3

The Direct Stiffness Method II

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§3.1. The Remaining DSM Steps

Chapter 2 covered the initial stages of the DSM. The three breakdown steps: *disconnection, localization* and *formation of member stiffness* take us down all the way to the generic truss element: the highest level of fragmentation. This is followed by the *assembly* process.

Assembly involves *merging* the stiffness equations of each member into the global stiffness equations. For this to make sense, the member equations must be referred to a common coordinate system, which for a plane truss is the global Cartesian system $\{x, y\}$. This is done through the globalization process covered in §2.8. On the computer the formation, globalization and merge steps are done concurrently, member by member. After all members are processed we have the *free-free master stiffness equations*.

Next comes the *solution*. This process embodies two substeps: *application of boundary conditions* and *solution* for the unknown joint displacements. To apply the BCs, the free-free master stiffness equations are modified by taking into account which components of the joint displacements and forces are given and which are unknown.

The modified equations are submitted to a linear equation solver, which returns the unknown joint (node) displacements. As discussed under **Notes and Bibliography**, on some FEM implementations — especially programs written in the 1960s and 1970s — one or more of the foregoing operations are done concurrently.

The solution step completes the DSM proper. *Postprocessing* steps may follow, in which derived quantities such as internal forces and stresses are recovered from the displacement solution.

§3.2. Assembly: Merge

§3.2.1. Governing Rules

The key operation of the assembly process is the "placement" of the contribution of each member to the master stiffness equations. The process is technically called *merge* of individual members. The merge operation can be physically interpreted as *reconnecting* that member in the process of fabricating the complete structure. For a truss structure, reconnection means inserting the pins back into the joints. See Figure 3.1.

Merging logic is mathematically governed by two rules of structural mechanics:

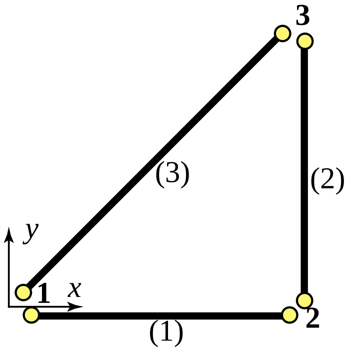

FIGURE 3.1. The disconnected example truss prior to merge. All member stiffness equations are in the global system. Reconnecting the truss means putting the pins back into the joints.

1. *Compatibility of displacements*: The displacement of all members meeting at a joint are the same.
2. *Force equilibrium*: The sum of forces exerted by all members that meet at a joint balances the external force applied to that joint. (3.1)

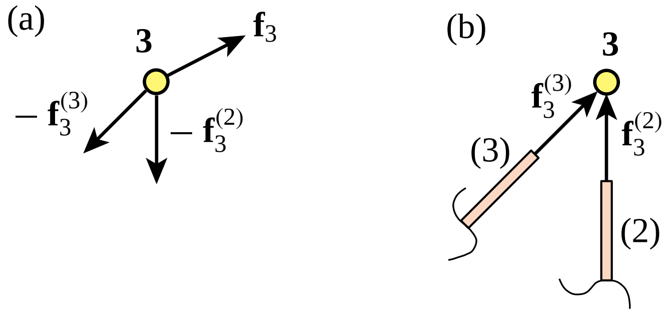

FIGURE 3.2. The force equilibrium of joint 3 of the example truss, depicted as a free body diagram in (a). Here $\mathbf{f}_3$ is the known external joint force applied on the joint. Joint forces $\mathbf{f}_3^{(2)}$ and $\mathbf{f}_3^{(3)}$ are applied *by the joint on the members*, as illustrated in (b). Consequently the forces applied *by the members on the joint* are $-\mathbf{f}_3^{(2)}$ and $-\mathbf{f}_3^{(3)}$. These forces would act in the directions shown in (a) if both members (2) and (3) were in tension. The free-body equilibrium statement is $\mathbf{f}_3 - \mathbf{f}_3^{(2)} - \mathbf{f}_3^{(3)} = \mathbf{0}$ or $\mathbf{f}_3 = \mathbf{f}_3^{(2)} + \mathbf{f}_3^{(3)}$. This translates into the two component equations: $f_{x3} = f_{x3}^{(2)} + f_{x3}^{(3)}$ and $f_{y3} = f_{y3}^{(2)} + f_{y3}^{(3)}$, of (3.2).

The first rule is physically obvious: reconnected joints must move as one entity. The second one can be visualized by considering a joint as a free body, although care is required in the interpretation of joint forces and their signs. Notational conventions to this effect are explained in Figure 3.2 for joint 3 of the example truss, at which members (2) and (3) meet. Application of the foregoing rules at this particular joint gives

$$\text{Rule 1:}\qquad u_{x3}^{(2)} = u_{x3}^{(3)},\quad u_{y3}^{(2)} = u_{y3}^{(3)}.$$

$$\text{Rule 2:}\quad f_{x3} = f_{x3}^{(2)} + f_{x3}^{(3)} = f_{x3}^{(1)} + f_{x3}^{(2)} + f_{x3}^{(3)},\quad f_{y3} = f_{y3}^{(2)} + f_{y3}^{(3)} = f_{y3}^{(1)} + f_{y3}^{(2)} + f_{y3}^{(3)}. \tag{3.2}$$

The addition of $f_{x3}^{(1)}$ to $f_{x3}^{(2)} + f_{x3}^{(3)}$ and of and $f_{y3}^{(1)}$ to $f_{y3}^{(2)} + f_{y3}^{(3)}$, respectively, changes nothing because member (1) is not connected to joint 3. We are just adding zeros. But this augmentation enables us to write the key matrix relation:

$$\boxed{\mathbf{f} = \mathbf{f}^{(1)} + \mathbf{f}^{(2)} + \mathbf{f}^{(3)}.} \tag{3.3}$$

§3.2.2. Hand Assembly by Augmentation and Merge

To directly visualize how the two rules (3.1) translate to merging logic, we first *augment* the member stiffness relations by adding zero rows and columns as appropriate to *complete* the force and displacement vectors.

For member (1):

$$\begin{bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \\ f_{x3}^{(1)} \\ f_{y3}^{(1)} \end{bmatrix} = \begin{bmatrix} 10 & 0 & -10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x1}^{(1)} \\ u_{y1}^{(1)} \\ u_{x2}^{(1)} \\ u_{y2}^{(1)} \\ u_{x3}^{(1)} \\ u_{y3}^{(1)} \end{bmatrix}. \tag{3.4}$$

For member (2):

$$\begin{bmatrix} f^{(2)}_{x1} \\ f^{(2)}_{y1} \\ f^{(2)}_{x2} \\ f^{(2)}_{y2} \\ f^{(2)}_{x3} \\ f^{(2)}_{y3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 & 5 \end{bmatrix} \begin{bmatrix} u^{(2)}_{x1} \\ u^{(2)}_{y1} \\ u^{(2)}_{x2} \\ u^{(2)}_{y2} \\ u^{(2)}_{x3} \\ u^{(2)}_{y3} \end{bmatrix}. \tag{3.5}$$

For member (3):

$$\begin{bmatrix} f^{(3)}_{x1} \\ f^{(3)}_{y1} \\ f^{(3)}_{x2} \\ f^{(3)}_{y2} \\ f^{(3)}_{x3} \\ f^{(3)}_{y3} \end{bmatrix} = \begin{bmatrix} 10 & 10 & 0 & 0 & -10 & -10 \\ 10 & 10 & 0 & 0 & -10 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & 0 & 10 & 10 \end{bmatrix} \begin{bmatrix} u^{(3)}_{x1} \\ u^{(3)}_{y1} \\ u^{(3)}_{x2} \\ u^{(3)}_{y2} \\ u^{(3)}_{x3} \\ u^{(3)}_{y3} \end{bmatrix}. \tag{3.6}$$

According to the first rule, we can *drop the member identifier* in the displacement vectors that appear in the foregoing matrix equations. Hence the reconnected member equations are

$$\begin{bmatrix} f^{(1)}_{x1} \\ f^{(1)}_{y1} \\ f^{(1)}_{x2} \\ f^{(1)}_{y2} \\ f^{(1)}_{x3} \\ f^{(1)}_{y3} \end{bmatrix} = \begin{bmatrix} 10 & 0 & -10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}, \tag{3.7}$$

$$\begin{bmatrix} f^{(2)}_{x1} \\ f^{(2)}_{y1} \\ f^{(2)}_{x2} \\ f^{(2)}_{y2} \\ f^{(2)}_{x3} \\ f^{(2)}_{y3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 & 5 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}, \tag{3.8}$$

$$\begin{bmatrix} f^{(3)}_{x1} \\ f^{(3)}_{y1} \\ f^{(3)}_{x2} \\ f^{(3)}_{y2} \\ f^{(3)}_{x3} \\ f^{(3)}_{y3} \end{bmatrix} = \begin{bmatrix} 10 & 10 & 0 & 0 & -10 & -10 \\ 10 & 10 & 0 & 0 & -10 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & 0 & 10 & 10 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}. \tag{3.9}$$

These three equations can be represented in direct matrix notation as

$$\mathbf{f}^{(1)} = \mathbf{K}^{(1)}\,\mathbf{u}, \qquad \mathbf{f}^{(2)} = \mathbf{K}^{(2)}\,\mathbf{u}, \qquad \mathbf{f}^{(3)} = \mathbf{K}^{(3)}\,\mathbf{u}. \tag{3.10}$$

According to the second rule, expressed in matrix form as (3.3), we have

$$\mathbf{f} = \mathbf{f}^{(1)} + \mathbf{f}^{(2)} + \mathbf{f}^{(3)} = \left(\mathbf{K}^{(1)} + \mathbf{K}^{(2)} + \mathbf{K}^{(3)}\right)\mathbf{u} = \mathbf{K}\,\mathbf{u}, \tag{3.11}$$

so all we have to do is add the three stiffness matrices that appear above, and we arrive at the master stiffness equations:

$$\boxed{\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix} = \begin{bmatrix} 20 & 10 & -10 & 0 & -10 & -10 \\ 10 & 10 & 0 & 0 & -10 & -10 \\ -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & -5 & 10 & 15 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}.} \tag{3.12}$$

Using this technique *member merging* becomes simply *matrix addition*.

This explanation of the assembly process is conceptually the easiest to follow and understand. It is virtually foolproof for hand computations. However, this is *not* the way the process is carried out on the computer because it would be enormously wasteful of storage for large systems. A computer-oriented procedure is discussed in §3.5.

§3.3. Solution

Having formed the master stiffness equations we can proceed to the solution phase. To prepare the equations for a linear solver we need to separate known and unknown components of **f** and **u**. In this Section a technique suitable for hand computation is described.

§3.3.1. Applying Displacement BCs by Reduction

If one attempts to solve the system (3.12) numerically for the displacements, surprise! The solution "blows up" because the coefficient matrix (the master stiffness matrix) is singular. The mathematical interpretation of this behavior is that rows and columns of **K** are linear combinations of each other (see Remark 3.1 below). The physical interpretation of singularity is that there are unsuppressed *rigid body motions*: the truss still "floats" in the $\{x, y\}$ plane.

To eliminate rigid body motions and render the system nonsingular we must apply the physical *support conditions* as *displacement boundary conditions*. From Figure 2.4(b) we observe that the support conditions for the example truss are

$$u_{x1} = u_{y1} = u_{y2} = 0, \tag{3.13}$$

whereas the known applied forces are

$$f_{x2} = 0, \quad f_{x3} = 2, \quad f_{y3} = 1. \tag{3.14}$$

When solving the overall stiffness equations by hand, the simplest way to account for support conditions is to *remove* equations associated with known joint displacements from the master system. To apply (3.13) we have to remove equations 1, 2 and 4. This can be systematically accomplished by *deleting* or “striking out” rows and columns number 1, 2 and 4 from **K** and the corresponding components from **f** and **u**. The reduced three-equation system is

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 10 \\ 0 & 10 & 15 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} f_{x2} \\ f_{x3} \\ f_{y3} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}. \tag{3.15}$$

Equation (3.15) is called the *reduced master stiffness system*. The coefficient matrix of this system is no longer singular.

Remark 3.1. In mathematical terms, the free-free master stiffness matrix **K** in (3.12) has order $N = 6$, rank $r = 3$ and a rank deficiency of $d = N - r = 6 - 3 = 3$ (these concepts are summarized in Appendix C.) The dimension of the null space of **K** is $d = 3$. This space is spanned by three independent rigid body motions: the two rigid translations along x and y and the rigid rotation about z.

Remark 3.2. Conditions (3.13) represent the simplest type of support conditions, namely specified zero displacements. More general constraint forms, such as prescribed nonzero displacements and multifreedom constraints, are handled as described in §3.6 and Chapters 8–9, respectively.

§3.3.2. Solving for Displacements

Solving the reduced system by hand (for example, via Gauss elimination) yields

$$\begin{bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.4 \\ -0.2 \end{bmatrix}. \tag{3.16}$$

This is called a *partial displacement solution* (also *reduced displacement solution*) because it excludes known displacement components. This solution vector is *expanded* to six components by including the three specified values (3.13) in the appropiate slots:

$$\mathbf{u} = \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.4 \\ -0.2 \end{bmatrix}. \tag{3.17}$$

This is the *complete displacement solution*, or simply the *displacement solution*.

§3.4. PostProcessing

The last processing step of the DSM is the solution for joint displacements. But often the analyst needs information on other mechanical quantities; for example the reaction forces at the supports, or the internal member forces. Such quantities are said to be *derived* because they are *recovered* from the displacement solution. The recovery of derived quantities is part of the so-called *postprocessing steps* of the DSM. Two such steps are described below.

§3.4.1. Recovery of Reaction Forces

Premultiplying the complete displacement solution (3.17) by **K** we get

$$\mathbf{f}=\mathbf{K}\mathbf{u}=\begin{bmatrix}20&10&-10&0&-10&-10\\10&10&0&0&-10&-10\\-10&0&10&0&0&0\\0&0&0&5&0&-5\\-10&-10&0&0&10&10\\-10&-10&0&-5&10&15\end{bmatrix}\begin{bmatrix}0\\0\\0\\0\\0.4\\-0.2\end{bmatrix}=\begin{bmatrix}-2\\-2\\0\\1\\2\\1\end{bmatrix}\tag{3.18}$$

This vector recovers the known applied forces (3.14) as can be expected. Furthermore we get three *reaction forces*: $f_{x1}=f_{y1}=-2$ and $f_{y2}=1$, which are associated with the support conditions (3.13). It is easy to check that the complete force system is in self equilibrium for the free-free structure; this is the topic of Exercise 3.1.

§3.4.2. Recovery of Internal Forces and Stresses

Often the structural engineer is not so much interested in displacements as in *internal forces* and *stresses*. These are in fact the most important quantities for preliminary structural design. In pin-jointed trusses the only internal forces are the *axial member forces*. For the example truss these forces, denoted by $F^{(1)}$, $F^{(2)}$ and $F^{(3)}$, are depicted in Figure 3.3. The average axial stress σ^e is obtained on dividing F^e by the cross-sectional area of the member.

The axial force F^e in member e can be obtained as follows. Extract the displacements of member e from the complete displacement solution $\mathbf{u}$ to form $\mathbf{u}^e$. Then recover local joint displacements from $\bar{\mathbf{u}}^e=\mathbf{T}^e\mathbf{u}^e$.

Compute the member elongation d^e (relative displacement) and recover the axial force from the equivalent spring constitutive relation:

$$d^e=\bar{u}^e_{xj}-\bar{u}^e_{xi},\qquad F^e=\frac{E^eA^e}{L^e}d^e.\tag{3.19}$$

Note that $\bar{u}^e_{yi}$ and $\bar{u}^e_{yj}$ are not needed in computing d^e.

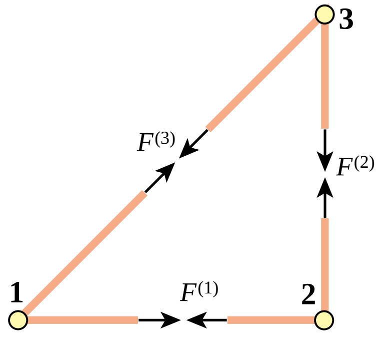

FIGURE 3.3. The internal forces in the example truss are the axial forces $F^{(1)}$, $F^{(2)}$ and $F^{(3)}$ in the members. Directions shown pertain to tension.

Example 3.1. Recover $F^{(2)}$ in example truss. Member (2) goes from node 2 to node 3 and $\varphi^{(2)}=90^\circ$. Extract the global displacements of the member from (3.17): $\mathbf{u}^{(2)}=[\,u_{x2}\;\;u_{y2}\;\;u_{x3}\;\;u_{y3}\,]^T=[\,0\;\;0\;\;0.4\;\;0.2\,]^T$. Convert to local displacements using $\bar{\mathbf{u}}^{(2)}=\mathbf{T}^{(2)}\mathbf{u}^{(2)}$:

$$\begin{bmatrix}\bar{u}_{x2}\\\bar{u}_{y2}\\\bar{u}_{x3}\\\bar{u}_{y3}\end{bmatrix}=\begin{bmatrix}\cos 90^\circ&\sin 90^\circ&0&0\\-\sin 90^\circ&\cos 90^\circ&0&0\\0&0&\cos 90^\circ&\sin 90^\circ\\0&0&-\sin 90^\circ&\cos 90^\circ\end{bmatrix}\begin{bmatrix}u_{x2}\\u_{y2}\\u_{x3}\\u_{y3}\end{bmatrix}=\begin{bmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{bmatrix}\begin{bmatrix}0\\0\\0.4\\-0.2\end{bmatrix}=\begin{bmatrix}0\\0\\-0.2\\-0.4\end{bmatrix}.\tag{3.20}$$

Then $d^{(2)}=\bar{u}_{x3}-\bar{u}_{x2}=-0.2-0=-0.2$, and $F^{(2)}=(50/10)\times-0.2=-1$ (compression).

Remark 3.3. An alternative interpretation of (3.19) is to regard $e^e=d^e/L^e$ as the (average) member axial strain, $\sigma^e=E^ee^e$ as (average) axial stress, and $F^e=A^e\sigma^e$ as the axial force. This is more in tune with the Theory of Elasticity viewpoint discussed in Exercise 2.6.

§3.5. *Computer Oriented Assembly and Solution

§3.5.1. *Assembly by Freedom Pointers

The practical computer implementation of the DSM assembly process departs significantly from the "augment and add" technique described in §3.1.4. There are two major differences:

(I) Member stiffness matrices are *not* expanded. Their entries are directly merged into those of **K** through the use of a "freedom pointer array" called the *Element Freedom Table* or EFT.

(II) The master stiffness matrix **K** is stored using a special format that takes advantage of symmetry and sparseness.

Difference (II) is a more advanced topic that is deferred to the last part of the book. For simplicity we shall assume here that **K** is stored as a *full square matrix*, and study only (I). For the example truss the freedom-pointer technique expresses the entries of **K** as the sum

$$K_{pq} = \sum_{e=1}^{3} K^e_{ij} \qquad \text{for} \quad i = 1, \ldots 4, \;\; j = 1, \ldots 4, \;\; p = \text{EFT}^e(i), \;\; q = \text{EFT}^e(j). \tag{3.21}$$

Here K^e_{ij} denote the entries of the 4×4 globalized member stiffness matrices in (2.19) through (2.21). Entries K_{pq} that do not get any contributions from the right hand side remain zero. EFT^e denotes the Element Freedom Table for member e. For the example truss these tables are

$$\text{EFT}^{(1)} = \{1, 2, 3, 4\}, \qquad \text{EFT}^{(2)} = \{3, 4, 5, 6\}, \qquad \text{EFT}^{(3)} = \{1, 2, 5, 6\}. \tag{3.22}$$

Physically these tables map local freedom indices to global ones. For example, freedom number 3 of member (2) is u_{x3}, which is number 5 in the master equations; consequently EFT$^{(2)}(3) = 5$. Note that (3.21) involves three nested loops: over e (outermost), over i, and over j. The ordering of the last two is irrelevant. Advantage may be taken of the symmetry of $\mathbf{K}^e$ and **K** to roughly halve the number of additions. Exercise 3.5 follows the scheme (3.21) by hand.

The assembly process for general structures using this technique is studied in Chapter 25.

§3.5.2. *Applying DBC by Modification

In §3.3.1 the support conditions (3.13) were applied by reducing (3.12) to (3.15). Reduction is convenient for hand computations because it cuts down on the number of equations to solve. But it has a serious flaw for computer implementation: the equations must be rearranged. It was previously noted that on the computer the number of equations is not the only important consideration. Rearrangement can be as or more expensive than solving the equations, particularly if the coefficient matrix is stored in sparse form or on secondary storage.

To apply support conditions without rearranging the equations we clear (set to zero) rows and columns corresponding to prescribed zero displacements as well as the corresponding force components, and place ones on the diagonal to maintain non-singularity. The resulting system is called the *modified* set of master stiffness equations. For the example truss this approach yields

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 & 10 \\ 0 & 0 & 0 & 0 & 10 & 15 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \tag{3.23}$$

in which rows and columns for equations 1, 2 and 4 have been cleared. Solving this modified system produces the complete displacement solution (3.17) directly.

Remark 3.4. In a “smart” stiffness equation solver the modified system need not be explicitly constructed by storing zeros and ones. It is sufficient to *mark* the equations that correspond to displacement BCs. The solver is then programmed to skip those equations. However, if one is using a standard solver from, say, a library of scientific routines or a commercial program such as *Matlab* or *Mathematica*, such intelligence cannot be expected, and the modified system must be set up explicitly.

§3.6. Prescribed Nonzero Displacements

The support conditions considered in the example truss resulted in the specification of zero displacement components; for example $u_{y2} = 0$. There are cases, however, where the known value is nonzero. This happens, for example, in the study of settlement of foundations of ground structures such as buildings and bridges, and in the analysis of motion-driven machinery components.

Mathematically these are called non-homogenous boundary conditions. The treatment of this generalization of the FEM equations is studied in the following subsections.

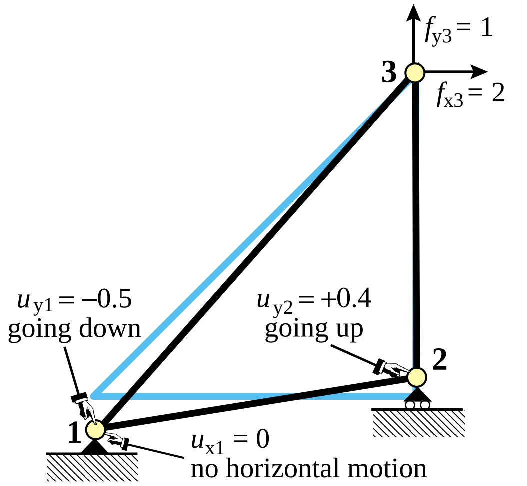

FIGURE 3.4. The example truss with prescribed nonzero vertical displacements at joints 1 and 2.

§3.6.1. Application of Nonzero-DBCs by Reduction

We describe first a matrix reduction technique, analogous to that used in §3.3.1, which is suitable for hand computations. Recall the master stiffness equations (3.12) for the example truss:

$$\begin{bmatrix} 20 & 10 & -10 & 0 & -10 & -10 \\ 10 & 10 & 0 & 0 & -10 & -10 \\ -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & -5 & 10 & 15 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix} \tag{3.24}$$

Suppose that the applied forces are again (3.14) but the prescribed displacements change to

$$u_{x1} = 0, \quad u_{y1} = -0.5, \quad u_{y2} = 0.4 \tag{3.25}$$

This means that joint 1 goes down vertically whereas joint 2 goes up vertically, as depicted in Figure 3.4. Inserting the known data into (3.24) we get

$$\begin{bmatrix} 20 & 10 & -10 & 0 & -10 & -10 \\ 10 & 10 & 0 & 0 & -10 & -10 \\ -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & -5 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & -5 & 10 & 15 \end{bmatrix} \begin{bmatrix} 0 \\ -0.5 \\ u_{x2} \\ 0.4 \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} f_{x1} \\ f_{y1} \\ 0 \\ f_{y2} \\ 2 \\ 1 \end{bmatrix} \tag{3.26}$$

The first, second and fourth rows of (3.26) are removed, leaving only

$$\begin{bmatrix} -10 & 0 & 10 & 0 & 0 & 0 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & -5 & 10 & 15 \end{bmatrix} \begin{bmatrix} 0 \\ -0.5 \\ u_{x2} \\ 0.4 \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \tag{3.27}$$

Columns 1, 2 and 4 are removed by transferring all known terms from the left to the right hand side:

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 10 \\ 0 & 10 & 15 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} (-10) \times 0 + 0 \times (-0.5) + 0 \times 0.4 \\ (-10) \times 0 + (-10) \times (-0.5) + 0 \times 0.4 \\ (-10) \times 0 + (-10) \times (-0.5) + (-5) \times 0.4 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ -2 \end{bmatrix}. \tag{3.28}$$

These are the *reduced stiffness equations*. Note that its coefficient matrix of (3.28) is exactly the same as in the reduced system (3.15) for prescribed zero displacements. The right hand side, however, is different. It consists of the applied joint forces *modified by the effect of known nonzero displacements*. These are called the *modified node forces* or *effective node forces*. Solving the reduced system yields

$$\begin{bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} 0 \\ -0.5 \\ 0.2 \end{bmatrix}. \tag{3.29}$$

Filling the missing entries with the known values (3.25) yields the complete displacement solution (listed as row vector to save space):

$$\mathbf{u} = [\,0 \quad -0.5 \quad 0 \quad 0.4 \quad -0.5 \quad 0.2\,]^T. \tag{3.30}$$

Taking the solution (3.30) and going through the postprocessing steps discussed in §3.4, we can find that *reaction forces and internal member forces do not change*. This is a consequence of the fact that the example truss is *statically determinate*. The force systems (internal and external) in such structures are insensitive to movements such as foundation settlements.

§3.6.2. *Application of Nonzero-DBCs by Modification

The computer-oriented modifification approach follows the same idea outlined in §3.5.2. As there, the main objective is to avoid rearranging the master stiffness equations. To understand the process it is useful to think of being done in two stages. First equations 1, 2 and 4 are modified so that they become trivial equations, as illustrated for the example truss and the displacement boundary conditions (3.25):

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -10 & -10 & 0 & 0 & 10 & 10 \\ -10 & -10 & 0 & -5 & 10 & 15 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{x2} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} 0 \\ -0.5 \\ 0 \\ 0.4 \\ 2 \\ 1 \end{bmatrix} \tag{3.31}$$

The solution of this system recovers (3.26) by construction (for example, the fourth equation is simply $1 \times u_{y2} = 0.4$). In the next stage, columns 1, 2 and 4 of the coefficient matrix are cleared by transferring all known terms to the right hand side, following the same procedure explained in (3.29). We thus arrive at

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 & 10 \\ 0 & 0 & 0 & 0 & 10 & 15 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{x2} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} 0 \\ -0.5 \\ 0 \\ 0.4 \\ -3 \\ -2 \end{bmatrix} \tag{3.32}$$

As before, these are called the *modified master stiffness equations*. Note that (3.32) retains the original number and order as well as matrix symmetry. Solving this system yields the complete displacement solution (3.30).

If all prescribed displacements are zero, forces on the right hand side are not modified, and one would get (3.23) as may be expected.

Remark 3.5. The modification is not actually programmed as discussed above. First the applied forces in the right-hand side are modified for the effect of nonzero prescribed displacements, and the prescribed displacements stored in the reaction-force slots. This is called the *force modification* step. Second, rows and columns of the stiffness matrix are cleared as appropriate and ones stored in the diagonal positions. This is called the *stiffness modification* step. It is essential that the procedural steps be executed in the indicated order, because stiffness terms must be used to modify forces before they are zeroed out.

§3.6.3. *Matrix Forms of Nonzero-DBC Application Methods

The reduction and modification techniques for applying DBCs can be presented in compact matrix form. First, the free-free master stiffness equations $\mathbf{K}\mathbf{u} = \mathbf{f}$ are partitioned as follows:

$$\begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix}. \tag{3.33}$$

In this matrix equation, subvectors $\mathbf{u}_2$ and $\mathbf{f}_1$ collect displacement and force components, respectively, that are *known*, *given* or *prescribed*. Subvectors $\mathbf{u}_1$ and $\mathbf{f}_2$ collect force and displacement components, respectively, that are *unknown*. Forces in $\mathbf{f}_2$ represent reactions on supports; consequently $\mathbf{f}_2$ is called the *reaction vector*.

On transferring the known terms to the right hand side the first matrix equation becomes

$$\mathbf{K}_{11}\mathbf{u}_1 = \mathbf{f}_1 - \mathbf{K}_{12}\mathbf{u}_2. \tag{3.34}$$

This is the *reduced master equation system*. If the support B.C.s are homogeneous (that is, all prescribed displacements are zero), $\mathbf{u}_2 = \mathbf{0}$, and we do not need to change the right-hand side:

$$\mathbf{K}_{11}\mathbf{u}_1 = \mathbf{f}_1. \tag{3.35}$$

Examples that illustrate (3.34) and (3.35) are (3.28) and (2.23), respectively.

The computer-oriented modification technique retains the same joint displacement vector as in (3.34) through the following rearrangement:

$$\begin{bmatrix} \mathbf{K}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 - \mathbf{K}_{12}\mathbf{u}_2 \\ \mathbf{u}_2 \end{bmatrix}. \tag{3.36}$$

This *modified system* is simply the reduced equation (3.35) augmented by the trivial equation $\mathbf{I}\mathbf{u}_2 = \mathbf{u}_2$. This system is often denoted as

$$\widehat{\mathbf{K}}\mathbf{u} = \hat{\mathbf{f}}. \tag{3.37}$$

Solving (3.37) yields the complete displacement solution including the specified displacements $\mathbf{u}_2$.

For the computer implementation it is important to note that the partitioned form (3.33) is only used to permit the use of compact matrix notation. In actual programming the equations are *not* explicitly rearranged: they retain their original numbers. For instance, in the example truss

$$\mathbf{u}_1 = \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{y2} \end{bmatrix} \equiv \begin{bmatrix} \text{DOF \#1} \\ \text{DOF \#2} \\ \text{DOF \#4} \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} u_{x2} \\ u_{x3} \\ u_{y3} \end{bmatrix} \equiv \begin{bmatrix} \text{DOF \#3} \\ \text{DOF \#5} \\ \text{DOF \#6} \end{bmatrix}. \tag{3.38}$$

The example shows that $\mathbf{u}_1$ and $\mathbf{u}_2$ are generally interspersed throughout $\mathbf{u}$. Thus, matrix operations such as $\mathbf{K}_{12}\mathbf{u}_2$ involve indirect (pointer) addressing so as to avoid explicit array rearrangement.

Notes and Bibliography

The coverage of the assembly and solution steps of the DSM, along with globalization and application of BCs, is not uniform across the wide spectrum of FEM books. Authors have introduced "quirks" although the overall concepts are not affected. The most common variations arise in two contexts:

(1) Some treatments apply support conditions *during* merge, explicitly eliminating known displacement freedoms as the elements are processed and merged into **K**. The output of the assembly process is what is called here a reduced stiffness matrix.[1]

(2) In the *frontal solution method* of Irons [102,103], assembly and solution are done concurrently. More precisely, as elements are formed and merged, displacement boundary conditions are applied, and Gauss elimination and reduction of the right hand side starts once the assembler senses (by tracking an "element wavefront") that no more elements contribute to a certain node.

Both variants appeared in FEM programs written during the 1960s and 1970s. They were motivated by computer resource limitations of the time: memory was scarce and computing time expensive.[2] On the negative side, interweaving leads to unmodular programming (which easily becomes "spaghetti code" in low-level languages such as Fortran). Since a frontal solver has to access the element library, which is typically the largest component of a general-purpose FEM program, it has to know how to pass and receive information about each element. A minor change deep down the element library can propagate and break the solver.

Squeezing storage and CPU savings on present computers is of less significance. Modularity, which simplifies scripting in higher order languages such as *Matlab* is desirable because it increases "plug-in" operational flexibility, allows the use of built-in solvers, and reduces the chance for errors. These changes reflect economic reality: human time is nowadays far more expensive than computer time.

A side benefit of modular assembly-solution separation is that often the master stiffness must be used in a different way than just solving $\mathbf{Ku} = \mathbf{f}$; for example in dynamics, vibration or stability analysis. Or as input to a model reduction process. In those cases the solution stage can wait.

Both the hand-oriented and computer-oriented application of boundary conditions have been presented here, although the latter is still considered an advanced topic. While hand computations become unfeasible beyond fairly trivial models, they are important from a instructional standpoint.

The augment-and-add procedure for hand assembly of the master stiffness matrix is due to H. Martin [116].

References

Referenced items have been moved to Appendix R.

[1] For the example truss, the coefficient matrix in (3.15) is a reduced stiffness whereas that in (3.23) is a modified one.

[2] As an illustration, the first computer used by the writer, the "classical mainframe" IBM 7094, had a magnetic-core memory of 32,768 36-bit words ($\approx$ 0.2 MB), and was as fast as an IBM PC of the mid 1980s. One mainframe had to serve the whole Berkeley campus, and Ph.D. students were allocated 2 CPU hours per semester. Getting a moderately complex FE model through involved heavy use of slower secondary storage such as disk or tape in batch jobs.

Homework Exercises for Chapter 3
The Direct Stiffness Method II

EXERCISE 3.1 [A:15] Draw a free body diagram of the nodal forces (3.18) acting on the free-free truss structure, and verify that this force system satisfies translational and rotational (moment) equilibrium.

EXERCISE 3.2 [A:15] Using the method presented in §3.4.2 compute the axial forces in the three members of the example truss. Partial answer: $F^{(3)} = 2\sqrt{2}$.

EXERCISE 3.3 [A:20] Describe an alternative method that recovers the axial member forces of the example truss from consideration of joint equilibrium, without going through the computation of member deformations. Can this method be extended to arbitrary trusses?

EXERCISE 3.4 [A:20] Suppose that the third support condition in (3.13) is $u_{x2} = 0$ instead of $u_{y2} = 0$. Rederive the reduced system (3.15) for this case. Verify that this system cannot be solved for the joint displacements u_{y2}, u_{x3} and u_{y3} because the reduced stiffness matrix is singular.[3] Offer a physical interpretation of this failure.

EXERCISE 3.5 [N:20] Construct by hand the free-free master stiffness matrix of (3.12) using the freedom-pointer technique (3.21). Note: start from **K** initialized to the null matrix, then cycle over $e = 1, 2, 3$.

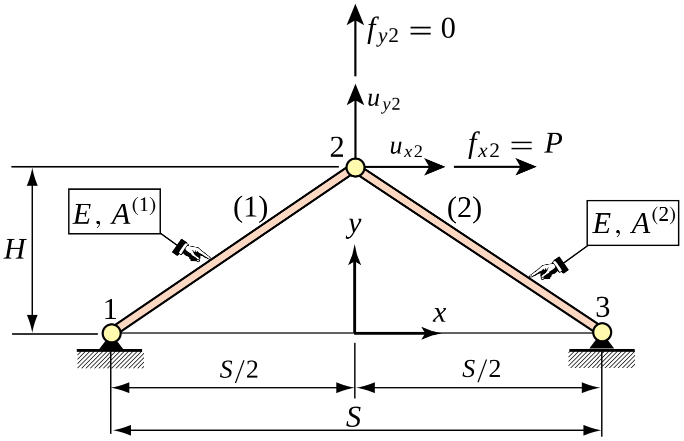

FIGURE E3.1. Truss structure for Exercise 3.7.

EXERCISE 3.6 [N:25] Consider the two-member arch-truss structure shown in Figure E3.1. Take span $S = 8$, height $H = 3$, elastic modulus $E = 1000$, cross section areas $A^{(1)} = 2$ and $A^{(2)} = 4$, and horizontal crown force $P = f_{x2} = 12$. Using the DSM carry out the following steps:

(a) Assemble the master stiffness equations. Any method: augment-and-add, or the more advanced “freedom pointer” technique explained in §3.5.1, is acceptable.

(b) Apply the displacement BCs and solve the reduced system for the crown displacements u_{x2} and u_{y2}. Partial result: $u_{x2} = 9/512 = 0.01758$.

(c) Recover the node forces at all joints including reactions. Verify that overall force equilibrium (x forces, y forces, and moments about any point) is satisfied.

(d) Recover the axial forces in the two members. Result should be $F^{(1)} = -F^{(2)} = 15/2$.

[3] A matrix is singular if its determinant is zero; cf. §C.2 of Appendix C for a “refresher” in that topic.

EXERCISE 3.7 [N:20] Resolve items (a) through (c) — omitting (d) — of the problem of Exercise 3.7 if the vertical right support "sinks" so that the displacement u_{y3} is now prescribed to be -0.5. Everything else is the same. Use the matrix reduction scheme of §3.6.1 to apply the displacement BCs.