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Math 109 Homework 1

Problem 1 (Eccles I.4, p.53)**.** Prove the following statements concerning positive integers a, b, and c.
(i) (a divides b) and (a divides c) $\implies$ a divides $(b + c)$.
(ii) (a divides b) or (a divides c) $\implies$ a divides bc.

Solution.

(i)

Proof. If (a divides b) and (a divides c) when a, b, and c are all positive integers, it is possible to claim that (b + c) is divisible by a.

let $b = na$ and $c = ma$, where n and m are all positive integers,

and b and c are divisible by a.

Then, $b + c = na + ma = (na + ma) = a(n + m)$,

and $a(n + m)$ is divisible by a.

Since n and m are positive integers, $n + m$ is also a positive integer.

These steps prove that $b + c$ is divisible by a.

Therefore, (a divides b) and (a divides c) $\implies$ a divides $(b + c)$, is true.

(ii)

Proof. If (a divides b) or (a divides c) when a, b, and c are all positive integers, it is possible to state that a divides bc.

When c is a constant positive integer and $c = ma$, where m is any positive,

then $bc = b(ma) = a(mb)$.

$a(mb)$ is divisible by a.

mb is a positive integer since m and b are both positive integers.

When b is a constant positive integer and $b = na$, where n is any positive integer, then $bc = (na)c = a(nc)$.

$a(nc)$ is divisible by a.

nc is a positive integer since n and b are both positive integers.

These two cases show that a divides bc.

Therefore, we can conclude that (a divides b) or (b divides c) $\implies$ a divides bc.

Problem 2 (Eccles I.6, p. 54)**.** Use the properties of addition and multiplication of real numbers given in Properties 2.3.1 to deduce that, for all real numbers a and b,
(i) $a \times 0 = 0 = 0 \times a$,
(ii) $(-a)b = -ab = a(-b)$,
(iii) $(-a)(-b) = ab$.

Solution.

(i)

Proof. In order to prove that $a \times 0 = 0 = 0 \times a$, where a can be any real number, there are couple steps to follow.

First, understand that $a + x = 0$ for all real numbers x, x is always $-a$.

Zero Law states that $0 = 0 + 0$.

Multiply a in both sides of the equation, then $a \times 0 = a \times (0 + 0)$.

Use distributive property, then $a \times 0 = a \times 0 + a \times 0$.

Use subtraction to minus $a \times 0$ on both sides of the equation,

then $a \times 0$ - $a \times 0 = a \times 0$.

This should result $0 = a \times 0$.

Next, use commutative property of multiplication to prove that the order

of numbers in multiplication does not really matter.

This means $a \times 0 = 0 \times a$.

Therefore, $a \times 0 = 0 = 0 \times a$ is a true statement.

(ii)

Proof. In order to prove that $(-a)b = -ab = a(-b)$ where a and b are any positive real number, we first need to know that $a + (-a) = 0$.

Multiply b on both sides of the equation, then $b(a + (-a)) = b(0)$.

Use distributive property then $ab + (-a)b = 0$.

Use subtraction to minus ab on both sides of the equation, then

$ab + (-a)b - ab = 0 - ab$.

This results $(-a)b = -ab$.

Similarly, $a(-b) = -ab$, if we started from $b + (-b) = 0$

Therefore, we can state that $(-a)b = -ab = a(-b)$ is a true statement.

(iii)

Proof. In order to prove that $(-a)(-b) = ab$, we first use $a + (-a) = 0$.

Then multiply b on both sides of the equation, then

$(-b)(a + (-a)) = (-b)0.$

Use distributive property, then $a(-b) + (-a)(-b) = 0$.

Use associative property of multiplication, then $-(ab) + (-a)(-b) = 0$.

Use Subtraction to minus ab on both sides of the equation, then

$-(ab) + (-a)(-b) + ab = 0 + ab.$

This will result $(-a)(-b) = ab$.

Therefore, we can conclude that $(-a)(-b) = ab$ is true.

Problem 3 (Eccles I.10, p. 54)**.** What is wrong with the following proof that 1 is the largest integer?

> Let n be the largest integer. Then, since 1 is an integer we must have $1 \leq n$. On the other hand, since n^2 is also an integer we must have $n^2 \leq n$ from which it follows that $n \leq 1$. Thus, since $1 \leq n$ and $n \leq 1$ we must have $n = 1$. Thus 1 is the largest integer as claimed.

What does this argument prove?

Solution. Suppose n is the largest integer.

Then, it is obvious that $1 \leq n$ since 1 is the smallest positive integer.

In this case, n^2 is smaller than n since the hypothesis claims that n is the largest integer.

This allows us to form an equality $n^2 \leq n$.

If $\frac{n^2}{n} \leq \frac{n}{n}$, then $n \leq 1$.

Because there are two equalities $1 \leq n$ and $n \leq 1$, the statement concludes that $n = 1$, and 1 is the largest integer.

If $n \leq 1$, n becomes a negative integer, and $n^2 \leq n$ becomes valid.

However, if $1 \leq n$, n becomes any positive integer greater than 1.

Thus, $n^2 \leq n$ becomes a false equality.

Therefore, it is not possible to state that n is the largest integer to conclude that 1 is the largest integer.

Problem 4 (Eccles I.19, p. 56)**.** Prove that

$$\prod_{i=2}^{n}\left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n}$$

for integers $n \geq 2$.

Solution. If

$$\prod_{i=2}^{n}\left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n}$$

for $n \geq 2$ is true, then

$$\prod_{i=2}^{n+1}\left(1 - \frac{1}{i^2}\right) = \frac{(n+1)+1}{2(n+1)}$$

for $n \geq 2$ must be true as well.

$$\prod_{i=2}^{n+1}\left(1-\frac{1}{i^2}\right) = (1-\frac{1}{(n+1)^2})(\prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right)) = (\frac{(n+1)^2-1}{(n+1)^2})(\frac{n+1}{2n})$$

$$= (\frac{n^2+2n+1-1}{n+1})(\frac{1}{2n}) = (\frac{n(n+2)}{n+1})(\frac{1}{2n}) = (\frac{n+2}{n+1})(\frac{1}{2}) = \frac{n+2}{2(n+1)} = \frac{(n+1)+1}{2(n+1)}.$$

Therefore, we can conclude that

$$\prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right) = \frac{n+1}{2n}$$

for $n \geq 2$ is true.

Problem 5 (Eccles I.21, p. 56)**.** Suppose that x is a real number such that $x + 1/x$ is an integer. Prove by induction on n that $x^n + 1/x^n$ is an integer for all positive integers n. [For the inductive step consider $(x^k + 1/x^k)(x + 1/x)$.]

Solution. x is a real number that allows $x + \frac{1}{x}$ to be an integer.

To prove $x^n + \frac{1}{x^n}$ is an integer for all positive integers n, understand that

$x + \frac{1}{x}$ is also an integer.

$f(n) = x^n + \frac{1}{x^n}$; 1 is the smallest positive integer.

$f(1) = x^1 + \frac{1}{x^1} = x + \frac{1}{x}$; therefore, $x + \frac{1}{x}$ is an integer.

Suppose $x^n + \frac{1}{x^n}$ is an integer.

$g(n) = (x^n + \frac{1}{x^n})(x + \frac{1}{x}) = x^{n+1} + \frac{x^n}{x} + \frac{x}{x^n} + \frac{1}{x^{n+1}} = x^{n+1} + x^{n-1} + \frac{1}{x^{n-1}} + \frac{1}{x^{n+1}}$.

If $n = 1$, $g(1) = x^{1+1} + x^{1-1} + \frac{1}{x^{1-1}} + \frac{1}{x^{1+1}} = x^2 + x^0 + \frac{1}{x^0} + \frac{1}{x^2}$.

0^{th} power of any real number has a numerical value of 1.

Therefore, $x^0 + \frac{1}{x^0} = 1 + 1 = 2$.

Back to $f(n) = x^n + \frac{1}{x^n}$.

If $n = 2$, then $f(2) = x^2 + \frac{1}{x^2}$; we assume this is an integer.

Therefore, $g(1) = x^2 + x^0 + \frac{1}{x^0} + \frac{1}{x^2} = x^2 + \frac{1}{x^2} + 2$, and it is an integer as well.

From these steps, we were able to find out $x^{n-1} + \frac{1}{xn-1}$, x + $\frac{1}{x}$, and

$x^{n+1} + \frac{1}{x^{n+1}}$ are all integers when $x \geq 1$.

Finally, we can conclude that $x^n + \frac{1}{x^n}$ is an integer for all positive integers

n.

Problem 6 (Eccles I.25, p. 57)**.** Let u_n be the nth Fibonacci number (Definition 5.4.2). Prove, by *induction on n* (without using the Binet formula Proposition 5.4.3), that

$$u_{m+n} = u_{m-1}u_n + u_m u_{n+1}$$

for all positive integers m and n.

Deduce, again using induction on n, that u_m divides u_{mn}.

Solution. *Proof.* 1 If $u_{m+n} = u_{m-1}u_n + u_m u_{n+1}$ is true,

then $u_{m+(n+1)} = u_{m-1}u_{n+1} + u_m u_{(n+1)+1}$.

Start by: $u_{m+n+1} = u_{m+n-1} + u_{m+n}$

$= u_{m-1}u_{n-1} + u_m u_n + u_{m-1}u_n + u_m u_{n+1}$

$= u_{m-1}(u_{n-1} + u_n) + u_m(u_n + u_{n+1})$

$= u_{m-1}(u_{n+1}) + u_m(u_{n+2})$

$= u_{m-1}(u_{n+1}) + u_m(u_{(n+1)+1})$

Definition 5.4.2 allows us to conclude with this equation because

$u_{n+1} = u_{n-1} + u_m u_n$ and $u_{n+2} = u_n + u_{n+1}$ by induction on n.

Proof. 2 If u_m divides u_{mn}, then u_m should divide $u_{m(n+1)}$.

$u_{m(n+1)} = u_{m+mn} = u_{m-1}u_{mn} + u_m(u_{mn+1})$

$= u_{m-1}(u_{mn+1} - u_{mn-1}) + u_m(u_{mn+1})$

$= u_{m-1}(u_{mn+1}) - u_{m-1}(u_{mn-1}) + u_m(u_{mn+1})$

$= u_{mn+1}(u_{m-1} + u_m) - u_{m-1}(u_{mn-1})$

$= u_{m+1}(u_{mn+1}) - u_{m-1}(u_{mn-1})$

$= u_{m+1}(u_{mn+1}) - (u_{m+1} - u_m)(u_{mn+1} - u_{mn})$

$= u_{m+1}(u_{mn+1}) - (u_{m+1}u_{mn+1} - u_{m+1}u_{mn} - u_m u_{mn+1} + u_m u_{mn})$

$= u_{m+1}u_{mn} + u_m u_{mn+1} - u_m u_{mn}$

Since u_m divides u_{mn}, $\frac{u_{mn}}{u_m}$ is a constant number k: [1,n].

And, $\frac{u_m}{u_m} = 1$.

Therefore, $u_{m(n+1)}$ is divisible by u_m.