Problem set 7

Saturday, November 12, 2016 12:27 AM

1. Suppose X be a non-empty set. For any subset A of X, let $\mathbb{1}_A$ be the characteristic function of A. That means,

$$\mathbb{1}_A : X \to \{0,1\}, \quad \mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

(a) Suppose X is finite. What is $\sum_{x \in X} \mathbb{1}_A(x)$?

(b) Use part (a) and $\mathbb{1}_{A \cup B} = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_{A \cap B}$ to conclude

$$|A \cup B| = |A| + |B| - |A \cap B| .$$

(c) Suppose $A_1, \dots, A_n$ are subsets of X. By induction on n, show that $\mathbb{1}_{(A_1 \cup A_2 \cup \dots \cup A_n)^c} = (1 - \mathbb{1}_{A_1})(1 - \mathbb{1}_{A_2}) \cdots (1 - \mathbb{1}_{A_n})$.

(Hint. Use $(A \cup B)^c = A^c \cap B^c$, $\mathbb{1}_{A^c} = 1 - \mathbb{1}_A$, $\mathbb{1}_{A^c \cap B^c} = \mathbb{1}_{A^c} \cdot \mathbb{1}_{B^c}$.)

(d) Use part (c) to conclude

$$\mathbb{1}_{(A_1 \cup \dots \cup A_n)^c} = 1 - (\mathbb{1}_{A_1} + \dots + \mathbb{1}_{A_n}) + (\overbrace{\mathbb{1}_{A_1 \cap A_2} + \dots + \mathbb{1}_{A_{n-1} \cap A_n}}^{\text{all } \mathbb{1}_{A_i \cap A_j}}) - (\overbrace{\mathbb{1}_{A_1 \cap A_2 \cap A_3} + \dots + \mathbb{1}_{A_{n-2} \cap A_{n-1} \cap A_n}}^{\text{all } \mathbb{1}_{A_i \cap A_j \cap A_k}}) + \dots$$

(signs alternate) $+ (-1)^n \, \mathbb{1}_{A_1 \cap A_2 \cap \dots \cap A_n}$.

Then, using (a), deduce

$$|(A_1 \cup A_2 \cup \dots \cup A_n)^c| = |X| - \sum_i |A_i| + \sum_{i_1 < i_2} |A_{i_1} \cap A_{i_2}| - \dots + (-1)^n |A_1 \cap \dots \cap A_n| .$$

(This is called inclusion-exclusion formula).

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2.(a) Suppose $f: X \to Y$ is a function. Prove that if $g: Y \to X$ is a left inverse of f and $h: Y \to X$ is a right inverse of f, then $g = h$.

(Hint ① You are allowed to use the fact that for three functions $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} X_4$ we have $f_3 \circ (f_2 \circ f_1) = (f_3 \circ f_2) \circ f_1$.

② Consider $g \circ f \circ h$.)

(b) Use part (a) and a theorem proved in class to show:

If a function $X \xrightarrow{f} Y$ is a bijection, then there is a unique function $Y \xrightarrow{g} X$ such that $g \circ f = I_X$ and $f \circ g = I_Y$.

(This function is called the inverse of f, and it is denoted by $f^{(-1)}$.)

3. Suppose $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ are two bijections.

(a) Prove that $g \circ f$ is a bijection.

(b) Prove that $f^{(-1)}$ is a bijection. ($f^{(-1)}$ is as above.)

4. Suppose $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ are two functions, and $g \circ f$ is a bijection. Prove that

$$g \text{ is injective} \iff f \text{ is surjective.}$$

(Hint. g bijective $\Rightarrow$ $f = \underbrace{g^{(-1)}}_{\text{bijective}} \circ \underbrace{(g \circ f)}_{\text{bijective}}$

f bijective $\Rightarrow$ $g = \underbrace{(g \circ f)}_{\text{bijective}} \circ \underbrace{f^{(-1)}}_{\text{bijective}}$.

∴ Use problem 3 and a theorem from class.)

Problem set 7

Saturday, November 12, 2016 1:44 AM

This is NOT part of your homework assignment. It is for your practice.

5. Suppose $X \xrightarrow{f} Y$ is a function. Prove that

(a) f has a <u>unique</u> left inverse $\iff$ f is bijective.

(b) f has a <u>unique</u> right inverse $\iff$ f is bijective.

(<u>Hint</u>. Look at the way we proved Lemma 1 and Lemma 2 in class.)