Math club modules

Topic: Geometry

Triangles

Armaan Hooda (Organizer)

Contents

1 Congruence of Triangles **1**
1.1 Different congruence criteria . . . 1

2 Angles of a Transversal **2**
2.1 Types of angles . . . 2
2.2 Properties . . . 2
2.3 Practice problems . . . 5

3 Area of Triangles **7**
3.1 Revision . . . 7
3.2 Properties . . . 7
3.3 Practice problems . . . 8

4 Similarity of Triangles **9**
4.1 Different similarity criteria . . . 9
4.2 Properties . . . 10
4.3 Practice problems . . . 11

1 Congruence of Triangles

Two triangles ΔABC and $\Delta A_1B_1C_1$ are said to be congruent when their **corresponding** sides and **corresponding** angles are equal.

1.1 Different congruence criteria

Criterion SSS (side-side-side)

If three pairs of corresponding sides are equal, then the triangles are congruent.

Criterion SAS (side-angle-side)

If two pairs of corresponding sides and the angles between them are equal, then the triangles are congruent.

Criterion ASA (angle-side-angle)

If two pairs of corresponding angles and the sides formed by the common rays of these angles are equal, then the triangles are congruent.

2 Angles of a Transversal

2.1 Types of angles

When two lines p and q are intersected by a third line t, we get 8 angles. The line t is called a transversal. The pairs of angles, depending on their position relative to the transversal and the two given lines are called:

corresponding angles: if they lie on the same side of the transversal and one of them is in the interior of the lines p and q, while the other one is in the exterior.

alternate angles: if they lie on different side of the transversal and both of them are either in the interior or in the exterior of the lines p and q;

consecutive interior angles: if they lie on the same side of the transversal and both of them are either in the interior or in the exterior of the lines p and q.

2.2 Properties

Property 2.1 (Sum of angles in a triangle): The sum of the interior angles in a triangle is 180 degrees.

Property 2.2: Find the sum of interior angles in an n-gon.

Property 2.3: In any triangle, a greater side subtends a greater angle.

Property 2.4 (Triangle Inequality): In any triangle, the sum of the lengths of any two sides is greater than the length of the third side.

Property 2.5 (Midsegment Theorem): In a triangle, the segment joining the midpoints of any two sides is parallel to the third side and half its length.

Property 2.6: Let M be a midpoint of the side AB in the triangle ABC. Prove that $\angle ABC = 90^\circ$ if and only if $\overline{MA} = \overline{MB} = \overline{MC}$.

2.3 Practice problems

1. Let ABC be an equilateral triangle. Let $D \in AB$ and $E \in BC$, such that $AD = BE$. Let $AE \cap CD = F$. Find $\angle CFE$.

2. In the triangle ABC, let A_1 be the midpoint of BC and let B_1 and C_1 be the feet of the altitudes from the vertices B and C, respectively. Prove that the triangle $A_1B_1C_1$ is equilateral if and only if $\angle BAC = 60^\circ$.

3. Let M and N be midpoints of the sides AB and AC, respectively, in a triangle ABC. Let P and Q be points outside the triangle, such that $PM \perp AB$, $PM = \frac{1}{2}AB$ and $QN \perp AC$, $QN = \frac{1}{2}AC$. If L is the midpoint of BC, prove that $LP = LQ$ and $\angle PLQ = 90^\circ$.

4. Let C be a point on the line segment AB. Let D be a point that doesn't lie on the line AB. Let$M$ and N be points on the angle bisectors of $\angle ACD$ and $\angle BCD$, respectively, such that $MN \parallel AB$. Prove that the line CD bisects MN.

5. Let ABC be a triangle and let M be a point on the ray AB beyond B, such that $\overline{BM} = \overline{BC}$. Prove that MC is parallel to the angle bisector of ABC.

6. **(IGO 2015, Elementary)** Let ABC be a triangle with $\angle A = 60^\circ$. The points M, N and K lie on BC, AC and AB, respectively, such that $\overline{BK} = \overline{KM} = \overline{MN} = \overline{NC}$. If $\overline{AN} = 2\overline{AK}$, find the values of $\angle B$ and $\angle C$.

7. Let $ABCD$ be a convex quadrilateral with right angle at the vertex C. Let $P \in CD$, such that $\angle APD = \angle BPC$ and $\angle BAP = \angle ABC$. Prove that
$$\overline{BC} = \frac{\overline{AP} + \overline{BP}}{2}$$

8. **(IGO 2017, Intermediate)** Let ABC be an acute-angled triangle with $\angle A = 60^\circ$. Let E, F be the feet of altitudes through B, C respectively. Prove that $\overline{CE}$-$\overline{BF} = \frac{3}{2}(\overline{AC}$-$\overline{AB})$.

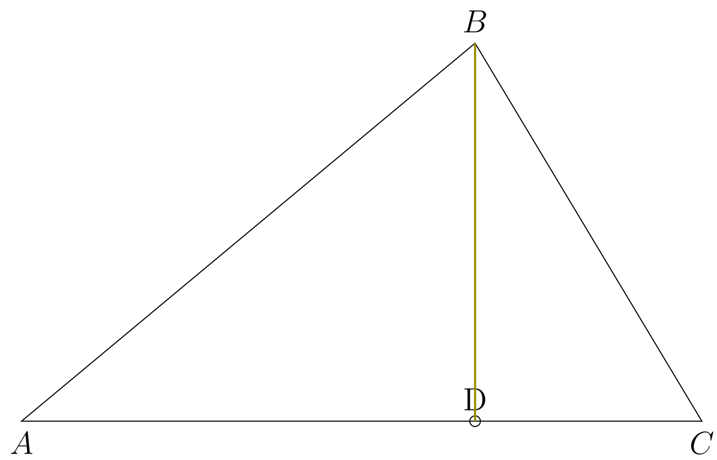
9. Prove that any point P that lies on the side bisector of a line segment AB is equidistant from the endpoints.

10. Let P_0 be the reflection of the point P with respect to the line AB. Prove that $\triangle PAB \cong \triangle P_0AB$.

3 Area of Triangles

3.1 Revision

Area of a triangle $= \frac{1}{2} \times \text{base} \times \text{altitude} = \frac{1}{2} \times AC \times BD$



3.2 Properties

Property 3.1

1. Two triangles that have base sides of equal length and a common altitude, have equal areas.
2. Two triangles that have a common base side and altitudes of equal length, have equal areas.

Property 3.2: Let A, P, B be collinear points in that order and let Q be a point that is not collinear with them. Then

$$\frac{P_{\triangle APQ}}{P_{\triangle BPQ}} = \frac{\overline{AP}}{\overline{PB}}$$

Property 3.3 (Thales' Proportionality Theorem): Let OAB be a triangle and let CD be a line that intersects its sides OA and OB at C and D, respectively. Prove that

$$AB \parallel CD \iff \frac{\overline{OC}}{\overline{CA}} = \frac{\overline{OD}}{\overline{DB}}$$

3.3 Practice problems

1. Let $ABCD$ be a parallelogram with area 1. Let M be the midpoint of the side AD. Let $BM \cap AC = P$. Find the area of $MPCD$.
2. Let $ABCD$ be a trapezoid ($AB \parallel CD$). Let its diagonals AC and BD intersect at P. Let the areas of the triangle $\triangle ABP$ and $\triangle CDP$ be m and n, respectively. Prove that area of $ABCD = (\sqrt{m} + \sqrt{n})^2$.
3. Let P be a point on the side AB in $\triangle ABC$, such that $AP = 3PB$. Let $Q \in AC$, such that $AQ = 4QC$. Prove that BQ bisects the line segment CP.

4 Similarity of Triangles

Two triangles ΔABC and $\Delta A_1B_1C_1$ are said to be similar when their **corresponding** angles are equal and their **corresponding** sides are proportional.

Question: What is the ratio of areas of two similar triangles if the ratio of their sides is k? (k is called the **ratio of similarity**)

4.1 Different similarity criteria

Criterion AA (angle-angle)

If two pairs of corresponding angles are equal, then the triangles are similar.

Criterion SSS (side-side-side)

If three pairs of corresponding sides are proportional, then the triangles are similar.

Criterion SAS (side-angle-side)

If two pairs of corresponding sides are proportional and the angles between them are equal, then the triangles are similar.

4.2 Properties

Property 4.1 (Euclid's laws): In a right triangle ABC, with the right angle at C, let D be the foot of the perpendicular from C to AB. Prove that:

$$\overline{CD} = \overline{AD} \cdot \overline{DB}$$
$$\overline{AC} = \overline{AD} \cdot \overline{AB}$$
$$\overline{BC} = \overline{BD} \cdot \overline{BA}$$

Property 4.2 (Pythagorean Theorem): Prove that the square of the hypothenuse in a right triangle is equal to the sum of the squares of the legs.

Property 4.3 (Angle Bisector Theorem): The angle bisector in a triangle divides the opposite side in segments proportional to the other two sides of the triangle.

4.3 Practice problems

1. Let $\triangle ABC$ be a right triangle. The angle bisector of $\angle ABC$ intersects AC at D. If $AD = 5$ and $CD = 3$, find AB.

2. In the triangle ABC, let BE and CF be perpendiculars to the angle bisector AD. Prove that $\overline{AE} \cdot \overline{DF} = \overline{AF} \cdot \overline{DE}$.

3. Let ABC be a right triangle ($\angle BCA = 90°$). Let CD be the altitude from the vertex C. Prove that the distances from the point D to the legs of the triangle are proportional to the lengths of the legs.

4. **(Sharygin 2011, Correspondence Round)**. The diagonals of a trapezoid $ABCD$ meet at point O. Point M of lateral side CD and points P, Q of bases BC and AD are such that segments MP and MQ are parallel to the diagonals of the trapezoid. Prove that line PQ passes through point O.