Your Name
Summer Academy 2016

Math 109 Homework 1

Problem 1 (Eccles I.4, p.53)**.** Prove the following statements concerning positive integers a, b, and c.

(i) (a divides b) and (a divides c) $\implies$ a divides $(b + c)$.

(ii) (a divides b) or (b divides c) $\implies$ a divides bc.

Solution. (i)

To prove the statement ((a divides b) and (a divides c) $\implies$ a divides $(b+c)$ to be true, we only need consider the case by assuming the hypothesis (a divides b and (a divides c) to be true and deduce the validity of the conclusion a divides $(b + c)$ to be true. We do so because the conditional statement could be true even if the hypothesis is false.

Using *definition 2.2.1*, we show that b is a multiple of a for certain integers q $(b = aq)$ and c is a multiple of a for some integer p $(c = ap)$.

Hence $b + c = aq + ap$ which $= a(q + p)$ using the distributivity operation. Since $b + c$ can be seen as a multiple of a, it is divisible by a. Also q and p are both integers, the addition of integers results in an integer.

Thus if (a divides b) and (a divides c) $\Rightarrow$ a divides $(b + c)$.$\square$

(ii)

To prove the statement (a divides b) or (b divides c) $\implies$ a divides bc to be true, we assume (a divides b) or (a divides c) to be true and deduce a divides bc to be true.

Using *definition 2.2.1*, we show that b is a multiple of a for some integer q$(b = aq)$ and c is a multiple of a for some integer p$(c = ap$

Since the hypothesis uses the 'or' connective, we can use either statement (a divides b) or (a divides c) to prove (a divides bc) to be true.

Using a divides b,

$$\frac{bc}{a} = \frac{(aq)c}{a} = \frac{a(qc)}{a}$$

The conclusion results in a integer as b is a multiple of a, which allows for a to divide b, which is clearly shown by associativity(of **Properties 2.3.1**). Furthermore, since q and c are some integers, the product of two integers produce an integer.

Thus if (a divides b) or (a divides c) then a divides bc. □

Problem 2 (Eccles I.6, p. 54)**.** Use the properties of addition and multiplication of real numbers given in Properties 2.3.1 to deduce that, for all real numbers a and b,

(i) $a \times 0 = 0 = 0 \times a$,

(ii) $(-a)b = -ab = a(-b)$,

(iii) $(-a)(-b) = ab$.

Solution. (i)

(ii)

By using the Associativity operation(of **Properties 2.3.1**), we would be able to clearly see the equivalence of the three statements with or without brackets without ambiguity.

$$(-a)b = (-ab) = -ab$$

Thus, (-a)b = -ab = a(-b).

(iii)

By using the Associativity operation(of **Properties 2.3.1**), we would not be hindered by the ambiguity that the parentheses provides.
Therefore, we would be able to show

$$(-a)(-b) = (-a * -b) = (-1 * a * -1 * b) = (-1 * -1 * a * b) = ab$$

Thus, $(-a)(-b) = ab$.

Problem 3 (Eccles I.10, p. 54)**.** What is wrong with the following proof that 1 is the largest integer?

> Let n be the largest integer. Then, since 1 is an integer we must have $1 \leq n$. On the other hand, since n^2 is also an integer we must have $n^2 \leq n$ from which it follows that $n \leq 1$. Thus, since $1 \leq n$ and $n \leq 1$ we must have $n = 1$. Thus 1 is the largest integer as claimed.

What does this argument prove?

Solution.

Problem 4 (Eccles I.19, p. 56)**.** Prove that

$$\prod_{i=2}^{n}\left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n}$$

for integers $n \geq 2$.

Solution. *Proof* We use induction on n.
Base case:

$$for\ n = 2, \prod_{i=2}^{2}(1 - \frac{1}{i^2}) = (1 - \frac{1}{2^2}) = \frac{3}{4}$$

and

$$\frac{n+1}{2n} = \frac{2+1}{2*2} = \frac{3}{4}$$

and so the result holds.

Inductive step:

Suppose as inductive hypothesis that

$$\prod_{i=2}^{2}(1 - \frac{1}{i^2}) = (1 - \frac{1}{2^2}) = \frac{3}{4}\ for\ integers\ n \leq 2.$$

Then,

$$\begin{aligned}
\prod_{i=2}^{n+1}(1-\frac{1}{i^2}) &= \prod_{i=2}^{n}(1-\frac{1}{i^2}) \ * \prod_{i=n+1}^{n+1}(1-\frac{1}{i^2}) \\
&= \frac{n+1}{2n} * (1-\frac{1}{n+1^2}) \\
&= \frac{n+1}{2n} * \frac{(n+1^2)-1}{(n+1^2)} \\
&= \frac{1}{2n} * \frac{n^2+2n+1-1}{n+1} \\
&= \frac{1}{2n} * \frac{n(n+2)}{n+1} \\
&= \frac{n+2}{2(n+2} \\
&= \frac{n+(1+1)}{2(n+2)} \\
&= \frac{(n+1)+1}{2n+2}
\end{aligned}$$

As required to prove the result for n+1.
Hence the result holds for all n $\leq$ 2 $\square$

Problem 5 (Eccles I.21, p. 56)**.** Suppose that x is a real number such that $x + 1/x$ is an integer. Prove by induction on n that $x^n + 1/x^n$ is an integer for all positive integers n. [For the inductive step consider $(x^k + 1/x^k)(x + 1/x)$.]

Solution. *Proof* We use strong induction on n to prove that the statement produces an integer.
Base case:
Since $x^1 + \frac{1}{x^1}$ is assumed to be an integer, case is proved.

Inductive step:

Now suppose as an inductive hypothesis that $x^k + \frac{1}{x^k}$ is an integer for all $k = 1, ..., n$.
Then,

$$(x^n + \frac{1}{x^n})(x + \frac{1}{x}) = x^{n+1} + \frac{1}{x^{n+1}} + x^{n-1} + \frac{1}{x^{n-1}}$$

Since $x^k + \frac{1}{x^k}$ is an integer for all $k = 1, ..., n$, $x^{n-1} + \frac{1}{x^{n-1}}$ and hence $x^{n+1} + \frac{1}{x^{n+1}}$ are integers.

Therefore, by strong induction, $x^n + \frac{1}{x^n}$ is an integer for all positive integers n. $\square$

Problem 6 (Eccles I.25, p. 57)**.** Let u_n be the nth Fibonacci number (Definition 5.4.2). Prove, by *induction on n* (without using the Binet formula Proposition 5.4.3), that

$$u_{m+n} = u_{m-1}u_n + u_m u_{n+1}$$

for all positive integers m and n.

Deduce, again using induction on n, that u_m divides u_{mn}.

Solution. *Base case*:
Since u_n is the *nth* Fibonacci number, we can make the base case so that

Inductive step:
suppose the formula holds true for all positive integers n and thus serves as our induction hypothesis. Then using the *Fibonacci definition*, we find the $n + 1$ case

$$\begin{aligned} U_{m+(n+1)} &= U_{(m+n)+1} = U_{(m+n)-1} + U_{m+n} \\ &= (U_{m-1}U_{n-1} + U_n U_n) + (U_{m-1}U_n + U_m U_{n+1}) \\ &= U_{m-1}U_{n+1} + U_m U_{n+2} \end{aligned}$$

as required to prove the formula for $n = k + 1$

Conclusion: Hence by induction, the following formula holds for all positive integers m and n.