Dynamic systems, I

Modelling Biological Systems, BIOS13

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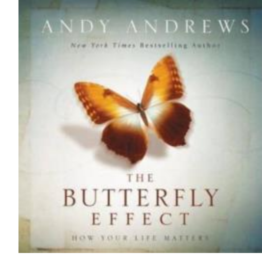
Definition

- A dynamic system *changes over time*

- It can usually be described by a set of *state variables* that change over time (t): $x(t)$, $y(t)$, …

- Example 1: A cup of coffee
  - State variables: *Temperature*(t), *pH*(t), …

- Example 2: A population
  - State variable(s): population density ($n(t)$), age structure, spatial structure, …



Properties

- Dynamic systems **change with time** according to a set of **rules** that determine how one **state** of a system change to another state in **state space**.

- Formulated by differential and integral calculus (Leibniz(1646-1716) and Newton (1643-1727))

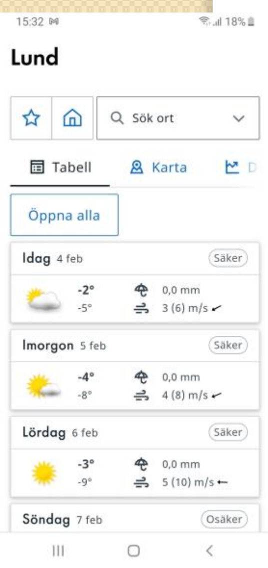
- Dynamical system can be classified by their properties:

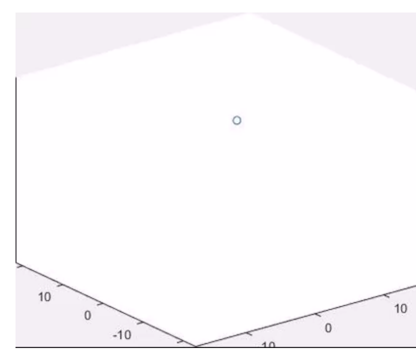
Discrete	Continuous
Linear	Non-linear
Deterministic	Stochastic
Autonomous	Non-autonomous

$$\frac{dx}{dt} = \sigma(y\text{-}x)$$

$$\frac{dy}{dt} = x(\rho - z) - y$$

$$\frac{dz}{dt} = xy - \beta z$$



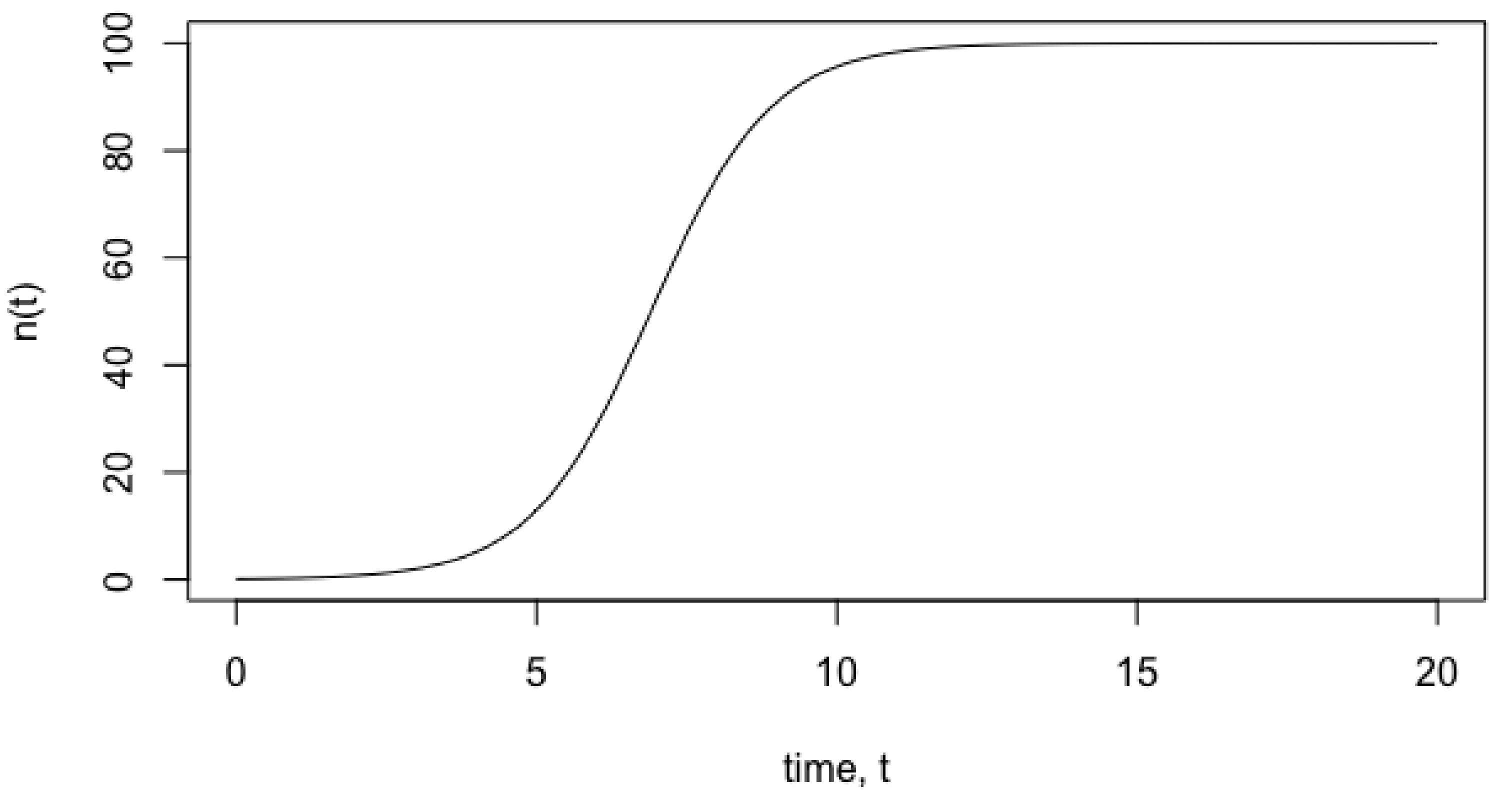


State variables

- The number of state variables = the number of dimensions of the system
- The state variables can be chosen in many different ways, depending on the application. What variables are interesting or relevant?
- The dimensionality of a system is therefore not given beforehand, but really is up to us.

System dynamics

- The dynamics of a system are often plotted as the value of the state variables over time



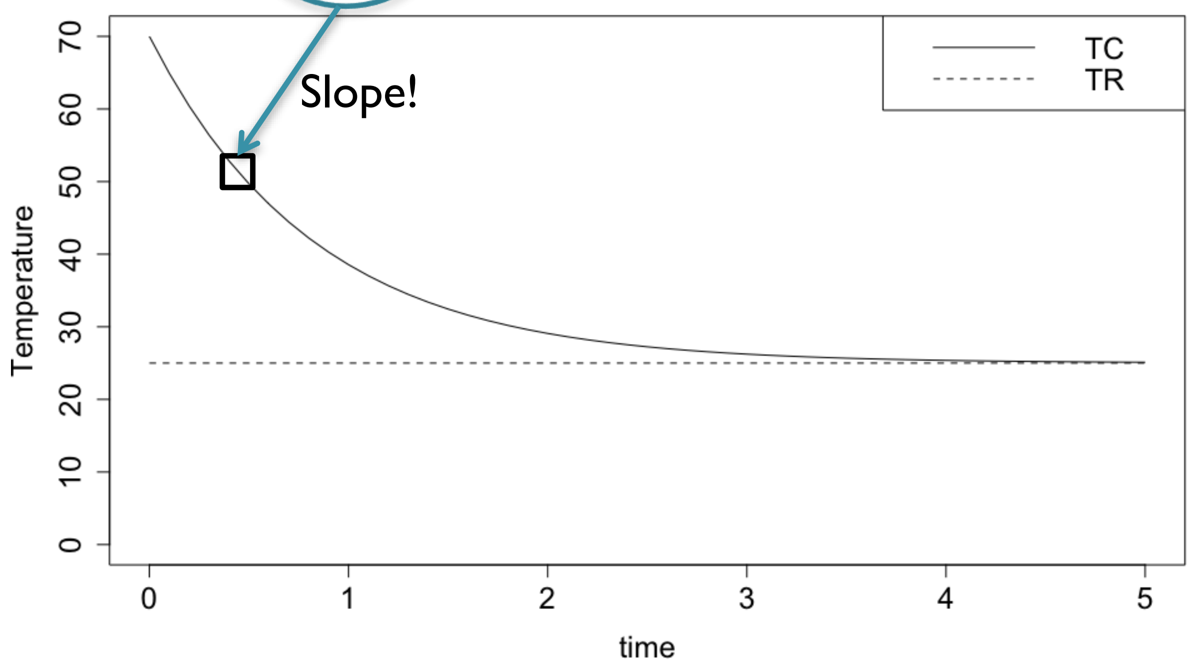
System dynamics

- The mathematical description is often a ***differential equation***, which expresses the rate of change of the state variable(s) as a function
- The function can be a function of the state variables themselves, and/or external variables.
- Example 1: Coffee cup temperature $T_C(t)$

Room temperature

$$\frac{dT_C(t)}{dt} = f(T_C(t), T_R) = -1.2(T_C(t) - T_R)$$

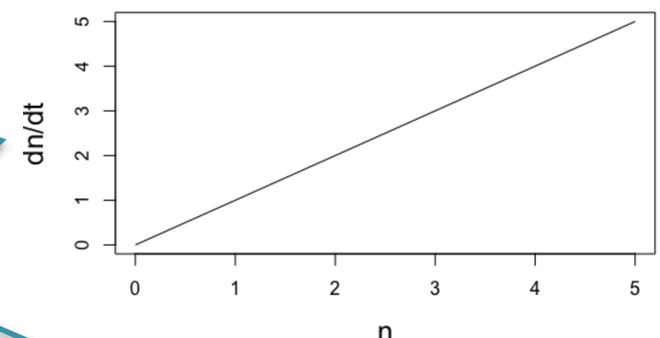
Slope!



Exponential growth

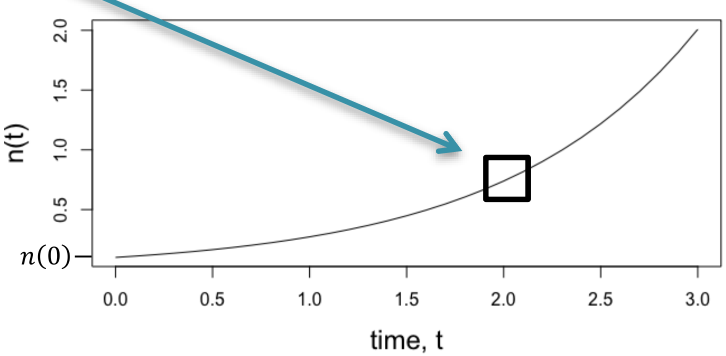
- A simple example is that of exponential growth, applicable to bacterial growth in a test tube (and many other things).
- The underlying assumption is that all individuals give birth at a fixed rate b and die at a fixed rate d.
- If the population has n individuals, the total population growth rate is Births – Deaths = $bn - dn = \underbrace{(b-d)}_{r}n = rn$
- The parameter r is the *per capita* growth rate, the growth rate per individual.

$$\frac{dn}{dt} = rn$$

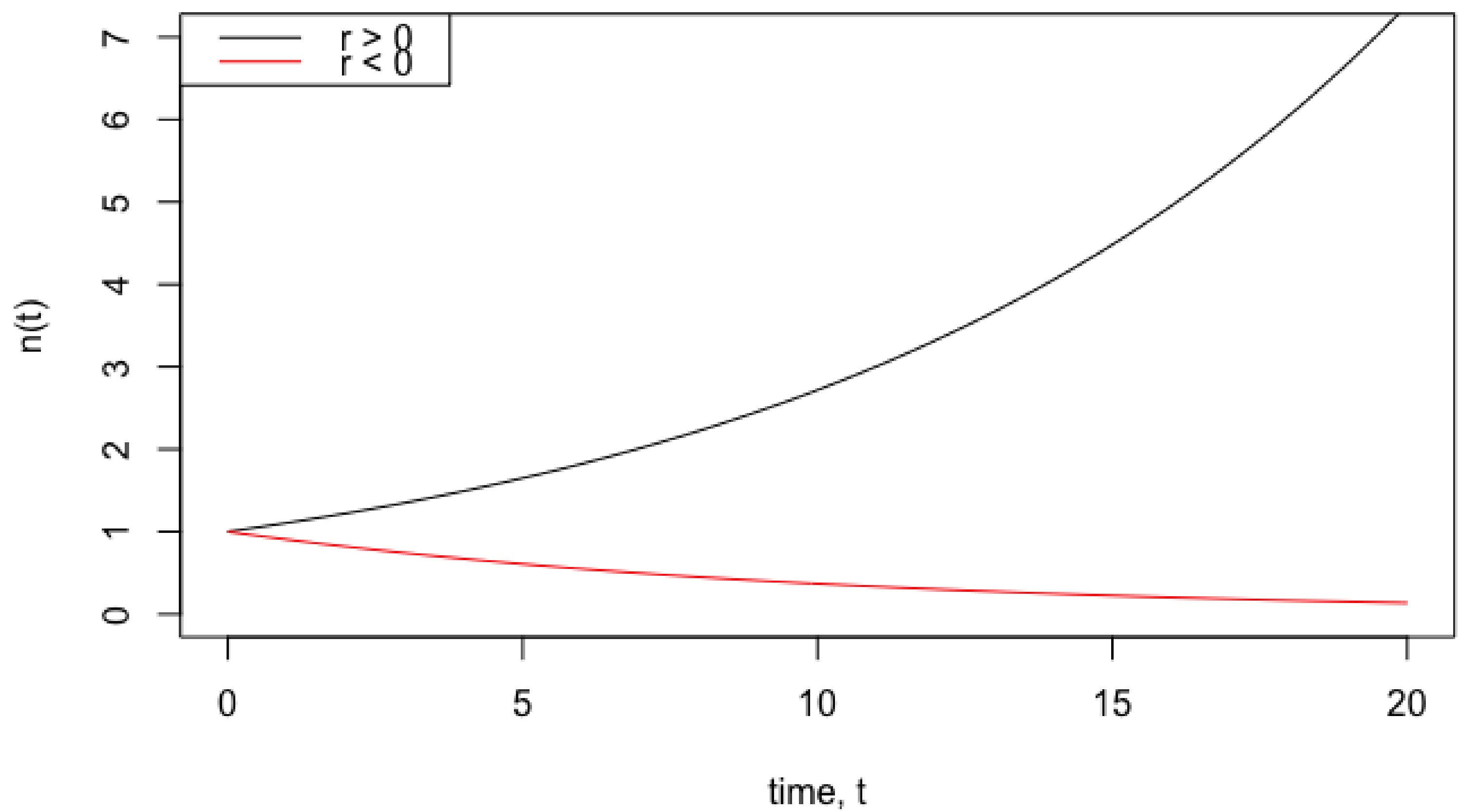


- Solution:

$$n(t) = n(0)e^{rt}$$



Exponential growth

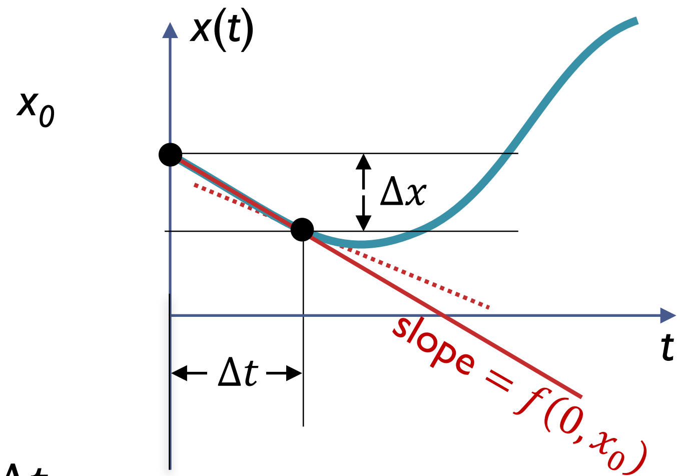


Numerical solutions to differential equations

Basic approach, solving: $\frac{dx}{dt} = f(t,x), \quad x(0) = x_0$

Idea: Use $\frac{\Delta x}{\Delta t} \approx \frac{dx}{dt}$

1. Start by setting $x = x_0, t = 0$
2. Calculate $f(t,x)$
3. Choose a small Δt
4. Calculate $\Delta x = f(t,x)\Delta t$
5. Update $x \leftarrow x + \Delta x, \; t \leftarrow t + \Delta t$
6. Repeat from 2, until reaching final *t*.



Numerical solutions to exponential growth

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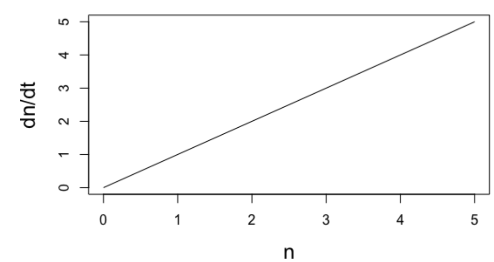
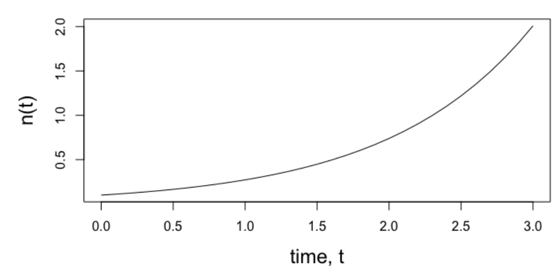
$$\frac{dn}{dt} = rn$$



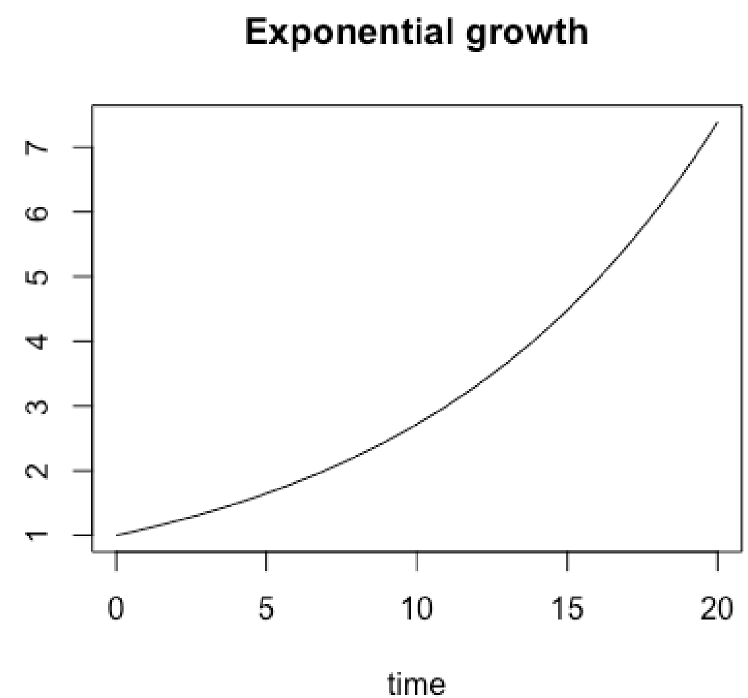


Differential equations in R

```
install.packages("deSolve", dependencies = TRUE)
# First load the deSolve package:
library(deSolve)

# define the growth function:
expGrowth <- function(t, n, r) {
  dndt <- r * n
  return(list( dndt ))  # the output has to be a list
}

# set up a vector of time-points for the output:
timevec <- seq( 0, 20, by=0.1 )

# set the necessary parameters:
r <- 0.1 # per capita growth rate
n0 <- 1  # initial population size

# call the ode function to solve
the differential equation:
out <- ode( y = n0, func = expGrowth,
         times = timevec, parms = r)

# plot the output:
plot( out , main='Exponential growth' )
```

Only once!

The ode function of deSolve solves differential equations of the type
dx/dt = f(t, x, P)
where P is a parameter of your choice.



Logistic growth

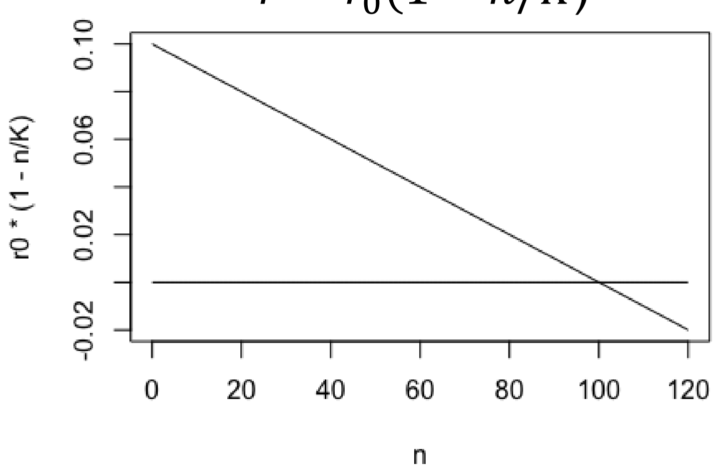
Population growth can not go on for ever.

It is reasonable to assume that the birth rate will decrease with population density, that the death rate will increase with population density, or all of the above.

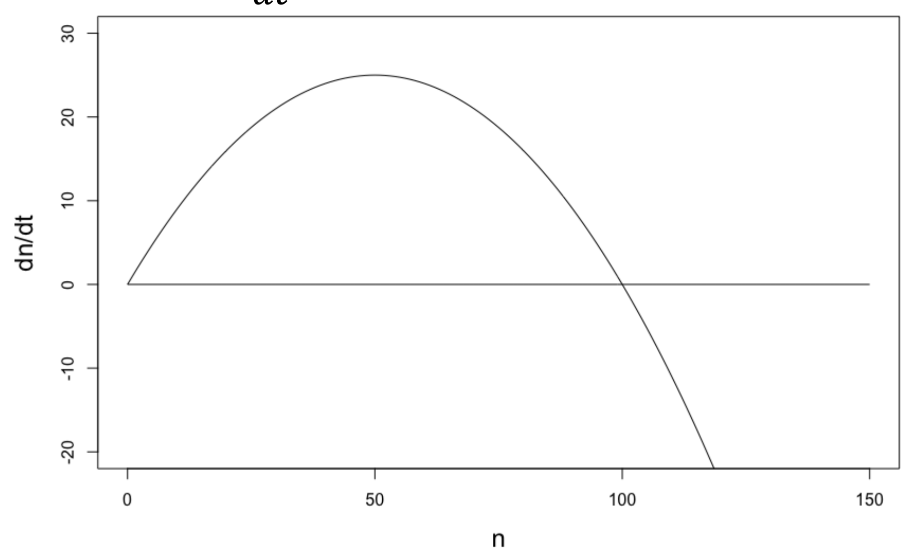
In conclusion, the per capita growth rate ($r = b - d$) will decrease with population size n.

The simplest assumption is a linear function:

$$r = r_0(1 - n/K)$$



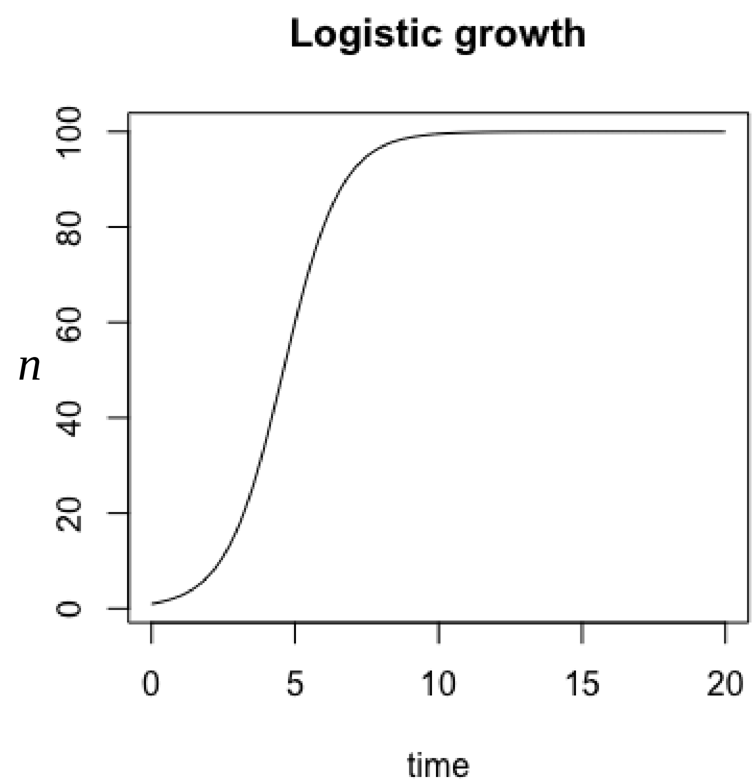
$$\frac{dn}{dt} = rn = r_0 n(1 - n/K)$$



Differential equations in R

More than one parameter: use a list

```
# First load the deSolve package:
library(deSolve)

# define the growth function:
logisticGrowth <- function(t, n, P) {
  dndt <- P$r0 * n * ( 1 - n / P$K )
  return(list( dndt ))
}

# set up a vector of time-points for the output:
timevec <- seq( 0, 20, by=0.1 )

# we need a list of parameters:
P <- list( r0=1, K=100 )

n0 <- 1  # initial population size

# call the ode function:
out <- ode( y = n0, func = logisticGrowth,
               times = timevec, parms = P)

# plot the output:
plot( out , main='Logistic growth' )
```



Equilibrium

- An equilibrium is defined as a state where no state variables change.
  dx/dt=0, dy/dt=0, …

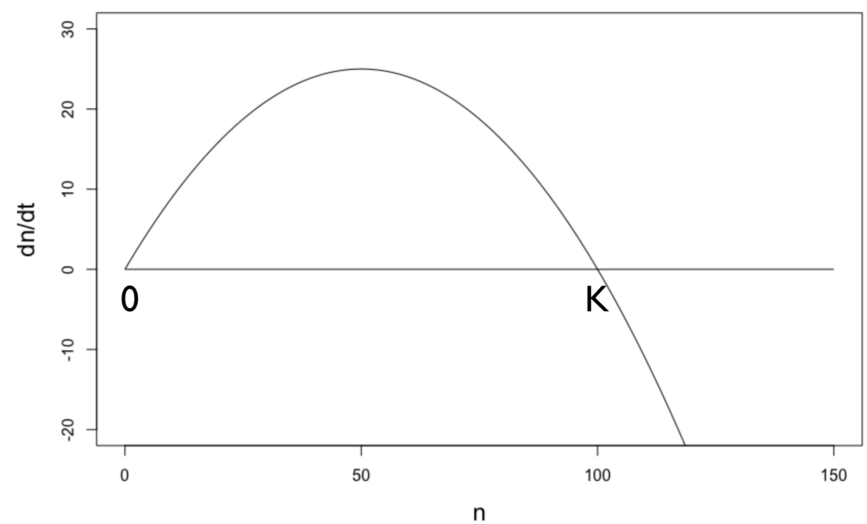
$$\frac{dn}{dt} = f(n) = r_0 n(1 - n/K)$$

Solve: $f(n) = 0$

$$r_0 n(1 - n/K) = 0$$

Solutions:

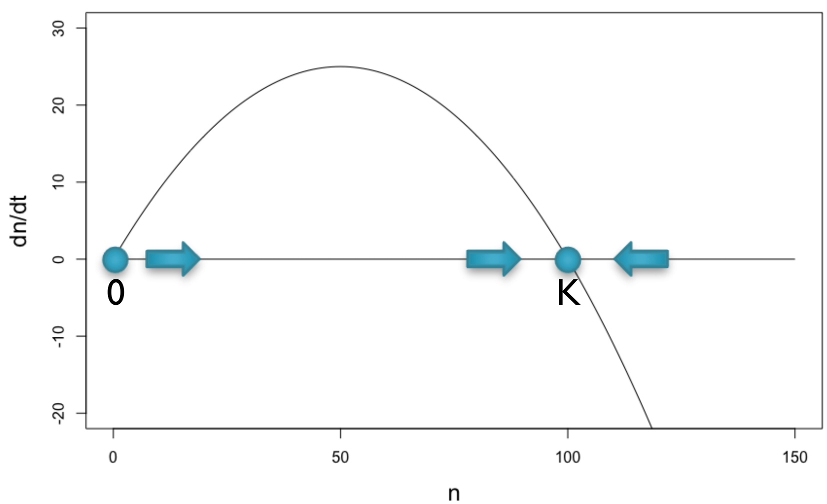
$n = 0$ or $n = K$

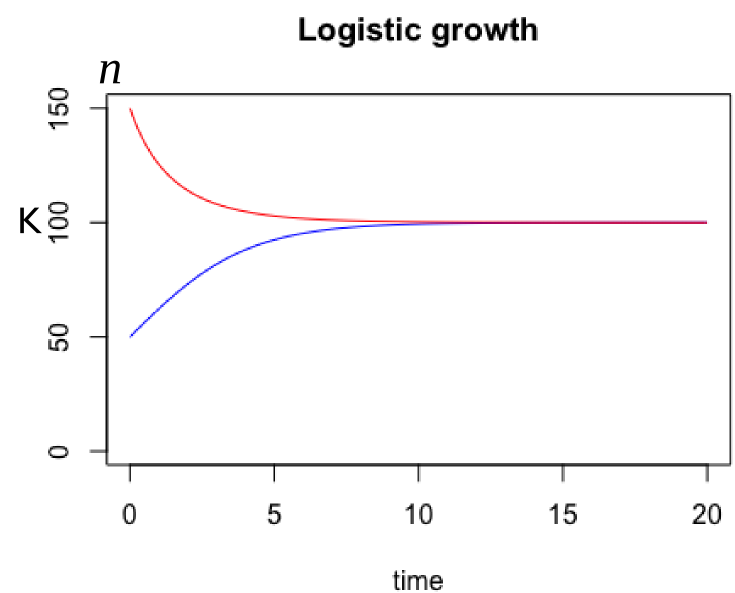


Stability

- An equilibrium is *stable* if the system will approach the equilibrium when started nearby

- For the logistic equation, *K* is stable, but 0 is not.





General condition for stability

- Given an arbitrary differential equation $\dfrac{dn}{dt} = f(n)$
- We find equilibrium points by solving $f(n^*) = 0$
- Close to n^*, the dynamics behave as $\dfrac{dx}{dt} = f'(n^*)x$ where $x = n - n^*$
- An equilibrium n^* is stable *if* $\boxed{f'(n^*) < 0}$
- The rate of the return to equilibrium is determined by the magnitude of $f'(n^*)$
- The *return time* is defined as $T_R = -1/f'(n^*)$

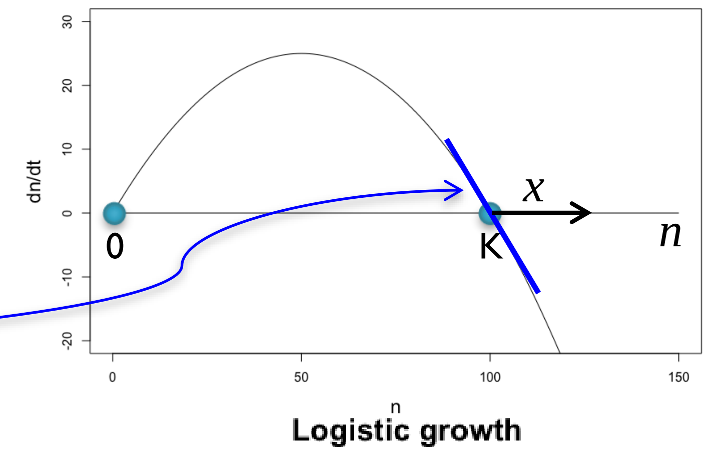
Analyzing stability

- We can linearize the growth function near K:

$$\frac{dn}{dt} = f(n) = r_0 n(1 - n/K)$$

$$f(n) \approx f(K) + f'(K)(n - K) =$$

$$= 0 + \underline{(-r_0)(n - K)}$$



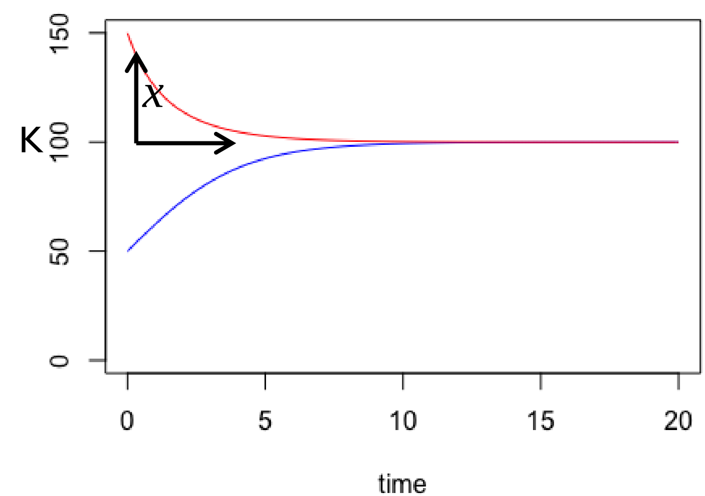
new variable: $x(t) = n(t) - K$

$$\boxed{\frac{dx}{dt}} = \frac{dn}{dt} = f(n) \approx -r_0(n - K) \boxed{= -r_0 x}$$

this is exponential growth!

$$\boxed{x \to 0 \iff n \to K}$$

(if we can show $x \to 0$, that means $n \to K$)



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