1. Divisibility

Definition 1.1. *The **integers** are the set of numbers of the form*

$$\{\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$$

So 3356 is an integer but 1/2 is not. Note that we can add, subtract and multiply two integers. For example

$$2 + 3 = 5, \qquad 6 - 8 = -2 \qquad \text{and} \qquad 2 \cdot 3 = 6.$$

But we cannot *always* divide. For example we cannot divide 2 into 1 (and get an integer).

Definition 1.2. *Let m and n be two integers.*

*We say m **divides** n, denoted $m \mid n$, if there is an integer k such that $n = km$.*

For example, 2 divides 6 as $6 = 3 \cdot 2$ (explicitly, $n = 6$, $m = 3$ and $k = 2$). 3 also divides 6 as $6 = 2 \cdot 3$ (explicitly, $n = 6$, $m = 2$ and $k = 3$). But 5 does not divide 7. How would we prove this?

Suppose that 5 does divide 7. Then we could find an integer k such that $7 = k5 = 5k$. What can we say about k? $k > 0$, that is, k is positive, as 5 and 7 are positive. If $k = 1$ then $k5 = 5 \neq 7$, too small. If $k \geq 2$ then

$$\begin{aligned} 5k &\geq 5 \cdot 2 \\ &= 10 \\ &> 7, \end{aligned}$$

too large. Note that we have exhausted all possible choices for k, since either $k \leq 0$, or $k = 1$, or $k > 1$. Thus there is no integer k such that $5k = 7$ and so 5 does not divide 7.

We record some basic properties of divisibility:

Lemma 1.3. *Every integer is divisible by* 1.

Proof. Let n be an integer. Then $n = n \cdot 1$, so that 1 divides n. □

Lemma 1.4. *Every integer divides* 0.

Proof. Let n be an integer. Then $0 = 0 \cdot n$, so that n divides 0. □

Here is a slightly more interesting result:

Lemma 1.5. *Let a and b be integers.*

If a divides b and b is non-zero then $|a| \leq |b|$.

Proof. By assumption we may find an integer k such that $b = ka$.

Claim 1.6. $k \neq 0$.

Proof of (1.6). Suppose not, suppose that $k = 0$.

Then

$$\begin{aligned} b &= ka \\ &= 0a \\ &= 0, \end{aligned}$$

which contradicts our assumption that b is non-zero. Thus $k \neq 0$. $\square$

We have

$$\begin{aligned} |b| &= |ka| \\ &= |k||a| \\ &\geq 1 \cdot |a| \\ &= |a|. \end{aligned} \qquad \square$$

Note that the proof of (1.5) hides something important. How do we know in the first place that we need to check $k \neq 0$?

Before we write down a clean proof of (1.5), we try a few things on a piece of scratch paper. The obvious thing to do is start with the equality $b = ka$ and take the absolute value, as in the last step of the proof. We are happy with this, until we realise that we don't know

$$|k||a| \geq |a|,$$

unless we know $k \neq 0$. So then we realise we need to check $k \neq 0$.

But when we write down the proof, we hide all the details of how we constructed the proof in the first place.