11

Variational Formulation of Bar Element

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§11.1. A New Beginning

This Chapter begins Part II of the course. This Part focuses on the construction of structural and continuum finite elements using a *variational formulation* based on the Total Potential Energy. Why only elements? Because the other synthesis steps of the DSM: globalization, merge, BC application and solution, remain the same as in Part I. Those operations are not element dependent.

Individual elements are constructed in this Part beginning with the simplest ones and progressing to more complicated ones. The formulation of 2D finite elements from a variational standpoint is discussed in Chapters 14 and following. Although the scope of that formulation is broad, exceeding structural mechanics, it is better understood by going through specific elements first.

From a geometrical standpoint the simplest finite elements are one-dimensional or *line elements*. This means that the *intrinsic dimensionality* is one, although these elements may be used in one, two or three space dimensions upon transformation to global coordinates as appropriate. The simplest one-dimensional structural element is the *two-node bar element*, which we have already encountered in Chapters 2, 3 and 5 as the truss member.

In this Chapter the bar stiffness equations are rederived using the variational formulation. For uniform properties the resulting equations are the same as those found previously using the physical or Mechanics of Materials approach. The variational method has the advantage of being readily extendible to more complicated situations, such as variable cross section or more than two nodes.

§11.2. Definition of Bar Member

In structural mechanics a *bar* is a structural component characterized by two properties:

(1) One preferred dimension: the *longitudinal dimension* or *axial dimension* is much larger that the other two dimensions, which are collectively known as *transverse dimensions*. The intersection of a plane normal to the longitudinal dimension and the bar defines the *cross sections*. The longitudinal dimension defines the *longitudinal axis*. See Figure 11.1.

(2) The bar resists an internal axial force along its longitudinal dimension.

In addition to trusses, bar elements are used to model cables, chains and ropes. They are also used as fictitious elements in penalty function methods, as discussed in Chapter 9.

We will consider here only *straight bars*, although their cross section may vary. The one-dimensional mathematical model assumes that the bar material is linearly elastic, obeying Hooke's law, and that displacements and strains are infinitesimal. Figure 11.2 pictures the relevant quantities for a fixed-free bar. Table 11.1 collects the necessary terminology for the governing equations.

Figure 11.3 displays the governing equations of the bar in a graphic format called a *Tonti diagram*. The formal similarity with the diagrams used in Chapter 5 to explain MoM elements should be noted, although the diagram of Figure 11.3 pertains to the continuum model rather than to the discrete one.

Table 11.1 Nomenclature for Mathematical Model of Axially Loaded Bar

Quantity	*Meaning*
x	Longitudinal bar axis*
$(.)'$	$d(.)/dx$
$u(x)$	Axial displacement
$q(x)$	Distributed axial force, given per unit of bar length
L	Total bar length
E	Elastic modulus
A	Cross section area; may vary with x
EA	Axial rigidity
$e = du/dx = u'$	Infinitesimal axial strain
$\sigma = Ee = Eu'$	Axial stress
$p = A\sigma = EAe = EAu'$	Internal axial force
P	Prescribed end load

* x is used in this Chapter instead of $\bar{x}$ (as in Chapters 2–3) to simplify the notation.

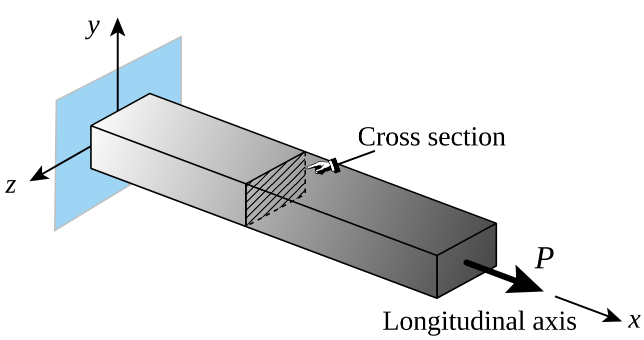

FIGURE 11.1. A fixed-free bar member.

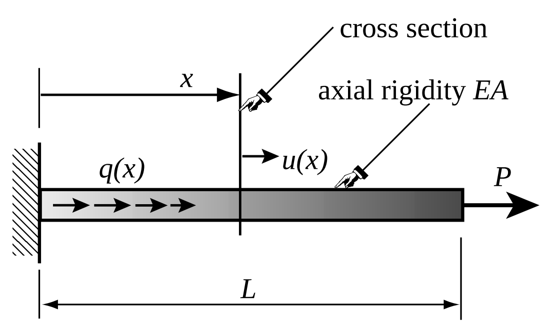

FIGURE 11.2. Quantities that appear in analysis of bar.

§11.3. Variational Formulation

To illustrate the variational formulation, the finite element equations of the bar will be derived from the Minimum Potential Energy principle.

§11.3.1. The Total Potential Energy Functional

In Mechanics of Materials it is shown that the *internal energy density* at a point of a linear-elastic material subjected to a one-dimensional state of stress σ and strain e is $\mathcal{U} = \frac{1}{2}\sigma(x)e(x)$, where σ is to be regarded as linked to the displacement u through Hooke's law $\sigma = Ee$ and the strain-displacement relation $e = u' = du/dx$. This $\mathcal{U}$ is also called the *strain energy density*. Integration

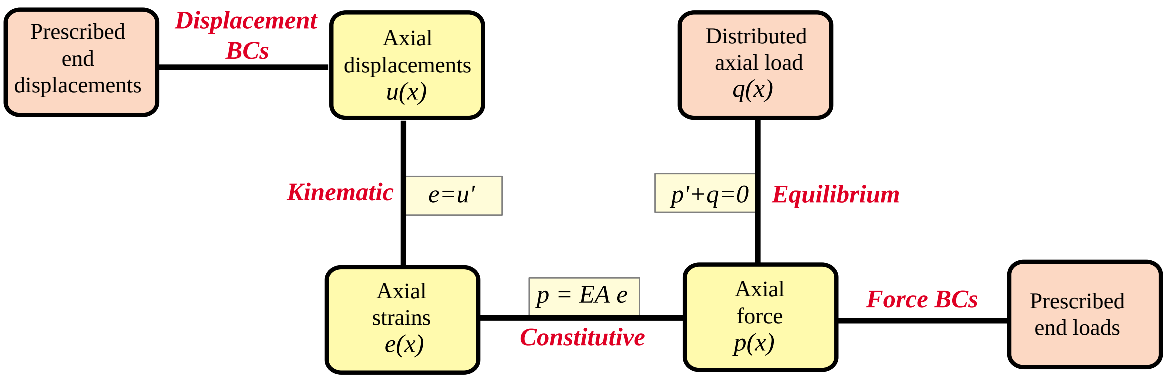

FIGURE 11.3. Tonti diagram for the continuum model of a bar member. Field equations and BCs are represented as lines connecting the boxes. Yellow (brown) boxes contain unknown (given) quantities.

over the volume of the bar gives the total internal energy

$$U = \tfrac{1}{2}\int_V \sigma\, e\, dV = \tfrac{1}{2}\int_0^L pe\, dx = \tfrac{1}{2}\int_0^L (EAu')u'\, dx = \tfrac{1}{2}\int_0^L u' EA\, u'\, dx, \tag{11.1}$$

in which all integrand quantities may depend on x.

The *external energy* due to applied mechanical loads pools contributions from two sources:

1. The distributed load $q(x)$. This contributes a cross-section density of $q(x)u(x)$ because q is assumed to be already integrated over the section.
2. Any applied end load(s). For the fixed-free example of Figure 11.2 the end load P would contribute $P\,u(L)$.

The second source may be folded into the first by conventionally writing any point load P acting at a cross section $x = a$ as a contribution $P\,\delta(a)$ to $q(x)$, where $\delta(a)$ denotes the one-dimensional Dirac delta function at $x = a$. If this is done the external energy can be concisely expressed as

$$W = \int_0^L qu\, dx. \tag{11.2}$$

The total potential energy of the bar is given by

$$\boxed{\Pi = U - W} \tag{11.3}$$

Mathematically this is a functional, called the *Total Potential Energy* functional or TPE. It depends only on the axial displacement $u(x)$. In variational calculus this is called the *primary variable* of the functional. When the dependence of Π on u needs to be emphasized we shall write $\Pi[u] = U[u] - W[u]$, with brackets enclosing the primary variable. To display both primary and independent variables we write, for example, $\Pi[u(x)] = U[u(x)] - W[u(x)]$.

Remark 11.1. According to the rules of Variational Calculus, the Euler-Lagrange equation for Π is

$$\frac{\partial \Pi}{\partial u} - \frac{d}{dx}\frac{\partial \Pi}{\partial u'} = -q - (E A\, u')' = 0 \tag{11.4}$$

This is the equation of equilibrium in terms of the axial displacement, usually written $(E A\, u')' + q = 0$, or $E A\, u'' + q = 0$ if $E A$ is constant. This equation is not explicitly used in the FEM development. It is instead replaced by $\delta \Pi = 0$, with the variation restricted over the finite element interpolation functions.

§11.3.2. Variation of an Admissible Function

The concept of *admissible variation* is fundamental in both variational calculus and the variationally formulated FEM. *Only the primary variable(s) of a functional may be varied.* For the TPE functional (11.4) this is the axial displacement $u(x)$. Suppose that $u(x)$ is changed to $u(x) + \delta\, u(x)$.[1]

This is illustrated in Figure 11.4, where for convenience $u(x)$ is plotted normal to x. The functional changes from Π to $\Pi + \delta\Pi$. The function $\delta\, u(x)$ and the scalar $\delta\Pi$ are called the *variations* of $u(x)$ and Π, respectively. The variation $\delta\, u(x)$ should not be confused with the ordinary differential $du(x) = u'(x)\, dx$ since on taking the variation the independent variable x is frozen; that is, $\delta x = 0$.

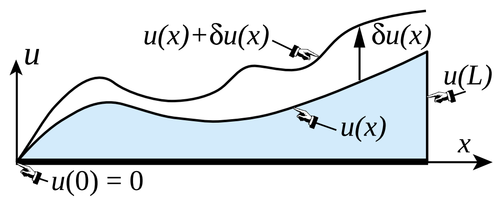

FIGURE 11.4. Concept of admissible variation of the axial displacement function $u(x)$. For convenience $u(x)$ is plotted normal to the longitudinal axis. Both depicted $u(x)$ and $u(x) + \delta u(x)$ are kinematically admissible, and so is the variation $\delta u(x)$.

A displacement variation $\delta u(x)$ is said to be *admissible* when both $u(x)$ and $u(x) + \delta\, u(x)$ are *kinematically admissible* in the sense of the Principle of Virtual Work (PVW). This agrees with the conditions stated in the classic variational calculus.

A *kinematically admissible* axial displacement $u(x)$ obeys two conditions:

(i) It is continuous over the bar length, that is, $u(x) \in \mathcal{C}_0$ in $x \in [0, L]$.

(ii) It satisfies exactly any displacement boundary condition, such as the fixed-end specification $u(0) = 0$ of Figure 11.2.

The variation $\delta\, u(x)$ depicted in Figure 11.4 is kinematically admissible because both $u(x)$ and $u(x) + \delta\, u(x)$ satisfy the foregoing conditions. The physical meaning of (i)–(ii) is the subject of Exercise 11.1.

§11.3.3. The Minimum Potential Energy Principle

The Minimum Potential Energy (MPE) principle states that the actual displacement solution $u^*(x)$ that satisfies the governing equations is that which renders Π stationary:[2]

$$\boxed{\delta\Pi = \delta U - \delta W = 0 \quad \text{iff} \quad u = u^*} \tag{11.5}$$

with respect to *admissible* variations $u = u^* + \delta u$ of the exact displacement field $u^*(x)$.

[1] The symbol δ not immediately followed by a parenthesis is not a delta function but instead denotes variation with respect to the variable that follows.

[2] The symbol “iff” in (11.5) is an abbreviation for “if and only if”.

Remark 11.2. Using standard techniques of variational calculus[3] it can be shown that if $EA > 0$ the solution $u^*(x)$ of (11.5) exists, is unique, and renders $\Pi[u]$ a minimum over the class of kinematically admissible displacements. The last attribute explains the "mininum" in the name of the principle.

§11.3.4. TPE Discretization

To apply the TPE functional (11.2) to the derivation of finite element equations we replace the continuum mathematical model by a discrete one consisting of a union of bar elements. For example, Figure 11.5 illustrates the subdivision of a bar member into four two-node elements.

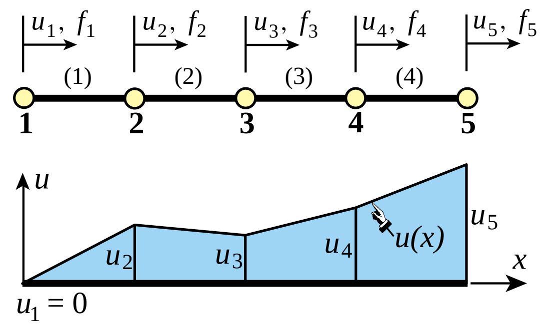

FIGURE 11.5. FEM discretization of bar member. A piecewise- linear admissible displacement trial function $u(x)$ is drawn underneath the mesh. It is assumed that the left end is fixed; thus $u_1 = 0$.

Functionals are scalars. Therefore, corresponding to a discretization such as that shown in Figure 11.5, the TPE functional (11.4) may be decomposed into a sum of contributions of individual elements:

$$\boxed{\Pi = \Pi^{(1)} + \Pi^{(2)} + \ldots + \Pi^{(N_e)}} \tag{11.6}$$

where N_e is the number of elements. The same decomposition applies to the internal and external energies, as well as to the stationarity condition (11.5):

$$\boxed{\delta\Pi = \delta\Pi^{(1)} + \delta\Pi^{(2)} + \ldots + \delta\Pi^{(N_e)} = 0.} \tag{11.7}$$

Using the fundamental lemma of variational calculus,[4] it can be shown that (11.7) implies that for a generic element e we may write

$$\boxed{\delta\Pi^e = \delta U^e - \delta W^e = 0.} \tag{11.8}$$

This *variational equation* is the basis for the derivation of element stiffness equations once the displacement field has been discretized over the element, as described next.

Remark 11.3. In mathematics (11.8) is called a *weak form*. In mechanics it also states the Principle of Virtual Work for each element: $\delta U^e = \delta W^e$, which says that the virtual work of internal and external forces on admissible displacement variations is equal if the element is in equilibrium [137].

[3] See references in **Notes and Bibliography** at the end of Chapter.

[4] See, e.g., Chapter II of Gelfand and Fomin [76].

§11.3.5. Bar Element Discretization

Figure 11.6 depicts a generic bar element e. It has two nodes, which are labeled 1 and 2. These are called the *local node numbers*.[5] The element is referred to its local axis $\bar{x} = x - x_1$, which measures the distance from its left end. The two degrees of freedom are u_1^e and u_2^e. (Bars are not necessary on these values since the directions of $\bar{x}$ and x are the same.) The element length is $\ell = L^e$.

The mathematical concept of bar finite elements is based on *approximation* of the axial displacement $u(x)$ over the element. The exact displacement u^* is replaced by an approximate displacement

$$u^*(x) \approx u^e(x) \tag{11.9}$$

over the finite element mesh. This approximate displacement, $u^e(x)$, taken over all elements $e = 1, 2, \ldots N^e$, is called the *finite element trial expansion* or simply *trial expansion*. See Figure 11.5.

This FE trial expansion must belong to the class of kinematically admissible displacements defined in §11.3.2. Consequently, it must be $\mathcal{C}_0$ continuous over and between elements.

§11.3.6. Shape Functions

In a two-node bar element the only possible variation of the displacement u^e that satisfies the interelement continuity requirement stated above is *linear*. It can be expressed by the interpolation formula

$$\boxed{u^e(x) = N_1^e u_1^e + N_2^e u_2^e = [\, N_1^e \quad N_2^e \,] \begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix} = \mathbf{N}\mathbf{u}^e.} \tag{11.10}$$

The functions N_1^e and N_2^e that multiply the node displacements u_1 and u_2 are called *shape functions*. These functions *interpolate* the internal displacement u^e directly from the node values.

See Figure 11.6. For the bar element, with $\bar{x} = x - x_1$ measuring the distance from the left node i, the shape functions are

$$N_1^e = 1 - \frac{\bar{x}}{\ell} = 1 - \zeta, \qquad N_2^e = \frac{\bar{x}}{\ell} = \zeta. \tag{11.11}$$

Here $\zeta = (x - x_1)/\ell = \bar{x}/\ell$ is a dimensionless coordinate, also known as a *natural coordinate*. Note that $dx = \ell\, d\zeta$ and $d\zeta = dx/\ell$. The shape function N_1^e has the value 1 at node 1 and 0 at node 2. Conversely, shape function N_2^e has the value 0 at node 1 and 1 at node 2. This is a general property of shape functions. It follows from the fact that element displacement interpolations such as (11.10) are based on physical node values.

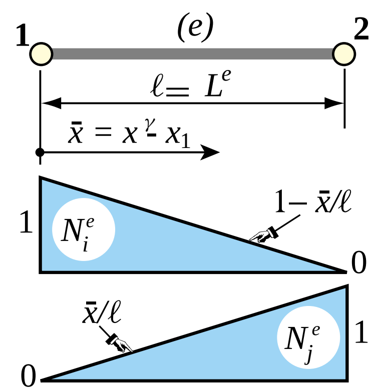

FIGURE 11.6. The shape functions of the generic bar element.

[5] Note the notational change from the labels i and j of Part I. This will facilitate transition to multidimensional elements.

Remark 11.4. In addition to continuity, shape functions must satisfy a *completeness* requirement with respect to the governing variational principle. This condition is stated and discussed in later Chapters. Suffices for now to say that the shape functions (11.11) do satisfy this requirement.

§11.3.7. The Strain-Displacement Equation

The axial strain over the element is

$$e = \frac{du^e}{dx} = (u^e)' = \left[\frac{dN_1^e}{dx} \quad \frac{dN_2^e}{dx} \right] \begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix} = \frac{1}{\ell} [\,-1 \quad 1\,] \begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix} = \mathbf{B}\mathbf{u}^e, \tag{11.12}$$

where

$$\mathbf{B} = \frac{1}{\ell} [\,-1 \quad 1\,] \tag{11.13}$$

is called the *strain-displacement* matrix.

§11.3.8. *Trial Basis Functions

Shape functions are associated with elements. A *trial basis function*, or simply *basis function*, is associated with a node. Suppose node i of a bar discretization connects elements $(e1)$ and $(e2)$. The trial basis function N_i is defined as

$$N_i(x) = \begin{cases} N_i^{(e1)} & \text{if } x \in \text{element } (e1) \\ N_i^{(e2)} & \text{if } x \in \text{element } (e2) \\ 0 & \text{otherwise} \end{cases} \tag{11.14}$$

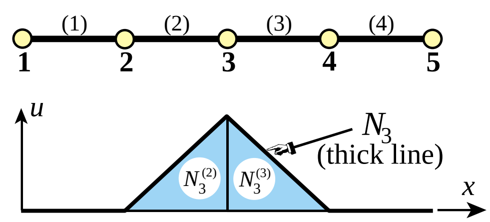

FIGURE 11.7. Trial basis function for node 3.

For a piecewise linear discretizations such as the two-node bar this function has the shape of a hat. Thus it is sometimes called a *hat function* or *chapeau function*. See Figure 11.7, in which $i = 3$, $e1 = 2$, $e2 = 3$. The concept is important in the variational interpretation of FEM as a Rayleigh-Ritz method.

§11.4. The Finite Element Equations

In linear FEM the discretization process for the TPE functional leads to the following algebraic form

$$\Pi^e = U^e - W^e, \quad U^e = \tfrac{1}{2}(\mathbf{u}^e)^T \mathbf{K}^e \mathbf{u}^e, \quad W^e = (\mathbf{u}^e)^T \mathbf{f}^e, \tag{11.15}$$

where $\mathbf{K}^e$ and $\mathbf{f}^e$ are called the *element stiffness matrix* and the *element consistent nodal force vector*, respectively. Note that in (11.15) the three energies are only function of the node displacements $\mathbf{u}^e$. U^e and W^e depend quadratically and linearly, respectively, on those displacements.

Taking the variation of the discretized TPE of (11.15) with respect to the node displacements gives[6]

$$\delta\Pi^e = \left(\delta\mathbf{u}^e\right)^T \frac{\partial \Pi^e}{\partial \mathbf{u}^e} = \left(\delta\mathbf{u}^e\right)^T \left[\mathbf{K}^e\mathbf{u}^e - \mathbf{f}^e\right] = 0. \tag{11.16}$$

Because the variations $\delta\mathbf{u}^e$ can be arbitrary, the bracketed quantity must vanish, which yields

$$\boxed{\mathbf{K}^e\mathbf{u}^e = \mathbf{f}^e} \tag{11.17}$$

These are the element stiffness equations. Hence the foregoing names given to $\mathbf{K}^e$ and $\mathbf{f}^e$ are justified *a posteriori*.

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[6] The $\frac{1}{2}$ factor disappears on taking the variation because U^e is quadratic in the node displacements. For a review on the calculus of discrete quadratic forms, see Appendix D.

§11.4.1. The Stiffness Matrix

For the two-node bar element, the internal energy U^e is

$$U^e = \tfrac{1}{2}\int_{x_1}^{x_2} e\,EA\,e\,dx = \tfrac{1}{2}\int_0^1 e\,EA\,e\,\ell\,d\zeta, \tag{11.18}$$

where the strain e is related to the nodal displacements through (11.12). This form is symmetrically expanded by inserting $e = \mathbf{B}\mathbf{u}^e$ into the second e and $e = e^T = (\mathbf{u}^e)^T\mathbf{B}^T$ into the first e:

$$U^e = \tfrac{1}{2}\int_0^1 [\,u_1^e \quad u_2^e\,]\,\frac{1}{\ell}\begin{bmatrix} -1 \\ 1 \end{bmatrix} EA\,\frac{1}{\ell}\,[\,-1 \quad 1\,]\begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix}\ell\,d\zeta. \tag{11.19}$$

The nodal displacements can be moved out of the integral, giving

$$U^e = \tfrac{1}{2}\,[\,u_1^e \quad u_2^e\,]\int_0^1 \frac{EA}{\ell^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\ell\,d\zeta\ \begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix} = \tfrac{1}{2}\big(\mathbf{u}^e\big)^T\mathbf{K}^e\mathbf{u}^e. \tag{11.20}$$

in which

$$\boxed{\mathbf{K}^e = \int_0^1 EA\,\mathbf{B}^T\mathbf{B}\,\ell\,d\zeta = \int_0^1 \frac{EA}{\ell^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\ell\,d\zeta.} \tag{11.21}$$

is the element stiffness matrix. If the rigidity EA is constant over the element,

$$\mathbf{K}^e = EA\,\mathbf{B}^T\mathbf{B}\int_0^1 \ell\,d\zeta = \frac{EA}{\ell^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\ell = \frac{EA}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{11.22}$$

This is the same element stiffness matrix of the prismatic truss member derived in Chapters 2 and 5 by a Mechanics of Materials approach, but now obtained through a variational argument.

§11.4.2. The Consistent Node Force Vector

The *consistent node force vector* $\mathbf{f}^e$ introduced in (11.15) comes from the element contribution to the external work potential W:

$$W^e = \int_{x_1}^{x_2} q\,u\,dx = \int_0^1 q\,\mathbf{N}^T\mathbf{u}^e\,\ell\,d\zeta = \big(\mathbf{u}^e\big)^T\int_0^1 q\begin{bmatrix} 1-\zeta \\ \zeta \end{bmatrix}\ell\,d\zeta = \big(\mathbf{u}^e\big)^T\,\mathbf{f}^e, \tag{11.23}$$

in which $\zeta = (x - x_1)/\ell$. Consequently

$$\boxed{\mathbf{f}^e = \int_{x_1}^{x_2} q\begin{bmatrix} 1-\zeta \\ \zeta \end{bmatrix}dx = \int_0^1 q\begin{bmatrix} 1-\zeta \\ \zeta \end{bmatrix}\ell\,d\zeta.} \tag{11.24}$$

If the force q is constant over the element, one obtains the same results as with the EbE load-lumping method of Chapter 7. See Exercise 11.3.

§11.5. *Accuracy Analysis

Low order 1D elements may give surprisingly high accuracy. In particular the lowly two-node bar element can display infinite accuracy under some conditions. This phenomenon is studied in this advanced section as it provides an introduction to modified equation methods and Fourier analysis along the way.

§11.5.1. *Nodal Exactness and Superconvergence

Suppose that the following two conditions are satisfied:

1. The bar properties are constant along the length (prismatic member).
2. The distributed load $q(x)$ is zero between nodes. The only applied loads are point forces at the nodes.

If so, a linear axial displacement $u(x)$ as defined by (11.10) and (11.11) is the exact solution over each element since constant strain and stress satisfy, element by element, all of the governing equations listed in Figure 11.3.[7] It follows that if the foregoing conditions are verified the FEM solution is *exact*; that is, it agrees with the analytical solution of the mathematical model.[8] Adding extra elements and nodes would not change the solution. That is the reason behind the truss discretizations used in Chapters 2–3: *one element per member is enough* if they are prismatic and loads are applied to joints. Such models are called *nodally exact*.

What happens if the foregoing assumptions are not met? Exactness is then generally lost, and several elements per member may be beneficial if spurious mechanisms are avoided.[9] For a 1D lattice of equal-length, prismatic two-node bar elements, an interesting and more difficult result is: *the solution is nodally exact for any loading if consistent node forces are used.* This is proven in the subsection below. This result underlies the importance of computing node forces correctly.

If conditions such as equal-length are relaxed, the solution is no longer nodally exact but convergence at the nodes is extremely rapid (faster than could be expected by standard error analysis) as long as consistent node forces are used. This phenomenon is called *superconvergence* in the FEM literature.

§11.5.2. *Fourier Patch Analysis

The following analysis is based on the modified differential equation (MoDE) method of Warming and Hyett [181] combined with the Fourier patch analysis approach of Park and Flaggs [126,127].

Consider a lattice of two-node prismatic bar elements of constant rigidity EA and equal length ℓ, as illustrated in Figure 11.8. The total length of the lattice is L. The system is subject to an arbitrary axial load $q(x)$. The only requirement on $q(x)$ is that it has a convergent Fourier series in the space direction.

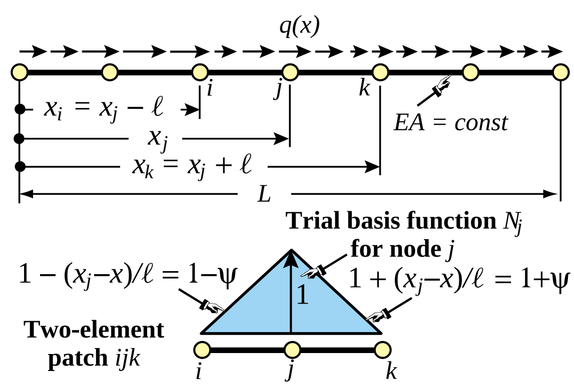

FIGURE 11.8. Superconvergence patch analysis.

From the lattice extract a patch[10] of two elements connecting nodes x_i, x_j and x_k as shown in Figure 11.8. The

[7] The internal equilibrium equation $p' + q = EA\,u'' + q = 0$ is trivially verified because $q = 0$ from the second assumption, and $u'' = 0$ because of shape function linearity.

[8] In variational language: the Green function of the $u'' = 0$ problem is included in the FEM trial space.

[9] These can happen when transforming such elements for 2D and 3D trusses. See Exercise E11.7.

[10] A patch is the set of all elements connected to a node; in this case j.

FEM patch equations at node j are

$$\frac{EA}{\ell}\left[\,-1 \quad 2 \quad -1\,\right]\begin{bmatrix} u_i \\ u_j \\ u_k \end{bmatrix} = f_i, \tag{11.25}$$

in which the node force f_j is obtained by consistent lumping:

$$f_j = \int_{x_i}^{x_k} q(x)N_j(x)\,dx = \int_{-1}^{0} q(x_j+\psi\ell)(1+\psi)\,\ell\,d\psi + \int_{0}^{1} q(x_j+\psi\ell)(1-\psi)\,\ell\,d\psi. \tag{11.26}$$

Here $N_j(x)$ is the "hat" trial basis function for node j, depicted in Figure 11.8, and $\psi = (x-x_j)/\ell$ is a dimensionless coordinate that takes the values -1, 0 and 1 at nodes i, j and k, respectively. If $q(x)$ is expanded in Fourier series

$$q(x) = \sum_{m=1}^{M} q_m e^{i\beta_m x}, \quad \beta_m = m\pi/L, \tag{11.27}$$

(the term $m = 0$ requires special handling) the exact solution of the continuum equation $EA\,u'' + q = 0$ is

$$u^*(x) = \sum_{m=1}^{M} u_m^* e^{i\beta_m x}, \quad u_m^* = \frac{q_m e^{i\beta_m x}}{EA\beta_m^2}. \tag{11.28}$$

Evaluation of the consistent force using (11.26) gives

$$f_j = \sum_{m=1}^{M} f_{jm}, \quad f_{jm} = q_m\ell\,\frac{\sin^2(\frac{1}{2}\beta_m\ell)}{\frac{1}{4}\beta_m^2\ell^2}\,e^{i\beta_m x_2}. \tag{11.29}$$

To construct a modified differential equation (MoDE), expand the displacement by Taylor series centered at node j. Evaluate at i and k: $u_i = u_j - \ell u_j' + \ell^2 u_j''/2! - \ell^3 u_j'''/3! + \ell^4 u_j^{iv}/4! + \ldots$ and $u_k = u_j + \ell u_j' + \ell^2 u_j''/2 + \ell^3 u_j'''/3! + \ell^4 u_j^{iv}/4! + \ldots$. Replace these series into (11.25) to get

$$-2EA\ell\left(\frac{1}{2!}u_j'' + \frac{\ell^2}{4!}u_j^{iv} + \frac{\ell^4}{6!}u_j^{vi} + \ldots\right) = f_j. \tag{11.30}$$

This is an ODE of infinite order. It can be reduced to an algebraic equation by assuming that the response of (11.30) to $q_m\,e^{i\beta_m x}$ is harmonic: $u_{jm}e^{i\beta_m x}$. If so $u_{jm}'' = -\beta_m^2 u_{jm}$, $u_{jm}^{iv} = \beta_m^4 u_{jm}$, etc, and the MoDE becomes

$$2EA\ell\beta_m^2\left(\frac{1}{2!} - \frac{\beta_m^2\ell^2}{4!} + \frac{\beta_m^4\ell^4}{6!} - \ldots\right)u_{jm} = 4EA\ell\sin^2(\tfrac{1}{2}\beta_m\ell)\,u_{jm} = f_{jm} = q_m\ell\,\frac{\sin^2(\frac{1}{2}\beta_m\ell)}{\frac{1}{4}\beta_m^2\ell^2}\,e^{i\beta_m x_j}. \tag{11.31}$$

Solving gives $u_{jm} = q_m e^{i\beta_m x_j}/(EA\beta_m^2)$, which compared with (11.28) shows that $u_{jm} = u_m^*$ for any $m > 0$. Consequently $u_j = u_j^*$. In other words, the MoDE (11.30) and the original ODE: $EAu'' + q = 0$ have the same value at $x = x_j$ for any load $q(x)$ developable as (11.27). This proves nodal exactness. In between nodes the two solutions will not agree.[11]

The case $m = 0$ has to be treated separately since the foregoing expressions become $0/0$. The response to a uniform $q = q_0$ is a quadratic in x, and it is not difficult to prove nodal exactness.

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[11] The FEM solution varies linearly between nodes whereas the exact one is generally trigonometric.

Notes and Bibliography

The foregoing development pertains to the simplest structural finite element: the two-node bar element. For bars this may be generalized in various directions.

Refined bar elements. Adding internal nodes we can pass from linear to quadratic and cubic shape functions. These elements are rarely useful on their own right, but as accessories to 2D and 3D high order continuum elements (for example, to model edge reinforcements.) For that reason they are not considered here. The 3-node bar element is developed in exercises assigned in Chapter 16. *Two- and three-dimensional truss structures*. The only additional ingredients are the transformation matrices discussed in Chapters 3 and 6. *Curved bar elements*. These can be derived using isoparametric mapping, a device introduced later.

Matrices for straight bar elements are available in any finite element book; for example Przemieniecki [140].

Tonti diagrams were introduced in the 1970s in papers now difficult to access, for example [171]. Scanned images are available, however, at `http://www.dic.units.it/perspage/discretephysics`

The fundamentals of Variational Calculus may be studied in the excellent textbook [76], which is now available in an inexpensive Dover edition. The proof of the MPE principle can be found in texts on variational methods in mechanics. For example: Langhaar [108], which is the most readable "old fashioned" treatment of the energy principles of structural mechanics, with a beautiful treatment of virtual work. (Out of print but used copies may be found via the web engines cited in §1.5.2.) The elegant treatment by Lanczos [107] is recommended as reading material although it is more oriented to physics than structural mechanics.

The first accuracy study of FEM discretizations using modified equation methods is by Waltz et. al. [179]; however their procedures were faulty, which led to incorrect conclusions. The first correct derivation of modified equations appeared in [181]. The topic has recently attracted interest from applied mathematicians because modified equations provide a systematic tool for *backward error analysis* of differential equations: the discrete solution is the exact solution of the modified problem. This is particularly important for the study of long term behavior of discrete dynamical systems, whether deterministic or chaotic. Recommended references along these lines are [82,86,160].

Nodal exactness of bar models for point node loads is a particular case of a theorem by Tong [170]. For arbitrary loads it was proven by Park and Flaggs [126,127], who followed a variant of the scheme of §11.5.2. A different technique is used in Exercise 11.8. The budding concept of superconvergence, which emerged in the late 1960s, is outlined in the book of Strang and Fix [155]. There is a monograph [180] devoted to the subject; it covers only Poisson problems but provides a comprehensive reference list until 1995.

References

Referenced items moved to Appendix R.

Homework Exercises for Chapter 11

Variational Formulation of Bar Element

EXERCISE 11.1 [D:10] Explain the kinematic admissibility requirements stated in §11.3.2 in terms of physics, namely ruling out the possibility of gaps or interpenetration as the bar material deforms.

EXERCISE 11.2 [A/C:15] Using (11.21), derive the stiffness matrix for a *tapered* bar element in which the cross section area varies linearly along the element length:

$$A = A_i(1-\zeta) + A_j\,\zeta, \tag{E11.1}$$

where A_i and A_j are the areas at the end nodes, and $\zeta = x^e/\ell$ is the dimensionless coordinate defined in §11.3.6. Show that this yields the same answer as that of a stiffness of a constant-area bar with cross section $\frac{1}{2}(A_i + A_j)$. Note: the following *Mathematica* script may be used to solve this exercise:[12]

```
ClearAll[Le,x,Em,A,Ai,Aj];
Be={{-1,1}}/Le; ζ=x/Le; A=Ai*(1-ζ)+Aj*ζ;
Ke=Integrate[Em*A*Transpose[Be].Be,{x,0,Le}];
Ke=Simplify[Ke];
Print["Ke for varying cross section bar: ",Ke//MatrixForm];
```

In this and following scripts `Le` stands for ℓ.

EXERCISE 11.3 [A:10] Using the area variation law (E11.1), find the consistent load vector $\mathbf{f}^e$ for a bar of constant area A subject to a uniform axial force $q = \rho g A$ per unit length along the element. Show that this vector is the same as that obtained with the element-by-element (EbE) "lumping" method of §8.4, which simply assigns half of the total load: $\frac{1}{2}\rho g A\ell$, to each node.

EXERCISE 11.4 [A/C:15] Repeat the previous calculation for the tapered bar element subject to a force $q = \rho g A$ per unit length, in which A varies according to (E11.1) whereas ρ and g are constant. Check that if $A_i = A_j$ one recovers $f_i = f_j = \frac{1}{2}\rho g A\ell$. Note: the following *Mathematica* script may be used to solve this exercise:[13]

```
ClearAll[q,A,Ai,Aj,ρ,g,Le,x];
ζ=x/Le; Ne={{1-ζ,ζ}}; A=Ai*(1-ζ)+Aj*ζ; q=ρ*g*A;
fe=Integrate[q*Ne,{x,0,Le}];
fe=Simplify[fe];
Print["fe for uniform load q: ",fe//MatrixForm];
ClearAll[A];
Print["fe check: ",Simplify[fe/.{Ai->A,Aj->A}]//MatrixForm];
```

EXERCISE 11.5 [A/C:20] A tapered bar element of length ℓ, end areas A_i and A_j with A interpolated as per (E11.1), and constant density ρ, rotates on a plane at uniform angular velocity ω (rad/sec) about node i. Taking axis x along the rotating bar with origin at node i, the centrifugal axial force is $q(x) = \rho A\omega^2 x$ along the length, in which $x \equiv x^e$. Find the consistent node forces as functions of ρ, A_i, A_j, ω and ℓ, and specialize the result to the prismatic bar $A = A_i = A_j$. Partial result check: $f_j = \frac{1}{3}\rho\omega^2 A\ell^2$ for $A = A_i = A_j$.

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[12] The `ClearAll[...]` at the start of the script is recommended programming practice to initialize variables and avoid "cell crosstalk." In a `Module` this is done by listing the local variables after the `Module` keyword.

[13] The `ClearAll[A]` before the last statement is essential; else `A` would retain the previous assignation.

EXERCISE 11.6 [A:15] (Requires knowledge of Dirac's delta function properties.) Find the consistent load vector $\mathbf{f}^e$ if the bar is subjected to a concentrated axial force Q at a distance $x = a$ from its left end. Use Equation (11.31), with $q(x) = Q\,\delta(a)$, in which $\delta(a)$ is the one-dimensional Dirac's delta function at $x = a$. Note: the following script does it by *Mathematica*, but it is overkill:

```
ClearAll[Le,q,Q,a,x];
ζ=x/Le; Ne={{1-ζ,ζ}}; q=Q*DiracDelta[x-a];
fe=Simplify[ Integrate[q*Ne,{x,-Infinity,Infinity}] ];
Print["fe for point load Q at x=a: ",fe//MatrixForm];
```

EXERCISE 11.7 [C+D:20] In a learned paper, Dr. I. M. Clueless proposes "improving" the result for the example truss by putting three extra nodes, 4, 5 and 6, at the midpoint of members 1–2, 2–3 and 1–3, respectively. His "reasoning" is that more is better. Try Dr. C.'s suggestion using the *Mathematica* implementation of Chapter 4 and verify that the solution "blows up" because the modified master stiffness is singular. Explain physically what happens.

EXERCISE 11.8 [A:35, close to research paper level]. Prove nodal exactness of the two-node bar element for arbitrary but Taylor expandable loading without using the Fourier series approach. Hints: expand $q(x) = q(x_j) + (\ell\psi)q'(x_j) + (\ell\psi)^2 q''(x_j)/2! + \ldots$, where $\ell\psi = x - x_j$ is the distance to node j, compute the consistent force $f_j(x)$ from (11.26), and differentiate the MoDE (11.30) repeatedly in x while truncating all derivatives to a maximum order $n \geq 2$. Show that the original ODE: $EAu'' + q = 0$, emerges as an identity regardless of how many derivatives are kept.