4

One-Parameter Residual Equations

TABLE OF CONTENTS

§4.1. Introduction

This Chapter continues on the topic of residual equations introduced in Chapter 3. The general residual force equation presented there is specialized, through the concept of staging introduced in §3.4, to the one-parameter form in which $\mathbf{r}$ is a function of $\mathbf{u}$ (the state) and λ (the control). Together these form the *control-state space*. The separable case in which $\mathbf{u}$ and λ can be segregated to both sides of the residual equations, is described.

Further insight into the structural response may be achieved with the help of constant-residual incremental flows. Paths and orthogonal hypersurfaces are introduced and interpreted geometrically. Finally, the concepts of arclength and scaling are discussed.

§4.2. Rate Forms and Incremental Velocity

In this section we study further the one-parameter residual equation (3.17), reproduced below for convenience:

$$\boxed{\mathbf{r}(\mathbf{u},\lambda) = \mathbf{0}.} \tag{4.1}$$

The corresponding residual-derivative equations are

$$\boxed{\dot{\mathbf{r}} = \mathbf{K}\dot{\mathbf{u}} - \mathbf{q}\dot{\lambda},} \tag{4.2}$$

$$\boxed{\ddot{\mathbf{r}} = \mathbf{K}\ddot{\mathbf{u}} + \dot{\mathbf{K}}\dot{\mathbf{u}} - \mathbf{q}\ddot{\lambda} - \dot{\mathbf{q}}\dot{\lambda},} \tag{4.3}$$

$$\mathbf{K} = \frac{\partial \mathbf{r}}{\partial \mathbf{u}}, \qquad \mathbf{q} = -\frac{\partial \mathbf{r}}{\partial \lambda} \tag{4.4}$$

where $\mathbf{K}$ is the tangent stiffness matrix introduced in §3.3, and $\mathbf{q}$ is the *incremental load vector*. The latter is the specialization of the control matrix $\mathbf{Q}$ defined in §3.3, to the one-parameter case. These equations will be used in the sequel instead of the more general (3.14)–(3.15) unless otherwise noted.

Rate forms of $\mathbf{r}(\mathbf{u},\lambda) = \mathbf{0}$ are obtained by equating the above derivatives to zero:

$$\dot{\mathbf{r}} = \mathbf{0}, \qquad \text{or} \qquad \mathbf{K}\dot{\mathbf{u}} = \mathbf{q}\dot{\lambda}, \tag{4.5}$$

$$\ddot{\mathbf{r}} = \mathbf{0}, \qquad \text{or} \qquad \mathbf{K}\ddot{\mathbf{u}} + \dot{\mathbf{K}}\mathbf{u} = \mathbf{q}\ddot{\lambda} + \dot{\mathbf{q}}\dot{\lambda}. \tag{4.6}$$

At *regular* points of the $(\mathbf{u},\lambda)$ space the tangent stiffness $\mathbf{K}$ is nonsingular. If so, we can solve the first-order rate form (4.5) for $\dot{\mathbf{u}}$:

$$\dot{\mathbf{u}} = \mathbf{K}^{-1}\mathbf{q}\dot{\lambda} = \mathbf{v}\dot{\lambda}, \qquad \text{or} \quad \frac{\partial \mathbf{u}}{\partial \lambda} = \mathbf{u}' = \mathbf{v}, \tag{4.7}$$

where

$$\boxed{\mathbf{v} = \mathbf{K}^{-1}\mathbf{q}.} \tag{4.8}$$

This vector is called the *incremental velocity vector* and is an important component of all solution methods based on continuation.

§4.3. Separable Residuals and Proportional Loading

The force-balance equivalent of (3.3) for a one-parameter residual equation is

$$\mathbf{p}(\mathbf{u}) = \mathbf{f}(\mathbf{u}, \lambda). \tag{4.9}$$

If the right hand side, which represents the external force vector, does not depend on the state parameters $\mathbf{u}$, that is

$$\boxed{\mathbf{p}(\mathbf{u}) = \mathbf{f}(\lambda),} \tag{4.10}$$

the system of equations (4.1) or (4.9) is called *separable*. Furthermore, if $\mathbf{f}$ is linear in λ the loading is said to be *proportional*. Obviously $\mathbf{q} = \partial\mathbf{f}/\partial\lambda$ is then a *constant* vector.

Remark 4.1. The more general system (3.3) containing multiple control parameters is said to be separable if

$$\mathbf{p}(\mathbf{u}) = \mathbf{f}(\boldsymbol{\Lambda}). \tag{4.11}$$

In this case the loading is called proportional if $\mathbf{f}$ is *linear* in all control parameters, thus giving a constant control matrix $\mathbf{Q}$.

Remark 4.2. If a separable system derives from a total potential energy $\Pi = U - P$, then the external work potential P must be linear in the state parameters u_i. Furthermore for the loading to be proportional, P must also be linear in λ.

§4.4. Response Visualization by Incremental Flow

§4.4.1. Diagrams for One Degree of Freedom

As discussed in Chapter 2, the solution of the one-parameter residual form

$$\mathbf{r}(\mathbf{u}, \lambda) = \mathbf{0}, \tag{4.12}$$

is often plotted on the u versus λ plane, where u is a representative component of $\mathbf{u}$.

One such diagram is illustrated in Figure 4.1. If λ is a load amplitude, this is called a *load-displacement response* curve or simply a *response* curve. It is common practice to make the curve pass through the origin $\lambda = 0$, $u = 0$. More general terms for this geometrization are *equilibrium path* or *equilibrium trajectory*. The path passing through the origin is called the *primary* or *fundamental* path because it usually represents the operation of the structure under normal service conditions.

A path can, of course, be traversed in two directions. These are identified as positive or + sense, and negative or − sense. As illustrated in Figure 4.2, we shall use the convention that the positive sense is associated with increasing values of the pseudo-time t when the path is parametrically described as $\mathbf{u} = \mathbf{u}(t)$ and $\lambda = \lambda(t)$.

A diagram such as that in Figure 4.1 gives of course only a partial picture of the structural behavior unless there is only a single degree of freedom. For a better understanding of the way numerical solution procedures work (or fail to) it is instructive to "look around" the equilibrium path by considering the perturbed residual equation

$$\boxed{\mathbf{r}(\mathbf{u}, \lambda) = \mathbf{r}_c,} \tag{4.13}$$

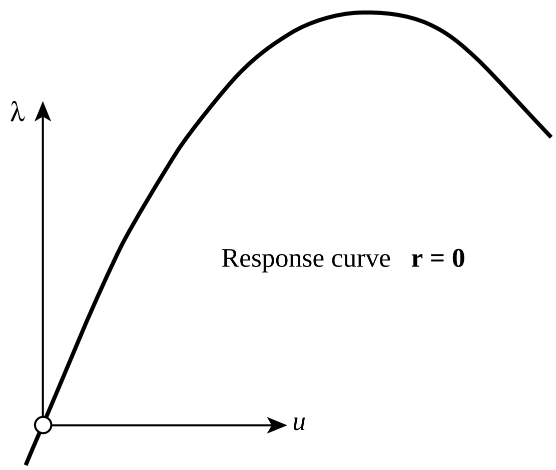

Figure 4.1. Typical response diagram showing primary equilibrium path.

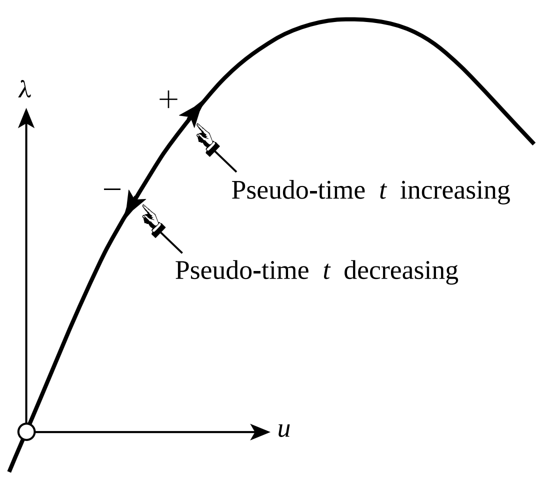

Figure 4.2. Positive and negative traversal senses on a path.

where $\mathbf{r}_c$ is a *constant* vector. This is the general solution of $\dot{\mathbf{r}} = \mathbf{0}$. Additional information can be conveyed by drawing the solutions of (4.13) for various values of the right-hand side near zero. This produces constant-residual paths as illustrated in Figure 4.3. Collectively these paths form the *incremental flow* whose differential equation is either $\dot{\mathbf{r}} = \mathbf{0}$, or, if we take $\lambda \equiv t$:

$$\mathbf{r}' = \frac{\partial \mathbf{r}}{\partial \lambda} = \mathbf{0}, \tag{4.14}$$

where primes denote derivatives with respect to λ. This can also be presented as

$$\mathbf{r}' = \mathbf{K}\frac{\partial \mathbf{u}}{\partial \lambda} + \frac{\partial \mathbf{r}}{\partial \lambda} = \mathbf{K}\mathbf{u}' - \mathbf{q} = \mathbf{0}. \tag{4.15}$$

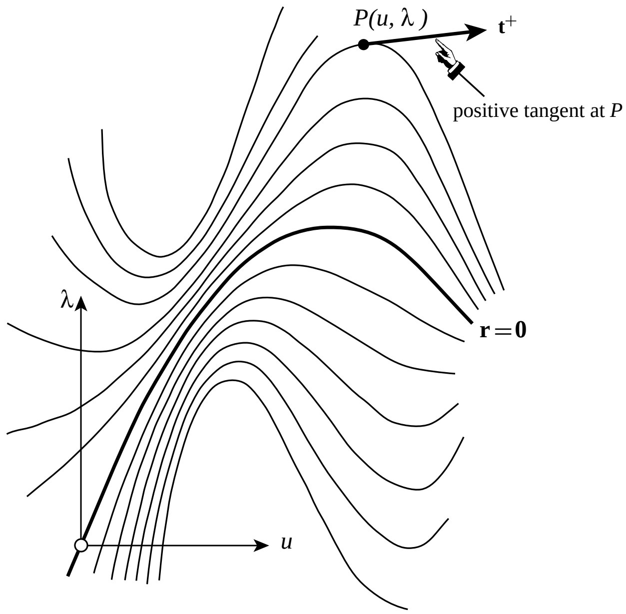

Figure 4.3. The incremental flow field as a family of constant-residual trajectories.

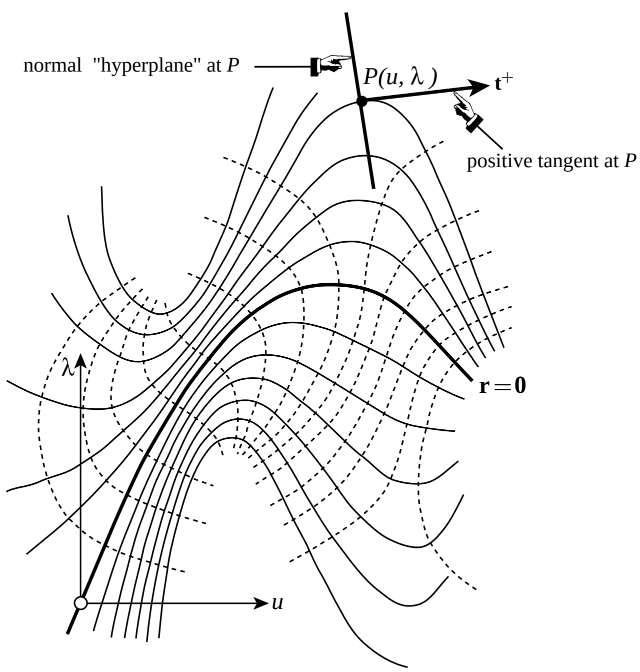

Figure 4.4. Incremental flow (full curves) and the flow-orthogonal envelope (dashed curves). This envelope reduces here to a family of curves because there is only one degree of freedom u.

If $\mathbf{K}$ is nonsingular, solving (4.15) yields $\mathbf{u}' = \mathbf{K}^{-1}\mathbf{q} = \mathbf{v}$. The incremental solution methods covered later exploit these forms, which explains the qualifier "incremental" applied to the flow.

Figure 4.3 also illustrates the construction of the tangent vector $\mathbf{t}^{+}$ at an arbitrary point $P(u, \lambda)$. This procedure is described more precisely in §4.5.

Figure 4.4 depicts a set of curves whose trajectories are orthogonal to the incremental flow. This set is called the *flow-orthogonal envelope*. It will be explained later in §4.4 that this set generally consists of a family of hypersurfaces. For a system with one degree of freedom, however, the envelope reduces to a family of curves, as in Figure 4.4. This concept will be useful later in explaining how incremental-iterative solution methods work.

Example 4.1. For simple one-degree of freedom systems it is easy to plot the incremental flow using standard graphic packages. As an example consider the following residual equation, which is obtained as solution of one of the Exercises of Chapter 6:

$$r(\mu, \lambda) = \alpha^3 \mu(1 - \mu)(2 - \mu) - \lambda. \tag{4.16}$$

Here μ is a dimensionless state parameter and α an angle in radians characterizing the reference position of the structure. The following `Mathematica` program produces the incremental flow plot for $\alpha = 30^\circ$ using the `ContourPlot` function:

```
alpha = Pi/6; r = alpha^3*mu*(1-mu)*(2-mu)-lambda;
ContourPlot[r,{mu,0,2},{lambda,-.1,.1},PlotPoints->30];
```

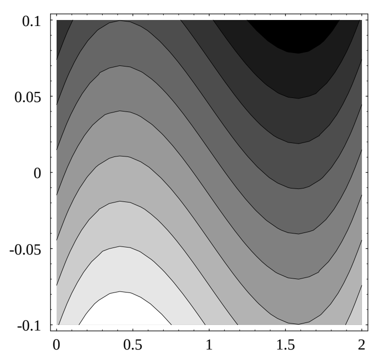

Figure 4.5. Incremental flow plot for the residual (4.16) produced by `Mathematica` via its `ContourPlot` function.

Examination of Figure 4.5 shows that the $r = \text{const}$ curves are simply translations of each other along the λ axis because λ appears simply as $-\lambda$. This is typical of proportional loading situations.

§4.4.2. Diagrams for Multiple Degrees of Freedom

If the number of degrees of freedom increases to $N > 1$ the incremental flow still remains a family of curves in the $N + 1$-dimensional control-state space space $(\mathbf{u}, \lambda)$. Visualization, however, is restricted to $N = 2$ as illustrated in Figure 4.6. For three or more degrees of freedom, only cross sections of the control-state space can be displayed, in which one or two representative degrees of freedom or functions of such are plotted. This "projection" requires some ingenuity and experience.

The flow-orthogonal envelope becomes a family of ordinary surfaces if $N = 2$, as illustrated in Figure 4.7. For three or more degrees of freedom, the envelope becomes a family of hypersurfaces.

§4.5. Intrinsic Geometry of Incremental Flow

§4.5.1. Tangent Vector

At a generic *regular* point P of coordinates $(\mathbf{u}, \lambda)$, not necessarily on the equilibrium path, we can construct an *unnormalized* tangent vector $\mathbf{t}$ defined by

$$\boxed{\mathbf{t} = \begin{bmatrix} \mathbf{u}' \\ \lambda' \end{bmatrix} = \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix},} \tag{4.17}$$

where $\mathbf{v} = \mathbf{K}^{-1}\mathbf{q}$ is the incremental velocity vector (4.8). Tangent vectors are illustrated in Figures 4.3 and 4.8 for one and two degrees of freedom, respectively.

The tangent vector *normalized to unit length* is

$$\mathbf{t}_u = \begin{bmatrix} \mathbf{v}/f \\ 1/f \end{bmatrix}, \tag{4.18}$$

where f is the scaling factor

$$f = |\mathbf{t}| = +\sqrt{||\mathbf{t}||_2} = +\sqrt{1 + \mathbf{v}^T\mathbf{v}}. \tag{4.19}$$

The *positive tangent direction* and the *positive unit tangent* are defined as

$$\mathbf{t}^+ = \pm \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix}, \qquad \mathbf{t}_u^+ = \frac{\mathbf{t}^+}{f} = \pm \begin{bmatrix} \mathbf{v}/f \\ 1/f \end{bmatrix}. \tag{4.20}$$

The positive tangent direction points in the positive sense of path traversal, as defined in §4.2 and Figure 4.2.

§4.5.2. Normal Hyperplane and Flow-Orthogonal Envelope

The hyperplane N_P normal to $\mathbf{t}$ at $P(\mathbf{u}, \lambda)$ has the equation

$$\mathbf{v}^T\, \Delta\mathbf{u} + \Delta\lambda = 0, \tag{4.21}$$

where $\Delta\mathbf{u} = \mathbf{u} - \mathbf{u}_P$ and $\Delta\lambda = \lambda - \lambda_P$ are increments from P. Dividing these increments by Δt and passing to the limit one obtains

$$\boxed{\mathbf{v}^T\dot{\mathbf{u}} + \dot{\lambda} = 0.} \tag{4.22}$$

For a one degree of freedom u the hyperplane reduce to a line in (u, λ) space, as illustrated in Figure 4.4. For two degrees of freedom the normal hyperplane is an ordinary plane in the 3D space (u_1, u_2, λ), as illustrated in Figure 4.8.

For one degree of freedom (4.22) is the differential equation of a flow orthogonal to the incremental flow, as illustrated in Figure 4.3; this flow is the envelope of the normals. For two degrees of freedom (4.22) represents a family of surfaces, see Figure 4.6. For more degrees of freedom (4.22) is a family of hypersurfaces. The orthogonality property plays an important role in corrective solution methods.

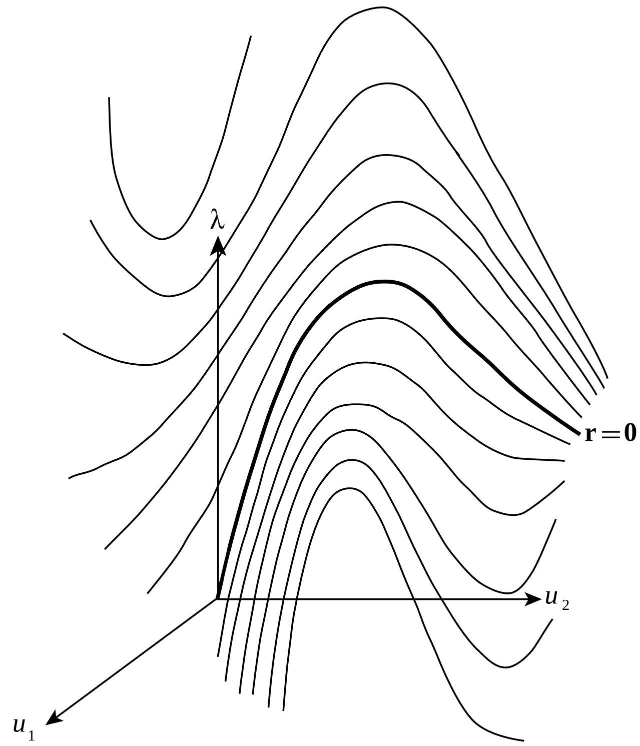

Figure 4.6. An incremental flow response diagram for two degrees of freedom. The plane paths of Figure 4.3 now become space curves. Only a few paths are shown to reduce clutter.

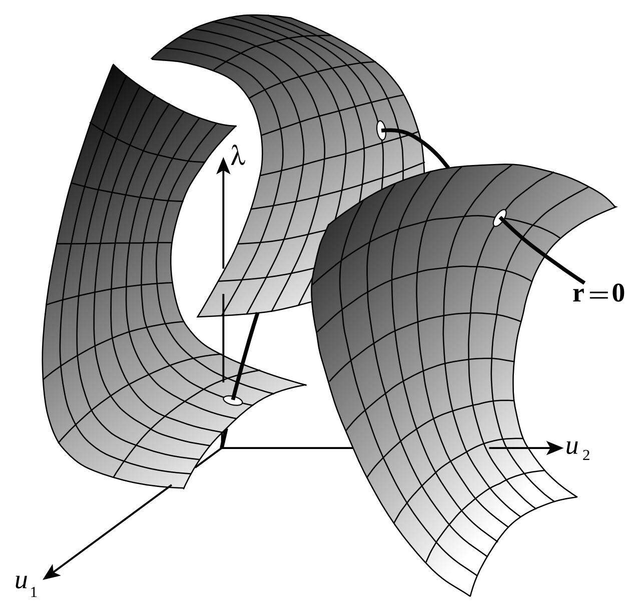

Figure 4.7. A response diagram for two degrees of freedom, showing some members of the flow-orthogonal envelope. Only the primary equilibrium path $\mathbf{r} = \mathbf{0}$ is shown to reduce clutter.

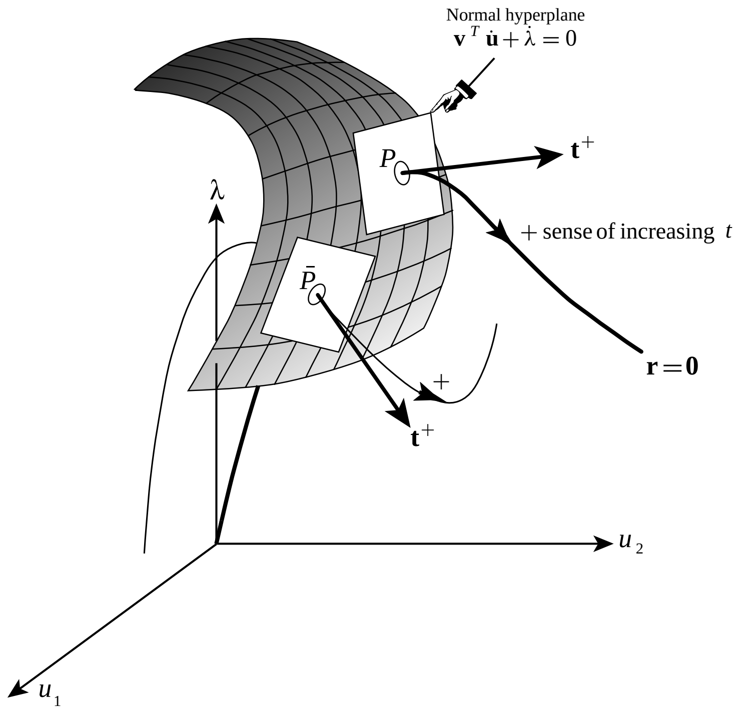

Figure 4.8. Illustrating the tangent vector and normal hyperplane in an incremental flow diagram for two degrees of freedom. Point P is on the primary equilibrium path but $\bar{P}$ is generic.

§4.5.3. ArcLength Distance

Theleft hand side of the hyperplane equation (4.21) normalized on dividing through by f

$$\Delta s = \frac{1}{f}(\mathbf{v}^T\ \Delta\mathbf{u} + \Delta\lambda), \tag{4.23}$$

acquires the following geometric meaning: Δs is the *signed distance* from the normal hyperplane at P to a point $Q(\Delta\mathbf{u},\ \Delta\lambda)$. For small increments $(\Delta\mathbf{u},\ \Delta\lambda)$, Δs may be considered as an approximation to the arclength s of the path that passes through P because

$$ds = \frac{1}{f}(\mathbf{v}^T\ d\mathbf{u} + d\lambda). \tag{4.24}$$

This important concept is illustrated in Figure 4.9.

Remark 4.3. At isolated limit points studied in Chapter 5, the normalization process (4.23) reduces the unit tangent to

$$\mathbf{t}_u = \begin{bmatrix}\mathbf{z}\\ 0\end{bmatrix}, \qquad \mathbf{t}_u^+ = \begin{bmatrix}\pm\mathbf{z}\\ 0\end{bmatrix}, \tag{4.25}$$

where $\mathbf{z}$ is the unit length null eigenvector of $\mathbf{K}$, that is, $\mathbf{K}\mathbf{z} = \mathbf{0}$. The sign ambiguity arises because $+\mathbf{z}$ and $-\mathbf{z}$ are both eigenvectors; one of them has to be chosen to satisfy the positive-traversal convention. At bifurcation points and non isolated limit points $\mathbf{t}$ is not unique.

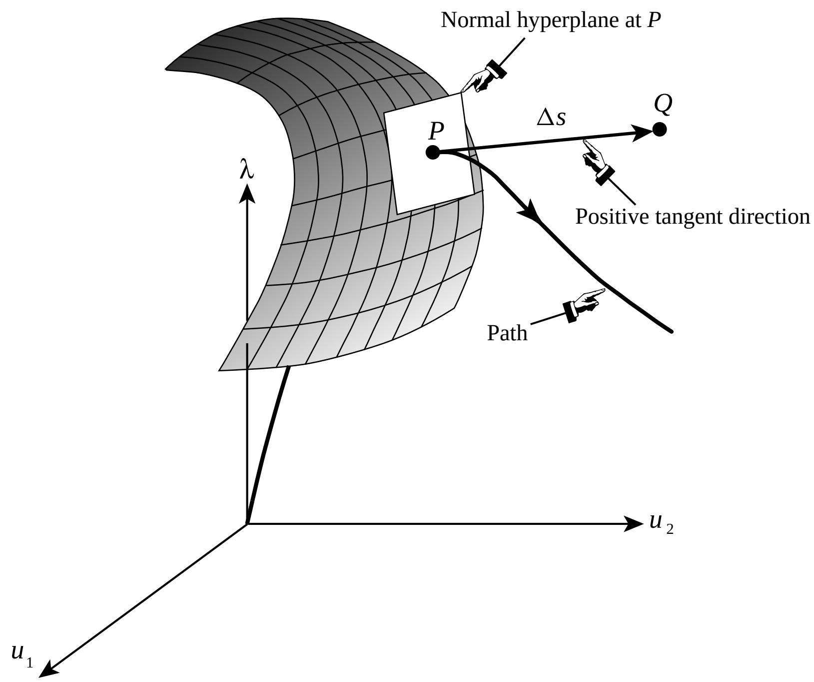

Figure 4.9. The concept of arclength distance Δs from point P to point Q. Note that the point order is important: the arclength distance from Q to P is not generally the same as that from P to Q.

Remark 4.4. From (4.19) and (4.24) we note the formulas

$$\frac{d\mathbf{u}}{ds} = \frac{\mathbf{v}}{f}, \qquad \frac{d\lambda}{ds} = \frac{1}{f}. \tag{4.26}$$

Remark 4.5. In the mathematical literature the incremental flow projected on the $\mathbf{u}$ state space is sometimes called a *Davidenko flow* in honor of the father of continuation methods, should λ be interpreted as a continuation parameter.

Remark 4.6. An alternative to plotting (4.13) for response visualization, is to consider the use of the constant-residual-norm equation

$$||\mathbf{r}(\mathbf{u}, \boldsymbol{\lambda})|| = C, \tag{4.27}$$

where $||\mathbf{r}||$ denotes a vector norm such as, for instance, the Euclidean norm $||\mathbf{r}||_2 = \mathbf{r}^T \mathbf{r}$, and C is a nonnegative numeric constant. This relation does not generally represent a family of curves but a family of tube-like hypersurfaces that for sufficiently small C "wrap around" equilibrium paths, as illustrated in Figure 4.10. Because of the visual clutter evident in that figure, equation (4.27) is less suitable than (4.13) to study what happens in the neighboorhood of equilibrium paths.

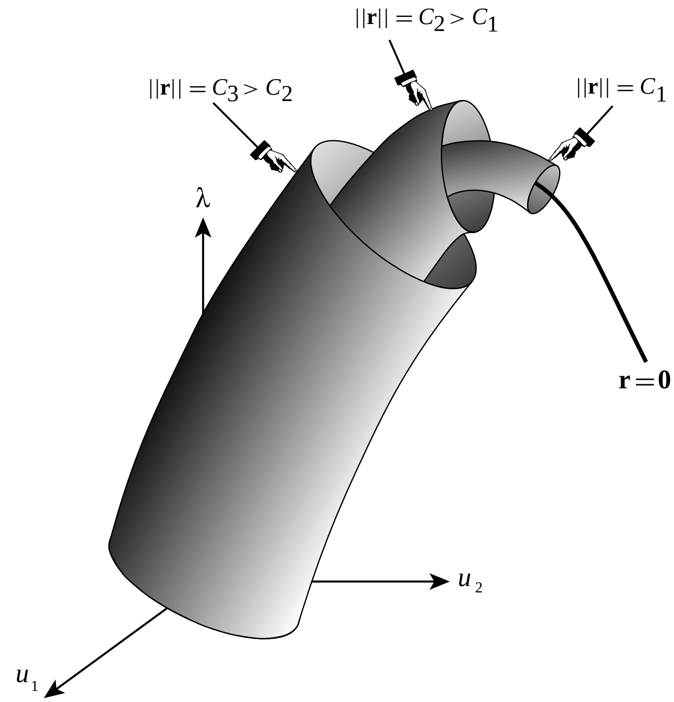

Figure 410. For Remark 4.6: illustrating that the constant residual-norm equation $||\mathbf{r}|| = \text{const}$ generally represents a family of tube-like surfaces "wrapping around" the equilibrium paths.

§4.6. *State Vector Scaling

In applying nonlinear equation solving techniques to structural mechanics (or, in general, to problems in engineering and physics) the issue of *scaling* often arises because of two aspects:

1. The residual $\mathbf{r}$ has two types of arguments: $\mathbf{u}$ and λ. Translational degrees of freedom collected in the state vector $\mathbf{u}$ have physical dimensions of length (displacement) whereas λ is dimensionless.

2. The degrees of freedom in $\mathbf{u}$ may have heterogeneous physical dimensions. For example, in the analysis of finite element models that account for bending effects $\mathbf{u}$ may contain both translations and rotations.

To reduce the sensitivity of solution procedures to these factors, it is often advisable to introduce a scaling of the state vector $\mathbf{u}$ to render it dimensionless and thus placed on an equal footing with λ:

$$\widetilde{\mathbf{u}} = \mathbf{S}\mathbf{u}. \tag{4.28}$$

Here the scaling matrix $\mathbf{S}$ is *diagonal*, and a superposed tilde identifies a scaled quantity. If all entries of $\mathbf{u}$ have homogeneous dimensions, one may take simply $\mathbf{S} = (1/u)\,\mathbf{I}$, where the scalar u has the dimension of $\mathbf{u}$.

The scaled versions of other quantities defined previously are

$$\Delta\widetilde{\mathbf{u}} = \mathbf{S}\,\Delta\mathbf{u}, \qquad \widetilde{\mathbf{q}} = \mathbf{S}^{-1}\mathbf{q}, \qquad \widetilde{\mathbf{K}} = \mathbf{S}^{-1}\mathbf{K}\mathbf{S}^{-1}, \tag{4.29}$$

$$\widetilde{\mathbf{v}} = \mathbf{S}\mathbf{v}, \qquad \widetilde{\mathbf{t}} = \begin{bmatrix} \mathbf{S}\mathbf{v} \\ 1 \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbf{v}} \\ 1 \end{bmatrix}, \qquad \widetilde{f} = \sqrt{1 + \mathbf{v}^T\mathbf{S}^2\mathbf{v}} = \sqrt{1 + \widetilde{\mathbf{v}}^T\widetilde{\mathbf{v}}}, \tag{4.30}$$

$$\widetilde{\mathbf{t}}_u = (1/\widetilde{f}) \begin{bmatrix} \widetilde{\mathbf{v}} \\ 1 \end{bmatrix}, \qquad \Delta\widetilde{s} = (\widetilde{\mathbf{v}}^T\,\Delta\widetilde{\mathbf{u}} + \Delta\lambda)/\widetilde{f} = (\mathbf{v}^T\mathbf{S}^2\,\Delta\mathbf{u} + \Delta\lambda)/\widetilde{f}. \tag{4.31}$$

Homework Exercises for Chapter 4

One-Parameter Residual Equations

EXERCISE 4.1 [A:5+15] Consider the residual force equations

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} u_1 + 3u_2^2 - 2\Lambda_1 \\ u_2 + 6u_1 u_2 - \Lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{E4.1}$$

(a) Is this system of equations separable in the sense discussed in §4.3?

(b) If so, can **f** and **p** be expressed as gradients of scalar functions U and P and what are these?

EXERCISE 4.2 [A:15+15] Suppose that (E4.1) is to be solved in two stages:

Stage 1. Start from $\Lambda_1 = \Lambda_2 = 0$ and go to $\Lambda_1 = 0$ and $\Lambda_2 = 5$. Parameter λ varies from 0 to 1.

Stage 2. Start from $\Lambda_1 = 0$, $\Lambda_2 = 5$ and go to $\Lambda_1 = \Lambda_2 = 10$. Again λ varies from 0 to 1.

(a) Express the residual in the one-parameter form (4.1) for each stage.

(b) Find the expression of the incremental load vector **q** in each stage. Is the loading proportional?

EXERCISE 4.3 [A:20] Suppose the first residual force above is replaced by $r_1 = u_1 + 3u_2^2 - 2\Lambda_1^2$.

(a) Is the system still separable?

(b) For the same two stages of the previous exercise, is the loading proportional?

EXERCISE 4.4 [A:25] For stage 1 of Exercise 4.2, write down the analytical expressions of the incremental velocity, the tangent vectors **t** and $\mathbf{t}_u$, the normal hyperplane equation, and the differential equations of the flow-orthogonal envelope. Note: explicit inversion of $\mathbf{K}^{-1}$ may be done using the formulas to invert a 2×2 matrix.

EXERCISE 4.5 [A:25] Verify the assertion of Remark 4.6 by using the Euclidean norm $||\mathbf{r}|| = \mathbf{r}^T\mathbf{r}$ of the residual vector.

EXERCISE 4.6 [A:25] Explain whether the unnormalized tangent vector **t** introduced in §4.5.1 may be defined as

$$\mathbf{t} = \begin{bmatrix} \dot{\mathbf{u}} \\ \dot{\lambda} \end{bmatrix}, \tag{E4.2}$$

and whether this definition is more general than (4.17).