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Math 109 Homework 2

Problem 1 (Eccles II.13, p.117)**.** Prove that, for sets A, B, C and D,
(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$,
(ii) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Solution. (i) A proof that two sets are equal requires us to prove two set inclusions. In this case we can do them together as follows.

$$\begin{aligned}(x,y) \in A \times (B \cap C) &\iff x \in A \textit{ and } y \in (B \cap C)\\ &\iff x \in A \textit{ and } (y \in B \textit{ or } y \in C)\\ &\iff (x,y) \in (A \times B) \textit{ or } (x,y) \in (A \times C)\\ &\iff (x,y) \in (A \times B) \cup (A \times C)\end{aligned}$$

Thus $(x,y) \in A \times (B \cup C) \iff (x,y) \in (A \times B) \cup (A \times C)$.

□

(I did it myself)

(ii) A proof that two sets are equal requires us to prove two set inclusions. In this case we can do them together as follows.

$$\begin{aligned}(x,y) \in (A \times B) \cap (C \times D) &\iff x \in A \textit{ and } y \in B \textit{ and } x \in C \textit{ and } y \in D\\ &\iff x \in A \textit{ and } x \in C \textit{ and } y \in B \textit{ and } y \in D\\ &\iff x \in (A \cap C) \textit{ and } y \in (B \cap D)\\ &\iff (A \cap C) \times (B \cap D)\end{aligned}$$

Since x is the object of set A and set C, they have precisely the same elements $x \in A \iff x \in B \, or \, A \subseteq B \, and \, B \subseteq A. The \, same \, logic \, happens \, for \, object \, y \, and \, set \, C \, and \, D. Thus, \; (x,y) \in (A \times B) \cap (C \times D) \iff (A \cap C) \times (B \cap D)$

□

(I did this myself)

Problem 2 (Eccles II.19, p.118)**.** Let $f : X \to Y$ be a function. Prove that there exists a function $g : Y \to X$ such that $f \circ g = I_Y$ if and only if f is a surjection. [g is called a *right inverse* of f.]

Solution. According to *Proposition 9.2.5*, $f \circ g = I_y$ if and only if f:X →Y and g:Y → $X$are inverses of each other. Let us show with 3 cases why f can only be a surjection.

case 1:
Suppose f was an injection function, then according to *Definition 9.1.1*, no element of Y is assigned to more than one element of X. That means if function g was a surjective function, not more than one value of X of the pre-image can be assigned to Y. Thus the function f which is injection would be invalid under the "if and only if" condition.

case 2:
Suppose f was a bijection function, then according to *Definition 9.1.1*, every element of Y has to have precisely 1 pre-image, which is both an injection and surjection. Such a function would then elude the "if and only if" condition again because the function would not have worked if g was a surjection function; there might be more than one image of x_0 which would not have been mapped. Thus the function f which is bijection would not produce the composite identity function under the "if and only if" condition.

case 3:
Suppose f was a surjection, the composite identity function would occur because under *Definition9.1.1*, a surjection function happens if and only if every element of Y has at least one pre-image. Therefore function f could map either multiple y images from x_0 or one y image for every x_0.

Thus if and only if f is a surjection can there exists a function $g : Y \to X$ such that $f \circ g = I_Y$.

(I did this myself except for 1st part of i), where I had some advice on the template for answering by Matthew.

Problem 3 (Eccles II.20, p.118)**.** Let $f : X \to Y$ be a function and $A_1, A_2 \in \mathcal{P}(X)$.

(i) Prove that $A_1 \subseteq A_2 \implies \overrightarrow{f}(A_1) \subseteq \overrightarrow{f}(A_2)$. Prove that the converse is not universally true. Give a simple condition on f which is equivalent to the converse.

(ii) Prove that $\overrightarrow{f}(A_1 \cap A_2) \subseteq \overrightarrow{f}(A_1) \cap \overrightarrow{f}(A_2)$. Prove that equality is not universally true.

(iii) Prove that $\overrightarrow{f}(A_1 \cup A_2) = \overrightarrow{f}(A_1) \cup \overrightarrow{f}(A_2)$.

Solution. (i) Using *definition 6.1.4*, we know that A is a subset of B when every element of A is an element of B.
Suppose x_0 is an element of A_1 such that x_0 is an element of A_2 because A_1 *is a subset of* A_2.Then $f(A_1)$ will be a subset of $f(A_2)$ as the image of x_0 for $f(A_1)$ would also be the same image as in $f(A_2)$.
$Therefore A_1 \subseteq A_2 \implies f(A_1) \subseteq f(A_1)$.

The converse is not universally true, Even though $f(A_1) \subseteq \overrightarrow{f}(A_2)$. Given that the power set of X has a set of subsets A_1 and A_2, we suppose A_1 be the singleton set $\{a_1\}$ and A_2 be the singleton set $\{a_2\}$. We also suppose that f:$\{a_2\}\rightarrow \{0, 2\}$ and f: $\{\mathrm{a}_1\} \rightarrow \{0\}$, to show that $f : (\mathrm{A}_1)$ is a subset of f:(A_2). Upon inspection, we would see that the subsets A_1 and A_2 would be different singleton set elements even though $f : (\mathrm{A}_1)$ is a subset of f:(A_2).

$Thus, even$ though f:(A_1) is a subset of f:(A_2), the converse would not be universally true if A_1 and A_2 are singleton sets with different elements.□

(ii) Suppose that A_1 and A_2 be singleton sets of the Power set of X and function be bijective. Therefore since $A_1 \cap A_2 = \emptyset$, the function will map the value to a value which is different than the two values from $f(A_1) \cap \overrightarrow{f}(A_2)$.

Thus under the impression that A_1 and A_2 be singleton sets of the Power set of X and function be bijective, $f(A_1 \cap A_2)$ would not be the subset of $f(A_1) \cap \overrightarrow{f}(A_2)$. Therefore the equality is not universally true. □

(iii) Suppose that A_1 and A_2 be separate sets of the Power set of X. Using *definition*$6.2.2$ and *definition*$8.1.1$, we see that the function f takes elements from either A_1 or A_2 and maps them. It is equivalent to the right-hand side, which represents the values mapped by function on $\mathrm{A}_1 or A_2$.

Thus both sides represent values mapped from the union of sets and are equivalent to each other. □

$(I did this myself)$

Problem 4 (Eccles IV.4, p.225)**.** Prove that there is no rational number whose square is 98.

Solution. Suppose for contradict that there's rational number q such that $q^2 = 98$. Write q as a fraction in lowest terms: $q = \frac{a}{b}$ such that a and b are integers such that $(a, b) = 1$. Now $q^2 = 98 \implies \frac{a^2}{b^2} = 98 \implies a^2 = 98b^2 \implies a^2 \ divisible\ by\ 98.\ It\ follows\ that\ a\ must\ be\ divisible\ by\ 98.$

Thus we can write $a = 98a_1$ for some a$_1 \in Z$. But then $a^2 = 98b^2 \implies 98^2a_1^2 = 98b^2 \implies 98a_1^2 = b^2 \implies b^2\ is\ divisible\ by\ 3 \implies b\ is\ divisible\ by\ 3,\ as\ above.$

Hence 98 is a common factor of a and b so that $(a, b) \neq 1,\ contradicting\ the\ choice\ of$ a and b $and\ giving\ the\ required\ contradiction.$

Hence there does not exist a rational number whose square is 98. □

Solve the linear diophantine equation

$$336m + 238n = 5558. \tag{4.1}$$

Prove that there is a unique pair of *positive* integers m and n satisfying this equation and find this solution.

Solution.

$$\begin{aligned} 336 &= 238 \times 1 + 98 \\ 238 &= 98 \times 2 + 42(-1) \\ 98 &= 42 \times 2 + 14(-2) \\ 42 &= 14 \times 3 + 0 \end{aligned}$$

(336,238) = 14, since 14 divides 5558, the equation has a solution.

$$\begin{aligned} 336 &= 336 \times 1 + 238 \times 0 \\ 238 &= 336 \times 0 + 238 \times 1(-1) \\ 98 &= 336 \times 1 + 238 \times (-1) \\ 42 &= 336 \times \frac{37}{24} + 238 \times (-2) \end{aligned}$$

This gives $336 \times \frac{37}{24} + 238 \times (-2) = 42$
multiply the equation above by $\frac{397}{3}$
$336 \times \frac{14689}{72} + 238 \times \frac{-794}{3} = 5558$, hence $m = \frac{14689}{72}, n = \frac{-794}{3} are the unique pair of positive integers.$

$$\begin{aligned} 336m + 238n = 5558 &\iff 336(m - \frac{14689}{72}) + 238(n + \frac{794}{3}) = 0 \\ &\iff (m - \frac{14689}{72}, n + \frac{794}{3}) = (24q, 17q) \\ &\iff (m, n) = (\frac{14689}{72} + 24q, \frac{-794}{3} + 17q) for some q \in Z \end{aligned}$$

(I did this myself.)

Problem 5 (Supplementary Problem 4)**.** Let a, b and c be integers such that a and b are coprime and c divides $a + b$. Prove that $\gcd(a, c) = \gcd(b, c) = 1$.

Solution.