Your Name
Summer Academy 2016

Math 109 Homework 1

Problem 1 (Eccles I.4, p.53)**.** Prove the following statements concerning positive integers a, b, and c.

(i) (a divides b) and (a divides c) $\implies$ a divides $(b+c)$.

(ii) (a divides b) or (b divides c) $\implies$ a divides bc.

Solution. (i)

We assume the hypothesis (a divides b) and (a divides c) to be true and deduce the validity of the conclusion a divides $(b + c)$ to be true.

Using *definition 2.2.1*, b is a multiple of a for certain integers q $(b = aq)$ and c is a multiple of a for some integer p $(c = ap)$.

Hence $b + c = aq + ap$ which $= a(q + p)$ using the distributivity operation. Since $b + c$ can be seen as a multiple of a, it is divisible by a. Also q and p are both integers, the addition of integers results in an integer.

Thus if (a divides b) and (a divides c) then a divides $(b + c)$.□

(ii)

Suppose (a divides b) or (a divides c) to be true and deduce a divides bc to be true. Since the hypothesis uses the 'or' connective, we would prove the argument to be true in two cases.

First, since a divides b, b would be a multiple a of a certain integer $q(b = aq)$. Thus if $b = aq$, then a divides $(aq)c$, a divides $a(qc)$. Therefore a divides bc, since the product of two integers q and c is an integer.

Second, since a divides c, c is a multiple of a for a certain integer q $(c = aq)$. Thus if $c = aq$, a divides $b(aq)$, a divides $a(bq)$. Therefore a divides bc, since the product of two integers b and q is an integer.

Therefore, if (a divides b) or (a divides c) then a divides bc. □

Problem 2 (Eccles I.6, p. 54)**.** Use the properties of addition and multiplication of real numbers given in Properties 2.3.1 to deduce that, for all real numbers a and b,

(i) $a \times 0 = 0 = 0 \times a$,

(ii) $(-a)b = -ab = a(-b)$,

(iii) $(-a)(-b) = ab$.

Solution. (i)

$$\begin{aligned} 0 &= 0 + 0 (\textit{zero law}) \\ a \times 0 &= a \times (0 + 0) \\ a \times 0 &= a \times 0 + a \times 0 (\textit{distributive property}) \\ a \times 0 - a \times 0 &= a \times 0 \\ 0 &= a \times 0 \\ 0 &= 0 \times a (\textit{commutative property}) \end{aligned}$$

Therefore it is proven, $a \times 0 = 0 = 0 \times a$ is a true statement.□

$(answer\ derived\ fromSangil\ Kim,\ traded\ homework\ 2\ question3\ part\ i\ for\ this.)$

(ii)

$$\begin{aligned} Let\ a + (-a) &= 0\ (\textit{subtraction property}) \\ b(a + (-a)) &= b(0) \\ ab + (-a)b &= 0\ (\textit{distribution property}) \\ ab + (-a)b - ab &= 0 - ab \\ (-a)b &= -ab \end{aligned}$$

Therefore it is proven, $(-a)b = -ab = a(-b)$.□

(iii)

$$\begin{aligned} Let\ a + (-a) &= 0\ (\textit{subtraction law}) \\ a(-b) + (-a)(-b) &= 0\ (\textit{distribution law}) \\ -(ab) + (-a)(-b) &= 0\ (\textit{associativity law}) \\ -(ab) + (-a)(-b) + ab &= 0 + ab \\ (-a)(-b) &= ab \end{aligned}$$

Therefore, it is proven $(-a)(-b) = ab$. $\square$

Problem 3 (Eccles I.10, p. 54)**.** What is wrong with the following proof that 1 is the largest integer?

> Let n be the largest integer. Then, since 1 is an integer we must have $1 \leq n$. On the other hand, since n^2 is also an integer we must have $n^2 \leq n$ from which it follows that $n \leq 1$. Thus, since $1 \leq n$ and $n \leq 1$ we must have $n = 1$. Thus 1 is the largest integer as claimed.

What does this argument prove?

Solution. Looking at the proof, we find that the hypothesis is false since there cannot be a largest integer and hence the conclusion that 1 is the largest integer is false even though the proof is valid(proof by contradiction that there is no largest integer).

Therefore, this argument does not proof anything. $\square$

Problem 4 (Eccles I.19, p. 56)**.** Prove that

$$\prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n}$$

for integers $n \geq 2$.

Solution. *Proof* We use induction on n.
Base case:

$$for\ n = 2, \prod_{i=2}^{2}(1 - \frac{1}{i^2}) = (1 - \frac{1}{2^2}) = \frac{3}{4}$$

and

$$\frac{n+1}{2n} = \frac{2+1}{2*2} = \frac{3}{4}$$

and so the result holds.

Inductive step:

Suppose as inductive hypothesis that

$$\prod_{i=2}^{2}(1-\frac{1}{i^2}) = (1-\frac{1}{2^2}) = \frac{3}{4} \; for\ integers\ n \leq 2.$$

Then,

$$\begin{aligned}
\prod_{i=2}^{n+1}(1-\frac{1}{i^2}) &= \prod_{i=2}^{n}(1-\frac{1}{i^2}) \;*\; \prod_{i=n+1}^{n+1}(1-\frac{1}{i^2}) \\
&= \frac{n+1}{2n} * (1-\frac{1}{n+1^2}) \\
&= \frac{n+1}{2n} * \frac{(n+1^2)-1}{(n+1^2)} \\
&= \frac{1}{2n} * \frac{n^2+2n+1-1}{n+1} \\
&= \frac{1}{2n} * \frac{n(n+2)}{n+1} \\
&= \frac{n+2}{2(n+2} \\
&= \frac{n+(1+1)}{2(n+2)} \\
&= \frac{(n+1)+1}{2n+2}
\end{aligned}$$

As required to prove the result for n+1.
Hence the result holds for all n $\leq$ 2. $\square$

Problem 5 (Eccles I.21, p. 56)**.** Suppose that x is a real number such that $x + 1/x$ is an integer. Prove by induction on n that $x^n + 1/x^n$ is an integer for all positive integers n. [For the inductive step consider $(x^k + 1/x^k)(x + 1/x)$.]

Solution. *Proof* We use strong induction on n to prove that the statement produces an integer.
Base case:
Since $x^1 + \frac{1}{x^1}$ is assumed to be an integer, case is proved.

Inductive step:

Now suppose as an inductive hypothesis that $x^k + \frac{1}{x^k}$ is an integer for all $k = 1, ..., n$.
Then,

$$(x^n + \frac{1}{x^n})(x + \frac{1}{x}) = x^{n+1} + \frac{1}{x^{n+1}} + x^{n-1} + \frac{1}{x^{n-1}}$$

Since $x^k + \frac{1}{x^k}$ is an integer for all $k = 1, ..., n$, $x^{n-1} + \frac{1}{x^{n-1}}$ and hence $x^{n+1} + \frac{1}{x^{n+1}}$ are integers.

Therefore, by strong induction, $x^n + \frac{1}{x^n}$ is an integer for all positive integers n. $\square$

Problem 6 (Eccles I.25, p. 57)**.** Let u_n be the nth Fibonacci number (Definition 5.4.2). Prove, by *induction on n* (without using the Binet formula Proposition 5.4.3), that

$$u_{m+n} = u_{m-1}u_n + u_m u_{n+1}$$

for all positive integers m and n.
Deduce, again using induction on n, that u_m divides u_{mn}.

Solution. *Base case*:
$U_1 = 1$, $U_2 = 1$, $U_3 = 2$ let n = 1

$U_{m+1} = U_{m-1}U_1 + U_{m-1}U_{1+1}$

$U_{m+1} = U_{m-1} + U_m$

therefore case n=1 is proved true
using n=2,

$$U_{m+2} = U_{m-1}U_2 + U_mU_3$$
$$U_{m+2} = U_{m-1}(1) + U_m(2)$$
$$U_{m+2} = U_{m-1} + U_m + U_m$$
$$U_{m+2} = U_{m+1} + U_m$$

Since n=1 and n=2 are proven true, base case is proved.

Inductive step:
suppose the formula holds true for all positive integers n and thus serves as our induction hypothesis. Then using the *Fibonacci definition*, we find the $n+1$ case

$$\begin{aligned} U_{m+(n+1)} = U_{(m+n)+1} &= U_{(m+n)-1} + U_{m+n} \\ &= (U_{m-1}U_{n-1} + U_nU_n) + (U_{m-1}U_n + U_mU_{n+1}) \\ &= U_{m-1}U_{n+1} + U_mU_{n+2} \end{aligned}$$

as required to prove the formula for $n = k + 1$

Conclusion: Hence by induction, the following formula holds for all positive integers m and n.$\square$
$(understanding\ and\ solution\ courtesy\ of\ \ Arden\ and\ Matthew)$

Base case for u_m divides u_{mn}:
let n= 1

$$u_m\ divides\ u_{m(1)} = u_m\ divides\ u_m = 1$$

since solution is an integer 1, u_m divides u_{mn}. Thus base case is proved.

Inductive step for u_m divides u_{mn}:

$$\begin{aligned}
U_{m(n+1)} = U_{m+mn} &= U_{m-1}U_{mn} + U_m(U_{mn+1}) \\
&= U_{m-1}(U_{mn+1}) - U_{m-1}(U_{mn-1}) + U_m(U_{mn+1}) \\
&= U_{mn+1}(U_{m-1} + U_m) - U_{m-1}(U_{mn-1}) \\
&= U_{m+1}(U_{mn+1}) - U_{m-1}(U_{mn-1}) \\
&= U_{m+1}(U_{mn+1}) - (U_{m+1} - U_m)(U_{mn+1} - U_{mn}) \\
&= U_{m+1}(U_{mn+1}) - (U_{m+1}U_{mn+1} - U_{m+1}U_{mn} - U_m Umn + 1 + U_m U_{mn}) \\
&= U_{m+1}U_{mn} + U_m U_{mn+1} - U_m U_{mn}
\end{aligned}$$

Therefore it is proven by strong induction that $U_{m(n+1)}$ is divisible by U_m.$\square$
$(from\ Sangil\ Kim)$