2

The Direct Stiffness Method I

TABLE OF CONTENTS

This Chapter begins the exposition of the Direct Stiffness Method (DSM) of structural analysis. The DSM is by far the most common implementation of the Finite Element Method (FEM). In particular, all major commercial FEM codes are based on the DSM.

The exposition is done by following the DSM steps applied to a simple plane truss structure. The method has two major stages: breakdown, and assembly+solution. This Chapter covers primarily the breakdown stage.

§2.1. Why A Plane Truss?

The simplest structural finite element is the 2-node bar (also called linear spring) element, which is illustrated in Figure 2.1(a). Perhaps the most complicated finite element (at least as regards number of degrees of freedom) is the curved, three-dimensional "brick" element depicted in Figure 2.1(b).

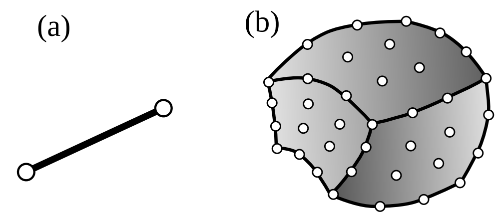

FIGURE 2.1. From the simplest through a highly complex structural finite element: (a) 2-node bar element for trusses, (b) 64-node tricubic, "brick" element for three-dimensional solid analysis.

Yet the remarkable fact is that, in the DSM, the simplest and most complex elements are treated alike! To illustrate the basic steps of this democratic method, it makes educational sense to keep it simple and use a structure composed of bar elements.

A simple yet nontrivial structure is the *pin-jointed plane truss*.[1] Using a plane truss to teach the stiffness method offers two additional advantages:

(a) Computations can be entirely done by hand as long as the structure contains just a few elements. This allows various steps of the solution procedure to be carefully examined and understood before passing to the computer implementation. Doing hand computations on more complex finite element systems rapidly becomes impossible.

(b) The computer implementation on any programming language is relatively simple and can be assigned as preparatory computer homework before reaching Part III.

§2.2. Truss Structures

Plane trusses, such as the one depicted in Figure 2.2, are often used in construction, particularly for roofing of residential and commercial buildings, and in short-span bridges. Trusses, whether two or three dimensional, belong to the class of *skeletal structures*. These structures consist of elongated structural components called *members*, connected at *joints*. Another important subclass of skeletal structures are frame structures or *frameworks*, which are common in reinforced concrete construction of buildings and bridges.

Skeletal structures can be analyzed by a variety of hand-oriented methods of structural analysis taught in beginning Mechanics of Materials courses: the Displacement and Force methods. They can also be analyzed by the computer-oriented FEM. That versatility makes those structures a good choice

[1] A one dimensional bar assembly would be even simpler. That kind of structure would not adequately illustrate some of the DSM steps, however, notably the back-and-forth transformations from global to local coordinates.

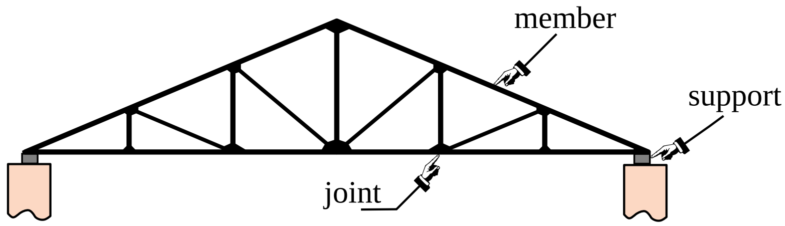

FIGURE 2.2. An actual plane truss structure. That shown is typical of a roof truss used in building construction.

to illustrate the transition from the hand-calculation methods taught in undergraduate courses, to the fully automated finite element analysis procedures available in commercial programs.

In this and the next Chapter we will go over the basic steps of the DSM in a "hand-computer" calculation mode. This means that although the steps are done by hand, whenever there is a procedural choice we shall either adopt the way which is better suited towards the computer implementation, or explain the difference between hand and computer computations. The actual computer implementation using a high-level programming language is presented in Chapter 5.

To keep hand computations manageable in detail we use just about the simplest structure that can be called a plane truss, namely the three-member truss illustrated in Figure 2.3. The *idealized* model of the example truss as a pin-jointed assemblage of bars is shown in Figure 2.4(a), which also gives its geometric and material properties. In this idealization truss members carry only axial loads, have no bending resistance, and are connected by frictionless pins. Figure 2.4(b) displays support conditions as well as the applied forces applied to the truss joints.

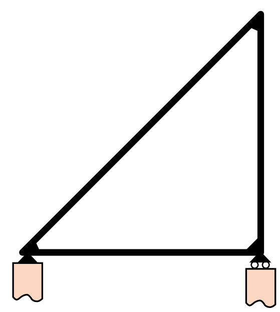

FIGURE 2.3. The three-member example truss.

It should be noted that as a practical structure the example truss is not particularly useful — the one depicted in Figure 2.2 is far more common in construction. But with the example truss we can go over the basic DSM steps without getting mired into too many members, joints and degrees of freedom.

§2.3. Idealization

Although the pin-jointed assemblage of bars (as depicted in Figure 2.4) is sometimes presented as an actual problem, it actually represents an *idealization* of a true truss structure. The axially-carrying members and frictionless pins of this structure are only an approximation of a real truss. For example, building and bridge trusses usually have members joined to each other through the use of gusset plates, which are attached by nails, bolts, rivets or welds. See Figure 2.2. Consequently members will carry some bending as well as direct axial loading.

Experience has shown, however, that stresses and deformations calculated for the simple idealized problem will often be satisfactory for overall-design purposes; for example to select the cross section of the members. Hence the engineer turns to the pin-jointed assemblage of axial force elements and uses it to carry out the structural analysis.

This replacement of true by idealized is at the core of the *physical interpretation* of the finite element method discussed in §1.4.

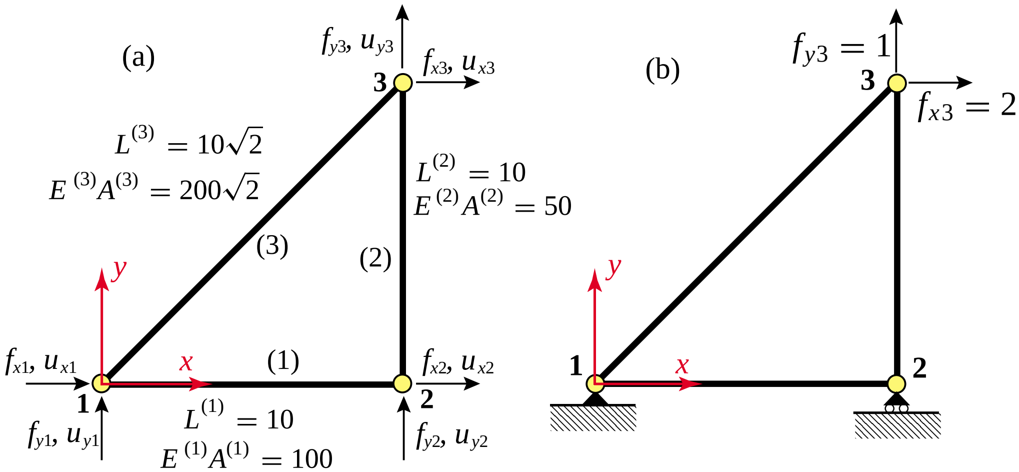

FIGURE 2.4. Pin-jointed idealization of example truss: (a) geometric and elastic properties, (b) support conditions and applied loads.

§2.4. Members, Joints, Forces and Displacements

The idealization of the example truss, pictured in Figure 2.4, has three *joints*, which are labeled 1, 2 and 3, and three *members*, which are labeled (1), (2) and (3). These members connect joints 1–2, 2–3, and 1–3, respectively. The member lengths are denoted by $L^{(1)}$, $L^{(2)}$ and $L^{(3)}$, their elastic moduli by $E^{(1)}$, $E^{(2)}$ and $E^{(3)}$, and their cross-sectional areas by $A^{(1)}$, $A^{(2)}$ and $A^{(3)}$. Note that an element number supercript is enclosed in parenthesis to avoid confusion with exponents. Both E and A are assumed to be constant along each member.

Members are generically identified by index e (because of their close relation to finite elements, see below). This index is placed as supercript of member properties. For example, the cross-section area of a generic member is A^e. The member superscript is *not* enclosed in parentheses in this case because no confusion with exponents can arise. But the area of member 3 is written $A^{(3)}$ and not A^3.

Joints are generically identified by indices such as i, j or n. In the general FEM, the name "joint" and "member" is replaced by *node* and *element*, respectively. The dual nomenclature is used in the initial Chapters to stress the physical interpretation of the FEM.

The geometry of the structure is referred to a common Cartesian coordinate system $\{x, y\}$, which is called the *global coordinate system*. Other names for it in the literature are *structure coordinate system* and *overall coordinate system*.

The key ingredients of the stiffness method of analysis are the *forces* and *displacements* at the joints.

In a idealized pin-jointed truss, externally applied forces as well as reactions *can act only at the joints*. All member axial forces can be characterized by the x and y components of these forces, denoted by f_x and f_y, respectively. The components at joint i will be identified as f_{xi} and f_{yi}, respectively. The set of all joint forces can be arranged as a 6-component column vector called $\mathbf{f}$.

The other key ingredient is the displacement field. Classical structural mechanics tells us that the displacements of the truss *are completely defined by the displacements of the joints*. This statement is a particular case of the more general finite element theory. The x and y displacement components will be denoted by u_x and u_y, respectively. The *values* of u_x and u_y at joint i will be called u_{xi} and u_{yi}. Like joint forces, they are arranged into a 6-component vector called $\mathbf{u}$. Here are the two vectors of nodal forces and nodal displacements, shown side by side:

$$\mathbf{f} = \begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}. \tag{2.1}$$

In the DSM these six displacements are the primary unknowns. They are also called the *degrees of freedom* or *state variables* of the system.[2]

How about the displacement boundary conditions, popularly called support conditions? This data will tell us which components of **f** and **u** are actual unknowns and which ones are known *a priori*. In pre-computer structural analysis such information was used *immediately* by the analyst to discard unnecessary variables and thus reduce the amount of hand-carried bookkeeping.

The computer oriented philosophy is radically different: *boundary conditions can wait until the last moment*. This may seem strange, but on the computer the sheer volume of data may not be so important as the efficiency with which the data is organized, accessed and processed. The strategy "save the boundary conditions for last" will be followed here also for the hand computations.

Remark 2.1. Often column vectors such as (2.1) will be displayed in row form to save space, with a transpose symbol at the end. For example, $\mathbf{f} = [\, f_{x1} \;\; f_{y1} \;\; f_{x2} \;\; f_{y2} \;\; f_{x3} \;\; f_{y3} \,]^T$ and $\mathbf{u} = [\, u_{x1} \;\; u_{y1} \;\; u_{x2} \;\; u_{y2} \;\; u_{x3} \;\; u_{y3} \,]^T$.

§2.5. The Master Stiffness Equations

The *master stiffness equations* relate the joint forces **f** of the complete structure to the joint displacements **u** of the complete structure *before* specification of support conditions.

Because the assumed behavior of the truss is linear, these equations must be linear relations that connect the components of the two vectors. Furthermore it will be assumed that if all displacements vanish, so do the forces.[3] If both assumptions hold the relation must be homogeneous and expressable in component form as

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix} = \begin{bmatrix} K_{x1x1} & K_{x1y1} & K_{x1x2} & K_{x1y2} & K_{x1x3} & K_{x1y3} \\ K_{y1x1} & K_{y1y1} & K_{y1x2} & K_{y1y2} & K_{y1x3} & K_{y1y3} \\ K_{x2x1} & K_{x2y1} & K_{x2x2} & K_{x2y2} & K_{x2x3} & K_{x2y3} \\ K_{y2x1} & K_{y2y1} & K_{y2x2} & K_{y2y2} & K_{y2x3} & K_{y2y3} \\ K_{x3x1} & K_{x3y1} & K_{x3x2} & K_{x3y2} & K_{x3x3} & K_{x3y3} \\ K_{y3x1} & K_{y3y1} & K_{y3x2} & K_{y3y2} & K_{y3x3} & K_{y3y3} \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}. \tag{2.2}$$

In matrix notation:

$$\boxed{\mathbf{f} = \mathbf{K}\,\mathbf{u}.} \tag{2.3}$$

[2] *Primary unknowns* is the correct mathematical term whereas *degrees of freedom* has a mechanics flavor: "any of a limited number of ways in which a body may move or in which a dynamic system may change" (Merrian-Webster). The term *state variables* is used more often in nonlinear analysis, material sciences and statistics.

[3] This assumption implies that the so-called *initial strain* effects, also known as *prestress* or *initial stress* effects, are neglected. Such effects are produced by actions such as temperature changes or lack-of-fit fabrication, and are studied in Chapter 29.

Here **K** is the *master stiffness matrix*, also called *global stiffness matrix*, *assembled stiffness matrix*, or *overall stiffness matrix*. It is a 6×6 square matrix that happens to be symmetric, although this attribute has not been emphasized in the written-out form (2.2). The entries of the stiffness matrix are often called *stiffness coefficients* and have a physical interpretation discussed below.

The qualifiers ("master", "global", "assembled" and "overall") convey the impression that there is another level of stiffness equations lurking underneath. And indeed there is a *member level* or *element level*, into which we plunge in the **Breakdown** section.

Remark 2.2. *Interpretation of Stiffness Coefficients*. The following interpretation of the entries of **K** is valuable for visualization and checking. Choose a displacement vector **u** such that all components are zero except the i^{th} one, which is one. Then **f** is simply the i^{th} column of **K**. For instance if in (2.3) we choose u_{x2} as unit displacement,

$$\mathbf{u} = [\,0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0\,]^T, \qquad \mathbf{f} = [\,K_{x1x2} \;\; K_{y1x2} \;\; K_{x2x2} \;\; K_{y2x2} \;\; K_{x3x2} \;\; K_{y3x2}\,]^T. \tag{2.4}$$

Thus K_{y1x2}, say, represents the y-force at joint 1 that would arise on prescribing a unit x-displacement at joint 2, while all other displacements vanish. In structural mechanics this property is called *interpretation of stiffness coefficients* as *displacement influence coefficients*. It extends unchanged to the general finite element method.

§2.6. The DSM Steps

The DSM steps, major and minor, are summarized in Figure 2.5 for the convenience of the reader. The two major processing steps are **Breakdown**, followed by **Assembly & Solution**. A postprocessing substep may follow, although this is not part of the DSM proper.

The first 3 DSM substeps are: (1) disconnection, (2) localization, and (3) computation of member stiffness equations. Collectively these form the *breakdown*. The first two are marked as *conceptual* in Figure 2.5 because they are not actually programmed as such. These subsets are implicitly carried out through the user-provided problem definition. Processing begins at the member-stiffness-equation forming substep.

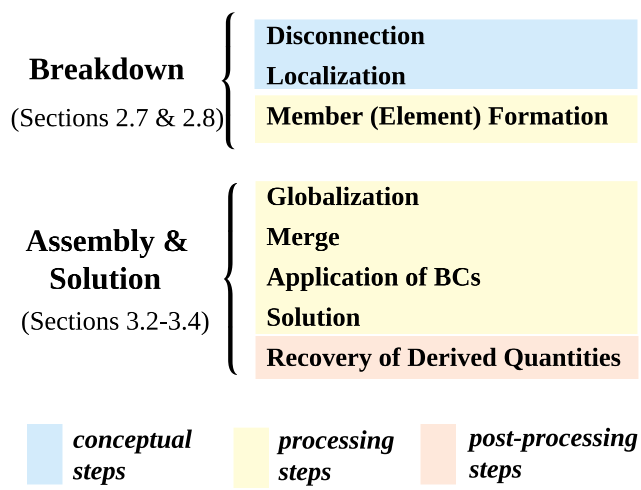

FIGURE 2.5. The Direct Stiffness Method steps.

§2.7. Breakdown

§2.7.1. Disconnection

To carry out the first breakdown step we proceed to *disconnect* or *disassemble* the structure into its components, namely the three truss members. This task is illustrated in Figure 2.6. To each member $e = 1, 2, 3$ assign a Cartesian system $\{\bar{x}^e, \bar{y}^e\}$. Axis $\bar{x}^e$ is aligned along the axis of the e^{th} member. Actually $\bar{x}^e$ runs along the member longitudinal axis; it is shown offset in that Figure for clarity.

By convention the positive direction of $\bar{x}^e$ runs from joint i to joint j, where $i < j$. The angle formed by $\bar{x}^e$ and x is the *orientation angle* φ^e. The axes origin is arbitrary and may be placed at the member midpoint or at one of the end joints for convenience.

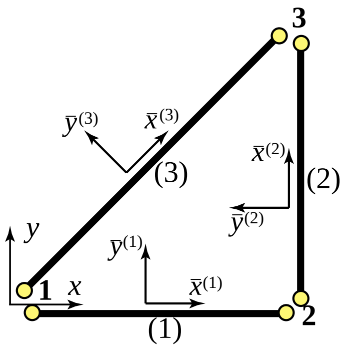

FIGURE 2.6. Breakdown of example truss into individual members (1), (2) and (3), and selection of local coordinate systems.

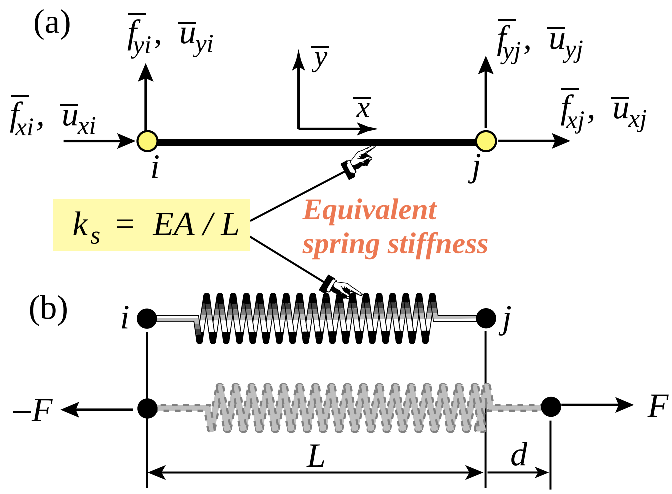

FIGURE 2.7. Generic truss member referred to its local coordinate system $\{\bar{x}, \bar{y}\}$: (a) idealization as bar element, (b) interpretation as equivalent spring.

Systems $\{\bar{x}^e, \bar{y}^e\}$ are called *local coordinate systems* or *member-attached coordinate systems*. In the general finite element method they also receive the name *element coordinate systems*.

§2.7.2. Localization

Next we drop the member identifier e so that we are effectively dealing with a *generic* truss member, as illustrated in Figure 2.7(a). The local coordinate system is $\{\bar{x}, \bar{y}\}$. The two end joints are i and j.

As shown in that figure, a generic truss member has four joint force components and four joint displacement components (the member degrees of freedom). The member properties are length L, elastic modulus E and cross-section area A.

§2.7.3. Computation of Member Stiffness Equations

The force and displacement components of the generic truss member shown in Figure 2.7(a) are linked by the *member stiffness relations*

$$\bar{\mathbf{f}} = \bar{\mathbf{K}}\,\bar{\mathbf{u}}, \tag{2.5}$$

which written out in full is

$$\begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix} = \begin{bmatrix} \bar{K}_{xixi} & \bar{K}_{xiyi} & \bar{K}_{xixj} & \bar{K}_{xiyj} \\ \bar{K}_{yixi} & \bar{K}_{yiyi} & \bar{K}_{yixj} & \bar{K}_{yiyj} \\ \bar{K}_{xjxi} & \bar{K}_{xjyi} & \bar{K}_{xjxj} & \bar{K}_{xjyj} \\ \bar{K}_{yjxi} & \bar{K}_{yjyi} & \bar{K}_{yjxj} & \bar{K}_{yjyj} \end{bmatrix} \begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix}. \tag{2.6}$$

Vectors $\bar{\mathbf{f}}$ and $\bar{\mathbf{u}}$ are called the *member joint forces* and *member joint displacements*, respectively, whereas $\bar{\mathbf{K}}$ is the *member stiffness matrix* or *local stiffness matrix*. When these relations are interpreted from the standpoint of the general FEM, “member” is replaced by “element” and “joint” by ”node.”

There are several ways to construct the stiffness matrix $\bar{\mathbf{K}}$ in terms of L, E and A. The most straightforward technique relies on the Mechanics of Materials approach covered in undergraduate courses. Think of the truss member in Figure 2.7(a) as a linear spring of equivalent stiffness k_s, an interpretation illustrated in Figure 2.7(b). If the member properties are *uniform* along its length, Mechanics of Materials bar theory tells us that[4]

$$k_s = \frac{EA}{L}, \tag{2.7}$$

Consequently the force-displacement equation is

$$F = k_s d = \frac{EA}{L} d, \tag{2.8}$$

where F is the internal axial force and d the relative axial displacement, which physically is the bar elongation. The axial force and elongation can be immediately expressed in terms of the joint forces and displacements as

$$F = \bar{f}_{xj} = -\bar{f}_{xi}, \qquad d = \bar{u}_{xj} - \bar{u}_{xi}, \tag{2.9}$$

which express force equilibrium[5] and kinematic compatibility, respectively. Combining (2.8) and (2.9) we obtain the matrix relation[6]

$$\bar{\mathbf{f}} = \begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix} = \bar{\mathbf{K}}\,\bar{\mathbf{u}}, \tag{2.10}$$

Hence

$$\boxed{\bar{\mathbf{K}} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.} \tag{2.11}$$

This is the truss stiffness matrix in local coordinates.

Two other methods for obtaining the local force-displacement relation (2.8) are covered in Exercises 2.6 and 2.7.

§2.8. Assembly: Globalization

The first substep in the assembly & solution major step, as shown in Figure 2.5, is *globalization*. This operation is done member by member. It refers the member stiffness equations to the global system $\{x, y\}$ so it can be merged into the master stiffness. Before entering into details we must establish relations that connect joint displacements and forces in the global and local coordinate systems. These are given in terms of *transformation matrices*.

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[4] See for example, Chapter 2 of [12].

[5] Equations $F = \bar{f}_{xj} = -\bar{f}_{xi}$ follow by considering the free body diagram (FBD) of each joint. For example, take joint i as a FBD. Equilibrium along x requires $-F - \bar{f}_{xi} = 0$ whence $F = -\bar{f}_{xi}$. Doing the same on joint j yields $F = \bar{f}_{xj}$.

[6] The matrix derivation of (2.10) is the subject of Exercise 2.3.

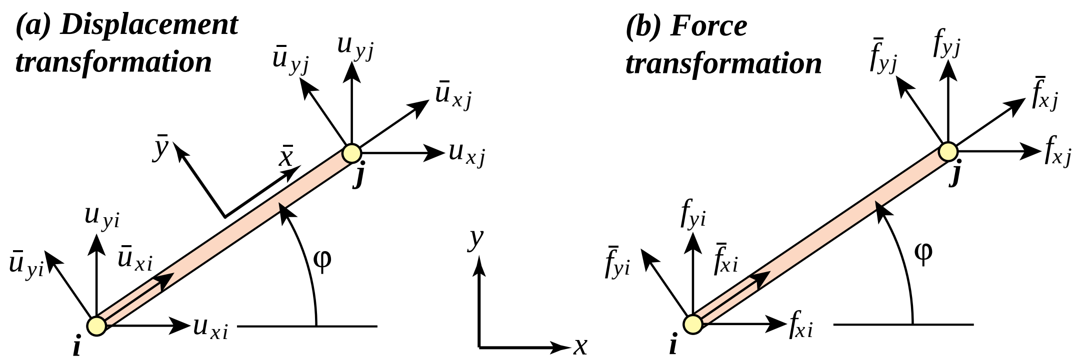

FIGURE 2.8. The transformation of node displacement and force components from the local system $\{\bar{x}, \bar{y}\}$ to the global system $\{x, y\}$.

§2.8.1. Coordinate Transformations

The necessary transformations are easily obtained by inspection of Figure 2.8. For the displacements

$$\begin{aligned} \bar{u}_{xi} &= u_{xi}c + u_{yi}s, \qquad & \bar{u}_{yi} &= -u_{xi}s + u_{yi}c, \\ \bar{u}_{xj} &= u_{xj}c + u_{yj}s, \qquad & \bar{u}_{yj} &= -u_{xj}s + u_{yj}c, \end{aligned} \quad . \tag{2.12}$$

where $c = \cos\varphi$, $s = \sin\varphi$ and φ is the angle formed by $\bar{x}$ and x, measured positive counterclockwise from x. The matrix form of this relation is

$$\begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix} = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix} \begin{bmatrix} u_{xi} \\ u_{yi} \\ u_{xj} \\ u_{yj} \end{bmatrix}. \tag{2.13}$$

The 4×4 matrix that appears above is called a *displacement transformation matrix* and is denoted[7] by $\mathbf{T}$. The node forces transform as $f_{xi} = \bar{f}_{xi}c - \bar{f}_{yi}s$, etc., which in matrix form become

$$\begin{bmatrix} f_{xi} \\ f_{yi} \\ f_{xj} \\ f_{yj} \end{bmatrix} = \begin{bmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & c & -s \\ 0 & 0 & s & c \end{bmatrix} \begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix}. \tag{2.14}$$

The 4×4 matrix that appears above is called a *force transformation matrix*. A comparison of (2.13) and (2.14) reveals that the force transformation matrix is the *transpose* $\mathbf{T}^T$ of the displacement transformation matrix $\mathbf{T}$. This relation is not accidental and can be proved to hold generally.[8]

Remark 2.3. Note that in (2.13) the local system (barred) quantities appear on the left-hand side, whereas in (2.14) they show up on the right-hand side. The expressions (2.13) and and (2.14) are discrete counterparts of what are called covariant and contravariant transformations, respectively, in continuum mechanics. The counterpart of the transposition relation is the *adjointness* property.

[7] This matrix will be called $\mathbf{T}_d$ when its association with displacements is to be emphasized, as in Exercise 2.5.

[8] A simple proof that relies on the invariance of external work is given in Exercise 2.5. However this invariance was only checked by explicit computation for a truss member in Exercise 2.4. The general proof relies on the Principle of Virtual Work, which is discussed later.

Remark 2.4. For this particular structural element $\mathbf{T}$ is square and orthogonal, that is, $\mathbf{T}^T = \mathbf{T}^{-1}$. But this property does not extend to more general elements. Furthermore in the general case $\mathbf{T}$ is not even a square matrix, and does not possess an ordinary inverse. However the congruential transformation relations (2.15)–(2.17) do hold generally.

§2.8.2. Transformation to Global System

From now on we reintroduce the member (element) index, e. The member stiffness equations in global coordinates will be written

$$\mathbf{f}^e = \mathbf{K}^e \mathbf{u}^e. \tag{2.15}$$

The compact form of (2.13) and (2.14) for the e^{th} member is

$$\bar{\mathbf{u}}^e = \mathbf{T}^e \mathbf{u}^e, \qquad \mathbf{f}^e = (\mathbf{T}^e)^T\, \bar{\mathbf{f}}^e. \tag{2.16}$$

Inserting these matrix expressions into $\bar{\mathbf{f}}^e = \overline{\mathbf{K}}^e\, \bar{\mathbf{u}}^e$ and comparing with (2.15) we find that the member stiffness in the global system $\{x, y\}$ can be computed from the member stiffness $\bar{\mathbf{K}}^e$ in the local system $\{\bar{x}, \bar{y}\}$ through the congruential transformation

$$\boxed{\mathbf{K}^e = (\mathbf{T}^e)^T\, \bar{\mathbf{K}}^e \mathbf{T}^e.} \tag{2.17}$$

Carrying out the matrix multiplications in closed form we get

$$\mathbf{K}^e = \frac{E^e A^e}{L^e} \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}, \tag{2.18}$$

in which $c = \cos\varphi^e$, $s = \sin\varphi^e$, with e superscripts of c and s suppressed to reduce clutter. If the angle is zero we recover (2.10), as may be expected. $\mathbf{K}^e$ is called a *member stiffness matrix in global coordinates*. The proof of (2.17) and verification of (2.18) is left as Exercise 2.8.

The globalized member stiffness matrices for the example truss can now be easily obtained by inserting appropriate values into (2.18). For member (1), with end joints 1–2, angle $\varphi = 0°$ and the member properties given in Figure 2.4(a) we get

$$\begin{bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \end{bmatrix} = 10 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x1}^{(1)} \\ u_{y1}^{(1)} \\ u_{x2}^{(1)} \\ u_{y2}^{(1)} \end{bmatrix}. \tag{2.19}$$

For member (2), with end joints 2–3, and angle $\varphi = 90°$:

$$\begin{bmatrix} f_{x2}^{(2)} \\ f_{y2}^{(2)} \\ f_{x3}^{(2)} \\ f_{y3}^{(2)} \end{bmatrix} = 5 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{x2}^{(2)} \\ u_{y2}^{(2)} \\ u_{x3}^{(2)} \\ u_{y3}^{(2)} \end{bmatrix}. \tag{2.20}$$

Finally, for member (3), with end joints 1–3, and angle $\varphi = 45^\circ$:

$$\begin{bmatrix} f_{x1}^{(3)} \\ f_{y1}^{(3)} \\ f_{x3}^{(3)} \\ f_{y3}^{(3)} \end{bmatrix} = 20 \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} u_{x1}^{(3)} \\ u_{y1}^{(3)} \\ u_{x3}^{(3)} \\ u_{y3}^{(3)} \end{bmatrix}. \tag{2.21}$$

In the following Chapter we will complete the main DSM steps by putting the truss back together through the merge step, and solving for the unknown forces and displacements.

Notes and Bibliography

The Direct Stiffness Method has been the dominant FEM version since the mid-1960s, and is the procedure followed by all major commercial codes in current use. DSM was invented and developed at Boeing in the early 1950s, through the leadership of Jon Turner [174–177], and had defeated its main competitor, the Force Method, by 1970 [59].

All applications-oriented FEM books cover the DSM, although the procedural steps are sometimes not clearly identified. In particular, the textbooks recommended in §1.7.6 offer adequate expositions.

Trusses, also called bar assemblies, are usually the first structures treated in Mechanics of Materials books written for undergraduate courses. Two widely used books at this level are [12] and [137].

Steps in the derivation of stiffness matrices for truss elements are well covered in a number of early treatment of finite element books, of which Chapter 5 of Przemieniecki [140] is a good example.

References

Referenced items have been moved to Appendix R.

Homework Exercises for Chapter 2
The Direct Stiffness Method I

EXERCISE 2.1 [D:10] Explain why *arbitrarily oriented* mechanical loads on an *idealized* pin-jointed truss structure must be applied at the joints. [Hint: idealized truss members have no bending resistance.] How about actual trusses: can they take loads applied between joints?

EXERCISE 2.2 [A:15] Show that the sum of the entries of each row of the master stiffness matrix $\mathbf{K}$ of any plane truss, before application of any support conditions, must be zero. [Hint: apply translational rigid body motions at nodes.] Does the property hold also for the columns of that matrix?

EXERCISE 2.3 [A:15] Using matrix algebra derive (2.10) from (2.8) and (2.9). Note: Place *all equations in matrix form first* and eliminate d and F by matrix multiplication. Deriving the final form with scalar algebra and rewriting it in matrix form gets no credit.

EXERCISE 2.4 [A:15] By direct multiplication verify that for the truss member of Figure 2.7(a), $\bar{\mathbf{f}}^T \bar{\mathbf{u}} = F\,d$. Intepret this result physically. (Hint: what is a force times displacement in the direction of the force?)

EXERCISE 2.5 [A:20] The transformation equations between the 1-DOF spring and the 4-DOF generic truss member may be written in compact matrix form as

$$d = \mathbf{T}_d\,\bar{\mathbf{u}}, \qquad \bar{\mathbf{f}} = F\,\mathbf{T}_f, \tag{E2.1}$$

where $\mathbf{T}_d$ is 1×4 and $\mathbf{T}_f$ is 4×1. Starting from the identity $\bar{\mathbf{f}}^T \bar{\mathbf{u}} = F\,d$ proven in the previous exercise, and using *compact matrix notation*, show that $\mathbf{T}_f = \mathbf{T}_d^T$. Or in words: *the displacement transformation matrix and the force transformation matrix are the transpose of each other*. (This can be extended to general systems)

EXERCISE 2.6 [A:20] Derive the equivalent spring formula $F = (EA/L)\,d$ of (2.8) by the Theory of Elasticity relations $e = d\bar{u}(\bar{x})/d\bar{x}$ (strain-displacement equation), $\sigma = Ee$ (Hooke's law) and $F = A\sigma$ (axial force definition). Here e is the axial strain (independent of $\bar{x}$) and σ the axial stress (also independent of $\bar{x}$). Finally, $\bar{u}(\bar{x})$ denotes the axial displacement of the cross section at a distance $\bar{x}$ from node i, which is linearly interpolated as

$$\bar{u}(\bar{x}) = \bar{u}_{xi}\left(1 - \frac{\bar{x}}{L}\right) + \bar{u}_{xj}\,\frac{\bar{x}}{L} \tag{E2.2}$$

Justify that (E2.2) is correct since the bar differential equilibrium equation: $d[A(d\sigma/d\bar{x})]/d\bar{x} = 0$, is verified for all $\bar{x}$ if A is constant along the bar.

EXERCISE 2.7 [A:20] Derive the equivalent spring formula $F = (EA/L)\,d$ of (2.8) by the principle of Minimum Potential Energy (MPE). In Mechanics of Materials it is shown that the total potential energy of the axially loaded bar is

$$\Pi = \tfrac{1}{2}\int_0^L A\,\sigma\,e\;d\bar{x} - Fd, \tag{E2.3}$$

where symbols have the same meaning as the previous Exercise. Use the displacement interpolation (E2.2), the strain-displacement equation $e = d\bar{u}/d\bar{x}$ and Hooke's law $\sigma = Ee$ to express Π as a function $\Pi(d)$ of the relative displacement d only. Then apply MPE by requiring that $\partial\Pi/\partial d = 0$.

EXERCISE 2.8 [A:20] Derive (2.17) from $\bar{\mathbf{K}}^e \bar{\mathbf{u}}^e = \bar{\mathbf{f}}^e$, (2.15) and (2.17). (*Hint*: premultiply both sides of $\bar{\mathbf{K}}^e \bar{\mathbf{u}}^e = \bar{\mathbf{f}}^e$ by an appropriate matrix). Then check by hand that using that formula you get (2.18). Falk's scheme is recommended for the multiplications.[9]

EXERCISE 2.9 [D:5] Why are disconnection and localization labeled as "conceptual steps" in Figure 2.5?

[9] This scheme is useful to do matrix multiplication by hand. It is explained in §B.3.2 of Appendix B.