau} \boldsymbol{\alpha}^T + \boldsymbol{\alpha} \boldsymbol{\tau}^T \right) + \boldsymbol{\alpha}^T \boldsymbol{\tau} \left(\boldsymbol{\alpha} \boldsymbol{\Phi}^T + \boldsymbol{\Phi} \boldsymbol{\alpha}^T \right) + \boldsymbol{\tau}^T \boldsymbol{\alpha} \boldsymbol{\alpha}^T \boldsymbol{\Phi} \mathbf{I} \right) \\
&\quad + c_4 \boldsymbol{\tau}^T \tilde{\boldsymbol{\alpha}} \boldsymbol{\Phi} \boldsymbol{\alpha} \boldsymbol{\alpha}^T + c_6 \boldsymbol{\tau}^T \boldsymbol{\alpha} \boldsymbol{\alpha}^T \boldsymbol{\Phi} \boldsymbol{\alpha} \boldsymbol{\alpha}^T, \\
\mathbf{V}_1^{\tau} &= c_2 \widetilde{\boldsymbol{\mathcal{M}}}_{\tau}^T + c_3 \boldsymbol{\alpha} \boldsymbol{\mathcal{M}}_{\tau}^T + c_5 \boldsymbol{\alpha} \boldsymbol{\alpha}^T \widetilde{\boldsymbol{\mathcal{M}}}_{\tau} + c_7 \left(\boldsymbol{\mathcal{M}}_{\tau} \boldsymbol{\alpha}^T + \boldsymbol{\alpha}^T \boldsymbol{\mathcal{M}}_{\tau} \mathbf{I} \right) + c_8 \boldsymbol{\alpha}^T \boldsymbol{\mathcal{M}}_{\tau} \boldsymbol{\alpha} \boldsymbol{\alpha}^T, \\
\mathbf{V}_2^{\tau} &= -c_3 \frac{d\boldsymbol{\alpha}}{dX}^T \boldsymbol{\mathcal{M}}_{\tau} \mathbf{I} - c_4 \frac{d\boldsymbol{\alpha}}{dX}^T \boldsymbol{\mathcal{M}}_{\tau} \boldsymbol{\alpha} \boldsymbol{\alpha}^T + c_5 \left(\widetilde{\frac{d\boldsymbol{\alpha}}{dX}}^T \boldsymbol{\mathcal{M}}_{\tau} \boldsymbol{\alpha}^T + \boldsymbol{\alpha} \widetilde{\frac{d\boldsymbol{\alpha}}{dX}} \boldsymbol{\mathcal{M}}_{\tau}^T + \boldsymbol{\alpha}^T \widetilde{\frac{d\boldsymbol{\alpha}}{dX}} \boldsymbol{\mathcal{M}}_{\tau} \mathbf{I} \right) \\
&\quad + c_6 \boldsymbol{\alpha}^T \widetilde{\frac{d\boldsymbol{\alpha}}{dX}} \boldsymbol{\mathcal{M}}_{\tau} \boldsymbol{\alpha} \boldsymbol{\alpha}^T + c_7 \left(\frac{d\boldsymbol{\alpha}}{dX} \boldsymbol{\mathcal{M}}_{\tau}^T + \boldsymbol{\mathcal{M}}_{\tau} \frac{d\boldsymbol{\alpha}}{dX}^T \right) + \\
&\quad + c_8 \frac{d\boldsymbol{\alpha}}{dX}^T \boldsymbol{\alpha} \left(\boldsymbol{\alpha} \boldsymbol{\mathcal{M}}_{\tau}^T + \boldsymbol{\mathcal{M}}_{\tau} \boldsymbol{\alpha}^T + \boldsymbol{\alpha}^T \boldsymbol{\mathcal{M}}_{\tau} \mathbf{I} \right) + c_9 \frac{d\boldsymbol{\alpha}}{dX}^T \boldsymbol{\alpha} \boldsymbol{\alpha}^T \boldsymbol{\mathcal{M}}_{\tau} \boldsymbol{\alpha} \boldsymbol{\alpha}^T,
\end{aligned}
\tag{11.73}
$$

in which

$$
\begin{aligned}
c_1 &= -\frac{\sin\alpha}{\alpha}, & c_2 &= \frac{1-\cos\alpha}{\alpha^2}, & c_3 &= \frac{\sin\alpha - \alpha\cos\alpha}{\alpha^3}, \\
c_4 &= -\frac{c_1 + 3c_3}{\alpha^2}, & c_5 &= -\frac{c_1 + 2c_2}{\alpha^2}, & c_6 &= -\frac{c_3 + 4c_5}{\alpha^2}, \\
c_7 &= \frac{1 + c_1}{\alpha^2}, & c_8 &= \frac{3c_3 - 2c_2}{\alpha^2}, & c_9 &= \frac{c_5 - 5c_8}{\alpha^2}.
\end{aligned}
\tag{11.74}
$$

A similar approach can be taken with (11.69), which defines $\mathcal{F}_{\sigma}$. The tangent stiffness matrix can be obtained by superposing all contributions.

§11.5.3. Transformation to Finite Element Freedoms

The final stage introduces a finite element representation for the degrees of freedom. The beam or beam assembly is divided into a set of two-node finite elements. Each of these nodes has three displacement degrees of freedom and three rotational degrees of freedom corresponding to the three $\{\alpha_X, \alpha_Y, \alpha_Z\}$ components of the rotational vector $\boldsymbol{\alpha}$. Each element in turn has twelve freedoms which are collected in the array $\mathbf{v}^T = \{\mathbf{u}_n \quad \boldsymbol{\alpha}_n\}^T$ where $\mathbf{d}_n$ collects the six translational freedoms while $\boldsymbol{\alpha}_n$ collects the six rotations. The cross-section state vector $\mathbf{z}$ is approximated inside each element by

$$
\mathbf{z} = \begin{bmatrix} \mathbf{N} & \mathbf{0} \\ \mathbf{0} & \dfrac{d\,\mathbf{N}}{d\,X} \\ 0 & \mathbf{N} \end{bmatrix} \begin{bmatrix} \mathbf{d}_n \\ \boldsymbol{\alpha}_n \end{bmatrix} = \mathbf{G}_z \begin{bmatrix} \mathbf{d}_n \\ \boldsymbol{\alpha}_n \end{bmatrix} = \mathbf{G}_z\,\mathbf{v}. \tag{11.75}
$$

where $\mathbf{N}$ is a matrix of linear shape functions. Since $\delta\mathbf{q} = \mathbf{G}_z \delta\mathbf{v}$ the final internal force vector $\mathbf{f}$ and tangent stiffness matrix $\mathbf{K}$ of each element are obtained through the DCCF transformations

$$
\mathbf{f} = \int_0^{L_0} \mathbf{G}_z^T \mathbf{f}_z \, dX, \qquad \mathbf{K} = \int_0^{L_0} \mathbf{G}_z^T \mathbf{K}_z \mathbf{G}_z \, dX. \tag{11.76}
$$

The choice of shape functions for the rotational vector poses some subtle questions. In small-deflection analysis it is common practice to select all Timoshenko beam shape functions to be linear in X. This choice obviously enforces nodal compatibility while preserving constant curvature states. But for finite deflections a linear interpolation for the rotational vector components cannot exactly represent a constant curvature state unless the rotations are about a single axis (plane rotations). The same is true if the rotation matrix $\mathbf{R}(X)$ is interpolated linearly. On the other hand, linear interpolation of Euler parameters does preserve the constant curvature state. This motivated the development of an interpolation scheme that starts from the 4 Euler parameters $\epsilon_i(X)$, $i = 0, 1, 2, 3$, $\sum_i \epsilon_i^2 = 1$ that orient the normal of a cross section at X. These are collected in the 4-vector $\boldsymbol{\epsilon} = [\,\epsilon_0 \quad \epsilon_1 \quad \epsilon_2 \quad \epsilon_3\,]^T$. Given the eight end values $\boldsymbol{\epsilon}(0)$ and $\boldsymbol{\epsilon}(L)$ the interpolation that can copy a constant curvature vector $\boldsymbol{\kappa}$ is found to be [1]

$$\epsilon(\zeta) = \cos(\zeta)\left(1 - \frac{\tan(\zeta)}{\tan(\zeta_L)}\right)\boldsymbol{\epsilon}(0) + \frac{\sin(\zeta)}{\sin(\zeta_L)}\boldsymbol{\epsilon}(L), \tag{11.77}$$

where $\zeta = \frac{1}{2}\kappa X$, $\zeta_L = \frac{1}{2}\kappa L$, $\kappa = \sqrt{\boldsymbol{\kappa}^T\boldsymbol{\kappa}}$. The constant curvature vector can be extracted from the end values through the formula

$$\boldsymbol{\kappa} = \frac{1}{\beta_2 L}\left[\left(\widetilde{\boldsymbol{\epsilon}(L)} - 2\epsilon_0(L)\mathbf{I}\right)\boldsymbol{\epsilon}(0) - \left(\widetilde{\boldsymbol{\epsilon}(0)} - 2\epsilon_0(0)\mathbf{I}\right)\boldsymbol{\epsilon}(0)\right], \tag{11.78}$$

This interpolation is then transformed to the variations in terms of the rotational vector. Details are provided in Reference [1].

§11.6. Equivalence of DCCF and Standard TL Formulation

The correspondence between the Direct Core Congruential Formulation (DCCF) and the Standard Formulation (SF) of the Total Lagrangian (TL) kinematic description is established below for 3D continuum finite elements. This connection was worked out in a course term project [4]. Such elements fit within the DCCF framework because their physical DOFs (node displacements) are of translational type.

The Standard Formulation is based on the same scheme used for linear finite elements: first interpolate, then vary. As in the linear case, the departure point is extremization of the Total Potential Energy functional (TPE) over the element domain:

$$\Pi = U - W = \int_{V_0} \mathbf{e}^T\mathbf{s}_0\, dV + \tfrac{1}{2}\int_{V_0} \mathbf{e}^T\mathbf{E}\mathbf{e}\, dV - \int_{V_0} \mathbf{u}^T\mathbf{b}\, dV - \int_{S_{t0}} \mathbf{u}^T\,\mathbf{t}\, dS, \tag{11.79}$$

where as usual conservative dead loading is assumed. In (11.79), $\mathbf{b}$ is the prescribed body force field, $\mathbf{t}$ are surface tractions prescribed over portion S_{t0} of the boundary in $\mathcal{C}_0$, and other quantities are as defined in Section 4. The weak equilibrium equations are obtained on making (11.79) stationary:

$$\delta\Pi = \delta U - \delta W = \int_{V_0} \delta\mathbf{e}^T\mathbf{s}_0\, dV + \int_{V_0} \delta\mathbf{e}^T\mathbf{E}\mathbf{e}\, dV - \int_{V_0} \delta\mathbf{u}^T\mathbf{b}\, dV - \int_{S_{t0}} \delta\mathbf{u}^T\,\mathbf{t}\, dS = \mathbf{0}. \tag{11.80}$$

The displacement and strain fields are interpolated in terms of the element degrees of freedom $\mathbf{v}$:

$$\mathbf{u} = \mathbf{N}\,\mathbf{v}, \qquad \delta\mathbf{u} = \mathbf{N}\,\delta\mathbf{v}, \qquad \delta\mathbf{e} = \mathbf{B}\,\delta\mathbf{v}, \tag{11.81}$$

where $\mathbf{B} = \mathbf{B}(\mathbf{v})$ depends in $\mathbf{v}$ but $\mathbf{N}$ does not. Substituting these interpolations into (11.80) yields the residual equilibrium equations

$$\delta\Pi = \delta\mathbf{v}^T\mathbf{r} = \delta\mathbf{v}^T(\mathbf{f} - \mathbf{p}) = \mathbf{0}. \tag{11.82}$$

where

$$\mathbf{f} = \int_{V_0} \mathbf{B}^T\,(\mathbf{s}_0 + \mathbf{E}\mathbf{e})\,dV = \int_{V_0} \mathbf{B}^T \mathbf{s}\,dV, \qquad \mathbf{p} = \int_{V_0} \mathbf{N}^T \mathbf{b}\,dV + \int_{S_{t0}} \mathbf{N}\mathbf{s}\,dS, \tag{11.83}$$

where $\mathbf{f}$ and $\mathbf{p}$ are the internal and external force vectors, respectively, and $\mathbf{s} = \mathbf{s}^0 + \mathbf{E}\mathbf{e}$ are the PK2 stresses in $\mathcal{C}$. Because the variations $\delta\mathbf{v}$ are arbitrary, the residual-force nonlinear equilibrium equation is $\mathbf{r} = \mathbf{f} - \mathbf{p} = \mathbf{0}$ or $\mathbf{f} = \mathbf{p}$. The tangent stiffness matrix is given by

$$\mathbf{K} = \frac{\partial \mathbf{r}}{\partial \mathbf{v}} = \frac{\partial \mathbf{f}}{\partial \mathbf{v}}, \tag{11.84}$$

because $\mathbf{p}$ (for conservative dead loading) does not depend on $\mathbf{v}$. Splitting $\mathbf{B} = \mathbf{B}_c + \mathbf{B}_v(\mathbf{v})$, where $\mathbf{B}_c$ is constant but $\mathbf{B}_v$ depends on $\mathbf{v}$, gives the well known decomposition

$$\mathbf{K} = \mathbf{K}_0 + \mathbf{K}_D + \mathbf{K}_G, \tag{11.85}$$

where $\mathbf{K}_0$, $\mathbf{K}_D$ and $\mathbf{K}_G$ denote the linear, initial-displacement and geometric stiffness matrices, respectively. These are given by

$$\begin{aligned} \mathbf{K}_0 &= \int_{V_0} \mathbf{B}_c^T \mathbf{E}\mathbf{B}_c\,dV, \\ \mathbf{K}_D &= \int_{V_0} (\mathbf{B}_c^T \mathbf{E}\mathbf{B}_v + \mathbf{B}_v^T \mathbf{E}\mathbf{B}_c + \mathbf{B}_v^T \mathbf{E}\mathbf{B}_v)\,dV, \\ \mathbf{K}_G\,\delta\mathbf{v} &= \int_{V_0} \delta\mathbf{B}^T \mathbf{s}\,dV. \end{aligned} \tag{11.86}$$

To correlate these standard forms with those produced by the DCCF, we note that the GL strains can be also split as $\mathbf{e} = \mathbf{e}_c + \mathbf{e}_v$, where $\mathbf{e}_c$ and $\mathbf{e}_v$ are linear and nonlinear in $\mathbf{v}$, respectively. The latter may be expressed in terms of the displacement gradients as

$$\mathbf{e}_v = \tfrac{1}{2}\mathbf{A}\mathbf{g}, \tag{11.87}$$

where $\mathbf{A}$ is the 6×9 matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{g}_1^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{g}_2^T & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{g}_3^T \\ \mathbf{0} & \mathbf{g}_3^T & \mathbf{g}_2^T \\ \mathbf{g}_3^T & \mathbf{0} & \mathbf{g}_1^T \\ \mathbf{g}_2^T & \mathbf{g}_1^T & \mathbf{0} \end{bmatrix} = \begin{bmatrix} g_1 & g_2 & g_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_4 & g_5 & g_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & g_7 & g_8 & g_9 \\ 0 & 0 & 0 & g_7 & g_8 & g_9 & g_4 & g_5 & g_6 \\ g_7 & g_8 & g_9 & 0 & 0 & 0 & g_1 & g_2 & g_3 \\ g_4 & g_5 & g_6 & g_1 & g_2 & g_3 & 0 & 0 & 0 \end{bmatrix}, \tag{11.88}$$

in which the displacement gradients are vector-arranged as

$$\mathbf{g}^T = [\, g_1 \quad g_2 \quad \cdots \quad g_8 \quad g_9 \,] = \left[\, \frac{\partial u_1}{\partial X_1} \quad \frac{\partial u_2}{\partial X_1} \quad \cdots \quad \frac{\partial u_2}{\partial X_3} \quad \frac{\partial u_3}{\partial X_3} \,\right]. \tag{11.89}$$

Comparing

$$\delta\mathbf{e}_v = \tfrac{1}{2}\,\delta\mathbf{A}\,\mathbf{g} + \tfrac{1}{2}\mathbf{A}\,\delta\mathbf{g} = \mathbf{A}\,\delta\mathbf{g}, \tag{11.90}$$

to the DCCF transformation relation $\delta\mathbf{g} = \mathbf{G}\,\delta\mathbf{v}$, in which $\mathbf{G}$ is independent of $\mathbf{v}$, we see that

$$\mathbf{B}_v = \mathbf{A}\mathbf{G}. \tag{11.91}$$

The other expression we require is $\delta\mathbf{A}^T\mathbf{s}$, which appears in the geometric stiffness matrix contracted with $\delta\mathbf{v}$:

$$\mathbf{K}_G\,\delta\mathbf{v} = \int_{V_0} \delta\mathbf{B}^T \mathbf{s}\,dV = \int_{V_0} \mathbf{G}^T\,\delta\mathbf{A}^T \mathbf{s}\,dV. \tag{11.92}$$

It is well known — see for instance Chapter 19 of Zienkiewicz [5] — that

$$\delta\mathbf{A}^T\mathbf{s} = \mathbf{M}\,\delta\mathbf{g} = \mathbf{M}\mathbf{G}\,\delta\mathbf{v}, \quad \text{with} \quad \mathbf{M} = \begin{bmatrix} s_1\mathbf{I} & s_4\mathbf{I} & s_5\mathbf{I} \\ s_4\mathbf{I} & s_2\mathbf{I} & s_6\mathbf{I} \\ s_5\mathbf{I} & s_6\mathbf{I} & s_3\mathbf{I} \end{bmatrix}, \tag{11.93}$$

where $\mathbf{I}$ is the 3×3 identity matrix and s_i, $i=1,\ldots 6$ are components of the PK2 stress tensor ordered $s_1 = s_{11}$, $s_2 = s_{22}$, $\ldots\, s_6 = s_{23}$. Using this relation, $\mathbf{K}_G$ can be placed in the standard form

$$\mathbf{K}_G = \int_{V_0} \mathbf{G}^T\mathbf{M}\mathbf{G}\,dV, \tag{11.94}$$

which by inspection is seen to be the DCCF-transformation of the core geometric stiffness $\mathbf{M} \equiv \mathbf{S}_G = s_i\mathbf{H}_i$, with the $\mathbf{H}_i$ matrices defined in (10.10).

To correlate other terms, write the linear part of the GL strains in terms of gradients as

$$\mathbf{e}_c = \mathbf{D}\mathbf{g} = \mathbf{D}\mathbf{G}\,\delta\mathbf{v}, \quad \text{with} \quad \mathbf{D} = \begin{bmatrix} 1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1\\ 0&1&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&1&0&0\\ 0&0&0&0&0&1&0&1&0 \end{bmatrix}. \tag{11.95}$$

The numerical $\mathbf{D}$ matrix can be easily related to the $\mathbf{h}_i$ vectors introduced in (10.10). Because both $\mathbf{D}$ and $\mathbf{G}$ are independent of $\mathbf{v}$ it follows that $\delta\mathbf{e}_c = \mathbf{D}\mathbf{G}\,\delta\mathbf{v}$ and consequently $\mathbf{B}_c = \mathbf{D}\mathbf{G}$. Partitioning $\mathbf{A}$ as $[\,\mathbf{a}_1^T \quad \mathbf{a}_2^T \quad \ldots \quad \mathbf{a}_6^T\,]$ one easily finds that $\mathbf{a}_i = \mathbf{H}_i\mathbf{g}$. Now the following identities can be verified through simple algebra:

$$\begin{aligned} \mathbf{D}^T\mathbf{E}\mathbf{D} &= E_{ij}\mathbf{h}_i\mathbf{h}_i^T = \mathbf{S}_0, \\ \mathbf{D}^T\mathbf{E}\mathbf{A} &= E_{ij}\mathbf{h}_i\mathbf{a}_j^T = E_{ij}\mathbf{h}_i\mathbf{g}^T\mathbf{H}_j = \mathbf{S}_1, \qquad \mathbf{A}^T\mathbf{E}\mathbf{D} = \mathbf{S}_1^T, \\ \mathbf{A}^T\mathbf{E}\mathbf{A} &= E_{ij}\mathbf{a}_i\mathbf{a}_j = E_{ij}\mathbf{H}_i\mathbf{g}_i\mathbf{g}_j^T\mathbf{H}_j = \mathbf{S}_2 = \mathbf{S}_2^T, \\ \mathbf{M} &= s_i^0\mathbf{H}_i + E_{ij}\mathbf{h}_i\mathbf{g}\mathbf{H}_j + \tfrac{1}{2}(\mathbf{g}^T\mathbf{H}_i\mathbf{g})\mathbf{H}_j = s_i^0\mathbf{H}_i + \mathbf{S}_1^* + \tfrac{1}{2}\mathbf{S}_2^* = s_i\mathbf{H}_i. \end{aligned} \tag{11.96}$$

Comparing these to the expressions of §10.4.3 we conclude that

$$\begin{aligned} \mathbf{K}_0 &= \int_{V_0} \mathbf{G}^T\mathbf{D}^T\mathbf{E}\mathbf{D}\mathbf{G}\,dV = \int_{V_0} \mathbf{G}^T\mathbf{S}_0\mathbf{G}\,dV, \\ \mathbf{K}_D &= \int_{V_0} \mathbf{G}^T(\mathbf{D}^T\mathbf{E}\mathbf{A} + \mathbf{A}^T\mathbf{E}\mathbf{D} + \mathbf{A}^T\mathbf{E}\mathbf{A})\mathbf{G}\,dV = \int_{V_0} \mathbf{G}^T\mathbf{S}_D\mathbf{G}\,dV, \\ \mathbf{K}_G &= \int_{V_0} \mathbf{G}^T\mathbf{M}\mathbf{G}\,dV = \int_{V_0} \mathbf{G}^T\mathbf{S}_G\mathbf{G}\,dV, \end{aligned} \tag{11.97}$$

which displays the equivalence of both formulations when no approximations are made. This proof may be extended without difficulty to the AGCCF in which case $\mathbf{G}$ is a function of $\mathbf{v}$, although as noted in the text that situation is sometimes mishandled in the Standard Formulation through the introduction of *a priori* kinematic approximations. The equivalence between DGCCF and SF is more difficult to prove because there is no TPE functional from which the latter can be derived, and such connection should be regarded as an open problem.

§11.7. References

[1] L. A. Crivelli, A Total-Lagrangian beam element for analysis of nonlinear space structures, *Ph. D. Dissertation*, Dept. of Aerospace Engineering Sciences, University of Colorado, Boulder, CO, 1990.

[2] C. A. Felippa and L. A. Crivelli, A congruential formulation of nonlinear finite elements, in *Nonlinear Computational Mechanics - The State of the Art*, ed. by P. Wriggers and W. Wagner, Springer-Verlag, Berlin, pp. 283–302, 1991.

[3] L. A. Crivelli and C. A. Felippa, A three-dimensional non-linear Timoshenko beam element based on the core-congruential formulation, *Int. J. Numer. Meth. Engrg.*, **36**, pp. 3647–3673, 1993.

[4] F. Abedzadeh Anaraki, A. Barzegar Mehrabi and H. R. Lofti, Correspondence between CC-TL and C-TL formulations, in *Term Projects in Nonlinear Finite Element Methods*, ed. by C. A. Felippa, Report CU-CSSC-91-12, Center for Space Structures and Controls, University of Colorado, Boulder, CO, May 1991.

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Homework Exercises for Chapter 11

Not Assigned

EXERCISE 11.1

Consider a two-node geometrically TL nonlinear bar in three dimensions built with the CCF. Write subroutines to compute the tangent stiffness matrix **K** and the PK2 stress s in the *current* configuration given the following input data:

(1) The coordinates of both element nodes in the reference configuration global Cartesian system $X,\ Y,\ Z$.

(2) The X, Y, Z displacements v_{Xi}, v_{Yi}, v_{Zi} of the two element end nodes $i = 1, 2$.

(3) The elastic modulus E and the reference bar cross section A_0.

(4) The bar axial stress s^0 in the reference configuration.

The subroutines you have to write (in Fortran, C or C++) have the following names and calling sequence interface:

```
BAR3F  (X0, v, E, A0, s0, F, status)
BAR3K  (X0, v, E, A0, s0, K, status)
BAR3S  (X0, v, E, s0, s, status)
```

`BAR3F` and `BAR3K` compute the internal force vector **f** and tangent stiffness matrix **K**, respectively, in the current configuration. Subroutine `BAR3S` computes the PK2 stress s in the current configuration.

Input variables (Fortran assumed in description):

Variable	*Declaration*	*Description*
`X0`	`double precision x0(3,2)`	Global coordinates of end nodes. The coordinates of node i go in the i^{th} column of `X0`
`v`	`double precision v(3,2)`	v_X, v_Y, v_Z displacements of end nodes. The displacements of node i go in the i^{th} column of `v`
`E`	`double precision e`	Elastic modulus
`A0`	`double precision a0`	Reference cross section area
`s0`	`double precision s0`	PK2 stress in reference configuration

The output variables are:

Variable	*Declaration*	*Description*
`F`	`double precision f(6)`	internal force vector
`K`	`double precision k(6,6)`	tangent stiffness matrix
`S`	`double precision s`	PK2 axial stress s in current configuration
`STAT`	`character*(*) stat`	Blank if no error; else error message

For intermediate manipulations, construct the local bar axis $\overline{X}$ joining nodes 1 to 2 in $\mathcal{C}_0$. Then construct $\overline{Y}$, and $\overline{Z}$ normal to $\overline{X}$ forming a right-handed system. As noted §11.2.1, there is some arbitrariness because $\overline{Y}$, $\overline{Z}$ may be "gyrated" around $\overline{X}$ without changing the final answer; select whatever orientation rule seem to be more computationally "robust" in the sense that it should not fail for arbitrary bar orientations.

The answer for this exercise should be a listing of `BAR3F`, `BAR3K` and `BAR3S`.

EXERCISE 11.2

Explain the rule you chose to orient $\overline{Y}$ and $\overline{Z}$. (Appropriate comments in the `BARK` source code should be sufficient to answer this one.)

EXERCISE 11.3

Test the subroutines on the following input data:

Argument	*Input value(s)*
`X0`	[1.23, 2.34, 3.45, 5.43, 4.32, 3.21]
`v`	[0.76, −2.12, 1.67, −2.45, 3.01, −3.28]
`E`	1.82
`A0`	0.765
`S0`	3.21

To feed these values write a short test driver that calls the two subroutines in turn; a suggested driver is listed under "Programming Recommendations."

Compare the output of `BAR3K` with the following tangent stiffness matrix:

$$\mathbf{K} = \begin{bmatrix} 0.935471 & 0.097502 & -0.071172 & -0.935471 & -0.097502 & 0.071172 \\ 0.097502 & 1.622137 & -0.511148 & -0.097502 & -1.622137 & 0.511148 \\ -0.071172 & -0.511148 & 1.295011 & 0.071172 & 0.511148 & -1.295011 \\ -0.935471 & -0.097502 & 0.071172 & 0.935471 & 0.097502 & -0.071172 \\ -0.097502 & -1.622137 & 0.511148 & 0.097502 & 1.622137 & -0.511148 \\ 0.071172 & 0.511148 & -1.295011 & -0.071172 & -0.511148 & 1.295011 \end{bmatrix} \tag{E11.1}$$

Compare the output of `BAR3F` to the following internal force vector:

$$\mathbf{f} = \begin{bmatrix} -0.912675 \\ -6.554668 \\ 4.784631 \\ 0.912675 \\ 6.554668 \\ -4.784631 \end{bmatrix} \tag{E11.2}$$

The computed PK2 stress s in the current configuration returned by `BAR3S` should be 5.603088.