25

Bifurcation: Linearized Prebuckling II

TABLE OF CONTENTS

§25.1. Introduction

This Chapter continues with the subject of linearized prebuckling (LPB) bifurcation analysis. It goes deeper than Chapter 24 in that it probes the assumptions (so far stated without proof) behind LPB, and the practical modeling implications that emanate from these assumptions.

To present some of the derivations in mathematical terms it is necessary to introduce the concept of *state decomposition* at the bifurcation point and to define homogeneous and particular solutions. This is done briefly in §24.2 primarily as a means of introducing notation for S24.3 and following. The detailed mathematical analysis of this decomposition is relegated to Chapter 26.

§25.2. State Decomposition at Bifurcation Point

Recall from previous Chapters that an isolated bifurcation point at λ_{cr} is characterized by a singular tangent stiffness at the equilibrium configuration,

$$\mathbf{K}(\mathbf{u}_{cr}, \lambda_{cr})\, \mathbf{z} = \mathbf{0}, \tag{25.1}$$

and by the normalized null eigenvector (buckling mode) $\mathbf{z} \neq \mathbf{0}$, $\|\mathbf{z}\| = 1$, being orthogonal to the incremental load vector:

$$\mathbf{q}^T \mathbf{z} = \mathbf{z}^T \mathbf{q} = 0. \tag{25.2}$$

Assume that we have located a bifurcation point B and computed the buckling mode $\mathbf{z}$. Our next task is to examine the structural behavior in the *neighborhood* of B. We shall be content with looking at the so-called *branching direction information*. This information characterizes the *tangents* to the equilibrium branches that cross at B.

To carry out this task we borrow from algebraic ODE theory. Consider the variation in the state vector $\mathbf{u}$ measured from its value $\mathbf{u}_B$ at buckling:

$$\Delta\mathbf{u} = \mathbf{u} - \mathbf{u}_B \tag{25.3}$$

Divide this increment by Δt, t being the timelike parameter introduced in Chapter 3, and pass to the limit:

$$\dot{\mathbf{u}} = \lim_{t\to 0} \frac{\Delta\mathbf{u}}{\Delta t}. \tag{25.4}$$

This variation rate $\dot{\mathbf{u}}$ from the bifurcation point can be decomposed into a *homogeneous solution* component $\sigma\mathbf{z}$ in the buckling mode direction, and a *particular solution* component $\mathbf{y}$, which is orthogonal to $\mathbf{z}$:

$$\dot{\mathbf{u}} = (\mathbf{y} + \sigma\mathbf{z})\dot{\lambda}, \qquad \mathbf{y}^T \mathbf{z} = \mathbf{z}^T \mathbf{y} = 0, \tag{25.5}$$

The particular solution solves the system

$$\mathbf{K}\mathbf{y} = \mathbf{q}, \qquad \mathbf{y}^T \mathbf{z} = 0, \tag{25.6}$$

which is simply the first-order incremental equation $\mathbf{K}\dot{\mathbf{u}} = \mathbf{q}\dot{\lambda}$ augmented by a normality constraint. Imposing this constraint removes the singularity (rank deficiency) of $\mathbf{K}$. The geometric interpretation of this decomposition on the $\mathbf{y}$, $\mathbf{z}$ plane is shown in Figure 24.1.

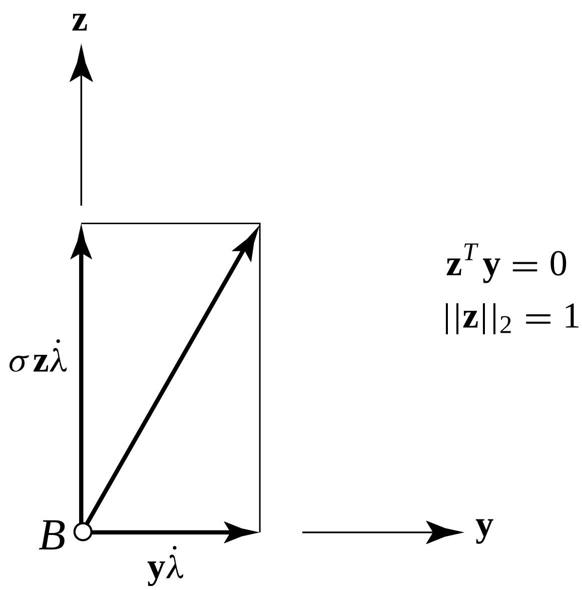

Figure 25.1. State decomposition at bifurcation point B.

§25.3. LPB Assumptions

With the notation introduced in §25.2 we may now state the key assumptions invoked in linearized prebuckling (LPB). (Those collected in item (II) have already been formally stated and used in Chapter 21.)

(I) The loading is conservative and proportional:

$$\mathbf{p} = \mathbf{q}_0 + \lambda\mathbf{q}. \tag{25.7}$$

and the structure is linearly elastic. Inother words, the residual equations are derivable from a potential energy function.

(II) The displacements and displacement gradients prior to the critical state are negligible in the sense that (a) the *material stiffness matrix can be evaluated at the reference configuration*, and (b) the geometric stiffness is proportional to the control parameter λ:

$$\mathbf{K}_M \equiv \mathbf{K}_0, \qquad \mathbf{K}_G \equiv \lambda\mathbf{K}_1, \tag{25.8}$$

in which $\mathbf{K}_0$ is the material matrix evaluated at the reference configuration, also called the *linear stiffness*, and $\mathbf{K}_1$ is the *reference geometric stiffness*. As discussed in the previous Chapter the singular stiffness criterion $\det\mathbf{K} = 0$ leads to the eigenproblem

$$(\mathbf{K}_0 + \lambda\mathbf{K}_1)\,\mathbf{z} = \mathbf{0}. \tag{25.9}$$

(III) The particular solution $\mathbf{y}$ defined in §25.2 is obtained by solving

$$(\mathbf{K}_0 + \lambda_{cr}\mathbf{K}_1)\,\mathbf{y} = \mathbf{q} \tag{25.10}$$

under the constraint $\mathbf{y}^T\mathbf{z} = 0$. Observe that from assumption (I) $\mathbf{q}$ is constant.

We now prove that if these assumptions hold, all critical points determined from the LPB eigenproblem are *bifurcation* points, that is, $\mathbf{z}^T\mathbf{q}$ vanishes. To show that, premultiply both sides of (25.10) by $\mathbf{z}^T$:

$$\mathbf{z}^T\mathbf{q} = \mathbf{z}^T(\mathbf{K}_0 + \lambda\mathbf{K}_1)\mathbf{y} = \mathbf{y}^T(\mathbf{K}_0 + \lambda\mathbf{K}_1)\mathbf{z} = \mathbf{y}^T(\mathbf{K}\mathbf{z}) = 0 \tag{25.11}$$

Note that the transformation $\mathbf{z}^T\mathbf{K}\mathbf{y} = \mathbf{y}^T\mathbf{K}\mathbf{z}$ holds because $\mathbf{K}_0$ and $\mathbf{K}_1$ are symmetric on account of the conservativeness assumption (I).

Remark 25.1. Bifurcation points are classified in later sections into various types: unsymmetric, stable-symmetric, stable-unsymmetric, and so on. It will be shown later that, under most common assumptions, LPB bifurcation points are generally of *symmetric* type. The LPB model does not provide, however, information as to the post-bifurcation stability, so we cannot say whether the bifurcation point is stable-symmetric or unstable-symmetric.

§25.4. Limitations of LPB

Linearized prebuckling (LPB) is used extensively in engineering design. Standard books in structural stability[1] concentrate upon it. In its finite element version LPB is a feature available in many finite element programs. Exercising this feature has the advantages of avoiding a full nonlinear analysis, which can be expensive and time-consuming. Given its practical importance, structure designers (and most especially aerospace designers) should be familiar with the range of applicability of LPB. The limitations are discussed next.

1. *Conservative loading*. LPB is a restricted form of the static criterion also known as Euler's test (see §24.2). If the loads are not conservative, the dynamic criterion should be used, at least to check out whether a flutter condition may occur. If the dynamic criterion shows that stability is lost by divergence, one may regress to the singular-stiffness test criterion.

2. *Loss of stability must be by symmetric bifurcation*. If the first critical point is a limit point or asymmetric bifurcation,[2] LPB is not strictly applicable although in some cases it may provide a sufficiently good approximation. Lacking experimental confirmation or *a priori* knowledge, the only practical way to check whether the first critical point is symmetric bifurcation is to go through a full nonlinear analysis.

3. *Prebuckling deformations must be small*. This assumption fits well many engineering structures because of the nature of construction materials. The structures that best fit these assumptions are straight columns, frameworks and flat plates, as illustrated in Figure 25.2. Care must be exercised for arches, shells, very thin members, and for imperfection-sensitive structures in general.

4. *Elastic material behavior*. If the material is inelastic the structure is not internally conservative. Then the tangent stiffness depends on the prior deformation history, and the LPB eigenproblem

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[1] For example, Timoshenko and Gere's *Theory of Elastic Stability*.

[2] Symmetric bifurcation occurs when bucking in the $\mathbf{z}$ and $-\mathbf{z}$ directions is equally likely. Asymmetric bifurcation occurs when one of the directions is physically more likely; for example axially compressed cylinders buckle inwards. This classification of critical points is covered in more detail in Chapter 11 and following.

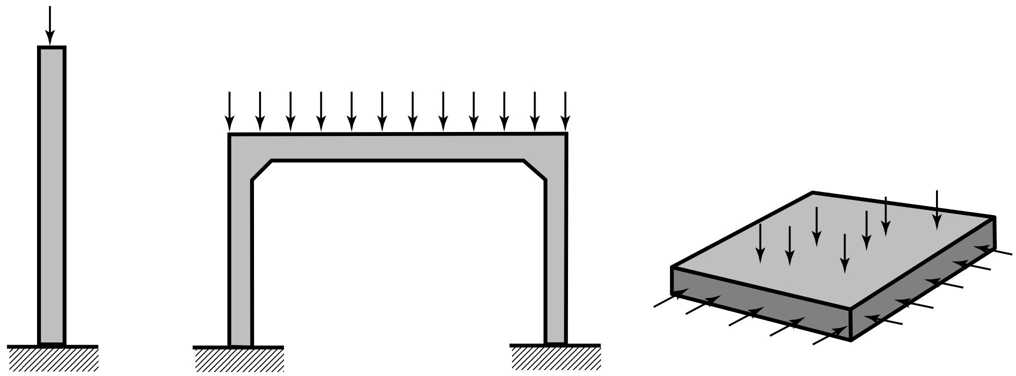

Figure 25.2. Structures that are adequately modeled by LPB assumptions.

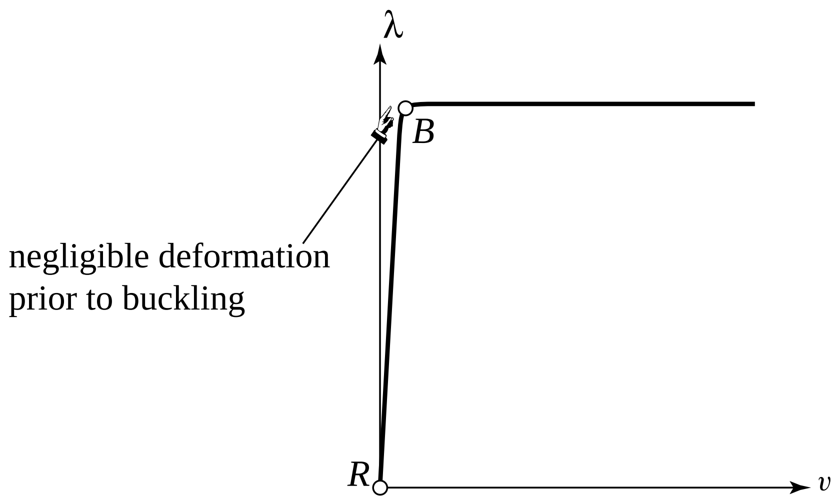

Figure 25.3. Type of response expected under LPB assumptions.
(Branch intersection at B not shown for clarity)

loses meaning. The topic of inelastic buckling (in particular creep and plastic buckling) is an enormous subject that falls outside the scope of this course.

5. *Applied loads should not depend nonlinearly on the displacements*. Such a dependence usually introduces nonconservative effects, thus voiding the conservative-loading assumptions. Even is the loads remain conservative, the reference geometric stiffness would depend on the load level, thus leading to a nonlinear eigenproblem.

6. *The effect of imperfections is negligible*. Some structures are highly *imperfection sensitive* in that the first critical load is strongly affected by the presence of imperfections. In such cases obviously LPB is of limited value or outright irrelevant.

§25.4.1. When LPB Works

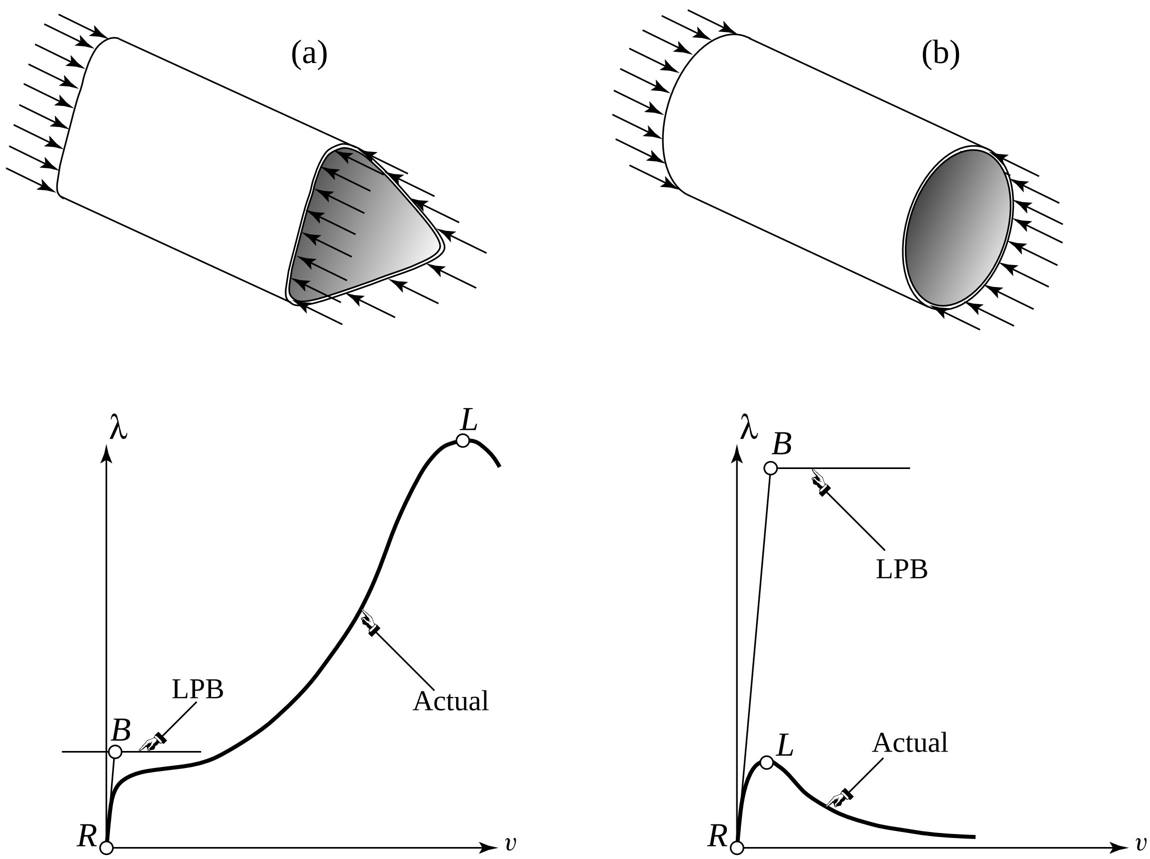

Figure 25.4. Two structures that fit the LPB assumptions poorly.

The systems that best fit the LPB model are *symmetrically loaded structures* such as straight columns and in-plane-loaded plates (laminas) which are not excessively thin. See Figure 25.2. The lateral buckling of such structures occurs following very small deformations, as typified by the response sketch in Figure 25.3.

§25.4.2. And When It Doesn't

Two examples of structures that are not properly treated by the LPB model are shown in Figure 25.4. The LPB predictions are way off in both cases, but for different reasons.

Case (a) is an axially compressed cylindrical shell made up of almost flat panels joined by curved panels, forming like a "curved triangle" cross section seen in some combat helicopters and the Space Shuttle fuselage. There is a substantial redistribution of stresses due to changes on geometry. The structure eventually collapses at a limit point substantially over the predicted LPB load. The latter is therefore overly safe.

On the other hand, the axially compressed circular cylinder of case (b) is highly imperfection-sensitive structure that fails at a substantially lower load than that predicted by LPB. Consequently the LPB prediction is highly unsafe.

§25.4.3. How to Extend the Applicability of LPB

One way to broaden the application of the LPB model is to *update the reference configuration*[3] so that the prebuckling deformations are reduced. If this is done the control parameter λ is of course measured from the latest reference configuration and consequently becomes a true stage control parameter. Limitations on the conservativeness of applied loads and types of critical point, however, cannot be readily circumvented by this "staging" technique.

[3] As naturally done in the CR description, in which the deformational displacements are measured from a continuously varying configuration, and also in the Updated Lagrangian description.

Homework Exercises for Chapter 25

Bifurcation: Linearized Prebuckling II

EXERCISE 25.1 [A:15] Find the particular solution $\mathbf{y}$ at the lowest bifurcation load of the two-bar example of Chapter 24.

EXERCISE 25.2 [A:15] Find the particular solution $\mathbf{y}$ at the symmetric and antisymmetric bifurcation loads of the one-element Euler column example of Exercise 24.4.

EXERCISE 25.3 [A:25] The first order residual rate equations is $\dot{\mathbf{r}} = \mathbf{0}$, where $\dot{\mathbf{r}}$ is given by

$$\dot{\mathbf{r}} = \mathbf{K}\dot{\mathbf{u}} - \mathbf{q}\dot{\lambda} = \mathbf{0}, \tag{E25.1}$$

(E25.1) holds at a bifurcation point where $\mathbf{K}$ and $\mathbf{q}$ are the tangent stiffness matrix and incremental load vector, respectively, at bifurcation. Decompose $\dot{\mathbf{u}} = (\mathbf{y} + \sigma\mathbf{z})\dot{\lambda}$, where $\mathbf{y}$ is the particular solution and $\mathbf{z} \neq \mathbf{0}$ the buckling mode normalized to length one. Show that the first-order differential equation system (E25.1) cannot give information on the "buckling mode amplitude" σ because one gets $\sigma = 0/0$. (Hint: premultiply that equation by an appropriate vector.)