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Math 109 Homework 2

Problem 1 (Eccles II.13, p.117)**.** Prove that, for sets A, B, C and D,
(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$,
(ii) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Solution. (i) A proof that two sets are equal requires us to prove two set inclusions. In this case we can do them together as follows.

$$\begin{aligned} (x,y) \in A \times (B \cap C) &\iff x \in A \textit{ and } y \in (B \cap C) \\ &\iff x \in A \textit{ and } (y \in B \textit{ or } y \in C) \\ &\iff (x,y) \in (A \times B) \textit{ or } (x,y) \in (A \times C) \\ &\iff (x,y) \in (A \times B) \cup (A \times C) \end{aligned}$$

Thus $(x,y) \in A \times (B \cup C) \iff (x,y) \in (A \times B) \cup (A \times C)$.

$\square$

(ii) A proof that two sets are equal requires us to prove two set inclusions. In this case we can do them together as follows.

$$\begin{aligned} (x,y) \in (A \times B) \cap (C \times D) &\iff x \in A \textit{ and } y \in B \textit{ and } x \in C \textit{ and } y \in D \\ &\iff x \in A \textit{ and } x \in C \textit{ and } y \in B \textit{ and } y \in D \\ &\iff x \in (A \cap C) \textit{ and } y \in (B \cap D) \\ &\iff (A \cap C) \times (B \cap D) \end{aligned}$$

Since x is the object of set A and set C, they have precisely the same elements $x \in A \iff x \in B\, or\, A \subseteq B\, and\, B \subseteq A. The\, same\, logic\, happens\, for\, object\, y\, and\, set\, C\, and\, D. Thus,\ (x,y) \in (A \times B) \cap (C \times D) \iff (A \cap C) \times (B \cap D)$

$\square$

Problem 2 (Eccles II.19, p.118)**.** Let $f : X \to Y$ be a function. Prove that there exists a function $g : Y \to X$ such that $f \circ g = I_Y$ if and only if f is a surjection. [g is called a *right inverse* of f.]

Solution. According to *Proposition 9.2.5*, $f \circ g = I_y$ if and only if f:X →Y and g:Y → $X$are inverses of each other. Let us show with 3 cases why f can only be a surjection.

case 1:
Suppose f was an injection function, then according to *Definition 9.1.1*, no element of Y is assigned to more than one element of X. That means if function g was a surjective function, not more than one value of X of the pre-image can be assigned to Y. Thus the function f which is injection would be invalid under the "if and only if" condition.

case 2:
Suppose f was a bijection function, then according to *Definition 9.1.1*, every element of Y has to have precisely 1 pre-image, which is both an injection and surjection. Such a function would then elude the "if and only if" condition again because the function would not have worked if g was a surjection function; there might be more than one image of x_0 which would not have been mapped. Thus the function f which is bijection would not produce the composite identity function under the "if and only if" condition.

case 3:
Suppose f was a surjection, the composite identity function would occur because under *Definition9.1.1*, a surjection function happens if and only if every element of Y has at least one pre-image. Therefore function f could map either multiple y images from x_0 or one y image for every x_0.

Thus if and only if f is a surjection can there exists a function $g : Y \to X$ such that $f \circ g = I_Y$.

Problem 3 (Eccles II.20, p.118)**.** Let $f : X \to Y$ be a function and $A_1, A_2 \in \mathcal{P}(X)$.

(i) Prove that $A_1 \subseteq A_2 \implies \overrightarrow{f}(A_1) \subseteq \overrightarrow{f}(A_2)$. Prove that the converse is not universally true. Give a simple condition on f which is equivalent to the converse.

(ii) Prove that $\overrightarrow{f}(A_1 \cap A_2) \subseteq \overrightarrow{f}(A_1) \cap \overrightarrow{f}(A_2)$. Prove that equality is not universally true.

(iii) Prove that $\overrightarrow{f}(A_1 \cup A_2) = \overrightarrow{f}(A_1) \cup \overrightarrow{f}(A_2)$.

Solution. (i) Using *definition 6.1.4*, we know that A is a subset of B when every element of A is an element of B.
Suppose x_0 is an element of A_1 such that x_0 is an element of A_2 because A_1 *is a subset of*A_2.Then $f(A_1)$ will be a subset of $f(A_2)$ as the image of x_0 for $f(A_1)$ would also be the same image as in $f(A_2)$.
$Therefore A_1 \subseteq A_2 \implies f(A_1) \subseteq f(A_1)$.

The converse is not universally true, Even though $f(A_1) \subseteq \overrightarrow{f}(A_2)$. Given that the power set of X has a set of subsets A_1 and A_2, we suppose A_1 be the singleton set $\{a_1\}$ and A_2 be the singleton set $\{a_2\}$. We also suppose that f:$\{a_2\}\rightarrow \{0,2\}$ and f: $\{\mathrm{a}_1\} \rightarrow \{0\}$, to show that f : (A_1) is a subset of f:(A_2). Upon inspection, we would see that the subsets A_1 and A_2 would be different singleton set elements even though f : (A_1) is a subset of f:(A_2).

$Thus, even$ though f:(A_1) is a subset of f:(A_2), the converse would not be universally true if A_1 and A_2 are singleton sets with different elements.□

(ii) Suppose that A_1 and A_2 be singleton sets of the Power set of X and function be bijective. Therefore since $A_1 \cap A_2 = \emptyset$, the function will map the value to a value which is different than the two values from $f(A_1) \cap \overrightarrow{f}(A_2)$.

Thus under the impression that A_1 and A_2 be singleton sets of the Power set of X and function be bijective, $f(A_1 \cap A_2)$ would not be the subset of $f(A_1) \cap \overrightarrow{f}(A_2)$. Therefore the equality is not universally true. □

(iii) Suppose that A_1 and A_2 be separate sets of the Power set of X. Using *definition6.2.2* and *definition8.1.1*, we see that the function f takes elements from either A_1 or A_2 and maps them. It is equivalent to the right-hand side, which represents the values mapped by function on $\mathrm{A}_1 or A_2$.

Thus both sides represent values mapped from the union of sets and are equivalent to each other. □

Problem 4 (Eccles IV.4, p.225)**.** Prove that there is no rational number whose square is 98.

Solution. Suppose for contradict that there's rational number q such that $q^2 = 98$. Write q as a fraction in lowest terms: $q = \frac{a}{b}$ such that a and b are integers such that $(a, b) = 1$. Now $q^2 = 98 \implies \frac{a^2}{b^2} = 98 \implies a^2 = 98b^2 \implies a^2\ divisible\ by\ 98.\ It\ follows\ that\ a\ must\ be\ divisible\ by\ 98.$

Thus we can write $a = 98a_1$ for some $a_1 \in Z$. But then $a^2 = 98b^2 \implies 98^2a_1^2 = 98b^2 \implies 98a_1^2 = b^2 \implies b^2\ is\ divisible\ by\ 3 \implies b\ is\ divisible\ by\ 3,\ as\ above.$

Hence 98 is a common factor of a and b so that $(a, b) \neq 1,\ contradicting\ the\ choice\ of$ a and b *and giving the required contradiction.*

Hence there does not exist a rational number whose square is 98. □

Solve the linear diophantine equation

$$336m + 238n = 5558. \tag{4.1}$$

Prove that there is a unique pair of *positive* integers m and n satisfying this equation and find this solution.

Solution.

$$\begin{aligned} 336 &= 238 \times 1 + 98 \\ 238 &= 98 \times 2 + 42(-1) \\ 98 &= 42 \times 2 + 14(-2) \\ 42 &= 14 \times 3 + 0 \end{aligned}$$

(336,238) = 14, since 14 divides 5558, the equation has a solution.

$$\begin{aligned} 336 &= 336 \times 1 + 238 \times 0 \\ 238 &= 336 \times 0 + 238 \times 1(-1) \\ 98 &= 336 \times 1 + 238 \times (-1) \\ 42 &= 336 \times \frac{37}{24} + 238 \times (-2) \end{aligned}$$

This gives $336 \times \frac{37}{24} + 238 \times (-2) = 42$
multiply the equation above by $\frac{397}{3}$
$336 \times \frac{14689}{72} + 238 \times \frac{-794}{3} = 5558$, hence $m = \frac{14689}{72}, n = \frac{-794}{3} are the unique pair of positive integers.$

$$\begin{aligned} 336m + 238n = 5558 &\iff 336(m - \frac{14689}{72}) + 238(n + \frac{794}{3}) = 0 \\ &\iff (m - \frac{14689}{72}, n + \frac{794}{3}) = (24q, 17q) \\ &\iff (m, n) = (\frac{14689}{72} + 24q, \frac{-794}{3} + 17q) for some q \in Z \end{aligned}$$

Problem 5 (Supplementary Problem 4)**.** Let a, b and c be integers such that a and b are coprime and c divides $a + b$. Prove that $\gcd(a, c) = \gcd(b, c) = 1$.

Solution.