Euclidean versus Non-Euclidean Geometry

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Contents

1 Introduction

In this paper, we would briefly go through the history of Euclidean and Non-Euclidean Geometry, try and see if we can prove the problematic fifth postulate of Euclid, the negation of the fifth postulate, that leads us to two conclusions, one that there may be no line passing through a point parallel to a given line, or there may be more than one line passing through such a point, that is parallel to a given line. This gives rise to Spherical and Hyperbolic Geometry. Under Spherical Geometry we go through the concepts of Geodesics, angles, spherical triangles, and their area. Next, we explore Hyperbolic Geometry, path integrals, distances in hyperbolic geometry, geodesics, failure of the fifth postulate, angles and calculation of area. This helps us understand the stark differences that arise between Euclidean and Non-Euclidean Geometry. Finally, we dive deep into the applications of Non-Euclidean Geometry, explaining Hyperbolic Neural Networks, Cosmology and geometry of the universe, how matter in the universe affects its geometry and use of spherical geometry in navigation.

2 Euclidean Geometry

1. **Euclidean geometry** is a mathematical system created by the ancient Greek mathematician Euclid, who documented it in his textbook called **the Elements**.

2. Euclid's method involves starting with a few simple and intuitive axioms (or postulates) and then using logical deduction to prove many other propositions (or theorems) based on these axioms.

3. He was the first to systematically organize these propositions into a logical system, where each result is derived from axioms and previously proven theorems.

3 History

1. In the early 19th century, significant progress was made in the development of non-Euclidean geometry. Russian mathematician Nikolai Ivanovich Lobachevsky and Hungarian mathematician János Bolyai independently published treatises on hyperbolic geometry in 1829-1830 and 1832 respectively.

2. Lobachevsky's approach involved negating the parallel postulate, while Bolyai developed a geometry where both Euclidean and hyperbolic geometries were possible depending on a parameter "k".

3. In 1854, Bernhard Riemann, in a famous lecture, established the field of Riemannian geometry. He discussed concepts such as manifolds, Riemannian metric, and curvature, which are fundamental to understanding non-Euclidean geometries.

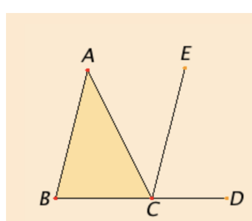
4. Riemann also constructed an infinite family of non-Euclidean geometries by providing a formula for a family of Riemannian metrics on the unit ball in Euclidean space, expanding the understanding and possibilities of non-Euclidean geometries.

4 Euclid's Postulates

1. A straight line segment can be drawn joining any two points.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight lines segment, a circle can be drawn having the segment as radius and one endpoint as center.
4. All Right Angles are congruent.
5. If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two Right Angles, then the two lines inevitably must intersect each other on that side if extended far enough. This postulate is equivalent to what is known as the Parallel Postulate.

5 The 5th Postulate

The 5th postulate is equivalent to the fact that the interior angles of a triangles add up to two right angles(180 degrees). Here is the proof for the same as given by Euclid himself:
Proposition: A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the sum of the interior angles on the same side equal to two right angles.
Proof Let ABC be a triangle, and let one side of it BC be produced to D.



I say that the exterior angle ACD equals the sum of the two interior and opposite angles CAB and ABC, and the sum of the three interior angles of the triangle ABC, BCA, and CAB equals two right angles.
Draw CE through the point C parallel to the straight line AB.

Since AB is parallel to CE, and AC falls upon them, therefore the alternate angles BAC and ACE equal one another.

Again, since AB is parallel to CE, and the straight line BD falls upon them, therefore the exterior angle ECD equals the interior and opposite angle ABC. .

But the angle ACE was also proved equal to the angle BAC. Therefore the whole angle ACD equals the sum of the two interior and opposite angles BAC and ABC.

Add the angle ACB to each. Then the sum of the angles ACD and ACB equals the sum of the three angles ABC, BCA, and CAB.

But the sum of the angles ACD and ACB equals two right angles. Therefore the sum of the angles ABC, BCA, and CAB also equals two right angles.

Therefore *in any triangle, if one of the sides is produced, then the exterior angle equals the sum of the two interior and opposite angles, and the sum of the three interior angles of the triangle equals two right angles.*

6 Axiomatic Basis Of Non-Euclidean Geometry

To obtain a non-Euclidean geometry, the parallel postulate (or its equivalent) must be replaced by its negation. Negating Playfair's axiom form, since it is a compound statement (... there exists one and only one ...), can be done in two ways:

1. In the first case, replacing the parallel postulate (or its equivalent) with the statement "In a plane, given a point P and a line l not passing through P, there exist two lines through P, which do not meet l" and keeping all the other axioms, yields hyperbolic geometry.

2. The second case is not dealt with as easily. Simply replacing the parallel postulate with the statement, "In a plane, given a point P and a line l not passing through P, all the lines through P meet l", does not give a consistent set of axioms. This follows since parallel lines exist in absolute geometry, but this statement says that there are no parallel lines. This problem was known (in a different guise) to Khayyam, Saccheri and Lambert and was the basis for their rejecting what was known as the "obtuse angle case". To obtain a consistent set of axioms that includes this axiom about having no parallel lines, some other axioms must be tweaked. These adjustments depend upon the axiom system used. Among others,

these tweaks have the effect of modifying Euclid's second postulate from the statement that line segments can be extended indefinitely to the statement that lines are unbounded. Riemann's elliptic geometry emerges as the most natural geometry satisfying this axiom. In this paper, we will be discussing about Spherical geometry, which is closely related to Elliptical geometry.

7 Spherical Geometry

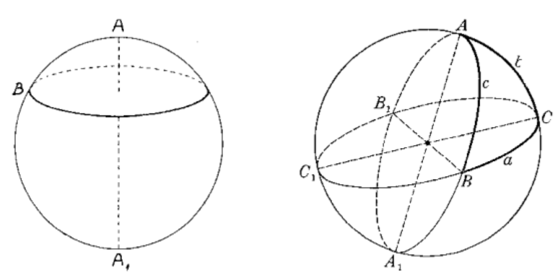
1. In Euclidean geometry, it is well-known that the shortest distance between two points is a straight line. However, in more complex two-dimensional surfaces, such as those studied in differential geometry, this concept becomes more complicated. Differential geometry is a branch of mathematics that deals with this area.

2. If the surface is represented by parameters, the theory of differential geometry can be used to derive a complex differential equation, whose solution represents the shortest path between two points on the surface.

3. Such a curve is called a **geodesic**. As **"geo"** means earth in Greek, the concept of a geodetic refers to the shortest path between two points on the surface of the earth.

7.1 Geodesic in Spherical Geometry

- When a sphere is cut with a plane, the resulting intersecting curve is a circle. However, if the plane passes through the center of the sphere, the intersection forms a great circle, which has a diameter equal to the diameter of the sphere. Both scenarios are illustrated in the figures above.
- In the field of differential geometry, it can be proven that the shortest path between two points on a sphere is a segment of the great circle that passes through those two points.



- In plane geometry, most figures are composed of straight line segments. Since great circles on a sphere can be considered as analogous to "straight lines" or geodesics, we will focus solely on the geometry of figures that consist of segments of great circles.

7.2 Angles in Spherical Geometry

- The points where two different great circles intersect are located exactly opposite to each other along a diameter, which also serves as the line where the planes that generate the two great circles intersect. The angle between two great circles is determined by the angle between their corresponding planes.
- The axis, or diameter, that is perpendicular to a plane belonging to a great circle intersects the sphere at two opposite points known as the poles of the great circle.
- When one great circle intersects the other at its poles, the two great circles are perpendicular to each other, as shown in the diagram below.

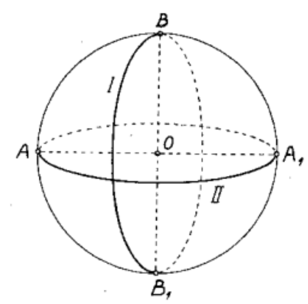


Fig. 49.

7.3 Spherical Triangles

- A spherical triangle is a portion of a sphere that is enclosed by the arcs of three great circles, which are referred to as the sides of the spherical triangle. These sides are typically measured in degrees or radians.
- On a sphere with a radius of R, the length of a side, denoted as a, can be calculated by multiplying its radian measure α by R:

$$a = \alpha R$$

- The angles of the spherical triangle are determined by the intersection angles between the great circles that form it.

- The concepts of isosceles triangles, height, bisecting lines, and bisector normal are similar to those in plane geometry.
- However, it's worth noting that the sum of the three angles in a spherical triangle is always greater than π

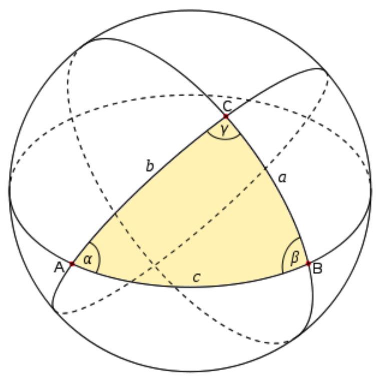
7.4 Area of Spherical Triangle

The surface area of any spherical triangle is given by :

$$A = R^2 E$$

where R is the radius of the sphere and E is the angular excess given by

$$E = A + B + C - \pi$$



Proof :

Consider the following three parts of the sphere: let P_A be the lune created from the triangle ABC , plus the triangle adjacent across the BC line segment, plus the opposite lune (on the opposite side of the sphere), and similarly for parts P_B and P_C.

the total area of the sphere is $4\pi R^2$, and the area of PA is certainly proportional to α

The area of PA is $4\frac{(2\alpha)}{2\pi}\pi R^2$

i.e. area of PA = $4\alpha R^2$

We notice now that $P_A \bigcup P_B \bigcup P_C$ is the entire sphere. We thus have:

area(P_A) + area(P_B) + area(P_C) = area of the sphere + 2 area of the triangle +2 area of the opposite triangle.

As the two triangles have the same area, X (say).

We get,

$$4\alpha R^2 + 4\beta R^2 + 4\gamma R^2 = 4\pi R^2 + 4X$$
$$X = (\alpha + \beta + \gamma - \pi)R^2$$

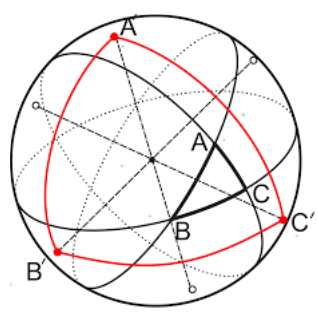
i.e.

$$X = ER^2$$

where E $= \alpha + \beta + \gamma - \pi$ is the angular excess

7.5 Perpendicularity of lines in Spherical triangles

- Every great circle has a pair of poles : meets of sphere with altitude line to the plane of the great circle through the center.
- **Perpendicularity** : Two lines are perpendicular iff one passes through the poles of the other.
- **Polar Of a Point :** For a point A on the sphere, its polar is the line *greatcircle* perpendicular to the diameter *axis* through A. So, every line has two poles, and every point has one polar line.
- **Perpendicularity of points :** Two points are perpendicular iff one lies on the polar of the other.
- **Polar Triangle :** If ABC is a spherical triangle , choose poles A', B', C' of lines BC, AC, AB respectively which are on the same sides as A,B,C. Then A$'$B$'$C$'$ is the polar triangle of ABC.



Theorem : Let △ ABC be a spherical triangle on the surface of a sphere whose center is O. Let the sides a,b,c of △ ABC be measured by the angles subtended at O, where a,b,c are opposite A,B,C respectively. Let △A′B′C′ be the polar triangle of △ ABC. Then A' is the supplement of a.That is:

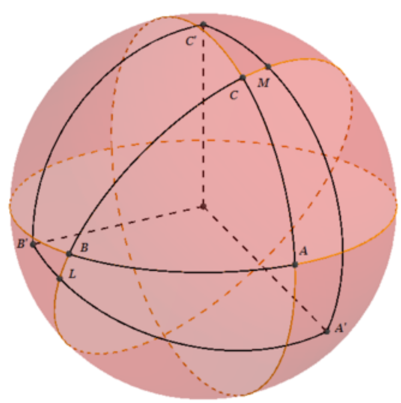
$$A' = \pi - a$$

and it follows by symmetry that:

$$B' = \pi - b$$

$$C' = \pi - c$$

.



Proof :

Let BC be produced to meet $A'B'$ and $A'C'$ at L and M respectively.

Because A' is the pole of the great circle LBCM, the spherical angle A' equals the side of the spherical triangle $A'LM$.

That is:

$$\angle A' = LM \tag{1}$$

From Spherical Triangle is Polar Triangle of its Polar Triangle, $\triangle ABC$ is also the polar triangle of △ A'B'C'.

That is, C is a pole of the great circle A'LB '.

Hence CL is a right angle.

Similarly, BM is also a right angle.

Thus we have:

$$
\begin{aligned}
LM &= LB + BM \\
&= LB + 90^\circ
\end{aligned} \tag{2}
$$

By definition, we have that:
$BC = a$

$$
\begin{aligned}
BC &= a \quad \text{by definition of } \triangle \text{ ABC} \\
LB + a &= LC \\
LB &= 90^\circ - a \quad \text{as LC} = 90^\circ
\end{aligned} \tag{3}
$$

Then :

$$
\begin{aligned}
\angle A &= LM \quad \text{from (1)} \\
&= LB + 90^\circ \quad \text{from (2)} \\
&= (90^\circ - a) + 90^\circ \quad \text{from (3)} \\
&= 180^\circ - a
\end{aligned} \tag{4}
$$

That is, A' is the supplement of a:

$$A' = \pi - a$$

By applying the same analysis to B' and C' , it follows similarly that:

$$B' = \pi - b$$

$$C' = \pi - c$$

Applying the above theorem about the sides and the angles in the polar triangle to a spherical triangle, we can prove that the sum of the three angles in a spherical triangle is always bigger than 180 °and less than 540°.

We assume that we have constructed the polar triangle to a spherical triangle ABC. According to the above theorem the sides in the polar triangle are $180^\circ - A$, $180^\circ - B$, $180^\circ - C$

Since the sum of the three sides must be less than 360 °, the following inequality is valid:

$$180^\circ - A + 180^\circ - B + 180^\circ - C < 360^\circ$$

$$\iff A + B + C > 180^\circ$$

Furthermore, the sum of the sides in the polar triangle must be greater than 0

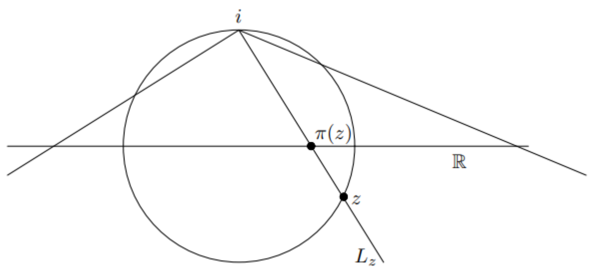
$$180^\circ - A + 180^\circ - B + 180^\circ - C > 0^\circ$$
$$\iff A + B + C < 540^\circ$$

8 Hyperbolic Geometry

There are various methods to create hyperbolic geometry, referred to as "models". Among these models, we will focus on one that is straightforward and practical, known as the upper half-plane model.

- The upper half-plane H is the set of complex numbers z with positive imaginary part:
$$\mathbb{H} = \{z \in \mathbb{C} | Im(z) > 0\}$$
- **Definition :** The circle at infinity or boundary of $\mathbb{H}$ is defined to be the set
$$\partial\mathbb{H} = \{z \in \mathbb{C} |\ \mathrm{Im(z)} = 0\} \bigcup \{\infty\}$$
. That is, $\partial\mathbb{H}$ is the real axis together with the point ∞.
- **Remarks :**
  (a) What does ∞ mean? It's just a point that we have 'invented' so that it makes sense to write things like $1/x \to \infty$ as $x \to 0$
  (b) We call $\partial\mathbb{H}$ the circle at infinity because topologically, it is a circle. We can see this using a process known as stereographic projection. Let
  $$K = \{z \in \mathbb{C} || z| = 1\}$$
  denote the unit circle in the complex plane $\mathbb{C}$.
  Now we define a map:
  $$\pi : K \to \mathbb{R} \bigcup \infty$$
  as follows.
  For z $\in$ K $\setminus$ $\{i\}$ let L_z be the (Euclidean) straight line passing through i and z; this line meets the real axis at a unique point, which we denote by $\pi(z)$. We define $\pi(i) = \infty$.
  (c) The map π is a homeomorphism from K to $\mathbb{R} \bigcup \infty$.
  We call ∂H the circle at infinity because (as we shall see) points on $\partial\mathbb{H}$ are at an infinite 'distance' from any point in $\mathbb{H}$.

Prior to establishing distances in $\mathbb{H}$, it is necessary to review the methodology for computing path integrals in $\mathbb{C}$ (or equivalently, in $\mathbb{R}^2$).



8.1 Path Integrals

A path σ in the complex plane $\mathbb{C}$ refers to the representation of a continuous function

$$\sigma(.) : [a, b] \to \mathbb{C}$$

where $[a, b] \subset \mathbb{R}$ is an interval. It is assumed that σ is differentiable and its derivative $\sigma'(t)$ is continuous. The points $\sigma(a)$ and $\sigma(b)$ are referred to as the end-points of the path σ. A function $\sigma : [a, b] \to \mathbb{C}$ that maps the interval to a given path is called a parametrization of that path. It is important to note that a path can have multiple parametrizations.

Let $f : \mathbb{C} \to \mathbb{R}$ be a continuous function. Then the integral of f along a path σ is defined to be:

$$\int_\sigma f = \int_a^b f(\sigma(t))\,|\sigma|'(t)\,dt$$

here $|\,.\,|$ denotes the usual modulus of a complex number, i.e.

$$|\sigma'(t)| = \sqrt{(Re(\sigma'(t)))^2 + (Im(\sigma'(t)))^2}$$

Remark : In order to calculate the integral of function f along the path σ, we need to select a specific parametrization for that path. Initially, it may seem that the definition of $\int_\sigma f$ depends on the choice of parametrization. However, we can demonstrate that this is not the case, as any two parametrizations of the same path will yield the same result. As a result, we may sometimes refer to a path by its parametrization.

Definition : A path σ with parametrisation $\sigma(.) : [a, b] \to \mathbb{C}$ is piecewise continuously differentiable if there exists a partition $a = t_0 < t_1 < \; < t_{n-1} < t_n = b$ of [a,b] such that $\sigma(.) : [a, b] \to \mathbb{C}$ is a continuous function and, for each j, $0 \leq j \leq n-1$, $\sigma : (t_j, t_{j+1}) \to \mathbb{C}$ is differentiable and has continuous derivative.

8.2 Distance in Hyperbolic Geometry

To define the hyperbolic metric in the upper half-plane model of hyperbolic space , we first define the length of an arbitrary piecewise continuously differentiable path in $\mathbb{H}$.

Let $\sigma : [a, b] \to \mathbb{H}$ be a path in the upper half-plane $\mathbb{H} = z \in \mathbb{C}|Im(z) > 0$. Then the hyperbolic length of σ is obtained by integrating the function f(z) = 1/ Im(z) along σ i.e.

$$length_{\mathbb{H}}(\sigma) = \int_{\sigma} \frac{1}{Im(z)} = \int_{a}^{b} \frac{|\sigma'(t)|}{Im(\sigma(t))}\, dt$$

We are now in a position to define the hyperbolic distance between two points in $\mathbb{H}$.

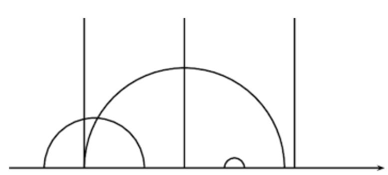
Definition : Let $z, z' \in \mathbb{H}$. We define the hyperbolic distance $d_{\mathbb{H}}(z, z')$ between z and z' as

$$d_{\mathbb{H}}(z, z') = \inf\{length_{\mathbb{H}}(\sigma)|\sigma \text{is a piecewise continuously differentiable path with end-points z and z}'\}$$

Remark: Therefore, we examine all paths that are piecewise continuously differentiable between points z and z', compute the hyperbolic length of each of these paths, and select the one with the shortest length. The smallest possible length (infimum) is attained by a specific path called a geodesic, and this geodesic is the only one that achieves this minimum length.

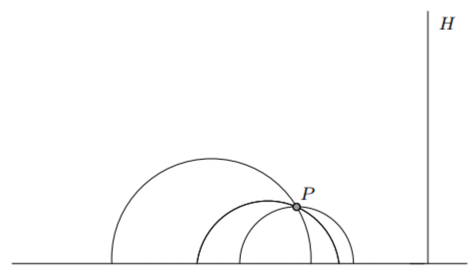
8.3 Geodesics in $\mathbb{H}$

The geodesics, or the shortest paths, in the half-plane model (denoted as $\mathbb{H}$) of hyperbolic geometry are either semi-circles that are perpendicular to the real axis, or vertical straight lines. Additionally, for any two points in H, there is a single geodesic that passes through them, and this geodesic is unique.



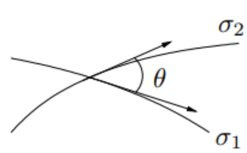
8.4 Failure of Euclid's parallel postulate

We can observe that Euclid's parallel postulate does not hold in the half-plane model of hyperbolic geometry. Specifically, if we have a geodesic (shortest path) and a point not on that geodesic, there are infinitely many other geodesics that pass through that point but do not intersect the given geodesic.



8.5 Angles

Assume that we have two paths, denoted as σ_1 and σ_2, which intersect at a point z in the half-plane model denoted as $\mathbb{H}$. By selecting suitable parameterizations for the paths, we can assume that z is the initial point of both paths, i.e., $z = \sigma_1(0) = \sigma_2(0)$. The angle between σ_1 and σ_2 at z is defined as the angle between their tangent vectors at the point of intersection.



8.6 Area

Let $A \subset \mathbb{H}$ be a subset of the upper half-plane. The hyperbolic area of A is defined to be the double integral

$$\mathrm{Area}_{\mathbb{H}}(A) = \int\int_A \frac{1}{y^2} dx\, dy = \int\int_A \frac{1}{Im(z)^2} dz$$

9 Applications of Non-Euclidean Geometry

9.1 Hyperbolic Deep Neural Networks

- In most of the current deep learning applications, the representation learning is conducted in the Euclidean space, which makes sense as the Euclidean space is the natural generalization of the visual three-dimensional space. However, recent research shows that many types of complex data exhibit a highly non-Euclidean latent anatomy. Also, it appears in several applications that the dissimilarity measures constructed by experts tend to have non-Euclidean behavior. In such cases, the Euclidean space does not provide the most powerful or meaningful geometrical representations.
- In many domains, data is with a tree-like structure or can be represented hierarchically. For example, social networks, human skeletons, sentences in natural language, and evolutionary relationships between biological entities in phylogenetics.
- Recently, hyperbolic spaces have been proposed as an alternative continuous approach to learn hierarchical representations from textual and graph-structured data. The negative-curvature of the hyperbolic space results in very different geometric properties, which makes it widely employed in many such areas. In the hyperbolic space, circle circumference $(2sinh(r))$ and disc area $(2\pi(cosh(r) - 1)$ grow exponentially with radius r, unlike the Euclidean space where they only grow linearly and quadratically. The exponential growth of the Poincare surface area with respect to its radius is analogous to the exponential growth of the number of leaves in a tree with respect to its depth, rather than polynomially as in the Euclidean case.
- Even when using an infinite number of dimensions, Euclidean space is unable to achieve comparable low distortion for tree data. Additionally, the smoothness of the hyperbolic spaces makes it possible to employ deep learning strategies that depend on differentiability. Hence, hyperbolic spaces have recently gained popularity in the context of deep neural networks to model embedded data into the space.

9.2 Cosmology and geometry of the universe

Einstein's general theory of relativity predicts that
ount of matter(or energy) in the universe.

In the general theory of relativity, as in the special theory of relativity, we deal with 4 dimensional manifold called space-time. In general, we can describe any such manifold by a metric or the line element given by ds^2. Einstein assumed the universe to be homogeneous, meaning that at big enough scales(typically at the scale of millions of galaxies) the universe "looks the same", meaning it has same properties at every point. Its also assumed to be isotropic, meaning there is no preferred direction. This is called the **Cosmological Principle** and is one of the most important underlying principles of all of cosmology.

Using the cosmological principle and the fact that space should have constant curvature, we can arrive at a line element most suitable to describe the whole universe's geometry. it is given by

$$\mathrm{d}s^2 = c^2\ \mathrm{d}t^2 - S^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\ \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)\right)$$

here, c is the speed of light, k can take values $-1, 0, 1$ we will shortly explain the significance of these. $S(t)$ is called the expansion factor or the scale factor which is a function of time, and its double derivative gives the rate of expansion of the universe as observed by astronomers. Let us take different cases of k.

9.2.1 k=0

When $k = 0$, we simply have

$$ds^2 = c^2 dt^2 - S^2(t)\left(\mathrm{d}r^2 + r^2\left(\ \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)\right)$$

If we now just look at the spatial part of this line element, call this $d\sigma^2$, we see that it is simply the Euclidean line element scaled by the constant factor S. Hence this case describes a flat(Euclidean) universe.

9.2.2 k=1

The line elements is,

$$\mathrm{d}s^2 = c^2\ \mathrm{d}t^2 - S^2(t)\left(\frac{\mathrm{d}r^2}{1-r^2} + r^2\left(\ \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)\right)$$

We notice that as $r \to 1$ there is a singularity, so we introduce a new coordinate $r = sin\chi$.

Then the metric becomes (let S_0 denote present-day scale factor)

$$\mathrm{d}\sigma^2 = S_0^2\left[\ \mathrm{d}\chi^2 + \sin^2\chi\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)\right].$$

We can now embed this 3 -surface in a four-dimensional Euclidean space with coordinates (w, x, y, z), where

$$\left.\begin{array}{l} w = S_0 \cos\chi, \\ x = S_0 \sin\chi \sin\theta \cos\phi, \\ y = S_0 \sin\chi \sin\theta \sin\phi, \\ z = S_0 \sin\chi \cos\theta. \end{array}\right\}$$

The embedding is possible because if substitute

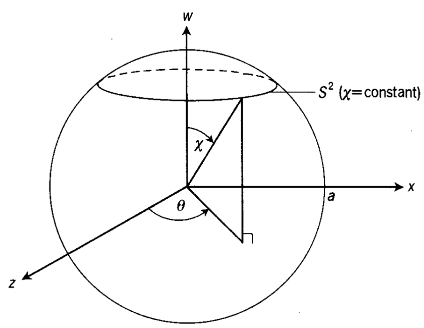
$$\mathrm{d}\sigma^2 = \mathrm{d}w^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = S_0^2 \left[\, \mathrm{d}\chi^2 + \sin^2\chi \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)\right]$$

we get

$$w^2 + x^2 + y^2 + z^2 = S_0^2$$

which shows that the surface can be regarded as a three-dimensional sphere in four-dimensional Euclidean space. This is depicted in the figure where one dimension ($y = 0$ or $\phi = 0$) is suppressed.

Figure 1: A surface of constant positive curvature embedded in a four-dimensional Euclidean space



($\phi = 0$)

The hypersurface is defined by the coordinate range

$$0 \leqslant \chi \leqslant \pi, \quad 0 \leqslant \theta \leqslant \pi, \quad 0 \leqslant \phi < 2\pi.$$

The 2-surfaces χ = constant, which appear as circles in the pictures, are 2-spheres of surface area

$$A_\chi = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \left(S_0 \sin\chi \mathrm{d}\theta\right)\left(R_0 \sin\chi \sin\theta \mathrm{d}\phi\right) = 4\pi S_0^2 \sin^2\chi$$

and (θ, ϕ) are the standard spherical polar coordinates of these 2-spheres.

Thus, the area of these 2 -spheres is zero at the North Pole, increases to a maximum at the equator, and decreases again to zero at the South Pole. The surface has 3-volume given by

$$
\begin{aligned}
V &= \int_{\chi=0}^{\pi} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \left(S_0 \ \mathrm{d}\chi\right) \left(S_0 \sin\chi \mathrm{d}\theta\right) \left(S_0 \sin\chi \sin\theta \mathrm{d}\phi\right) \\
&= 2\pi^2 S_0^3 = 2\pi^2 S_0^3 \left(t_0\right),
\end{aligned}
$$

which is why $S\left(t_0\right)$ is often referred to as the 'radius of the universe'.

So this case describes a universe with spherical geometry. We say that in this case we have positive curvature.

9.2.3 k=-1

We introduce a new coordinate $r = sinh\chi$. So the spatial part of the metric becomes,

$$\mathrm{d}\sigma^2 = S_0^2 \left[\, \mathrm{d}\chi^2 + \sin^2\chi \left(\mathrm{d}\theta^2 + \sinh^2\theta \mathrm{d}\phi^2\right)\right]$$

We can no longer embed this 3-surface in a four-dimensional Euclidean space, but it can be embedded in a flat Minkowski space, $ds^2 = -dw^2 + dx^2 + dy^2 + dz^2$ where

$$\left.\begin{array}{l} w = S_0 \cosh\chi \\ x = S_0 \sinh\chi \sin\theta\cos\phi \\ y = S_0 \sinh\chi \sin\theta \sin\phi \\ z = S_0 \sinh\chi \cos\theta \end{array}\right\}$$

These equations imply that

$$w^2 - x^2 - y^2 - z^2 = S_0^2,$$

so that the 3-surface is a three-dimensional hyperboloid in four-dimensional Minkowski space. This is depicted in the figure where one dimension ($y = 0$ or $\phi = 0$) is suppressed. The hypersurface is defined by the coordinate range

$$0 \leqslant \chi < \infty, \quad 0 \leqslant \theta \leqslant \pi, \quad 0 \leqslant \phi < 2\pi.$$

The 2-surfaces $\chi =$ constant, which appear as circles in the figure, are 2-spheres of surface area

$$A_\chi = 4\pi R_0^2 \sinh^2\chi$$

where (θ, ϕ) are the standard spherical polar coordinates on these 2-spheres.

Hence this case gives us a universe that is described by hyperbolic geometry. We say that in this case we have negative curvature.

9.3 How matter in the universe affect its geometry

The central equation in the general theory of relativity is the Einstein's field equation given by,

$$G_{ab} = \frac{8\pi G}{c^4} T_{ab}$$

Where G_{ab} is the Einstein tensor which is derived solely from the metric given above, in essence, it gives us the information about the geometry, whereas the T_{ab} is called the energy-momentum tensor which gives us the description of the matter distribution. This is a tensor equation, but in our case, the homogeneity and isotropy assumptions simplify it and we just have a diagonal 4×4 matrix. Solving this equation in our case gives us what are called the Friedmann equations, which relates the rate of change the scale factor, or in other words, the rate of expansion of the universe to the matter(or energy) density of the universe. The Friedmann equations for pressure-less matter(i.e galaxies that moves at constant velocity) are given by,

$$2\frac{\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2} = 0$$
$$\frac{\dot{S}^2 + kc^2}{S^2} = \frac{8\pi G\rho_0}{3}\frac{S_0^3}{S^3}$$

. where ρ is the matter/energy density. As explained earlier, k here describes the kind of geometry, in the simplest case, we take $k = 0$, it turns out that this is a good approximation according to observations. So the second equation becomes,

$$\frac{\dot{S}^2}{S^2} = \frac{8\pi G\rho_0}{3}\frac{S_0^3}{S^3}$$

Here

$$\left.\frac{\dot{S}}{S}\right|_{t_0} = H_0$$

is called the Hubble's Constant. Hence, applying this to the present epoch, we get

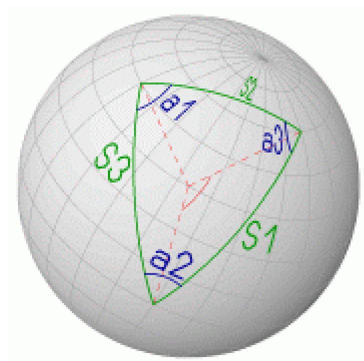
$$\rho_0 = \frac{3H_0^2}{8\pi G} \equiv \rho_{\mathrm{c}}.$$

This is called the critical density or the closure density. If we extend our analysis to other cases of $k = 1, -1$ it is easy to see that for $k = 1$ we get the matter density, is such that $\rho_0 > \rho_c$, this implies that if we have matter density greater than the critical density, we get a spherical universe, or in Einstein's words, the universes "closes".

In the case of $k = -1$, we get that $\rho_0 < \rho_c$, which means for matter density less than the critical density the universe has negative curvature, i.e it will have hyperbolic geometry.

9.4 Spherical Geometry and Navigation

A spherical triangle is defined by three sides with length S1, S2 and S3 and three including angles A1, A2 and A3.
The sides are segments of great circles and the length of each sides is defined by an angle. The angles of the sides are measured at the center of the sphere between the starting and ending "legs" of the great circle segment (shown red in the picture below).



The angles of the triangle A1, A2 and A3 are measured in the horizontal plane (on the surface of the sphere) of the vertex points of the spherical triangle. Notice that since all elements (sides and intersection angles) of the triangle are defined as angles, the values of these elements do not depend on the radius of the underlying sphere.
In the next identities the following convention is assumed: A1 is the angle opposite to side A1, A2 is the angle opposite to S2 and A3 is the angle opposite to S3.

Law of Sines:

$$sin(A1)/sin(S1) = sin(A2)/sin(S2) = sin(A3)/sin(S3)$$

Law of Cosines of Sides:

$$cos(S1) = cos(S2) \cdot cos(S3) + sin(S2) \cdot sin(S3) \cdot cos(a1)$$
$$cos(S2) = cos(S3) \cdot cos(S1) + sin(S3) \cdot sin(S1) \cdot cos(a2)$$
$$cos(S3) = cos(S1) \cdot cos(S2) + sin(S1) \cdot sin(S2) \cdot cos(a3)$$

Law of Cosines of Angles:

$$cos(a1) = -cos(a2) \cdot cos(a3) + sin(a2) \cdot sin(a3) \cdot cos(S1)$$
$$cos(a2) = -cos(a3) \cdot cos(a1) + sin(a3) \cdot sin(a1) \cdot cos(S2)$$
$$cos(a3) = -cos(a1) \cdot cos(a2) + sin(a1) \cdot sin(a2) \cdot cos(S3)$$

NOTE: Solving one of the above equations for a value of sides or angles of the spherical triangle, will require the inverse trigonometric functions Arcsine ($asin(x)$) and Arccosine ($acos(x)$). These "Arcus"-functions are defined uniquely only in a restricted range of resulting angles. This should be considered while applying these functions:

- $asin(x)$ is defined for x in the range between -1.0 to $+1.0$ and returns values between $-90°$ to $90°$. If y=asin(x) then also the angle "$180° - y$" is a valid value for the Arcsine of x: $sin(180° - y) = sin(y) = x$.
- $acos(x)$ is defined for x in the range between -1.0 to $+1.0$ and returns values between $180°$ to $0°$. If $y = acos(x)$ then also the angle "$360° - y$" is a valid value for the Arccosine of x: $cos(360 - y) = cos(-y) = cos(-y) = x$.

With a combination of the "Law of Cosines for Angles" and the "Law of Sines" the following identities can be deduced:

- $tan(A1) = sin(S1) \cdot sin(A3)/[cos(S1) \cdot sin(S2) - cos(a3) \cdot sin(S1) \cdot cos(S2)]$
- $tan(A2) = sin(S2) \cdot sin(A1)/[cos(S2) \cdot sin(S3) - cos(a1) \cdot sin(S2) \cdot cos(S3)]$
- $tan(A3) = sin(S3) \cdot sin(A2)/[cos(S3) \cdot sin(S1) - cos(a2) \cdot sin(S3) \cdot cos(S1)]$

A location or position on the surface of the Earth is uniquely defined by means of Latitude (Lat) and Longitude (Lon), where Latitude is measured from the Equator to the North (positive) or to the South (negative) to the position and Longitude is measured from the Prime Meridian of Greenwich to the East (positive) or to the West (negative) to the position.

The identities for spherical triangles can be directly applied to the special-case triangle setup shown. It consists of two arbitrary locations L0 (Lat0, Lon0) and L1 (Lat1, Lon1) and the North Pole (NP) as vertex points. These three sides are great-circle segments.
The vertex angles A0 and A1 are the angles under which the great-circle segment "L0 - L1" intercepts with the local Meridians in L0 and L1. The third vertex

angle - at the North Pole - is the difference in Longitude of the two locations (Lon1-Lon0).

Since the North Pole is one of the vertex points, two of the sides are meridian segments and thus great-circle segments. The third side **D** is the **great-circle distance** between the locations L0 and L1. The meridian segments of the triangle are the complementary angles of the Latitudes of the positions L0 and L1: $90°-$ Latx.
Two principal navigational problems can be solved with this spherical triangle arrangement:

- The "distance problem":
  If the locations L0 and L1 are given to us, the great-circle distance D between the locations as well as the angles A0 and A1 (true bearings) can be calculated.
- The "destination problem":
  If the location L0 (departure) and the great-circle distance D as well as the angle A0 are known, the position of the destination location L1 can be calculated.

9.4.1 The Distance Problem

With the "Law of Cosines for Sides" applied for side D the following result is obtained:

$$\begin{aligned} cos(D) = {} & cos(90\deg - Lat0)cos(90\deg - Lat1)+ \\ & sin(90\deg - Lat0)sin(90\deg - Lat1)cos(Lon1 - Lon0) \end{aligned}$$

This can be reduced to:

$$cos(D) = sin(Lat0)sin(Lat1) + cos(Lat0)cos(Lat1)cos(Lon1 - Lon0)$$
$$D[\deg] = acos[sin(Lat0)sin(Lat1) + cos(Lat0)cos(Lat1)cos(Lon1 - Lon0)]$$

This identity gives the "angular" distance D between the locations L0 and L1 in degrees. The distance in terms of nautical miles can also be found as, one minute-of-arc corresponds to one nautical mile, along a great circle. Hence, each degree of angular distance corresponds to 60 nautical miles:
$D[Nm] = 60 \cdot acos[sin(Lat0) \cdot sin(Lat1) + cos(Lat0) \cdot cos(Lat1) \cdot cos(Lon1 - Lon0)]$

The values of the vertex angles A0 and A1 can be best obtained by the "Law of Tangents":

- $$\begin{aligned} tan(A0) &= sin(90\deg - Lat1) \cdot sin(Lon1 - Lon0)/[cos(90\deg - Lat1) \cdot sin(90\deg - Lat0) - \\ &\quad cos(Lon1 - Lon0) \cdot sin(90\deg - Lat1) \cdot cos(90\deg - Lat0)] \\ &= cos(Lat1) \cdot sin(Lon1 - Lon0)/[sin(Lat1) \cdot cos(Lat0) - \\ &\quad cos(Lon1 - Lon0) \cdot cos(Lat1) \cdot sin(Lat0)] \end{aligned}$$

- $$\begin{aligned} tan(A1) &= sin(90\deg - Lat0) \cdot sin(Lon1 - Lon0)/[cos(90\deg - Lat0) \cdot sin(90\deg - Lat1) - \\ &\quad cos(Lon1 - Lon0) \cdot sin(90\deg - Lat0) \cdot cos(90\deg - Lat1)] \\ &= cos(Lat0) \cdot sin(Lon1 - Lon0)/[sin(Lat0) \cdot cos(Lat1) - \\ &\quad cos(Lon1 - Lon0) \cdot cos(Lat0) \cdot sin(Lat1)] \end{aligned}$$

9.4.2 The Destination Problem

If a great-circle journey is initiated from a location L0 in an initial direction A0 and the distance travelled is D, the coordinates of the destination location L1 can be found by solving the spherical triangle for the side "90°-Lat1" and the angle "Lon1-Lon0".

Applying the "Law of Cosines for sides" for "90°-Lat1" gives us the latitude of L1:

$$\begin{aligned} cos(90\deg - Lat1) &= cos(90\deg - Lat0) \cdot cos(D) + sin(90\deg - Lat0) \cdot sin(D) \cdot cos(A0) \\ sin(Lat1) &= sin(Lat0) \cdot cos(D) + cos(Lat0) \cdot sin(D) \cdot cos(A0) \\ Lat1 &= asin[sin(Lat0) \cdot cos(D) + cos(Lat0) \cdot sin(D) \cdot cos(A0)] \end{aligned}$$

The Longitude of L1 is obtained by applying the "Law of Tangents" to "Lon1-Lon0":

$$\begin{aligned} tan(Lon1 - Lon0) &= sin(D) \cdot sin(A0)/[cos(D) \cdot sin(90\deg - Lat0) - \\ &\quad cos(A0) \cdot sin(D) \cdot cos(90\deg - Lat0)] \\ &= sin(D) \cdot sin(A0)/[cos(D) \cdot cos(Lat0) - cos(A0) \cdot sin(D) \cdot sin(Lat0)] \end{aligned}$$

The summarized results giving us the coordinates of a destination (Lat0,Lon0) for a given departure (Lat0,Lon0), initial bearing (A0) and distance (D):

$$\begin{aligned} Lat1 &= sin^{-1}[sin(Lat0) \cdot cos(D) + cos(Lat0) \cdot sin(D) \cdot cos(A0)] \\ Lon1 &= Lon0 + tan^{-1} \frac{sin(D) \cdot sin(A0)}{[cos(D) \cdot cos(Lat0) cos(A0) \cdot sin(D) \cdot sin(Lat0)]} \end{aligned}$$

10 References

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