ON RATES, STATES, AND THE SIR MODEL

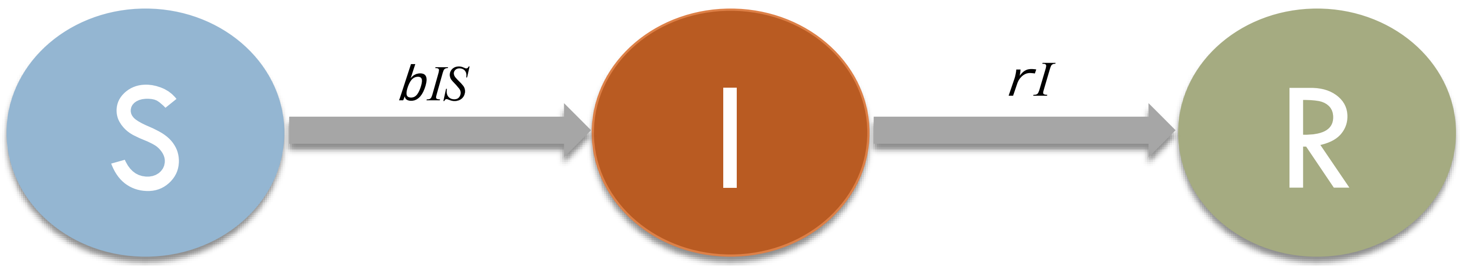
Modelling Biological Systems, BIOS13

Dept. of Biology, Lund

The SIR model

- The SIR model is a classic model from epidemiology (it actually is a class of models)
- The model describes the spread of a disease in a population
- It has three state variables: S, I and R
  - **S : Susceptibles** – the number of susceptible individuals
  - **I : Infected** – the number of infected individuals
  - **R : Recovered** – the number of recovered (immune) individuals

The basic SIR Model



$$\begin{cases} \dfrac{dS}{dt} = -bIS \\[2ex] \dfrac{dI}{dt} = bIS - rI \\[2ex] \dfrac{dR}{dt} = rI \end{cases}$$

S : Susceptibles

I : Infected

R : Recovered

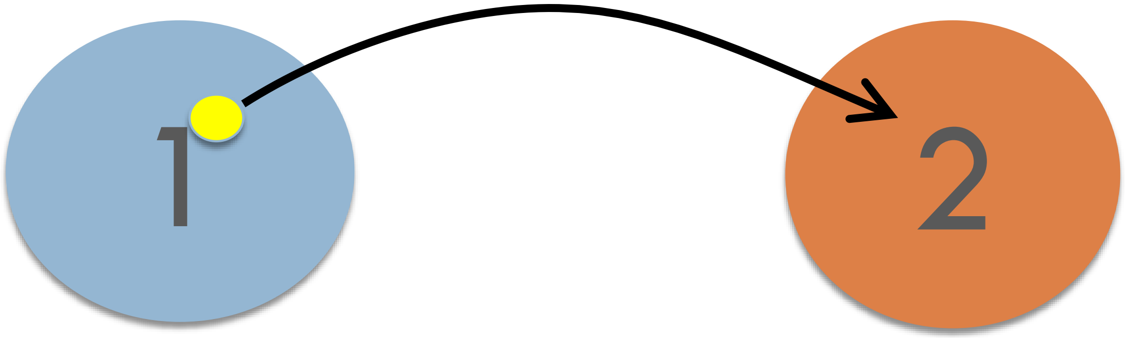
β : contact rate

ρ : recovery rate

Transition rates

- A *rate* is a 'speed of change'. It it often a direct rate of change, such as a growth rate, dn/dt, but can also be a *transition rate*.

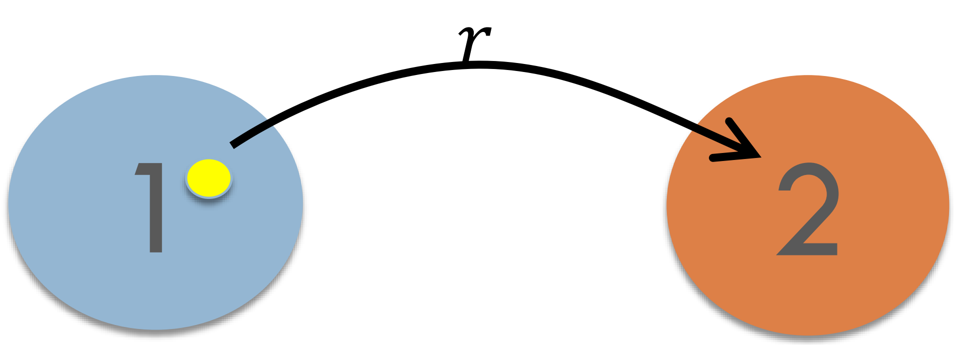
- *Transition rates* describe how frequently the transition from one state to another occurs.
  Examples: death rate (alive $\longrightarrow$ dead), dispersal rate (here $\longrightarrow$ there), or rate of a chemical reaction (compound 1 $\longrightarrow$ compound 2).



Transition rates, interpretation

Transition probability

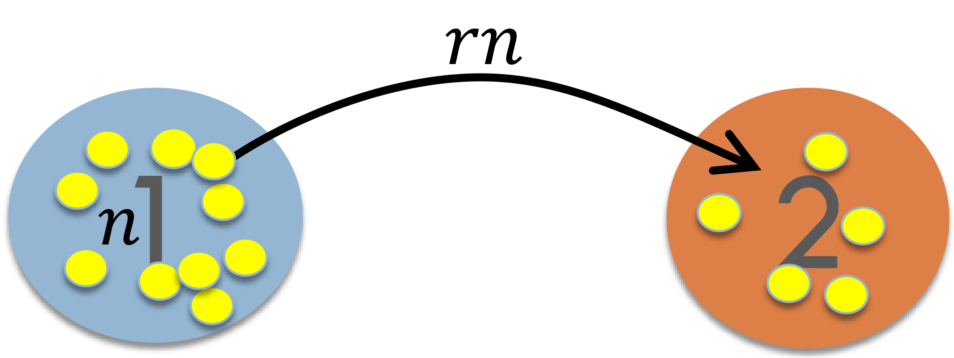
- Transition rates are (most often) probabilistic.
  The probability that a transition from state 1 to state 2 at rate r will occur in a time interval Δt is $r\Delta t$ (if Δt is small).
  Example: The probability that an individual with death rate 2/y (two per year) will die in a time interval $\Delta t = 0.01y$ is $2 \cdot 0.01 = 0.02$.
- An alternative, equivalent, interpretation: On average $r\Delta t$ transitions occur in the time interval Δt.
  Example: The individual above will die on average 0.02 times during that time interval.



Total transition rate

- Rates are often given *per item* (individual, or other object) or *per unit* (per mass unit, volume unit, or other).
  Examples: death rate per individual, decay rate per kg.

- The total rate at which a transition occurs is thus the per item rate multiplied by the number of items (e.g. individuals).
  Example: The total death rate of a population of size 1000 with a *per capita* death rate of 2/y is 2000/y. The average number of deaths in a time interval of 0.01y is thus 2000·0.01 = 20. (the probability interpretation fails here)



Average dynamics

- Even though the outcome is stochastic, it can be approximated by its average for large populations.
  For example, the group of infected in an SIR model decrease at a rate $-\rho I$.

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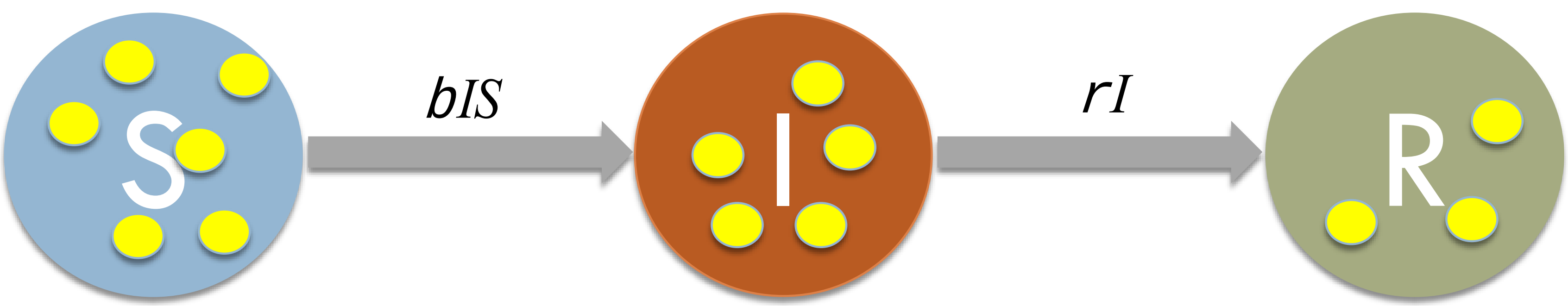
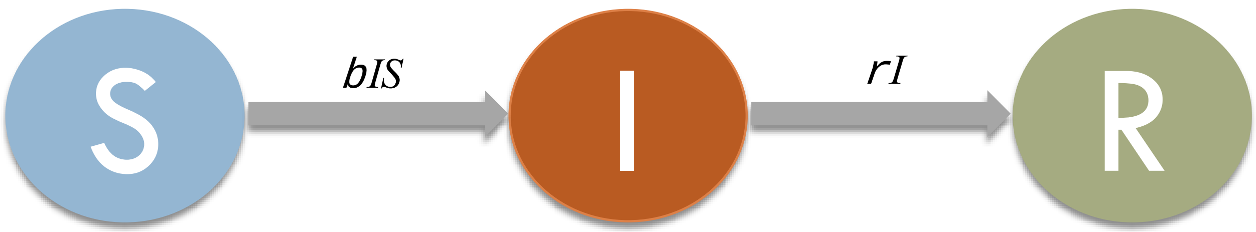


Average time to transition and R_0

- The average time an individual (or mass unit, or whatever) will stay in a state is $1/r$ if *r* is the transition rate *from* that state. Example: The SIR model



$$\begin{cases} \dfrac{dS}{dt} = -bIS \\ \dfrac{dI}{dt} = bIS - rI \\ \dfrac{dR}{dt} = rI \end{cases}$$

The average time to recovery is $1/\rho$, i.e. the average time in illness. The higher the recovery rate, the shorter is the time an individual remains infected.

Additionally, each infected individual infects βS new individuals per time unit (the total infection rate is βIS).

The total number of new infected per infected individual is then $\frac{1}{\rho} \cdot \beta S = R_0$.

R_0 is a central concept in epidemiology. A disease will not spread as long as $\mathrm{R}_0 < 1$.

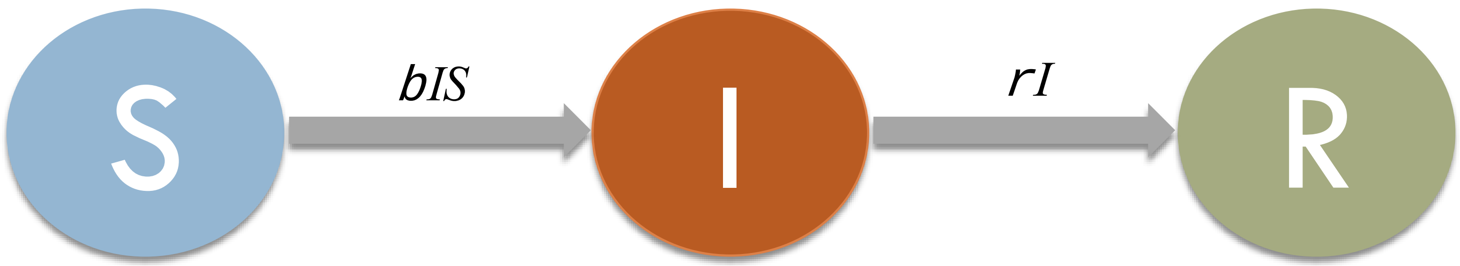
Some Covid-19 data

- R_0 has been estimated to between1.5 and 3.5
- Incubation period (from exposure to symptoms): 1-14 days (average 5.2)
- Recovery in 2 weeks (mild cases)

$$R_0 = \frac{1}{\rho} \cdot \beta S \qquad => \qquad \rho R_0 / S = \beta$$

Sources:
https://www.worldometers.info/coronavirus/
https://www.ecdc.europa.eu/en/covid-19/facts/questions-answers-basic-facts
https://ourworldindata.org/coronavirus

The basic SIR Model



$$
\begin{cases}
\dfrac{dS}{dt} = -bIS \\
\dfrac{dI}{dt} = bIS - rI \\
\dfrac{dR}{dt} = rI
\end{cases}
$$

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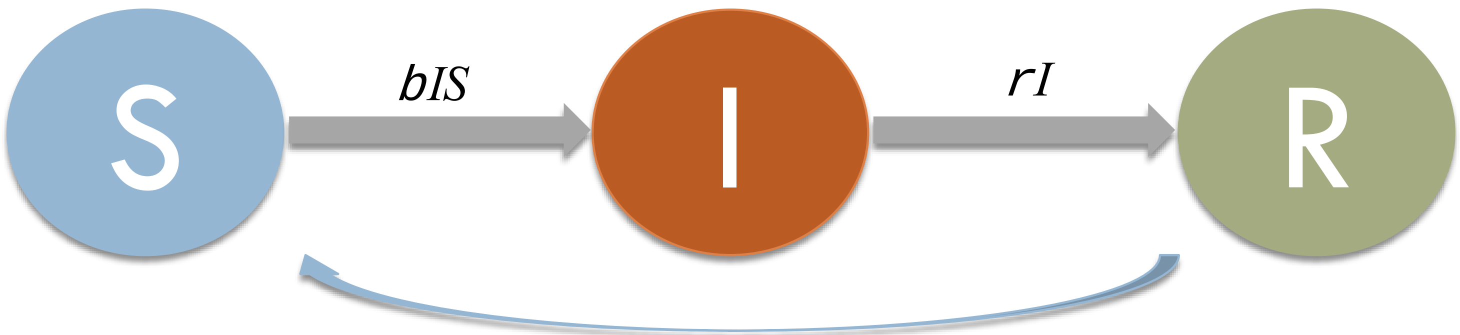
The basic SIR Model (limited resistance)



$$\frac{ds}{dt} = \beta IS + \gamma R$$

$$\frac{dI}{dt} = \beta IS - \rho I$$

$$\frac{dR}{dt} = \rho I - \gamma R$$

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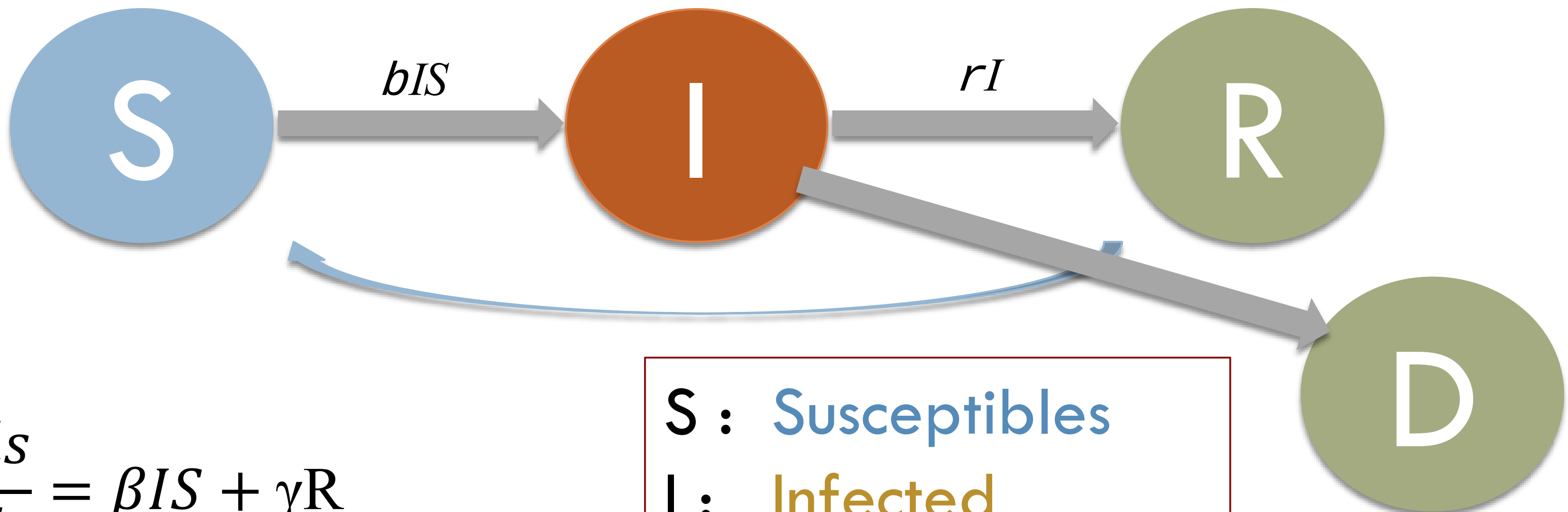
The basic SIR Model (death)



$$\frac{ds}{dt} = \beta IS + \gamma R$$

$$\frac{dI}{dt} = \beta IS - \rho I - \mu I$$

$$\frac{dR}{dt} = \rho I - \gamma R$$

S : Susceptibles

I : Infected

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β : contact rate

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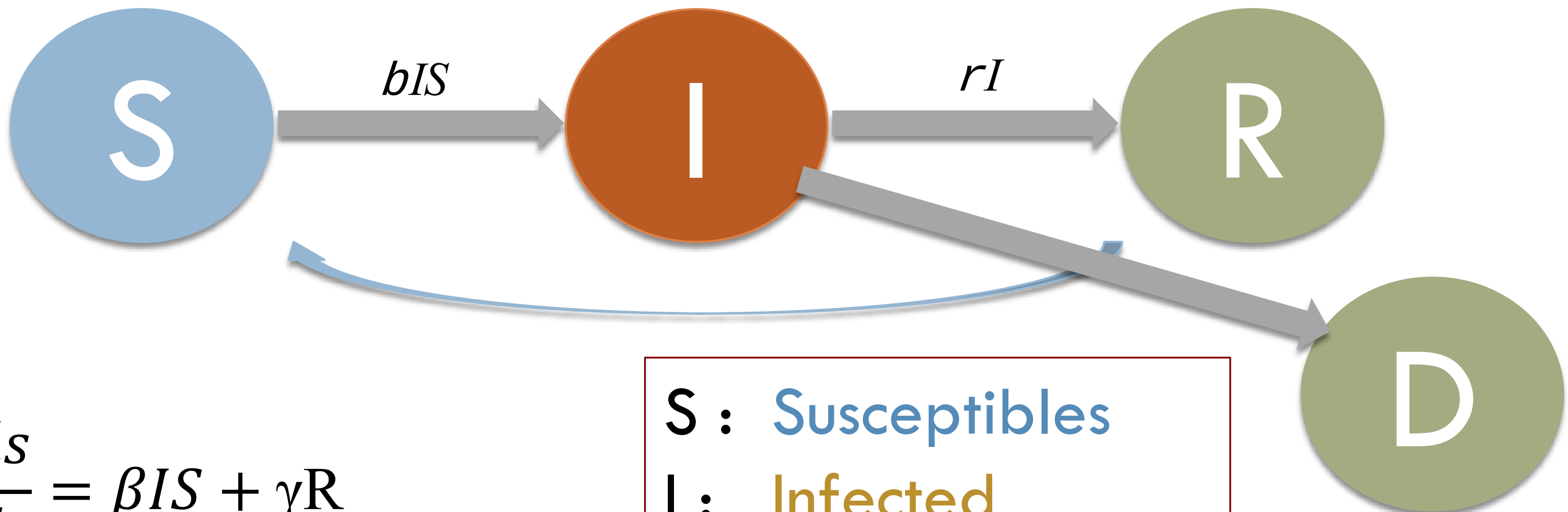
The basic SIR Model (death)



$$\frac{ds}{dt} = \beta IS + \gamma \mathrm{R}$$

$$\frac{dI}{dt} = \beta IS - \rho \mathrm{I} - \mu \mathrm{I}$$

$$\frac{dR}{dt} = \rho \mathrm{I} - \gamma \mathrm{R}$$

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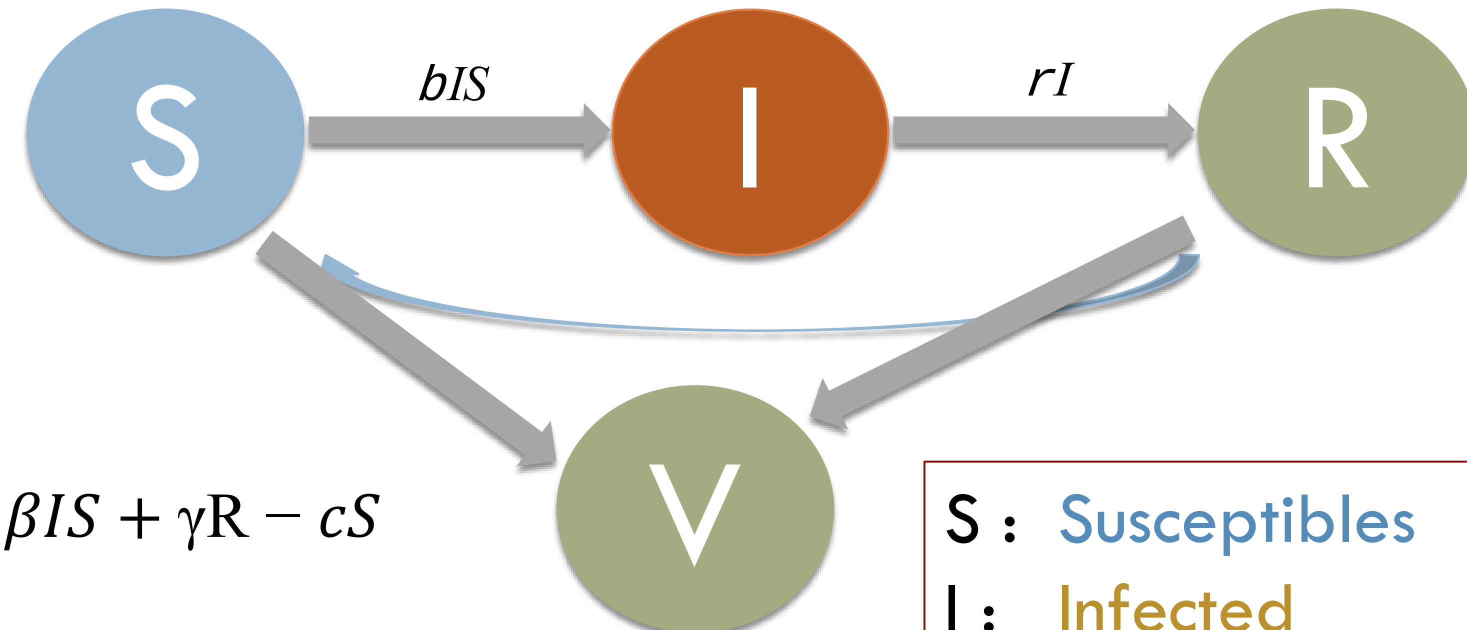
… human birth, non-infected deaths, ect, ect

The basic SIR Model (vaccination)



$$\frac{ds}{dt} = \beta IS + \gamma \mathrm{R} - cS$$

$$\frac{dI}{dt} = \beta IS - \rho \mathrm{I}$$

$$\frac{dR}{dt} = \rho \mathrm{I} + \gamma \mathrm{R} - \mathrm{cR}$$

$$\frac{dV}{dt} = c(S + R)$$

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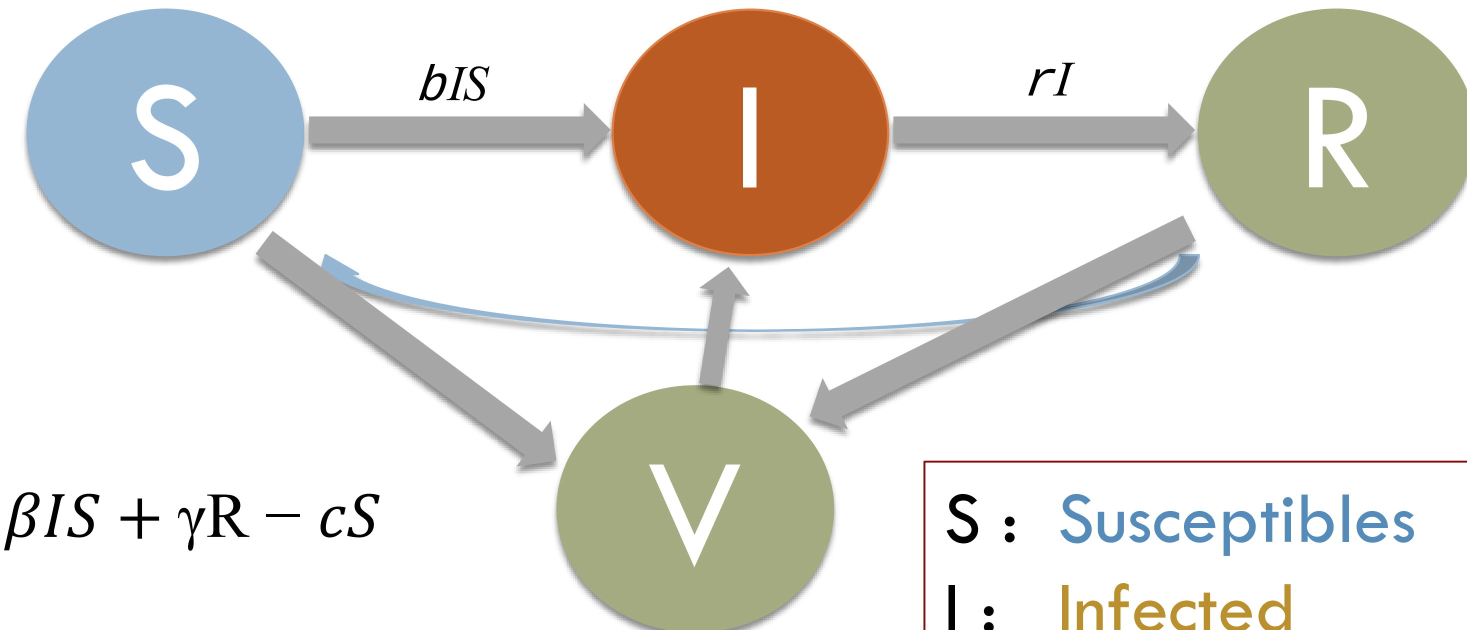
The basic SIR Model (vaccination)



$$\frac{ds}{dt} = \beta IS + \gamma R - cS$$

$$\frac{dI}{dt} = \beta IS - \rho I + e\beta IV$$

$$\frac{dR}{dt} = \rho I + \gamma R - cR$$

$$\frac{dV}{dt} = c(S + R) - e\beta IV$$

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