7

Review of Continuum Mechanics

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§7.1. The FEM Discretization

In Chapters 3 through 6 we have studied some general properties of the governing force-equilibrium equations of geometrically nonlinear structural systems with *finite* number of degrees of freedom (DOF). The DOFs are collected in the state variable vector $\mathbf{u}$. Those residual equations, being algebraic, are well suited for numerical computation.

Mathematical models of real structures, however, possess an *infinite* number of DOFs. As such they cannot be handled by numerical computations. The reduction to a finite number is accomplished by *discretization* methods. As noted in Chapter 1, for nonlinear problems in solid and structural mechanics the *finite element method* (FEM) is the most widely used discretization method.

This section provides background material for the derivation of geometrically nonlinear finite elements. The material gives a review of kinematic, kinetic and constitutive concepts from the three-dimensional continuum mechanics of an elastic deformable body, as needed in following Chapters. Readers familiar with continuum mechanics should peruse it to grab notation.

§7.2. Notation: Continuum vs. Discrete Mechanics

Continuum mechanics deals with vector and tensor fields such as displacements, strains and stresses. Four types of notation are in common use:

1. *Indicial Notation*. Also called *component notation*. This notation uses indexed components along with abbreviation rules such as commas for partial derivatives and Einstein's summation convention. It is a powerful notation, and as such is preferred in journals and monographs. It has the advantage of readily handling arbitrary tensors of any order, arbitrary coordinate systems and nonlinear relations. It sharply distinguishes between covariant and contravariant quantities, which is necessary in non-Cartesian coordinates. Tends to conceal or mask intrinsic properties, however, and as such is not suitable for basic instruction.

2. *Direct Notation*. Sometimes called *algebraic notation*. Vectors and tensors are represented by single symbols, usually bold letters. Has the advantage of compactness and quick visualization of intrinsic properties. Some operations correspond to matrix notations while others do not. This fuzzy overlap can lead to confusion in FEM work.

3. *Matrix Notation*. This is similar to the previous one, but entities are rearranged as appropriate so that only matrix operations are used. It can be translated directly to discrete equations as well as matrix-oriented programming languages such as *Matlab*. It has the disadvantage of losing contact with the original physical entities along the way. For example, stress is a symmetric second-order tensor that is rearranged as a 6-component vector for FEM developments. This change loses essential properties. For instance it makes sense to talk of principal stresses as eigenvalues of the stress tensor. But those get lost (as or least moved to the background) when rearranged as a stress vector.

4. *Full Notation*. In the full-form notation every term is spelled out. No ambiguities of interpretation can arise; consequently this works well as a notation of last resort, and also as a "comparison template" against one can check out the meaning of more compact expressions. It is also useful for programming in low-order languages.

As an example, consider the well known dot product between two physical vectors in 3D space, $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ done in the four different notations:

$$\underbrace{a_i b_i}_{\text{indicial}} = \underbrace{\mathbf{a}.\mathbf{b}}_{\text{direct}} = \underbrace{\mathbf{a}^T\mathbf{b}}_{\text{matrix}} = \underbrace{a_1 b_1 + a_2 b_2 + a_3 b_3}_{\text{full}}. \tag{7.1}$$

In the following review the direct, matrix and full notation are preferred, whereas the indicial notation is avoided. Usually the expression is first given in direct form and confirmed by full form if feasible. Then it is transformed to matrix notation for later used in FEM developments. The decision leads to possible ambiguities against reuse of vector symbols in two contexts: continuum mechanics and FEM discretizations. Such ambiguities are resolved in favor of keeping FEM notation simple.

§7.3. Particles, Motions, Displacements, Configurations

In the present section a structure is mathematically treated as a continuum body $\mathcal{B}$. The body is considered as being formed by a set of points P called *particles*, which are endowed with certain mechanical properties. For FEM analysis the body is divided into elements.

Particles displace or move in response to external actions characterized by control parameters Λ_i or, following the stage reduction discussed in Chapter 3, the single stage parameter λ. A one-parameter series of positions occupied by the particles as they move in space is called a *motion*. The motion may be described by the *displacement* $\underline{\mathbf{u}}(P) \equiv \underline{\mathbf{u}}(\mathbf{x})$ of the particles with respect to a *base* or *reference state* in which particle P is labelled P_0. [The underlining is used to distinguish the physical displacement vector from the finite element node displacement array, which is a computational vector.]

The displacements of all particles $\underline{\mathbf{u}}(x, y, z)$ such that $\mathbf{x} \equiv x, y, z \in \mathcal{B}$, constitutes the *displacement field*.

The motion is said to be *kinematically admissible* if:

1. Continuity of particles positions is preserved so that no gaps or voids appear. (The mathematical statement of this condition is given later.)

2. Kinematic constraints on the motion (for example, support conditions) are preserved.

A kinematically admissible motion along a stage will be called a *staged motion*. For one such motion the displacements $\underline{\mathbf{u}}(\mathbf{x})$ characterize the *state* and the stage control parameter λ characterizes the *control* or *action*. Both will be generally parametrized by the pseudo-time t introduced in Chapter 3, so that a staged motion can be generally represented by

$$\boxed{\lambda = \lambda(t), \qquad \underline{\mathbf{u}} = \underline{\mathbf{u}}(\mathbf{x}, t), \quad \mathbf{x} \in \mathcal{B}.} \tag{7.2}$$

If in these equations we freeze t, we have a *configuration* of the structure. Thus a configuration is formally the union of state and control. It may be informally viewed as a "snapshot" taken of the structure and actions upon it when the pseudotime is frozen. If the configuration satisfies the equilibrium equations, it is called an *equilibrium* configuration. In general, however, a randomly given configuration is not in equilibrium unless artificial body and surface forces are applied to it.

A *staged response*, or simply *response*, can be now mathematically defined as a series of equilibrium configurations obtained as λ is continuously varied.

§7.3.1. Distinguished Configurations

A particular characteristic of geometrically nonlinear analysis is the need to carefully distinguish among different configurations of the structure.

As noted above, each set of kinematically admissible displacements $\underline{\mathbf{u}}(\mathbf{x})$ plus a staged control parameter λ defines a configuration. This is not necessarily an equilibrium configuration; in fact it will not usually

Table 7.1 Distinguished Configurations in Nonlinear Analysis

Name	*Alias*	*Definition*	*Equilibrium Required?*	*Identification*
Admissible		A kinematically admissible configuration	No	$\mathcal{C}$
Perturbed		Kinematically admissible variation of an admissible configuration.	No	$\mathcal{C} + \delta\mathcal{C}$
Deformed	Current Spatial	Actual configuration taken during the analysis process. Contains others as special cases.	No	$\mathcal{C}^D$ or $\mathcal{C}(t)$
Base*	Initial Undeformed Material	The configuration defined as the origin of displacements. Strain free but not necessarily stress free.	Yes	$\mathcal{C}^0$, $\mathcal{C}^B$ or $\mathcal{C}(0)$
Reference		Configuration to which stepping computations are referred	TL,UL: Yes. CR: $\mathcal{C}^R$ no, $\mathcal{C}^0$ yes	TL: $\mathcal{C}^0$, UL: $\mathcal{C}^{n-1}$, CR: $\mathcal{C}^R$ and $\mathcal{C}^0$
Iterated†		Configuration taken at the k^{th} iteration of the n^{th} increment step	No	$\mathcal{C}^n_k$
Target†		Equilibrium configuration accepted at the n^{th} increment step	Yes	$\mathcal{C}^n$
Corotated‡	Shadow Ghost	Body or element-attached configuration obtained from $\mathcal{C}^0$ through a rigid body motion (CR description only)	No	$\mathcal{C}^R$
Aligned	Preferred Directed	A fictitious body or element configuration aligned with a particular set of axes (usually global axes)	No	$\mathcal{C}^A$

* $\mathcal{C}^0$ is often the same as the *natural state* in which body (or element) is undeformed and stress-free.
† Used only in the description of solution procedures.
‡ In dynamic analysis $\mathcal{C}^0$ and $\mathcal{C}^R$ are called the inertial and dynamic-reference configurations, respectively, when they apply to the entire structure.

be one. It is also important to realize that an equilibrium configuration is not necessarily a physical configuration assumed by the actual structure.[1]

Some configurations that are important in geometrically nonlinear analysis receive special qualifiers:

admissible, perturbed, deformed, base, reference, iterated, target, corotated, aligned

This terminology is collected in Table 7.1.

Remark 7.1. A great number of names can be found for these configurations in the literature in finite elements and continuum mechanics. To further compound the confusion, here are some of these alternative names.

Perturbed configuration: *adjacent, deviated, disturbed, incremented, neighboring, varied, virtual*.

[1] Recall the suspension bridge under zero gravity of Chapter 3.

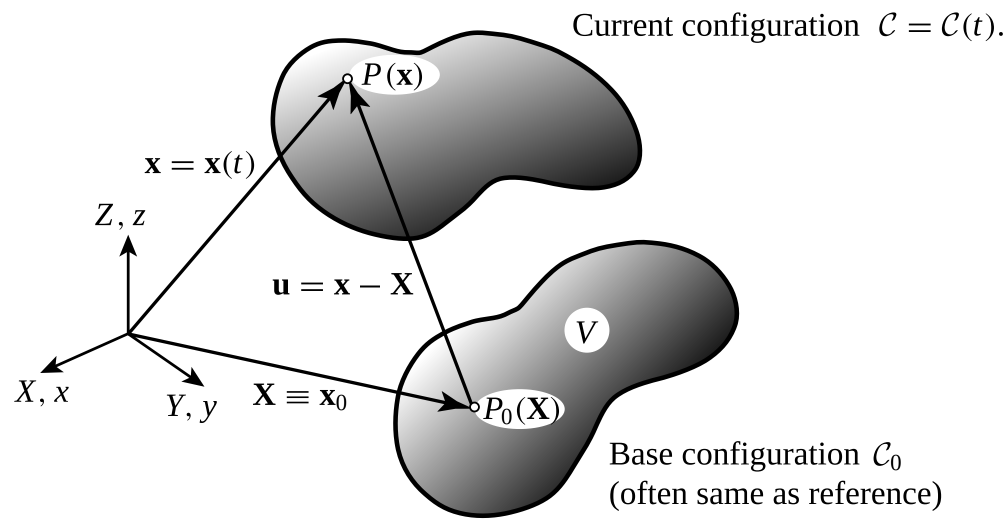

Figure 7.1. The geometrically nonlinear problem in a Lagrangian kinematic description: coordinate systems, reference and current configurations, and displacements. For many (but not all) problems, the base configuration, which is not shown separately in the Figure, would be the same as the reference configuration.

Deformed configuration: *arbitrary*, *current*, *distorted*, *moving*, *present*, *spatial*, *varying*.

Base configuration: *baseline*, *initial*, *material*, *global*, *natural*, *original*, *overall*, *undeformed*, *undistorted*.

Reference configuration: *fixed*, *frozen*, *known*.

Iterated configuration: *intermediate*, *stepped*.

Target configuration: *converged*, *equilibrated spatial*, *unknown*.

Corotated configuration: *attached*, *convected*, *ghost*, *phantom*, *shadow*.

Aligned configuration: *directed*, *preferred*.

In FEM treatments of nonlinear analysis, confusion often reigns supreme. A common scenario is to identify base and reference configurations in Total Lagrangian descriptions.

§7.3.2. Kinematic Descriptions

Three kinematic descriptions of geometrically nonlinear finite element analysis are in current use in programs that solve nonlinear structural problems. They can be distinguished by the choice of reference configuration.

1. *Total Lagrangian description* (TL). The reference configuration is seldom or never changed: often it is kept equal to the base configuration throughout the analysis. Strains and stresses are measured with respect to this configuration.

2. *Updated Lagrangian description* (UL). The last target configuration, once reached, becomes the next reference configuration. Strains and stresses are redefined as soon as the reference configuration is updated.

3. *Corotational description* (CR). The reference configuration is "split." Strains and stresses are measured from the corotated configuration whereas the base configuration is maintained as reference for measuring rigid body motions.

Remark 7.2. The TL formulation remains the most widely used in continuum-based finite element codes. The CR formulation is gaining in popularity for structural elements such as beams, plates and shells. The UL formulation is primarily used in treatments of vary large strains and flow-like behavior.

§7.3.3. Coordinate Systems

Configurations taken by a body or element during the response analysis are linked by a Cartesian *global frame*, to which all computations are ultimately referred.[2] There are actually two such frames:

(i) The *material global frame* with axes $\{X_i\}$ or $\{X, Y, Z\}$.

(ii) The *spatial global frame* with axes $\{x_i\}$ or $\{x, y, z\}$.[3]

The material frame tracks the base configuration whereas the spatial frame tracks all others. This distinction agrees with the usual conventions of classical continuum mechanics. In the present work both frames are taken to be *identical*, as nothing is gained by separating them. Thus only one set of global axes, with dual labels, is drawn in Figure 7.1.

In stark contrast to global frame uniqueness, the presence of elements means there are many *local frames* to keep track of. More precisely, each element is endowed with two local Cartesian frames:

(iii) The element *base frame* with axes $\{\tilde{X}_i\}$ or $\{\tilde{X}, \tilde{Y}, \tilde{Z}\}$.

(iv) The element *reference frame* with axes $\{\bar{x}_i\}$ or $\{\bar{x}, \bar{y}, \bar{z}\}$.

The base frame is attached to the base configuration. It remains fixed if the base is fixed. It is chosen according to usual FEM practices. For example, in a 2-node spatial beam element, $\tilde{X}_1$ is defined by the two end nodes whereas $\tilde{X}_2$ and $\tilde{X}_3$ lie along principal inertia directions. The origin is typically placed at the element centroid.

The meaning of the reference frame depends on the description chosen:

1. *Total Lagrangian* (TL). The reference frame and base frame coalesce.

2. *Updated Lagrangian* (UL). The reference frame is attached to the reference configuration, and recomputed when the reference configuration (the previous converged solution) is updated. It remains fixed during an iterative process.

3. *Corotational description* (CR). The reference frame is renamed *corotated frame* or *CR frame.* It remains attached to the element and continuosly moves with it.

The transformation

$$\mathbf{x} = \mathbf{X} + \underline{\mathbf{u}}, \tag{7.3}$$

maps the location of base particle $P(X, Y, Z)$ to $P(x, y, z)$; see Figure 7.1. Consequently the particle displacement vector is defined as

$$\underline{\mathbf{u}} = \begin{bmatrix} u_X \\ u_Y \\ u_Z \end{bmatrix} = \begin{bmatrix} x - X \\ y - Y \\ z - Z \end{bmatrix} = \mathbf{x} - \mathbf{X}. \tag{7.4}$$

in which (X, Y, Z) and (x, y, z) pertain to the same particle.

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[2] In dynamic analysis the global frame may be moving in time as a Galilean or inertial frame. This is convenient to track the motion of objects such as aircraft or satellites.

[3] The choice between $\{X_1, X_2, X_3\}$ versus $\{X, Y, Z\}$ and likewise $\{x_1, x_2, x_3\}$ versus $\{x, y, z\}$ is a matter of convenience. For example, when developing specific finite elements it is preferable to use $\{X, Y, Z\}$ or $\{x, y, z\}$ so as to reserve coordinate subscripts for node numbers.

Remark 7.3. Variations of this notation scheme are employed as appropriate to the subject under consideration. For example, the coordinates of P in a target configuration $\mathcal{C}_n$ may be called (x_n, y_n, z_n).

Remark 7.4. In continuum mechanics, coordinates (X, Y, Z) and (x, y, z) are called *material* and *spatial* coordinates, respectively. In general treatments both systems are curvilinear and need not coalesce. The foregoing relation (7.4) is restrictive in two ways: the base coordinate systems for the reference and current configurations coincide, and that system is Cartesian. This assumption is sufficient, however, for the problems treated in this course.

Remark 7.5. The dual notation $(X, Y, Z) \equiv (x_0, y_0, z_0)$ is introduced on two accounts: (1) the use of (x_0, y_0, z_0) sometimes introduces a profusion of additional subscripts, and (2) the notation agrees with that traditional in continuum mechanics for the material coordinates as noted in the previous remark. The identification $X \equiv x_0$, $Y \equiv y_0$, $Z \equiv z_0$ will be employed when it is convenient to consider the reference configuration as the initial target configuration (cf. Remark 7.1).

§7.3.4. Configurations and Staged Analysis

The meaning of some special configurations can be made more precise if the nonlinear analysis process is viewed as a sequence of *analysis stages*, as discussed in Chapter 3. We restrict attention to the *Total Lagrangian* (TL) and *Corotational* (CR) kinematic descriptions, which are the only ones covered in this course. In a staged TL nonlinear analysis, two common choices for the reference configuration are:

(1) *Reference* $\equiv$ *base*. The base configuration is maintained as reference configuration for *all* stages.

(2) *Reference* $\equiv$ *stage start*. The configuration at the start of an analysis stage, *i.e.* at $\lambda = 0$, is chosen as reference configuration.

A combination of these two strategies can be of course adopted. In a staged CR analysis the reference is split between base and corotated. The same update choices are available for the base. This may be necessary when rotations exceed 2π; for example in aircraft maneuvers.

The *admissible* configuration is a "catch all" concept that embodies all others as particular cases. The *perturbed* configuration is an admissible variation from a admissible configuration. An ensemble of perturbed configurations is used to establish *incremental* or *rate* equations.

The *iterated* and *target* configurations are introduced in the context of incremental-iterative solution procedures for numerically tracing equilibrium paths. The target configuration is the "next solution". More precisely, an equilibrium solution (assumed to exist) which satisfies the residual equations for a certain value of the stage control parameter λ. While working to reach the target, a typical solution process goes through a sequence of *iterated* configurations that are not in equilibrium.

The *corotated* configuration is a rigid-body rotation of the reference configuration that "follows" the current configuration like a "shadow". It is used in the corotational (CR) kinematic description of nonlinear finite elements. Strains measured with respect to the corotated configuration may be considered "small" in many applications, a circumstance that allows linearization of several relations and efficient treatment of stability conditions.

§7.4. Kinematics

This section cover the essential kinematics necessary for finite displacement analysis.

§7.4.1. Deformation and Displacement Gradients

The derivatives of (x, y, z) with respect to (X, Y, Z), arranged in Jacobian format, constitute the so-called *deformation gradient* matrix:

$$\mathbf{F} = \frac{\partial(x, y, z)}{\partial(X, Y, Z)} = \begin{bmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{bmatrix}. \tag{7.5}$$

The inverse relation gives the derivatives of (X, Y, Z) with respect to (x, y, z) as

$$\mathbf{F}^{-1} = \frac{\partial(X,Y,Z)}{\partial(x,y,z)} = \begin{bmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} & \frac{\partial X}{\partial z} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} & \frac{\partial Y}{\partial z} \\ \frac{\partial Z}{\partial x} & \frac{\partial Z}{\partial y} & \frac{\partial Z}{\partial z} \end{bmatrix}. \tag{7.6}$$

These matrices can be used to relate the coordinate differentials

$$d\mathbf{x} = \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \mathbf{F} \begin{bmatrix} dX \\ dY \\ dZ \end{bmatrix} = \mathbf{F}\, d\mathbf{X}, \qquad d\mathbf{X} = \mathbf{F}^{-1}\, d\mathbf{x}. \tag{7.7}$$

Similarly, the *displacement gradients* with respect to the reference configuration can be presented as the 3×3 matrix

$$\mathbf{G} = \mathbf{F} - \mathbf{I} = \begin{bmatrix} \frac{\partial x}{\partial X} - 1 & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} - 1 & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} - 1 \end{bmatrix} = \begin{bmatrix} \frac{\partial u_X}{\partial X} & \frac{\partial u_X}{\partial Y} & \frac{\partial u_X}{\partial Z} \\ \frac{\partial u_Y}{\partial X} & \frac{\partial u_Y}{\partial Y} & \frac{\partial u_Y}{\partial Z} \\ \frac{\partial u_Z}{\partial X} & \frac{\partial u_Z}{\partial Y} & \frac{\partial u_Z}{\partial Z} \end{bmatrix} = \nabla \mathbf{u}. \tag{7.8}$$

Displacement gradients with respect to the current configuration are given by

$$\mathbf{J} = \mathbf{I} - \mathbf{F}^{-1} = \begin{bmatrix} 1 - \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} & \frac{\partial Z}{\partial x} \\ \frac{\partial Y}{\partial x} & 1 - \frac{\partial Y}{\partial y} & \frac{\partial Y}{\partial z} \\ \frac{\partial Z}{\partial x} & \frac{\partial Z}{\partial y} & 1 - \frac{\partial Z}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_X}{\partial x} & \frac{\partial u_X}{\partial y} & \frac{\partial u_X}{\partial z} \\ \frac{\partial u_Y}{\partial x} & \frac{\partial u_Y}{\partial y} & \frac{\partial u_Y}{\partial z} \\ \frac{\partial u_Z}{\partial x} & \frac{\partial u_Z}{\partial y} & \frac{\partial u_Z}{\partial z} \end{bmatrix}. \tag{7.9}$$

For the treatment of the Total Lagrangian description it will found to be convenient to arrange the displacement gradients of (7.8) as a 9-component vector (printed as row vector to save space):

$$\begin{aligned} \mathbf{g}^T &= [\, g_1 \quad g_2 \quad g_3 \quad g_4 \quad g_5 \quad g_6 \quad g_7 \quad g_8 \quad g_9 \,] \\ &= \left[\, \frac{\partial u_X}{\partial X} \quad \frac{\partial u_Y}{\partial X} \quad \frac{\partial u_Z}{\partial X} \quad \frac{\partial u_X}{\partial Y} \quad \frac{\partial u_Y}{\partial Y} \quad \frac{\partial u_Z}{\partial Y} \quad \frac{\partial u_X}{\partial Z} \quad \frac{\partial u_Y}{\partial Z} \quad \frac{\partial u_Z}{\partial Z} \,\right]. \end{aligned} \tag{7.10}$$

Remark 7.6. For arbitrary rigid-body motions (motions without deformations) $\mathbf{F}^T\mathbf{F} = \mathbf{F}\mathbf{F}^T = \mathbf{I}$, that is, $\mathbf{F}$ is an orthogonal matrix.

Remark 7.7. Displacement gradient matrices are connected by the relations

$$\mathbf{G} = (\mathbf{I} - \mathbf{J})^{-1} - \mathbf{I}, \qquad \mathbf{J} = \mathbf{I} - (\mathbf{I} + \mathbf{G})^{-1}. \tag{7.11}$$

For small deformations $\mathbf{G} \approx \mathbf{J}^{-1}$ and $\mathbf{J} \approx \mathbf{G}^{-1}$.

Remark 7.8. In nonlinear continuum mechanics, displacement gradients play an important role that is absent in the infinitesimal theory. This is even more so in the Total-Lagrangian core-congruential formulation covered in Chapters 8–11.

Remark 7.9. The ratio between infinitesimal volume elements $dV = dx\,dy\,dz$ and $dV_0 = dX\,dY\,dZ$ in the current and reference configuration appears in several continuum mechanics relations. Because of (7.7) this ratio may be expressed as

$$\frac{dV}{dV_0} = \frac{\rho_0}{\rho} = \det \mathbf{F}, \tag{7.12}$$

where ρ and ρ_0 denote the mass densities in the current and reference configuration, respectively. This equation expresses the law of conservation of mass.

§7.4.2. Stretch and Rotation Tensors

Tensors $\mathbf{F}$ and $\mathbf{G}$ are the building blocks of various deformation measures used in nonlinear continuum mechanics. The whole subject is dominated by the *polar decomposition theorem*: any particle deformation can be expressed as a pure deformation followed by a rotation, or by a rotation followed by a pure deformation. Mathematically this is written as multiplicative decompositions:

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}. \tag{7.13}$$

Here $\mathbf{R}$ is an orthogonal rotation tensor, whereas $\mathbf{U}$ and $\mathbf{V}$ are symmetric positive definite matrices called the right and left stretch tensors, respectively. If the deformation is a pure rotation, $\mathbf{U} = \mathbf{V} = \mathbf{I}$. Premultiplying (7.13) by $\mathbf{F}^T = \mathbf{U}\mathbf{R}^T$ gives $\mathbf{U}^2 = \mathbf{F}^T\mathbf{F}$ and consequently $\mathbf{U} = \sqrt{\mathbf{F}^T\mathbf{F}}$. Postmultiplying (7.13) by $\mathbf{F}^T = \mathbf{R}^T\mathbf{V}$ gives $\mathbf{V}^2 = \mathbf{F}\mathbf{F}^T$ and consequently $\mathbf{V} = \sqrt{\mathbf{F}\mathbf{F}^T}$. Upon taking the square roots, the rotation is then computed as either $\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}$ or $\mathbf{R} = \mathbf{V}^{-1}\mathbf{F}$. Obviously $\mathbf{U} = \mathbf{R}\mathbf{V}\mathbf{R}^T$ and $\mathbf{V} = \mathbf{R}^T\mathbf{U}\mathbf{R}$.

The combinations $\mathbf{C}_R = \mathbf{F}^T\mathbf{F}$ and $\mathbf{C}_L = \mathbf{F}\mathbf{F}^T$ are symmetric positive definite matrices that are called the right and left Cauchy-Green strecht tensors, respectively. To get $\mathbf{U}$ and $\mathbf{V}$ as square roots it is necessary to solve the eigensystem of $\mathbf{C}_R$ and $\mathbf{C}_L$, respectively.

To convert a stretch tensor to a strain tensor one substracts $\mathbf{I}$ from it or takes its log, so as to have a measure that vanishes for rigid motions. Either $\mathbf{U} - \mathbf{I}$ or $\mathbf{V} - \mathbf{I}$ represent proper strain measures. These are difficult, however, to express analytically in terms of the displacement gradients because of the intermediate eigenproblem. A more convenient strain measure is described next.

§7.4.3. Green-Lagrange Strain Measure

A convenient finite strain measure is the Green-Lagrange[4] strain tensor. Its three-dimensional expression in Cartesian coordinates is

$$\underline{\mathbf{e}} = \tfrac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right) = \tfrac{1}{2}(\mathbf{G} + \mathbf{G}^T) + \tfrac{1}{2}\mathbf{G}^T\mathbf{G} = \begin{bmatrix} e_{XX} & e_{XY} & e_{XZ} \\ e_{YX} & e_{YY} & e_{YZ} \\ e_{ZX} & e_{ZY} & e_{ZZ} \end{bmatrix}, \tag{7.14}$$

[4] A more proper name would be Green-St.Venant strain tensor. In fact Lagrange never used it but his name appears because of its strong connection to the Lagrangian kinematic description. Many authors call this measure simply the Green strain tensor.

Identifying the components of $\mathbf{F}^T\mathbf{F}-\mathbf{I}$ or $\frac{1}{2}(\mathbf{G}+\mathbf{G}^T)+\frac{1}{2}\mathbf{G}^T\mathbf{G}$ with the tensor conponents we get

$$
\begin{aligned}
e_{XX} &= \frac{\partial u_X}{\partial X} + \tfrac{1}{2}\left[\left(\frac{\partial u_X}{\partial X}\right)^2 + \left(\frac{\partial u_Y}{\partial X}\right)^2 + \left(\frac{\partial u_Z}{\partial X}\right)^2\right]\\
e_{YY} &= \frac{\partial u_Y}{\partial Y} + \tfrac{1}{2}\left[\left(\frac{\partial u_X}{\partial Y}\right)^2 + \left(\frac{\partial u_Y}{\partial Y}\right)^2 + \left(\frac{\partial u_Z}{\partial Y}\right)^2\right]\\
e_{ZZ} &= \frac{\partial u_Z}{\partial Z} + \tfrac{1}{2}\left[\left(\frac{\partial u_X}{\partial Z}\right)^2 + \left(\frac{\partial u_Y}{\partial Z}\right)^2 + \left(\frac{\partial u_Z}{\partial Z}\right)^2\right]\\
e_{YZ} &= \tfrac{1}{2}\left(\frac{\partial u_Y}{\partial Z} + \frac{\partial u_Z}{\partial Y}\right) + \tfrac{1}{2}\left[\frac{\partial u_X}{\partial Y}\frac{\partial u_X}{\partial Z} + \frac{\partial u_Y}{\partial Y}\frac{\partial u_Y}{\partial Z} + \frac{\partial u_Z}{\partial Y}\frac{\partial u_Z}{\partial Z}\right] = e_{ZY},\\
e_{ZX} &= \tfrac{1}{2}\left(\frac{\partial u_Z}{\partial X} + \frac{\partial u_X}{\partial Z}\right) + \tfrac{1}{2}\left[\frac{\partial u_X}{\partial Z}\frac{\partial u_X}{\partial X} + \frac{\partial u_Y}{\partial Z}\frac{\partial u_Y}{\partial X} + \frac{\partial u_Z}{\partial Z}\frac{\partial u_Z}{\partial X}\right] = e_{XZ},\\
e_{XY} &= \tfrac{1}{2}\left(\frac{\partial u_X}{\partial Y} + \frac{\partial u_Y}{\partial X}\right) + \tfrac{1}{2}\left[\frac{\partial u_X}{\partial X}\frac{\partial u_X}{\partial Y} + \frac{\partial u_Y}{\partial X}\frac{\partial u_Y}{\partial Y} + \frac{\partial u_Z}{\partial X}\frac{\partial u_Z}{\partial Y}\right] = e_{YX}.
\end{aligned}
\tag{7.15}
$$

If the nonlinear portion (that enclosed in square brackets) of these expressions is neglected, one obtains the *infinitesimal strains* $\epsilon_{xx},\ \epsilon_{yy},\ \ldots\ \epsilon_{zx} = \frac{1}{2}\gamma_{zx},\ \epsilon_{xy} = \frac{1}{2}\gamma_{xy}$ encountered in linear finite element analysis. For future use in finite element work we shall arrange the components (7.15) as a 6-component strain vector $\mathbf{e}$ constructed as follows:

$$
\mathbf{e} = \begin{bmatrix} e_1\\ e_2\\ e_3\\ e_4\\ e_5\\ e_6 \end{bmatrix} = \begin{bmatrix} e_{XX}\\ e_{YY}\\ e_{ZZ}\\ e_{YZ}+e_{ZY}\\ e_{ZX}+e_{XZ}\\ e_{XY}+e_{YX} \end{bmatrix} = \begin{bmatrix} e_{XX}\\ e_{YY}\\ e_{ZZ}\\ 2e_{YZ}\\ 2e_{ZX}\\ 2e_{XY} \end{bmatrix}. \tag{7.16}
$$

Remark 7.10. Several other finite strain measures are used in nonlinear continuum mechanics. The common characteristic of all measures is that they must predict zero strains for arbitrary rigid-body motions, and must reduce to the infinitesimal strains if the nonlinear terms are neglected. This topic is further explored in Exercise 7.5.

§7.4.4. Strain-Gradient Matrix Expressions

For the development of the TL core-congruential formulation presented in following sections, it is useful to have a compact matrix expression for the Green-Lagrange strain components of (7.16) in terms of the displacement gradient vector (7.12). To that end, note that (7.15) may be rewritten as

$$
\begin{aligned}
e_1 &= g_1 + \tfrac{1}{2}(g_1^2+g_2^2+g_3^2),\\
e_2 &= g_5 + \tfrac{1}{2}(g_4^2+g_5^2+g_6^2),\\
e_3 &= g_9 + \tfrac{1}{2}(g_7^2+g_8^2+g_9^2),\\
e_4 &= g_6 + g_8 + g_4g_7 + g_5g_8 + g_6g_9,\\
e_5 &= g_3 + g_7 + g_1g_7 + g_2g_8 + g_3g_9,\\
e_6 &= g_2 + g_4 + g_1g_4 + g_2g_5 + g_3g_6.
\end{aligned}
\tag{7.17}
$$

These relations may be collectively embodied in the quadratic form

$$\boxed{e_i = \mathbf{h}_i^T \mathbf{g} + \tfrac{1}{2}\mathbf{g}^T \mathbf{H}_i \mathbf{g},} \tag{7.18}$$

where $\mathbf{h}_i$ are sparse 9×1 vectors:

$$\mathbf{h}_1 = \begin{bmatrix} 1\\0\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix}, \quad \mathbf{h}_2 = \begin{bmatrix} 0\\0\\0\\0\\1\\0\\0\\0\\0 \end{bmatrix}, \quad \mathbf{h}_3 = \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\1 \end{bmatrix}, \quad \mathbf{h}_4 = \begin{bmatrix} 0\\0\\0\\0\\0\\1\\0\\1\\0 \end{bmatrix}, \quad \mathbf{h}_5 = \begin{bmatrix} 0\\0\\1\\0\\0\\0\\1\\0\\0 \end{bmatrix}, \quad \mathbf{h}_6 = \begin{bmatrix} 0\\1\\0\\1\\0\\0\\0\\0\\0 \end{bmatrix}, \tag{7.19}$$

and $\mathbf{H}_i$ are very sparse 9×9 symmetric matrices:

$$\mathbf{H}_1 = \begin{bmatrix} 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{bmatrix}, \qquad \text{etc.} \tag{7.20}$$

Remark 7.11. For strain measures other than Green-Lagrange's, expressions similar to (7.14) may be constructed. But although the $\mathbf{h}_i$ remain the same, the $\mathbf{H}_i$ become complicated functions of the displacement gradients.

§7.4.5. Pull Forward and Pull Back

Most of the foregoing material is classical continuum mechanics as covered in dozens of scholarly books. Next is a kinematic derivation scheme that is quintaessential FEM. Consider the motion of an elastic bar element in the 2D plane as depicted in Figure 7.2.

(To be expanded, Chapter posted as is)

§7.5. Stress Measure

Associated with each finite strain measure is a corresponding stress measure that is conjugate to it in the sense of virtual work. That corresponding to the Green-Lagrange strain is the second Piola-Kirchhoff symmetric stress tensor, often abbreviated to "PK2 stress." The three-dimensional component expression of this tensor in Cartesian coordinates is

$$\underline{\mathbf{s}} = \begin{bmatrix} s_{XX} & s_{XY} & s_{XZ} \\ s_{YX} & s_{YY} & s_{YZ} \\ s_{ZX} & s_{ZY} & s_{ZZ} \end{bmatrix}, \tag{7.21}$$

in which $s_{XY} = s_{YX}$, etc. As in the case of strains, for future use in finite element work it is convenient to arrange the components (7.21) as a 6-component stress vector $\mathbf{s}$:

$$\mathbf{s}^T = [\, s_1 \quad s_2 \quad s_3 \quad s_4 \quad s_5 \quad s_6 \,] = [\, s_{XX} \quad s_{YY} \quad s_{ZZ} \quad s_{YZ} \quad s_{ZX} \quad s_{XY} \,]. \tag{7.22}$$

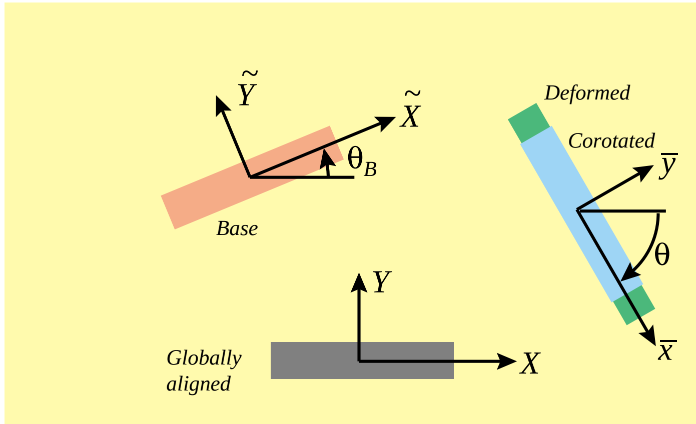

Figure 7.2. Rigid motion of bar in 2D illustrating concept of globally aligned configuration.

Remark 7.12. The physical meaning of the PK2 stresses is as follows: s_{ij} are stresses "pulled back" to the reference configuration $\mathcal{C}^0$ and referred to area elements there.

Remark 7.13. The PK2 stresses are related to the Cauchy (true) stresses σ_{ij} through the transformation

$$\begin{bmatrix} s_{XX} \\ s_{YY} \\ s_{ZZ} \\ s_{YZ} \\ s_{ZX} \\ s_{XY} \end{bmatrix} = \frac{\rho_0}{\rho} \begin{bmatrix} \frac{\partial X}{\partial x}\frac{\partial X}{\partial x} & \frac{\partial X}{\partial y}\frac{\partial X}{\partial y} & \frac{\partial X}{\partial z}\frac{\partial X}{\partial z} & \frac{\partial X}{\partial y}\frac{\partial X}{\partial z} & \frac{\partial X}{\partial z}\frac{\partial X}{\partial x} & \frac{\partial X}{\partial x}\frac{\partial X}{\partial y} \\ \frac{\partial Y}{\partial x}\frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y}\frac{\partial Y}{\partial y} & \frac{\partial Y}{\partial z}\frac{\partial Y}{\partial z} & \frac{\partial Y}{\partial y}\frac{\partial Y}{\partial z} & \frac{\partial Y}{\partial z}\frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial x}\frac{\partial Y}{\partial y} \\ \frac{\partial Z}{\partial x}\frac{\partial Z}{\partial x} & \frac{\partial Z}{\partial y}\frac{\partial Z}{\partial y} & \frac{\partial Z}{\partial z}\frac{\partial Z}{\partial z} & \frac{\partial Z}{\partial y}\frac{\partial Z}{\partial z} & \frac{\partial Z}{\partial z}\frac{\partial Z}{\partial x} & \frac{\partial Z}{\partial x}\frac{\partial Z}{\partial y} \\ \frac{\partial Y}{\partial x}\frac{\partial Z}{\partial x} & \frac{\partial Y}{\partial y}\frac{\partial Z}{\partial y} & \frac{\partial Y}{\partial z}\frac{\partial Z}{\partial z} & \frac{\partial Y}{\partial y}\frac{\partial Z}{\partial z} & \frac{\partial Y}{\partial z}\frac{\partial Z}{\partial x} & \frac{\partial Y}{\partial x}\frac{\partial Z}{\partial y} \\ \frac{\partial Z}{\partial x}\frac{\partial X}{\partial x} & \frac{\partial Z}{\partial y}\frac{\partial X}{\partial y} & \frac{\partial Z}{\partial z}\frac{\partial X}{\partial z} & \frac{\partial Z}{\partial y}\frac{\partial X}{\partial z} & \frac{\partial Z}{\partial z}\frac{\partial X}{\partial x} & \frac{\partial Z}{\partial x}\frac{\partial X}{\partial y} \\ \frac{\partial X}{\partial x}\frac{\partial Y}{\partial x} & \frac{\partial X}{\partial y}\frac{\partial Y}{\partial y} & \frac{\partial X}{\partial z}\frac{\partial Y}{\partial z} & \frac{\partial X}{\partial y}\frac{\partial Y}{\partial z} & \frac{\partial X}{\partial z}\frac{\partial Y}{\partial x} & \frac{\partial X}{\partial x}\frac{\partial Y}{\partial y} \end{bmatrix} \begin{bmatrix} \sigma_{XX} \\ \sigma_{YY} \\ \sigma_{ZZ} \\ \sigma_{YZ} \\ \sigma_{ZX} \\ \sigma_{XY} \end{bmatrix}, \tag{7.23}$$

$$\begin{bmatrix} \sigma_{XX} \\ \sigma_{YY} \\ \sigma_{ZZ} \\ \sigma_{YZ} \\ \sigma_{ZX} \\ \sigma_{XY} \end{bmatrix} = \frac{\rho}{\rho_0} \begin{bmatrix} \frac{\partial x}{\partial X}\frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y}\frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z}\frac{\partial x}{\partial Z} & \frac{\partial x}{\partial Y}\frac{\partial x}{\partial Z} & \frac{\partial x}{\partial Z}\frac{\partial x}{\partial X} & \frac{\partial x}{\partial X}\frac{\partial x}{\partial Y} \\ \frac{\partial y}{\partial X}\frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y}\frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z}\frac{\partial y}{\partial Z} & \frac{\partial y}{\partial Y}\frac{\partial y}{\partial Z} & \frac{\partial y}{\partial Z}\frac{\partial y}{\partial X} & \frac{\partial y}{\partial X}\frac{\partial y}{\partial Y} \\ \frac{\partial z}{\partial X}\frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y}\frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z}\frac{\partial z}{\partial Z} & \frac{\partial z}{\partial Y}\frac{\partial z}{\partial Z} & \frac{\partial z}{\partial Z}\frac{\partial z}{\partial X} & \frac{\partial z}{\partial X}\frac{\partial z}{\partial Y} \\ \frac{\partial y}{\partial X}\frac{\partial z}{\partial X} & \frac{\partial y}{\partial Y}\frac{\partial z}{\partial Y} & \frac{\partial y}{\partial Z}\frac{\partial z}{\partial Z} & \frac{\partial y}{\partial Y}\frac{\partial z}{\partial Z} & \frac{\partial y}{\partial Z}\frac{\partial z}{\partial X} & \frac{\partial y}{\partial X}\frac{\partial z}{\partial Y} \\ \frac{\partial z}{\partial X}\frac{\partial x}{\partial X} & \frac{\partial z}{\partial Y}\frac{\partial x}{\partial Y} & \frac{\partial z}{\partial Z}\frac{\partial x}{\partial Z} & \frac{\partial z}{\partial Y}\frac{\partial x}{\partial Z} & \frac{\partial z}{\partial Z}\frac{\partial x}{\partial X} & \frac{\partial z}{\partial X}\frac{\partial x}{\partial Y} \\ \frac{\partial x}{\partial X}\frac{\partial y}{\partial X} & \frac{\partial x}{\partial Y}\frac{\partial y}{\partial Y} & \frac{\partial x}{\partial Z}\frac{\partial y}{\partial Z} & \frac{\partial x}{\partial Y}\frac{\partial y}{\partial Z} & \frac{\partial x}{\partial Z}\frac{\partial y}{\partial X} & \frac{\partial x}{\partial X}\frac{\partial y}{\partial Y} \end{bmatrix} \begin{bmatrix} s_{XX} \\ s_{YY} \\ s_{ZZ} \\ s_{YZ} \\ s_{ZX} \\ s_{XY} \end{bmatrix}. \tag{7.24}$$

The density ratios that appears in these equations may be obtained from (7.13). If all displacement gradients are small, both transformations reduce to the identity, and the PK2 and Cauchy stresses coalesce.

§7.6. Constitutive Equations

Throughout this course we restrict our attention to constitutive behavior in which conjugate strains and stresses are *linearly* related. For the Green-Lagrange and PK2 measures used here, the stress-strain relations will be written, with the summation convention implied,

$$s_i = s_i^0 + E_{ij}\, e_j, \tag{7.25}$$

where e_i and s_i denote components of the strain and stress vectors defined by (7.16) and (7.22), respectively, s_i^0 are stresses in the reference configuration (also called prestresses) and E_{ij} are constant elastic moduli with $E_{ij} = E_{ji}$. In full matrix notation,

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \end{bmatrix} = \begin{bmatrix} s_1^0 \\ s_2^0 \\ s_3^0 \\ s_4^0 \\ s_5^0 \\ s_6^0 \end{bmatrix} + \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & E_{16} \\ E_{12} & E_{22} & E_{23} & E_{24} & E_{25} & E_{26} \\ E_{13} & E_{23} & E_{33} & E_{34} & E_{35} & E_{36} \\ E_{14} & E_{24} & E_{34} & E_{44} & E_{45} & E_{46} \\ E_{15} & E_{25} & E_{35} & E_{45} & E_{55} & E_{56} \\ E_{16} & E_{26} & E_{36} & E_{46} & E_{56} & E_{66} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{bmatrix}, \tag{7.26}$$

or in compact form,

$$\mathbf{s} = \mathbf{s}^0 + \mathbf{E}\mathbf{e}. \tag{7.27}$$

Remark 7.14. For an invariant reference configuration, PK2 and Cauchy (true) prestresses obviously coincide (see Remark 7.14). Thus $\boldsymbol{\sigma}^0 \equiv \mathbf{s}^0$ in such a case. However if the reference configuration is allowed to vary often, as in the UL description, things get more complicated.

§7.7. Strain Energy Density

We conclude this review by giving the expression of the strain energy density $\mathcal{U}$ in the current configuration reckoned per unit volume of the reference configuration:

$$\mathcal{U} = s_i^0 e_i + \tfrac{1}{2}(s_i - s_i^0)e_i = s_i^0 e_i + \tfrac{1}{2} e_i E_{ij} e_j, \tag{7.28}$$

or, in matrix form

$$\mathcal{U} = \mathbf{e}^T \mathbf{s}^0 + \tfrac{1}{2}\mathbf{e}^T \mathbf{E}\mathbf{e}. \tag{7.29}$$

If the current configuration coincides with the reference configuration, $\mathbf{e} = \mathbf{0}$ and $\mathcal{U} = 0$. It can be observed that the strain energy density is quadratic in the Green-Lagrange strains. To obtain this density in terms of displacement gradients, substitute (7.18) into (7.29) to get

$$\boxed{\mathcal{U} = s_i^0(\mathbf{h}_i^T\mathbf{g} + \mathbf{g}^T\mathbf{H}_i\mathbf{g}) + \tfrac{1}{2}\left[(\mathbf{g}^T\mathbf{h}_i + \tfrac{1}{2}\mathbf{g}^T\mathbf{H}_i\mathbf{g})E_{ij}(\mathbf{h}_j^T\mathbf{g} + \tfrac{1}{2}\mathbf{g}^T\mathbf{H}_j\mathbf{g})\right].} \tag{7.30}$$

Because $\mathbf{h}_i$ and $\mathbf{H}_i$ are constant, this relation shows that the strain energy density is *quartic* in the displacement gradients collected in $\mathbf{g}$.

The strain energy in the current configuration is obtained by integrating this energy density over the reference configuration:

$$U = \int_{V_0} \mathcal{U}\, dX\, dY\, dZ. \tag{7.31}$$

This expression forms the basis for deriving finite elements based on the Total Lagrangian description.

Homework Exercises for Chapter 7
Review of Continuum Mechanics

EXERCISE 7.1 [A:15] Obtain the expressions of $\mathbf{H}_3$ and $\mathbf{H}_5$.

EXERCISE 7.2 [A:15] Derive (7.28) by integrating $s_i\, de'_i$ from $\mathcal{C}_0$ ($e'_i = 0$) to $\mathcal{C}$ ($e'_i = e_i$) and using (7.25).

EXERCISE 7.3 [A:20] A bar of length L_0 originally along the $X \equiv x$ axis (the reference configuration $\mathcal{C}_0$) is rigidly rotated 90° to lie along the $Y \equiv y$ axis while retaining the same length (the current configuration $\mathcal{C}$). Node 1 at the origin $X = Y = 0$ stays at the same location.

(a) Verify that the motion from $\mathcal{C}_0$ to $\mathcal{C}$ is given by

$$x = -Y, \qquad y = X, \qquad z = Z. \tag{E7.1}$$

(b) Obtain the displacement field $\underline{\mathbf{u}}$, the deformation gradient matrix $\mathbf{F}$, the displacement gradient matrix $\mathbf{G}$ and the Green-Lagrange axial strain $e = e_{XX}$. Show that the Green-Lagrange measure correctly predicts zero axial strain whereas the infinitesimal strain measure $\epsilon = \epsilon_{XX} = \partial u_X/\partial X$ predicts the absurd value of -100% strain.

EXERCISE 7.4 [A:20] Let L_0 and L denote the length of a bar element in the reference and current configurations, respectively. The Green-Lagrange finite strain $e = e_{XX}$, if constant over the bar, can be defined as

$$e = \frac{L^2 - L_0^2}{2L_0^2}. \tag{E7.2}$$

Show that the definitions (E7.2) and of $e = e_{XX}$ in (7.15) are equivalent. (*Hint*: express L_0 and L in terms of the coordinates and displacements in the bar system.)

EXERCISE 7.5 [A:25] The Green-Lagrange strain measure is not the only finite strain measure used in structural and solid mechanics. For the uniaxial case of a stretched bar that moves from a length L_0 in $\mathcal{C}_0$ to a length L in $\mathcal{C}$, some of the other measures are defined as follows:

(a) Uniaxial Almansi strain:

$$e_A = \frac{L^2 - L_0^2}{2L^2}. \tag{E7.3}$$

(b) Uniaxial Hencky strain, also called logarithmic or "true" strain:

$$e_H = \log(L/L_0), \tag{E7.4}$$

where log denotes the natural logarithm.

(c) Uniaxial midpoint strain[5]

$$e_M = \frac{L^2 - L_0^2}{2[(L + L_0)/2]^2}. \tag{E7.5}$$

(d) Uniaxial engineering strain:

$$e_E = \epsilon = \frac{L - L_0}{L_0}. \tag{E7.6}$$

If $L = (1 + \epsilon)L_0$, show by expanding e_A, e_H and e_M in Taylor series in ϵ (about $\epsilon = 0$) that these measures, as well as the Green-Lagrange axial strain (E7.2), agree with each other to first order [*i.e.*, they differ by $O(\epsilon^2)$] as $\epsilon \to 0$.

[5] The midpoint strain tensor, which is a good approximation of the Hencky strain tensor but more easily computable, is frequently used in finite element plasticity or viscoplasticity calculations that involve large deformations, for example in metal forming processes.

EXERCISE 7.6 [A:30] (Advanced). Extend the definition of the Almansi, Hencky, midpoint and midpoint strains to a three dimensional strain state. *Hint:* use the spectral decomposition of $\mathbf{F}^T\mathbf{F}$ and the concept of function of a symmetric matrix.

EXERCISE 7.7 [A:35] (Advanced). Extend the definition of engineering strain to a three-dimensional strain state. The resulting measures (there are actually two) are called the *stretch tensors*. *Hint*: use either the spectral decomposition of $\mathbf{F}^T\mathbf{F}$, or the polar decomposition theorem of tensor calculus.

EXERCISE 7.8 [A:40] (Advanced). Define the stress measures conjugate to the Almansi, Hencky, midpoint and engineering strains.