Quantifiers and Logical Equivalences

Reading

The reading associated to this lecture is contained in Rosen, sections 1.3-1.5.

I recommend reading everything in 1.4 before "Logical Equivalences Involving Quantifiers." Then everything in 1.5 before "Negating Nested Quantifiers." Then read 1.3 (everything starting from the section on "Propositional Satisfiability" is optional). Then finish reading 1.4 (everything starting from "Using Quantifiers in System Specifications" is optional) and 1.5.

Summary: Quantifiers

A *predicate* is just a statement of some property. For example, "is a part of the United States" is a predicate. Often we use a variable to take the place of a subject in such a predicate; we say things like let $P(x)$ be the predicate "x is a part of the United States." Then replacing the variable x with various actual subjects gives us propositions. For example $P(\text{California})$ is a proposition that is true, and $P(\text{British Columbia})$ is a proposition that is false. One can also have predicates involving multiple variables. For example, consider the predicate $P(x, y)$ which is "x is a famous y." Then $P(\text{Steinbeck, novelist})$ is a true proposition, $P(\text{Steinbeck, painter})$ is false, and $P(\text{Monet, painter})$ is true.

Quantifiers are ways of taking predicates and turning them into propositions. There are two quantifiers used in mathematics.

- If $P(x)$ is a predicate, its *universal quantification*, denoted $\forall x P(x)$, is the proposition that "$P(x)$ for all x is the domain," where the *domain* is some pre-established collection of possible values.

  For example, consider the predicate $P(x)$ which is "x is even." If the domain is the set of all multiples of 10, then $\forall x P(x)$ is true. If the domain is the set of all integers, then $\forall x P(X)$ is false.

- If $P(x)$ is a predicate, its *existential quantification*, denoted $\exists x P(x)$, is the proposition that "there is an x in the domain such that $P(x)$."

  For example, consider the predicate $P(x)$ which is "$x^2 > 20$." If the domain is the numbers $1, 2, 3, 4$, then $\exists x P(x)$ is false. If the domain is $1, 2, 3, 4, 5$, then $\exists x P(x)$ is true.

You can also nest quantifiers over all of the variables.

Summary: Logical Equivalences

A *compound proposition* is built out of propositions, propositional variables, $\neg$, $\wedge$, $\vee$, $\oplus$, $\rightarrow$, $\leftrightarrow$, and quantifiers over some predicate variables, in a sensible way. This is vague and not precise: to make it precise, we would need to learn something about languages and formal grammars. In any case, these are some examples of compound propositions.

- $p \rightarrow q$
- $p \wedge$ "apples are fruit"
- ("oranges are fruit" $\vee p) \rightarrow (q \wedge \exists x P(x))$

Hopefully it is clear from these examples what a compound proposition is.

Two compound propositions and are *logically equivalent* if they always have the same truth value no matter what propositions are substituted in place of the propositional variables, what predicates are substituted in place of the predicate variables, and what domains the quantifiers quantify over. Here are some important examples of logical equivalence.

- $p \rightarrow q \equiv (\neg q) \rightarrow (\neg p)$
- $(\neg p) \vee q \equiv p \rightarrow q$
- $\neg(p \wedge q) \equiv (\neg p) \vee (\neg q)$

- $(\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$
- $\neg(\neg p) \equiv p$
- $\neg \forall x P(x) \equiv \exists x \neg P(x)$
- $\neg \exists x P(x) \equiv \forall x \neg P(X)$

You should be able to prove all of these. A non-example of a logical equivalence is that $p \to q$ is not equivalent to $q \to p$.

Comments

(1) Whenever you see a quantifier, make sure you know what domain it is quantifying over.

(2) Again, you should know that there are precedence rules regarding quantifiers, but its best to disambiguate using parentheses so that no one is confused.

(3) Make sure you understand the relationship between universal quantification over a finite domain and conjunction, and between existential quantification over a finite domain and disjunction.

(4) The order of quantifiers matters, and it matters a lot! For example, consider the propositional function $P(x, y)$ which is "x is friends with y" and suppose the domain is students in this class. Then $\forall x \exists y P(x, y)$ says that every student in this class has a friend in this class, whereas $\exists x \forall y P(x, y)$ says that there is a student in this class who is friends with everyone in the class! Always be very careful to not switch the order of quantifiers.

(5) You don't need to memorize all of the logical equivalences and their names. The most important thing you can do to understand logical equivalences is to go through the list Rosen gives and convince yourself why they all make sense.

(6) One way of showing that two compound propositions are logically equivalent is to write down truth tables for both and show that the truth tables are the same, at least when they don't have any quantifiers over predicate variables. Usually, it's better to use the examples of logical equivalences Rosen gives in his tables to derive more complicated equivalences.

(7) De Morgan's laws for quantifiers are very useful to keep in mind. If you go on to take analysis, you'll have to deal with a lot of statements with nested quantifiers and negating them, and its very convenient to be fluent with De Morgan's laws.