30

Dynamic Stability Analysis

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§30.1. Introduction

If the loading is nonconservative the loss of stability may not show up by the system going into another equilibrium state but by going into unbounded *motion*. To encompass this possibility we must consider the *dynamic* behavior of the system because stability is essentially a dynamic concept (recall the definition in §25.1).

The essential steps are as follows. We investigate the motion that occurs after some initial perturbation is applied to the equilibrium state being tested, and from the properties of the motion we can infer or deny stability. It if turns out that the perturbed motion consists of oscillations of increasing amplitude, or is a rapidly increasing departure from the equilibrium state, the equilibrium is unstable; otherwise it is stable.

The practicality of this approach depends crucially on the *linearization* of the equations of motion of the perturbation. Thus we avoid having to trace the ensemble of time histories for every conceivable dynamic departure from equilibrium — which for a system with many degrees of freedom would clearly be a computationally forbidding task.

By linearizing we can express the perturbation motion as the superposition of *complex exponential* elementary solutions. The characteristic exponents of these solutions can be determined through a characteristic value problem or eigenproblem. This problem includes the free-vibration natural frequency eigenproblem as particular case when the system is conservative and the tangent stiffness matrix is symmetric. Through the stability criterion discussed in §29.3, the set of characteristic exponents gives complete information on the linearized stability of the system at the given equilibrium configuration.

In practical studies the characteristic exponents are functions of the control parameter λ. Assuming that the system is stable for sufficiently small λ values, say $\lambda = 0$, we are primarily concerned with finding the first occurrence of λ at which the system loses stability. The transition to instability may occur in two different ways, which receive the names *divergence* and *flutter*, respectively.[1]

The distinction between divergence and flutter instability is important in that the singular-stiffness test discussed in Chapter 26 *remains valid if the stability loss occurs by divergence*, although of course the tangent stiffness is not necessarily symmetric. Therefore it follows that in that case we may fall back upon the static criterion, which is simpler to apply because it does not involves information about mass and damping. Such a regression is not possible, however, if the loss of stability occurs by flutter.

§30.2. The Linearized Equations of Motion

The structure is in static equilibrium under a given value of the control parameter λ. The equilibrium state is defined by the state vector $\mathbf{u}$. At time[2] $\tau = 0$ apply a dynamic input (*e.g.*, an impulse) to this configuration and examine the subsequent motion of the system. Roughly speaking if the motion is unbounded (remains bounded) as τ tends to infinity the system is dynamically unstable (stable).

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[1] These names originated in aeronautical engineering applications, more specifically the investigation of sudden airplane "blow ups" during the period 1910-1930. In the mathematical literature flutter goes by the name 'Hopf bifurcation.'

[2] The symbol τ denotes real time because t is used throughout the course to denote a pseudo-time parameter. Real time is considered only in this Chapter.

As noted in the Introduction, to simplify the mathematical treatment we consider only the *local stability* condition, in which the imparted excitation is so tiny that the subsequent motion can be viewed as a linearizable perturbation. We are effectively dealing with *small perturbations* about the equilibrium position.

Let $\mathbf{M}$ be the symmetric mass matrix, which is assumed positive definite, and $\mathbf{K}$ the tangent stiffness matrix, which is real but generally *unsymmetric* because of load nonconservativeness. The perturbation motion is denoted as

$$\mathbf{d}(\tau) = \mathbf{u}(\tau) - \mathbf{u}(0), \qquad \tau \geq 0^+ \tag{30.1}$$

The discrete, unforced, undamped governing equations of motion are

$$\mathbf{M}\ddot{\mathbf{d}} + \mathbf{K}\mathbf{d} = \mathbf{0}, \tag{30.2}$$

in which a superposed dot — unlike previous Chapters — denotes differentiation with respect to real time. The ordinary differential equations (30.2) express the linearized dynamic equilibrium between stiffness and inertial forces. The stiffness forces generally include nonconservative loading effects.

Remark 30.1. In structural with rotational DOFs, $\mathbf{M}$ might be only nonnegative definite because of the presence of zero rotational masses. If so it is assumed that those DOFs have been eliminated by a static condensation process.

The assumption of positive definiteness also excludes the presence of Lagrange multipliers in the state vector $\mathbf{u}$, because the associated masses of such degrees of freedom are zero. Again the stability criteria can be extended by eliminating the multipliers in the linearized equation of motion.

Remark 30.2. We shall ignore damping effects because of two reasons:

(1) The effect of diagonalizable, light viscous structural damping does not generally affect stability results (it certainly does not when stability loss is by divergence). See also Remark 29.4.

(2) The effect of more complicated nonlinear damping mechanisms such as dry friction may not be amenable to linearization.

Thus cases when damping effects are significant lead to mathematics beyond the scope of this course. Readers interested in pursuing this topic are referred to the vast literature on the subject of dynamic stability.

§30.3. The Characteristic Problem

The linear ODE system (30.2) can be treated by assuming the eigenmodal expansion

$$\mathbf{d}(\tau) = \sum_i \mathbf{d}_i(\tau) = \sum_i \mathbf{z}_i e^{p_i \tau}, \tag{30.3}$$

where i ranges over the number of degrees of freedom (number of state parameters). The p_i are generally complex numbers called the *characteristic exponents* whereas the corresponding column vectors $\mathbf{z}_i$ are the *characteristic modes* or *characteristic vectors*.[3]

[3] In his classical treatise *Nonconservative Problems of the Theory of Elastic Stability*, (Pergamon, 1963), Bolotin employs s for what we call here p, and so do many other authors. This notation connects well to the common use of the Laplace transform to do more complicated systems. However, we have already reserved s for Piola-Kirchhoff stresses as well as arclength.

Replacing $\ddot{\mathbf{d}}_i = p_i^2 \mathbf{d}$ into (30.2) yields

$$(\mathbf{K} + p_i^2 \mathbf{M})\,\mathbf{z}_i = \mathbf{0}, \tag{30.4}$$

which is the *characteristic problem* or *eigenproblem* that governs dynamic stability. This equation befits the generalized unsymmetric eigenproblem of linear algebra

$$\mathbf{A}\mathbf{x}_i = \mu_i \mathbf{B}\mathbf{x}_i \tag{30.5}$$

in which matrix $\mathbf{A} \equiv \mathbf{K}$ is real and generally unsymmetric whereas $\mathbf{B} \equiv \mathbf{M}$ is real symmetric positive definite. The eigenvalues $\mu_i \equiv -p_i^2$ of this eigenproblem may be either real or complex; if the latter, they occur in conjugate pairs. The square roots of these eigenvalues yield the characteristic exponents p_i of the eigenmodal expansion (30.3).

§30.3.1. Connection with the Free-Vibration Eigenproblem

If the system is *conservative* and *stable*, $\mathbf{K}$ is symmetric and positive definite. If so all roots p_i^2 of (30.4) are negative real and their square roots are *purely imaginary* numbers:

$$p_i = \pm j\omega_i, \tag{30.6}$$

where $j = \sqrt{-1}$, and the nonnegative real numbers ω_i are the natural frequencies of free vibration. Because $p^2 = -\omega_i^2$, (30.4) reduces to the usual vibration eigenproblem

$$(\mathbf{K} - \omega_i^2 \mathbf{M})\,\mathbf{z}_i = \mathbf{0}. \tag{30.7}$$

Thus for the conservative case we regress to a well studied problem. In such a case the system will simply *vibrate*, that is, perform *harmonic oscillations* about the equilibrium position because each root is associated with the solution

$$e^{j\omega_i\tau} = \cos\omega_i\tau + j\sin\omega_i\tau. \tag{30.8}$$

The presence of positive damping will of course damp out these oscillations and the system eventually returns to the static equilibrium position.

§30.4. Characteristic Exponents and Stability

The characteristic exponents are generally complex numbers:

$$p_i = \alpha_i + j\omega_i, \tag{30.9}$$

where α_i and ω_i are real numbers, and $j = \sqrt{-1}$. The component representation of the square of p_i is

$$p_i^2 = (\alpha_i^2 - \omega_i^2) + 2j\alpha_i\omega_i, \tag{30.10}$$

The exponential of a complex number has the component representation

$$e^{p_i\tau} = e^{(\alpha_i + j\omega_i)\tau} = e^{\alpha_i\tau}(\cos\omega_i\tau + j\sin\omega_i\tau), \tag{30.11}$$

On the basis of this representation we can classify the growth behavior of the subsequent motion and consequently the stability of the system as examined in the next 3 subsections.

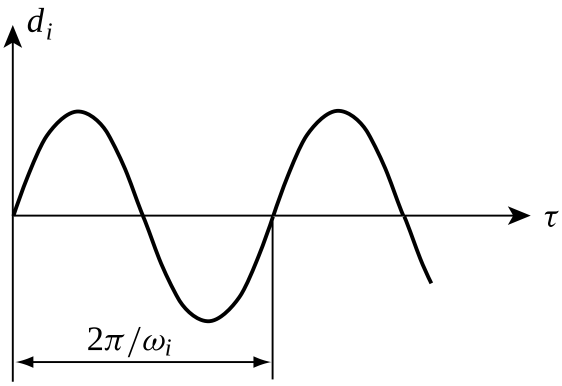

Figure 30.1. Harmonic oscillatory motion for the case where root p_i^2 of (30.4) is negative real. Equivalently, $p_i = \pm j\omega_i$ where ω_i is the circular frequency.

§30.4.1. Negative Real Case: Harmonic Oscillations

If

$$p_i = \pm j\omega_i, \qquad d(\tau) = \sum d_i(\tau), \quad d_i(\tau) = A_i \cos\omega_i\tau + B_i \sin\omega_i\tau. \tag{30.12}$$

where A_i and B_i are determined by initial conditions. The motion d_i associated with $\pm j\omega_i$ is harmonic and bounded, as illustrated in Figure 30.1. The system is dynamically stable for this individual eigenvalue.

If *all* eigenvalues are negative real and distinct, the system is dynamically stable because any superposition of harmonic motions of different periods is also a harmonic motion. If two or more eigenvalues coalesce the analysis becomes more complicated because of the appearance of secular terms that grow linearly in time. These effects can be studied in more detail in treatises in mechanical vibrations.

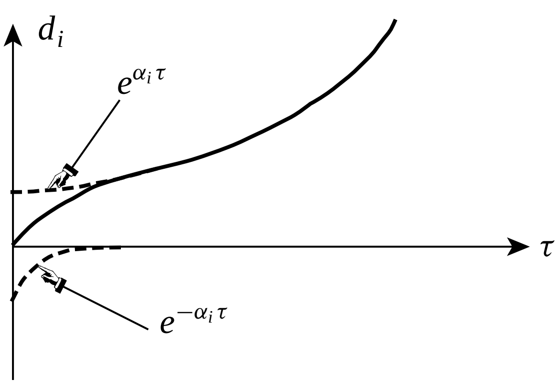

Figure 30.2. Aperiodic, exponentially growing motion for the real root case $p_i^2 = \alpha_i^2$, $p_i = \pm\alpha_i$. Transition to this kind of instability is called *divergence*.

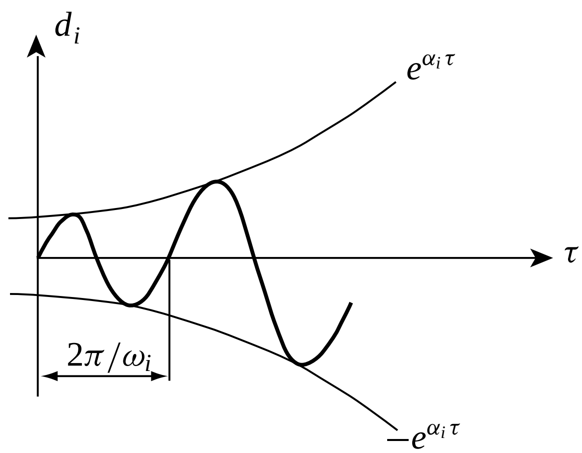

Figure 30.3. Periodic, exponentially growing motion for case $p_i = +\alpha_i \pm j\omega_i$ with nonzero α_i. Transition to this kind of instability is called *flutter*.

§30.4.2. Positive Real Case: Divergence

If p_i^2 is positive real,

$$p_i = \pm\alpha_i. \tag{30.13}$$

The $+\alpha_i$ square root will give rise to an *aperiodic*, exponentially growing motion. The other root will give rise to an exponentially decaying motion. When the two solutions are combined the exponentially growing one will dominate for sufficiently large τ as sketched in Figure 30.2, and the system is then exponentially unstable.

As noted above p_i^2 is generally a function of λ. The transition from stability (in which all roots are negative real) to this type of instability necessarily occurs when a eigenvalue $p_i^2(\lambda)$, moving from left to right as λ varies, passes through the origin $p^2 = 0$ of the p^2 complex plane. This type of instability is called *divergence*.

§30.4.3. Complex Case: Flutter

If p_i^2 is complex, solutions of the eigenproblem (30.4) occur in *conjugate pairs* because both matrices **M** and **K** are real. Consequently, if $p_i^2 = (\alpha_i^2 - \omega_i^2) + j\,(2\alpha_i\omega_i)$ is a complex eigenvalue so is its conjugate $\overline{(p_i^2)} = (\alpha_i^2 - \omega_i^2) - j\,(2\alpha_i\omega_i)$. On taking the square root of this pair we find *four* characteristic exponents

$$\pm\alpha_i \pm j\omega_i. \tag{30.14}$$

Two of these square roots will have positive real parts ($+\alpha$) and for sufficiently large τ they will eventually dominate the other pair, yielding exponentially growing oscillations; see Figure 30.3. This is called *periodic exponential instability* or *flutter instability*.

If the system is initially stable (i.e., all roots are negative real) then transition to this type of instability occurs when at a certain value of λ *two real roots coalesce* on the real axis and "branch out" into the complex p^2 plane. This loss of stability is called *flutter*.

Remark 30.3. Frequency coalescence is necessary but not sufficient for flutter. It is possible for two frequencies to pass by other "like ships crossing in the night" without merging. This happens if there is no mechanism by which the two associated eigenmodes can exchange energy.

Remark 30.4. The fact that all characteristic motions are either harmonic or exponentially growing is a consequence of the neglect of *damping* in setting up the stability problem. As noted in Remark 30.2, the presence of damping or, in general, dissipative forces, introduces additional mathematical complications that will not be elaborated upon here. Suffices to say that the addition of damping to a *conservative* system has always a *stabilizing* effect (Rayleigh's theorem). For non-conservative systems, the preceding statement is no longer true, and indeed several counterexamples involving *destabilizing damping* have been constructed over the past 40 years. In spite of this the effect is not often observed in practice.

Remark 30.5. The occurrence of flutter requires the coalescence of *two* natural frequencies. Consequently, flutter cannot occur in systems with one degree of freedom ("it takes two to flutter"). The physical interpretation of the flutter phenomenon is that one vibration mode absorbs energy and feeds it into another; this transference or "energy resonance" becomes possible when the two modes have the same frequency.

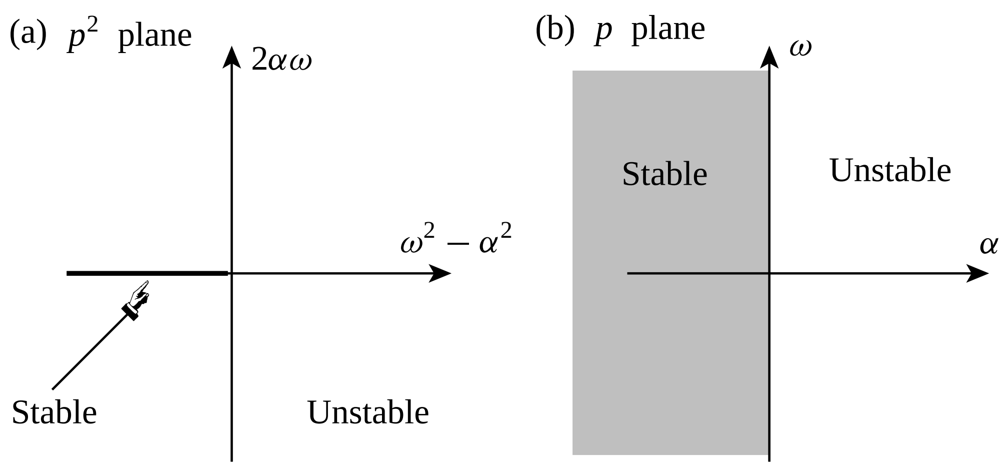

Figure 30.4. Stable and unstable regions in (a) the complex p^2 plane, (b) the complex p plane. For the latter the stable region is the left-half plane $\alpha = \Re(p) \leq 0$. For (a) it is the negative real axis.

§30.4.4. Stable and Unstable Regions in the Complex Plane

From the preceding study it follows that the only stable region in the complex p^2-plane is the negative real axis:

$$\Re(p^2) < 0, \quad \Im(p^2) = 0. \tag{30.15}$$

The rest of the p^2 complex plane is unstable; see Figure 30.4(a).

On the complex p-plane, the stable region is the left-hand plane

$$\alpha = \Re(p) \leq 0. \tag{30.16}$$

which includes the imaginary axis $\alpha = 0$ as stability boundary. The right-hand p-plane $\alpha > 0$ is unstable. See Figure 30.4(b).

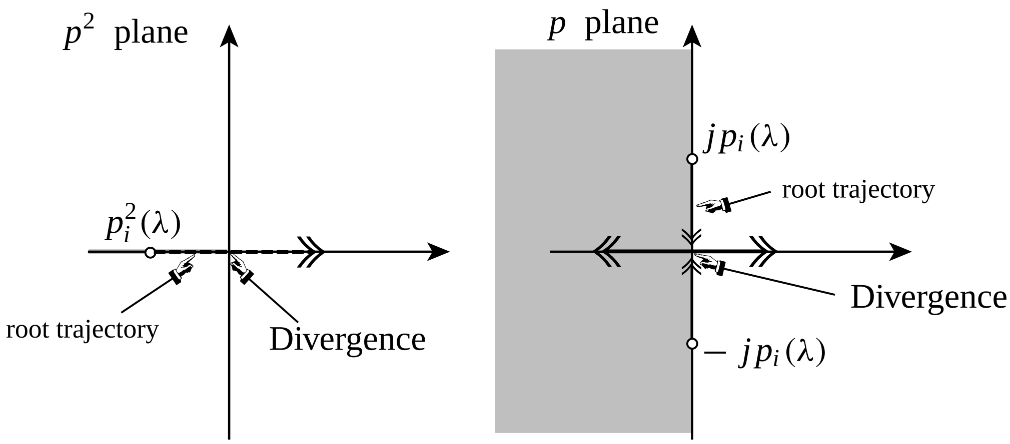

Figure 30.5. Root locus plots on the complex p^2 and p planes for divergence instability.

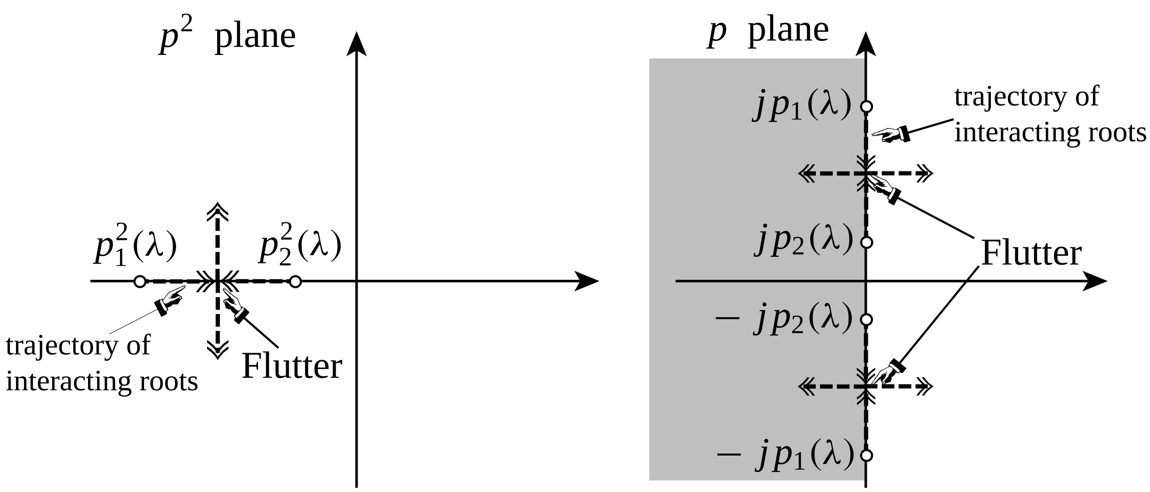

Figure 30.6. Root locus plots on the complex p^2 and p planes for flutter instability.

§30.5. Graphical Representations

§30.5.1. Root locus plots

Graphical representations of the "trajectories" of the eigenvalues $p_i(\lambda)$ as λ is varied on the complex p^2 or p planes are valuable insofar as enhancing the understanding of the differences between divergence and flutter. These are called *root locus plots*[4] and are illustrated in Figures 30.5 and 30.6.

Figure 30.5 illustrates loss of stability by divergence. As λ is varied, eigenvalue p_i^2 passes from the left-hand plane to the right-hand plane through the origin $p^2 = 0$. Stability loss occurs at the λ for which p_i^2 vanishes. The right-hand diagram depicts the same phenomenon on the p plane, for the root pair $\pm p_i$.

[4] The word *root* in root-locus is used as abbreviation for characteristic root or eigenvalue

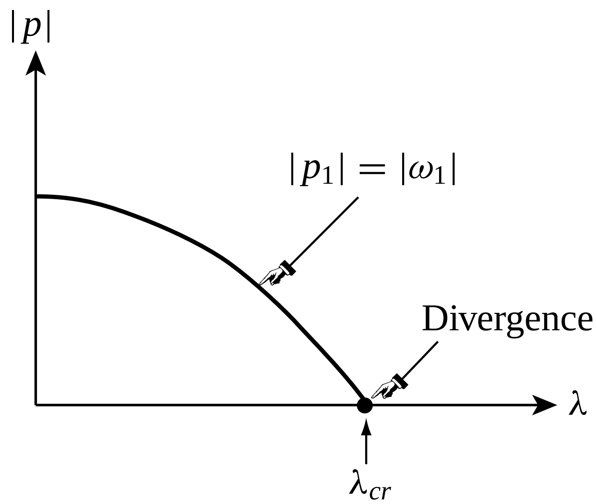

Figure 30.7. Root amplitude plot illustrating loss of stability by divergence at λ_{cr}.

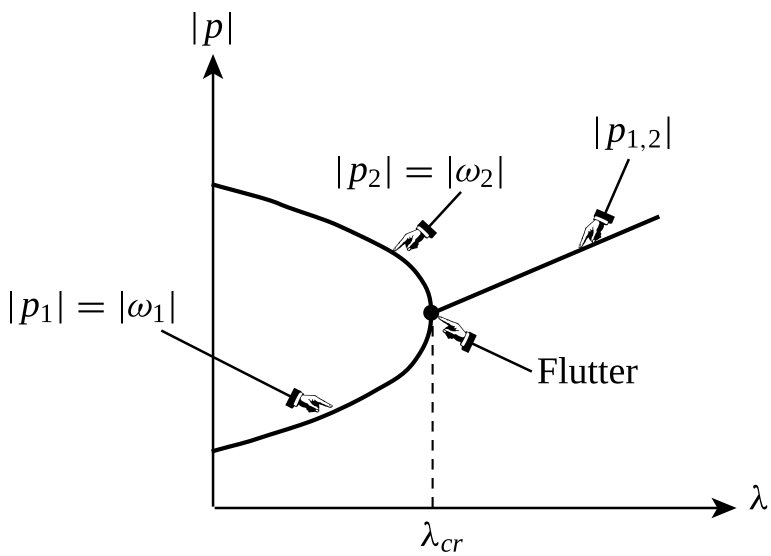

Figure 30.8. Root amplitude plot illustrating loss of stability by flutter at λ_{cr}.

Figure 30.6 illustrates loss of stability by flutter. As λ is varied, two interacting eigenvalues, labeled as p_1^2 and p_2^2, coalesce on the negative real axis of the p^2 plane and branch out into the unstable region. The right-hand diagram depicts the same phenomenon on the p plane for the interacting roots, which appears in complex-conjugate pairs.

§30.5.2. Amplitude Plots

Another commonly used visualization technique is the *characteristic root amplitude* or simply *root amplitude* plots. These plots show the magnitude of $p_i(\lambda)$, that is $|p_i(\lambda)|$ on the vertical axis against λ on the horizontal axis. If the eigenvalue is real, $|p_i|$ is simply its absolute value whereas if it is complex $|p_i|$ is its modulus.

This graphical representation enjoys the following advantages: (a) the critical value of λ is displayed more precisely than with a locus or trajectory plot, (b) all related square roots such as $\pm\alpha_i \pm \omega_i$ "collapse" into a single value, and (c) the variation of several important roots (for several values of i) may be shown without cluttering the picture.

Figures 30.7 and 30.8 illustrate typical root-amplitude plots in loss of stability by divergence and flutter, respectively.

§30.6. Regression to Zero Frequency and Static Tests

The stability loss by *divergence* occurs when an eigenvalue p_i vanishes. Because $\omega_i = 0$ if $p_i = 0$, this is equivalent to a zero-frequency test on the eigenproblem

$$(-\omega_i^2 \mathbf{M} + \mathbf{K})\, \mathbf{z}_i = \mathbf{0}. \tag{30.17}$$

But if $\omega_i = 0$ and $\mathbf{M}$ is positive definite, which we assume, then $\mathbf{K}$ must be singular. Therefore we can regress to the static criterion or singular tangent stiffness test

$$\det \mathbf{K}(\lambda) = 0, \tag{30.18}$$

which allows us to discard the mass matrix. This regression may be useful if one is solving a series of closely related problems, for example during the design of a structure which is known *a priori* to become unstable by divergence.

It should be cautioned, however, that the tangent stiffness matrix $\mathbf{K}$ for nonconservative systems is generally unsymmetric (Chapter 29), and that the test for singularity must take account of that property.

Homework Exercises for Chapter 30

Dynamic Stability Analysis

EXERCISE 30.1 (A+C:25) This Exercise studies the stability of the “follower load” nonconservative system shown in Figure E30.1.

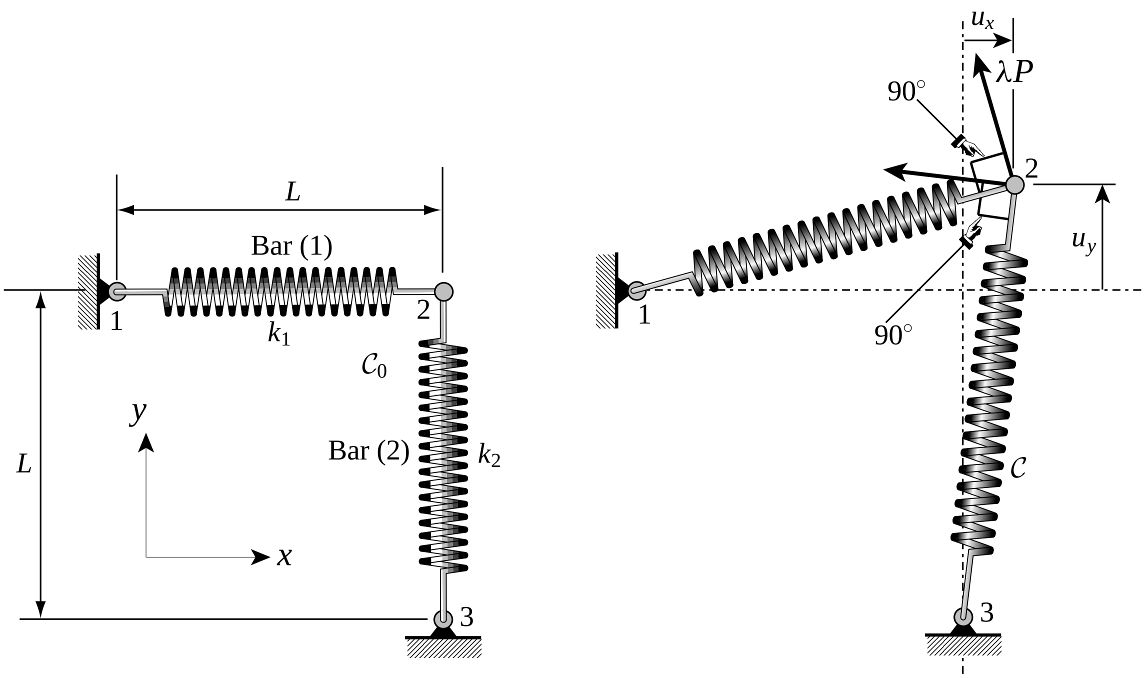

Figure E30.1. Structure for Exercise 30.1.

Two elastic bars, (1) and (2), are supported at 1 and 3 and hinged at 2. The bars have length L, axial stiffnesses k_1 and k_2, respectively, and can only move in the x, y plane. Bar (1) is loaded at node 2 by a force λP_1, directed upwards, that stays normal to bar (1) as it displaces. Bar (2) is loaded at node 2 by a force λP_2, directed leftwards, that stays normal to bar (2) as it displaces.

For the present exercise set $P_1 = P_2 = P$. Furthermore the following simplfying assumptions are to be made:

(A1) The displacements from the reference configuration are so small that $\mathcal{C} \equiv \mathcal{C}_0$ insofar as setting up the stability eigensystem[5]

(A2) The contribution of the geometric stiffness is neglected.

(a) Show that under the simplifying assumptions (A1)–(A2), the tangent stiffness at $\mathcal{C} \equiv \mathcal{C}_0$ in terms of the two degrees of freedom $u_x = u_{x2}$ and $u_y = u_{y2}$, is

$$\mathbf{K} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} + \frac{\lambda P}{L} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{E30.1}$$

[5] This is similar to LPB (Chapters 24-25), but here a dynamic analysis is involved.

The first component of $\mathbf{K}$ is the material stiffness whereas the second component is the load stiffness. Hint for the latter: use the results of Remark 29.4

The linearized dynamic eigenproblem (30.4) is

$$(p_i^2\mathbf{M} + \mathbf{K})\mathbf{z}_i = \mathbf{0}, \qquad i = 1, 2. \tag{E30.2}$$

The exponents p_i (the square roots of p_i^2) are generally complex numbers:

$$p_i = \alpha_i + j\omega_i, \tag{E30.3}$$

where α and ω are the real and imaginary part of p_i, respectively, $\mathbf{z}_i$ are associated eigenmodes, and $\mathbf{M}$ is the diagonal mass matrix

$$\mathbf{M} = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}, \tag{E30.4}$$

where M is the lumped mass at node 2 (half of the sum of the bar masses). By appropriate normalization show that the eigenproblem can be reduced to the *dimensionless* form

$$\left\{ \bar{p}^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \kappa & 0 \\ 0 & 1 \end{bmatrix} + \bar{\lambda} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\} \bar{\mathbf{z}}_i = \mathbf{0}, \tag{E30.5}$$

where $\kappa = k_1/k_2$, $\bar{p}$ and $\bar{\lambda}$ are dimensionless.

(c) Show that the critical positive $\bar{\lambda}_{cr}$ at which the eigenvalues $\bar{p}_i^2$ coalesce is given by the relation

$$\bar{\lambda}_{cr} = \frac{|1-\kappa|}{2}. \tag{E30.6}$$

Further show that if $\bar{\lambda} > \bar{\lambda}_{cr}$ the roots $\bar{p}_i$ become complex and hence explain whether loss of stability occurs. Is it divergence or flutter?

(d) For $\kappa = 0.01$, 1.0, 4.0 and 100 plot the dependence of $|\bar{p}_i|$ $(i = 1, 2)$ (where $|.|$ denotes the modulus of a complex number) on λ using

$$|\bar{p}|/\sqrt{\kappa}, \qquad \lambda/\sqrt{\kappa}, \tag{E30.7}$$

as vertical and horizontal axes, respectively. Go from $\lambda = 0$ up to $2\lambda_{cr}$ or 1.0, whichever is greater, and use sufficient steps to get reasonable graphical accuracy.

EXERCISE 30.2 (A+C:25) Do the previous exercise removing assumption (A2), that is, considering now the effect of the geometric stiffness $\mathbf{K}_G$ but still assuming $\mathcal{C} \equiv \mathcal{C}_0$. Is there any difference with the critical load result (E30.6)?

EXERCISE 30.3 (A+C:30) Beck's column[6] is the simplest follower-load problem involving a cantilevered beam-column.[7] This problem is shown in Figure E30.2.

The beam-column has length L, elastic modulus E and smallest moment of inertia I. It is loaded by a compressive force λP which after deformation rotates with the end section of the column and remains tangential to its deformed axis (see Figure above). The mass M (half of the column mass) is lumped at its free end.

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[6] M. Beck, Die Knicklast des eiseiting eigenspannen, tangential gedrückten Stabes, *Z. angew. Math. Phys.*, **3**, No. 3, 1952.

[7] It is sometimes used as a very simple model to illustrate stability analysis of rockets against the “pogo” effect.

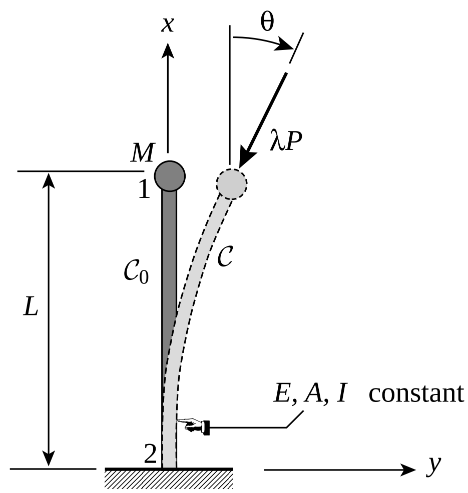

Figure E30.2. Beck's column: structure for Exercise 30.3.

If this problem is treated by the static criterion (Euler's method) one erroneously concludes that the beam column cannot lose stability for any value of the load λP[8] A dynamic stability analysis, first carried out by Beck (*loc.cit.*), shows that stability is lost by flutter at the critical load

$$\lambda P_{cr} = 20.05093 \frac{EI}{L^2}. \tag{E30.8}$$

(a) Find the critical dynamic load given by the finite element method if one Euler-Bernoulli beam-column element is used along the length of Beck's column. Lateral displacements may be considered infinitesimal; hence $\sin\theta \approx \theta$, $\cos\theta \approx 1$, and the axial force is simply λP. The degrees of freedom are u_{x1}, u_{y1} and θ_{z1}. Use the material and geometric stiffness matrices given in equations (E24.2) and (E24.3), respectively, to which an unsymmetric load stiffness matrix $\mathbf{K}_L$, which couples the θ_{z1} and u_{y1} degrees of freedom, should be added.

(b) Repeat the analysis for two and four elements of equal length along the column. For two elements the three nodes are 1 (top), 2 (middle of column) and 3 (root). Use lumped masses with $M_{X2} = M_{Y2}$ equal to one half of the total column mass and $M_{X1} = M_{Y1} = M_{X2}/2 = M_{Y2}/2$. For four elements there are five nodes, etc. Use of *Mathematica* or a similar program is recommended.

[8] See for example, pp. 7–8 of Bolotin's book cited in footnote 3.

Solution of Exercise 30.3(a) for one-element discretization:

The dynamic matrix perturbation equation taking $\mathcal{C} \approx \mathcal{C}_0$ is

$$\begin{bmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{u}_x \\ \ddot{u}_y \\ \ddot{\theta}_z \end{bmatrix} + \left(\begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} - \frac{P}{30L} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 36 & -3L \\ 0 & -3L & 4L^2 \end{bmatrix} + P \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} u_x \\ u_y \\ \theta_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{E30.9}$$

where for simplicity $u_x = u_{x1}$, $u_y = u_{y1}$, $\theta_z = \theta_{z1}$. The first dynamic equation in u_x uncouples and has no effect in the analysis. The last equation is static in nature because the rotational mass is zero. Thus, we can solve for θ_z in terms of u_y:

$$\theta_z = \frac{-\dfrac{6EI}{L^2} + \dfrac{P}{10}}{\dfrac{4EI}{L} - \dfrac{4PL}{30}} \, u_y = \frac{N}{D} \, u_y \tag{E30.10}$$

where N and D denote the numerator and denominator, respectively, of the relation that links θ_z to u_y. The eigenvalue equation becomes

$$\left(p^2 M + \frac{12EI}{L^3} - \frac{6EI}{L^2} \, \frac{N}{D} - \frac{36P}{30L} + 3L \, \frac{N}{D} \, \frac{P}{30L} + P \, \frac{N}{D} \right) u_y = 0. \tag{E30.11}$$

One of the bending eigenvalues p^2 of (E30.9) is always ∞ because the rotational mass is zero. Flutter occurs when the two beding eigenvalues coalesce at infinity. The finite p^2 becomes infinite if $D = 0$ while $N \neq 0$. Thus the critical load for "flutter at infinity" is

$$\boxed{P_{cr} = \frac{30EI}{L^2}} \tag{E30.12}$$

which is about 50% in error with respect to the analytical value $20.05093EI/L^2$ quoted in the exercise statement.

EXERCISE 30.4 (A:25) Do the previous exercise for a one-element discretization if the line of action of the applied end load is forced to pass through the cantilever root (point 2). Does the structure loses stability dynamically or statically?