Chromatic homotopy theory at height 1 and the image of J

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Key players at height 1

Formal group law:

Let $F_{\mathrm{m}}(x,y)$ be the p-typification of the multiplicative formal group law $x+y+xy$ over $\mathbb{F}_p$. Then the p-series of F_{m} is

$$[p]_{F_{\mathrm{m}}}(x) = x^p$$

Thus, F_{m} is exactly the height 1 Honda formal group law.

Morava E-theory

In general, we have that

$$E(k,\Gamma) = \mathbb{W}k[\![u_1,\ldots,u_{n-1}]\!]$$

In this case,

$$\mathbb{W}\mathbb{F}_p = \mathbb{Z}_p$$

and

$$E(\mathbb{F}_p, F_{\mathrm{m}}) = \mathbb{Z}_p$$

Adjoining an invertible class in degree -2 to make this into an even periodic theory, we have that the first Morava E-theory E_1 has coefficients $\mathbb{Z}_p[u^{\pm 1}]$. Furthermore, we may take F as a universal deformation of itself. Turning it into a degree -2 formal group law, we have

$$F(x,y) = u^{-1}F(ux, uy)$$

E_1 is a model for p-complete complex K-theory (it has the same coefficients and the same formal group law).

Morava stabilizer group:

We are interested in the group of endomorphisms of the multiplicative formal group law, F. First, note that it must contain $\mathbb{Z}$: given an integer $n \in \mathbb{Z}$, we send it to the n-series $[n]_F(x)$. We may extend this to $\mathbb{Z}_p$, since a p-adically convergent sequence of integers

$n_1, n_2, \ldots$ gives a p-adically convergent sequence of power series $[n_1](x), [n_2](x), \ldots$ (this requires checking $[p^r](x)$ mod x^{pr+1}). It turns out that there are no other endomorphisms, i.e. $\mathrm{End}(F) \cong \mathbb{Z}_p$.

S_1, the group of automorphisms of F, is the group of units in the p-adics. The reduction mod p map

$$\mathbb{Z}_p^\times \to \mathbb{F}_p^\times$$

sits in a short exact sequence

$$1 \to 1 + p\mathbb{Z}_p \to \mathbb{Z}_p^\times \to \mathbb{F}_p^\times$$

Thinking of $\mathbb{F}_p^\times$ as the group μ_{p-1} of $(p-1)$st roots of unity over $\mathbb{F}_p$, we may use Hensel's Lemma to construct a splitting $\mu_{p-1} \to \mathbb{Z}_p^\times$ so that

$$\mathbb{Z}_p^\times \cong \mu_{p-1} \times (1 + p\mathbb{Z}_p)$$

For odd primes, the above is topologically cyclic, generated by any element $g = (\zeta, \alpha)$ such that ζ is a primitive $(p-1)$st root of unity and $\alpha \notin 1 + p^2\mathbb{Z}_p$. For $p = 2$, we have

$$\mathbb{Z}_2^\times \cong \{\pm 1\} \times (1 + 4\mathbb{Z}_2)$$

and while $\mathbb{Z}_2^\times$ is not topologically cyclic, $1 + 4\mathbb{Z}_2$ is.

S_1 acts on the homology theory E_1 as follows. Given an automorphism $g(x)$ of the multiplicative formal group law over $\mathbb{F}_p$, we lift the coefficients to $\mathbb{Z}_p$ and adjust for the grading to get $\tilde{g}(x) = u^{-1}g(ux)$. The induced map $\psi : \mathbb{Z}_p = E(\mathbb{F}_p, F) \to E(\mathbb{F}_p, F) = \mathbb{Z}_p$ must be the identity, and so $\psi^* F = F$. We extend ψ to $\mathbb{Z}_p[u^{\pm 1}]$ by defining

$$\psi(u^{-1}) = g'(0)u^{-1}$$

This is the action of S_1 on the coefficients of E_1. When $g(x) = [n]_F(x) = (1+x)^n - 1$, then $\tilde{g}(x) = u^{-1}((1+ux)^n - 1)$, $g'(0) = n$, and

$$\psi(u) = nu$$

The action on the homology theory E_1 is given by applying Landweber exactness. $\psi : \mathbb{Z}_p[u^{\pm 1}] \to \mathbb{Z}_p[u^{\pm 1}]$ is a map of MU_*-modules and so we have an automorphism of $E_1 = \mathbb{Z}_p[u^{\pm 1}] \otimes_{MU_*} MU_*(-)$.

Defining the image of J

Let $\mathcal{H}(n)$ denote the monoid of homotopy self-equivalences of S^n that preserve the basepoint. It sits inside $\Omega^n S^n$ as the union of two components. There is an obvious map $O(n) \to \mathcal{H}(n)$ (There is also a map from $U(n)$ to $\mathcal{H}(2n)$ which factors through $O(2n)$). The composition

$$O(n) \to \mathcal{H}(n) \to \Omega^n S^n$$

induces

$$\pi_i(O(n)) \to \pi_i(\Omega^n S^n) = \pi_{n+i} S^n$$

and we can check that these maps commute with the maps $O(n) \to O(n+1)$ and $\mathcal{H}(n) \to \mathcal{H}(n+1)$ to yield a map of colimits

$$\phi : O \to \mathcal{H} \to \Omega^\infty \Sigma^\infty S^0$$

and a map of stable homotopy groups

$$\pi_i O \to \pi_i^s$$

We call this map the J-homomorphism and denote its image by $J(S^i)$.
The map $O(n) \to \mathcal{H}(n)$ sits in a fiber sequence

$$O(n) \to \mathcal{H}(n) \to \mathcal{H}(n)/O(n) \to BO(n) \to B\mathcal{H}(n)$$

Notice that $BO(n) \to B\mathcal{H}(n)$ is the map that classifies the underlying spherical fibration of the universal bundle over $BO(n)$.

Stable Adams operations

Recall that the Adams operations $\psi^k : K(X) \to K(X)$ are the unique natural ring homorphisms such that $\psi^k(L) = L^k$ whenever L is a line bundle. They are unstable in the sense that the diagram

$$\begin{array}{ccc} \Sigma^2 BU & \xrightarrow{B} & BU \\ {\scriptstyle 1\wedge\psi^k}\downarrow & & \downarrow{\scriptstyle \psi^k} \\ \Sigma^2 BU & \xrightarrow{B} & BU \end{array}$$

does not commute. If we invert k, we can fix this by defining $\tilde{\psi}^k$ on the $2n$th space of KU by $\tilde{\psi}^k = \frac{\psi^k}{k^n}$. It maps to $KU[\frac{1}{k}]$. Since ψ^k acts on the Bott class $\beta \in \pi_2(BU)$ by $\psi^k(\beta) = k\beta$ and the map $\Sigma^2 KU_0 \to KU_2$ is just multiplication by β. Our definition of $\tilde{\psi}^k$ adjusts for this.
If we complete at a prime p, then $\tilde{\psi}^k$ is defined for k coprime to p and from here on we drop the tilde and refer to these stable Adams operations as ψ^k. So $\mathbb{Z}$ sits inside $[K_p, K_p]$. One can show that $[K_p, K_p]$ is complete, so that the Adams operations extend to $\mathbb{Z}_p$. This actually turns out to be an isomorphism.

Notice that the action of the stable Adams operations on

$$\pi_* KU_p = \mathbb{Z}_p[\beta^{\pm 1}]$$

is exactly the action of the Morava stabilizer group on E_1.

The Adams conjecture

Adams Conjecture: If $k \in \mathbb{N}$, then for any $x \in K(X)$, we have $k^n(\psi^k(x) - x) = 0$ in the image of J for some $n \gg 0$.

This gives us an upper bound on the image of J.

The image of J completed at p

If we complete at a prime p, the Adams conjecture implies that the composition of the map $1-\psi^k$ with $BU_p \to B\mathcal{H}_p$ is nullhomotopic whenever k is coprime to p. This induces a map $\mathrm{hofib}(1-\psi^k) \to \mathcal{H}$ such that the following diagram commutes

$$\begin{tikzcd}
U_p \arrow[r] \arrow[d, "="'] & \mathrm{hofib}(1-\psi^k) \arrow[r] \arrow[d] & BU_p \arrow[r, "(1-\psi^k)"] \arrow[d] & BU_p \arrow[d, "="'] \\
U_p \arrow[r, "J"] & \mathcal{H}_p \arrow[r] & \mathcal{H}_p/U_p \arrow[r] & BU_p
\end{tikzcd}$$

Thus, we have shown that the Adams conjecture implies that the J homomorphism factors through the homotopy of the fiber of $1-\psi^k$. If p is odd, let g be a generator of $\mathbb{Z}_p^\times$. Then in fact $\mathrm{hofib}(1-\psi^g)$ is a split summand of $\mathcal{H}_p$. We will come back to this later. For now, let's assume

Theorem: The map $\pi_n(\mathrm{hofib}(1-\psi^g)) \to \pi_n B\mathcal{H}_p = \pi_n S_p^0$ is the inclusion of a split summand of $\pi_n S_p^0$ for $n \geq 0$.

hofib$(1-\psi^g)$ and $L_{K(1)}S$

The theorem of Devinatz-Hopkins that

$$L_{K(n)}S^0 = E_n^{hS_n}$$

in this case says that

$$L_{K(1)}S^0 = K_p^{h\mathbb{Z}_p\times}$$

Proposition: Let g be a topological generator of $\mathbb{Z}_p$. Then $K_p^{h\mathbb{Z}_p\times}$ =hofib$(1-\psi^g)$.

Proof: Consider the diagram

$$\begin{tikzcd}
K_p^{\mathbb{Z}} \arrow[r] \arrow[d] & K_p \arrow[r] \arrow[d] & * \arrow[d] \\
K_p \arrow[r, "(1,-\psi^g)"] & K_p \times K_p \arrow[r, "\mathrm{diff}"] & K_p
\end{tikzcd}$$

Both squares are pullbacks. This implies that the fibers of the horizontal compositions are equivalent. That is, $\mathrm{hofib}(1-\psi^g) = K_p^{h\mathbb{Z}}$.

It remains to show that $K_p^{h\mathbb{Z}} = K_p^{h\mathbb{Z}_p\times}$. To see this we use the homotopy fixed point spectral sequence. There is a map $K_p^{h\mathbb{Z}_p} \to K_p^{h\mathbb{Z}}$ given by inclusion of fixed points. It induces a map of homotopy fixed point spectral sequences, and

$$H_c^*(\mathbb{Z}_p^\times, \pi_*(K_p)) \to H_c^*(\mathbb{Z}, \pi_*(K_p))$$

is an isomorphism of E_2-terms. This can be seen by computing both of them.

Computing $\pi_* L_{K(1)} S$

Since $L_{K(1)}S = \mathrm{hofib}(1-\psi^g)$, we may use the long exact sequence of the fibration

$$L_{K(1)}S \to K_p \to K_p$$

to compute the homotopy of $L_{K(1)}S$. Recall that g is chosen to be a generator of $\mathbb{Z}_p^\times = \mu_{p-1} \times (1+p\mathbb{Z}_p)$ so that $g = (\zeta, y)$ where p divides $g-1$ but p^2 does not.
Firstly, since ψ^g acts on $\pi_0 K_p = \mathbb{Z}_p$ by the identity, $1-\psi^g$ vanishes on π_0 and since $\pi_{2k+1}K_p = 0$, we have

$$\pi_0 L_{K(1)}S \cong \mathbb{Z}_p$$

and

$$\pi_{-1} L_{K(1)}S \cong \mathbb{Z}_p$$

On $\pi_{2k}K_p$, ψ^g acts by g^k. Thus, $1-\psi^g$ is injective for $k \neq 0$ and so

$$\pi_{2k} L_{K(1)}S = 0$$

and

$$\pi_{2k-1} L_{K(1)}S \cong \mathbb{Z}_p/(1-g^k)$$

Now, if $p-1$ does not divide k, then $g^k - 1$ is a unit mod p and $\pi_{2k-1}L_{K(1)}S = 0$. If $k = (p-1)m$, then $g^k = (g^{p-1})^m$, and g^{p-1} topologically generates $1+p\mathbb{Z}_p$. If $m = p^r l$ where l is coprime to p, then $(g^{p-1})^m = ((g^{p-1})^{p^r})^m$ topologically generates the cyclic subgroup $1+p^{r+1}\mathbb{Z}_p$
so that $1-g^k$ generates $p^{r+1}\mathbb{Z}_p$ topologically. Thus, if $k = (p-1)p^r l$, then

$$\pi_{2k-1} = \mathbb{Z}/p^{r+1}$$

That is,

$$\pi_n L_{K(1)}S = \begin{cases} \mathbb{Z}_p & : n = 0, -1 \\ \mathbb{Z}/p^{r+1}\mathbb{Z} & : n+1 = 2(p-1)p^r l \ l \not\equiv 0 \bmod p \\ 0 & : \text{otherwise} \end{cases}$$

Aside: Bernoulli numbers

The Bernoulli numbers β_t are given by the power series of the function $x/(e^x - 1)$:

$$\frac{x}{e^x - 1} = \Sigma_{t=0}^{\infty} \beta_t \frac{x^t}{t!}$$

Since $\frac{x}{e^x-1} - 1 + \frac{x}{2}$ is an even function, $\beta_{2t+1} = 0$ for $t > 0$. Also, $\beta_1 = -\frac{1}{2}$.

This definition of Bernoulli numbers will come up in the lower bound of the image of J. What we will really be interested in are the denominators of $\frac{\beta_{2s}}{4s}$ when the fraction is expressed in lowest terms. Call this $m(2s)$. Adams describes $m(2s)$ by giving its p-adic evaluation:

Proposition: For p odd, $\nu_p(m(t)) = 1 + \nu_p(t)$ if $(p-1)$ divides t and is zero otherwise. For $p = 2$, $\nu_2(m(t)) = 2 + \nu_2(t)$ if t is even and 1 otherwise.

Notice that the order of $\pi_n L_{K(1)}S$ is exactly $\nu_p(\frac{n+1}{2})$, that is, the denominator of $\beta_{(n+1)/2}/(n+1)$.

Computing $\pi_* L_{E(1)} S$

To compute $\pi_* L_{E(1)} S$, we use the pullback square

$$\begin{array}{ccc} L_{E(1)}S & \longrightarrow & L_{K(1)}S \\ \downarrow & & \downarrow \\ L_{E(0)}S & \longrightarrow & L_{E(0)}L_{K(1)}S \end{array}$$

which gives a long exact sequence in homotopy

$$\cdots \to \pi_{n+1} L_{E(0)} L_{K(1)} S \to \pi_n L_{E(1)} S \to \pi_n L_{K(1)} S \oplus \pi_n L_{E(0)} S \to \pi_n L_{E(0)} L_{K(1)} S \to \dots$$

Recall that $L_{E(0)}S \simeq H\mathbb{Q} \simeq S\mathbb{Q}$ (where the right hand side is the rational Eilenberg-Moore spectrum). There is a universal coefficients theorem for $\pi_* SG$

$$0 \to G \otimes \pi_* X \to \pi_*(SG \wedge X) \to \mathrm{Tor}(G, \pi_{*-1} X) \to 0$$

which for $G = \mathbb{Q}$ implies that

$$\pi_* L_{E(0)} L_{K(1)} S \cong \pi_*(L_{E(0)} S \wedge L_{K(1)} S) \cong \mathbb{Q} \otimes \pi_*(L_{K(1)} S)$$

Thus, $\pi_n(L_{E(0)} L_{K(1)} S) \cong \mathbb{Q}_p$ for $n = 0, -1$ and is zero otherwise.

Then for $n \neq 0, -1, -2$ we have $\pi_n L_{E(1)} S \cong \pi_n L_{K(1)} S$. For the remaining groups, we have the exact sequence

$$0 \to \pi_0 L_{E(1)} S \to \mathbb{Z}_p \oplus \mathbb{Q} \to \mathbb{Q}_p \to \pi_{-1} L_{E(1)} S \to \mathbb{Z}_p \to \mathbb{Q}_p \to \pi_{-2} L_{E(1)} S \to 0$$

It follows that

$$\pi_n L_{E(1)} S = \begin{cases} \mathbb{Z} & : n = 0 \\ \mathbb{Q}_p/\mathbb{Z}_p & : n = -2 \\ \mathbb{Z}/p^{r+1}\mathbb{Z} & : n+1 = 2(p-1)p^r l \;\; l \not\equiv 0 \bmod p \\ 0 & : \text{otherwise} \end{cases}$$

Adams' lower bound on the image of J

The e-invariant

Adams computed a lower bound on the image of J and showed that it is the same as the upper bound. The computation consists of defining a homomorphism

$$e : \pi_k^s \to \mathbb{Q}/\mathbb{Z}$$

such that the composition

$$\pi_{2k-1} U(n) \xrightarrow{J} \pi_{2n+2k-1} S^{2n} \xrightarrow{e} \mathbb{Q}/\mathbb{Z}$$

when evaluated on a generator of $\pi_{2k-1} U(n)$ has denominator $m(k)$.

Given a map $g : S^{2m-1} \to S^{2n}$, let C_g denote the cofiber. Then we have a short exact sequence in K-theory

$$0 \to \tilde{K}(S^{2m}) \to \tilde{K}(C_f) \to \tilde{K}(S^{2n}) \to 0$$

Applying the Chern character, we have a homomorphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{K}(S^{2m}) & \longrightarrow & \tilde{K}(C_g) & \longrightarrow & \tilde{K}(S^{2n}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \tilde{H}^*(S^{2m};\mathbb{Q}) & \longrightarrow & \tilde{H}^*(C_g;\mathbb{Q}) & \longrightarrow & \tilde{H}^*(S^{2n};\mathbb{Q}) & \longrightarrow & 0 \end{array}$$

Let α, β denote elements of $\tilde{K}(C_g)$ mapping from and to generators of $\tilde{K}(S^{2m})$ and $\tilde{K}(S^{2n})$, respectively. Similarly, let $a, b \in \tilde{H}^*(C_g;\mathbb{Q})$ be elements mapping from and to generators of $H^{2m}(S^{2m};\mathbb{Z})$ and $H^{2n}(S^{2n};\mathbb{Z})$, respectively. We may assume $\mathrm{ch}(\alpha) = a$ and $\mathrm{ch}(\beta) = b + ra$ for some $r \in \mathbb{Q}$. β and b are not uniquely determined, but if we vary them by integer multiples of α and a, we change r by an integers. So r is well-defined in $\mathbb{Q}/\mathbb{Z}$. We define $e(g) = r$. One can check that it is a homomorphism.

Bounding the image of J below

The key to evaluating $e(Jf)$ is the following lemma.

Lemma: C_{Jf} is the Thom space of the bundle $E_f \to S^{2k}$ determined by the clutching function $f : S^{2k-1} \to U(n)$.Under this identification, $\beta \in \tilde{K}(C_{Jf})$ corresponds to the Thom class of E_f.

$K(T(E_f))$ is a free one-dimensional module over $K(S^{2k})$. It can be identified with a submodule of $K(P(E_f))$ generated by a specific relation corresponding to the Thom class. One may then apply the splitting principle and compute the value of the Chern character.

Theorem: (Atiyah?) Let E be any n-dimensional complex vector bundle with base B. Let U denote the Thom class in $\tilde{H}^*(T(E);\mathbb{Q})$ which corresponds to $1 \in H^*(B;\mathbb{Q})$ under the Thom isomorphism $\Phi : H^*(B;\mathbb{Q}) \to \tilde{H}^*(T(E);\mathbb{Q})$. Let bh_E denote the image of the characteristic class in $H^*(BU(n);\mathbb{Q})$ whose image in $H^*(BU(1)^n;\mathbb{Q})$ is

$$\prod_{1 \le r \le n} \frac{e^{x_r} - 1}{x_r}$$

Then

$$\Phi^{-1}\mathrm{ch}(U) = bh_E$$

After some manipulation of power series, this implies

$$e(Jf) = \alpha_k = \beta_k/k$$

The α-family

Theorem: Let p be an odd prime, $m = p^f$, and $r = (p-1)p^f$. Then there exists $\alpha \in \pi^s_{2r-1}$ such that

(i) $m\alpha = 0$

(ii) $e(\alpha) = -\frac{1}{m}$, and

(iii) The Toda bracket $\{m, \alpha, m\}$ is zero mod $m\pi^s_{2r}$.

For q large, we have

$$\alpha : S^{2q+2r-2} \to S^{2q-1}$$

and the Toda bracket gives a map

$$S^{2q+2r-1} \to S^{2q-1}$$

Let Y denote the cofiber of $m : S^{2q-1} \to S^{2q-1}$. Then since $m\{m, \alpha, m\} = 0$, the Toda bracket induces a map on the cofiber of m

$$A : \Sigma^{2r} Y \to Y$$

and we have a diagram

$$\begin{array}{ccc} \Sigma^{2r}Y & \xrightarrow{A} & Y \\ {\scriptstyle i}\downarrow & & \uparrow{\scriptstyle j} \\ S^{2q+r-1} & \xrightarrow{\alpha} & S^{2q} \end{array}$$

Adams defines the d invariant of a map $f : X \to Y$ as $f^* \in \mathrm{Hom}(K^*(Y), K^*(X))$. The e invariant may be viewed as an element of a certain Ext group, and Adams shows that

$$d(jA) = -me(\alpha)$$

which in this case implies

$$d(jA) = 1$$

Thus, A must be an isomorphism in K-theory.

Now, since A is induces an isomorphism in K-theory, so does any composite

$$A \circ \Sigma^{2r} A \circ \Sigma^{4r} A \circ \cdots \circ \Sigma^{2r(s-1)} A : \Sigma^{2rs} Y \to Y$$

We may now construct a map α_s via the following diagram.

$$\begin{array}{ccc} \Sigma^{2rs}Y & \longrightarrow & Y \\ {\scriptstyle i}\uparrow & & \downarrow{\scriptstyle j} \\ S^{2q+2rs-1} & \xrightarrow{\alpha_s} & S^{2q} \end{array}$$

An argument like the one above shows that

$$e(\alpha_s) = -\frac{1}{m}$$

which shows that α_s is essential.