Modelling biological systems

BIOS13 2023 HT

Lund university

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Agenda

Learning goals stated in the original course design:

“the students should be able to handle and analyze biological problems that are dependent on mathematical techniques for their solution”

01 Basic Algebra

02 Basic Calculus

03 Basic Linear Algebra

04 Linear Algebra in R

01 Basic Algebra

01 Basic Algebraic Rules

- Double Sums: $\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$
- Arithmetic sequence: $x_{i+1} - x_i = constant$
  - arithmetic sequence is a sequence of numbers such that the difference between the consecutive terms is constant (example: $2,5,8,11,14, \ldots$).
- Sum of an arithmetic sequence: $\sum_{i=1}^{n} x_i = n\frac{(x_1+x_n)}{2}$
  - $(number\ of\ terms) \times \frac{first\ term + last\ term}{2}$
- Geometric sequence: $\frac{x_{i+1}}{x_i} = constant$
  - Geometric sequence is a sequence of numbers such that the ratio between the consecutive terms is constant (example: $2,6,18,54, with\ the\ common\ ratio\ 3$).
- Sum of a geometric sequence: $\sum_{i=1}^{n} x_i = x_1 \frac{1-a^n}{1-a}$
  - $(first\ term) \times \frac{1-common\ ratio^{(number\ of\ terms)}}{1-common\ ratio}$

Can you derive them yourself?
It could be easier to memorise afterwards!

01 Basic Algebraic Rules

- Distributive law:
  - $a(b+c) = ab + ac$
- Laws of squares:
  - $(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$
  - $(a-b)^2 = (a-b)(a-b) = a^2 - 2ab + b^2$
- Conjugate rule:
  - $(a+b)(a-b) = a^2 - b^2$
- Laws of powers:
  - $a^n = a \times a \times \cdots \times a$ (n times)
  - $a^{\frac{1}{2}} = \sqrt{a}$
  - $(\sqrt{a})^2 = a$
  - $a^{\frac{1}{n}} = \sqrt[n]{a}$
  - $(\sqrt[n]{a})^n = a$
  - $a^m a^n = a^{m+n}$
  - $(a^m)^n = a^{mn}$
  - $(ab)^n = a^n b^n$
  - $a^{-1} = \frac{1}{a}$
  - $a^{-n} = \frac{1}{a^n}$
  - $\frac{a^m}{a^n} = a^m a^{-n} = a^{m-n}$
  - $a^0 = 1$

01 Basic Algebraic Rules

- Sums: $\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$
- Arithmetic sequence: $x_{i+1} - x_i = constant$
  - arithmetic sequence is a sequence of numbers such that the difference between the consecutive terms is constant (example: $2,5,8,11,14, \ldots$).
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01 Basic Algebraic Rules

- Arithmetic series derivation:

$\boldsymbol{X_n} = x_1 + x_2 + \cdots + x_{n-1} + x_n$ since there is a constant difference a between each term,

$\boldsymbol{X_n} = x_1 + (x_1 + a) + (x_1 + 2a) + \cdots + x_n$ (Eq1)

$\boldsymbol{X_n}$ can also be written in the opposite direction as:

$\boldsymbol{X_n} = x_n + (x_n - a) + (x_n - 2a) + \cdots + x_1$ (Eq2)

Adding (Eq1) and (Eq2) gives:

$2\,\boldsymbol{X_n} = n(x_1 + \mathrm{x_n})$

So, $\boldsymbol{X_n} = \frac{n(x_1 + x_n)}{2}$

How about the Geometric series?

01 Basic Algebraic Rules

- More rules for sums :
  - $\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = (x_{11} + x_{12} + x_{13} + \cdots + x_{1n}) + (x_{21} + \cdots + x_{2n}) + \cdots + x_{mn}$
  - $\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{m} x_{ij} = (x_{11} + x_{21} + x_{31} + \cdots + x_{m1}) + (x_{12} + \cdots + x_{m2}) + \cdots + x_{mn}$
  - $\sum_{i=1}^{m}\sum_{j=1}^{n}(x_{ij} + y_{ij}) = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{n} y_{ij}$
- Products:
  - $\prod_{i=1}^{n} x_i = x_1 x_2 \ldots x_n$
  - $\prod_{i=1}^{n} a x_i = a x_1 a x_2 \ldots a x_n = a^n \prod_{i=1}^{n} x_i$

01 Solving first degree (linear) polynomial

General rule: rearrange to isolate the unknown variable

- Number of unknown variable: 1
- Highest degree: 1
- Examples:

$$3x + 4 = 10$$

$$3x = 10 - 4 = 6$$

$$x = \frac{6}{3} = 2$$

- $\frac{3}{z-1} + 4 = 5$
- $\frac{4y+a}{1-y} = a - 1$

Try to solve by yourself !
The answers and the steps can be found in the canvas video "solving polynomial".

01

Solving second degree (quadratic) polynomial

- Number of unknown variable: 1
- Highest degree: 2
- General form:

$$ax^2 + bx + c = 0, a \neq 0$$

- General solution:

$$x = -\frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

- Examples:

$$x^2 - 2x - 3 = 0$$

$$x = -\frac{(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot (-3)}}{2 \cdot 1}$$

$$x = \frac{2 \pm \sqrt{4 - 4 \cdot 1 \cdot (-3)}}{2} = \frac{2 \pm 4}{2}$$

$$x_1 = -1, x_2 = 3$$

- $4x - 3 = \frac{5}{2-\mathrm{x}}$

Try to solve by yourself !
The answers and the steps can be found in the canvas video "solving polynomial".

01 Imaginary and Complex numbers

- The imaginary unit : $i = \sqrt{-1}$
  - $i^2 = -1$
  - $\frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$
- Imaginary numbers: $ia \text{ } where\ a\ is\ a\ real\ number$
  - $(ia)^2 = i^2a^2 = -a^2$
  - $\sqrt{-x} = \sqrt{(-1)x} = \sqrt{-1}\sqrt{x} = i\sqrt{x}$
- Complex numbers have a real and an imaginary part:
  - $z = a + bi, a = Re(z), b = Im(z)$

- Addition and multiplication of complex numbers follow the common algebraic rules:

  Let

  $z_1 = a_1 + b_1 i$

  $z_2 = a_2 + b_2 i,$

  - $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$
  - $z_1 z_2 = (a_1 + b_1 i)(a_2 + b_2 i)$

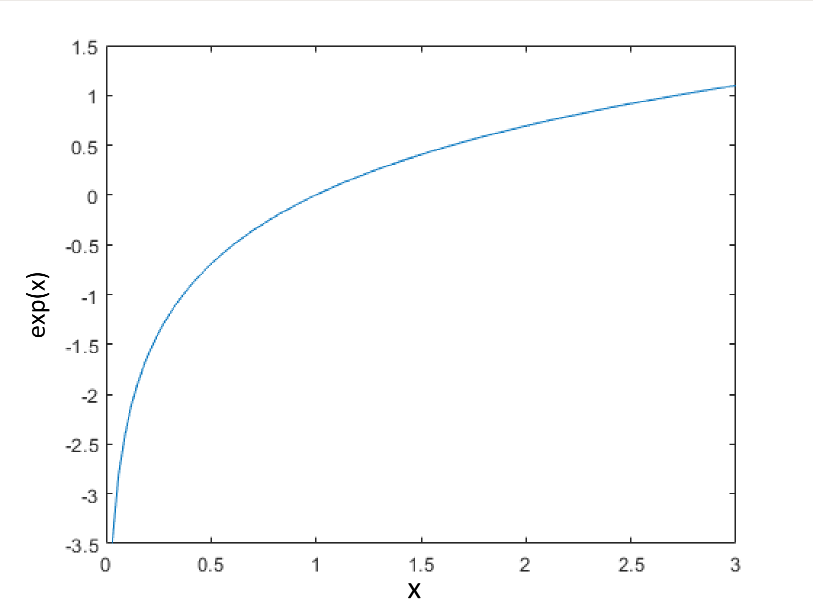
    $= a_1a_2 + a_1b_2i + b_1a_2i + b_1b_2i^2$

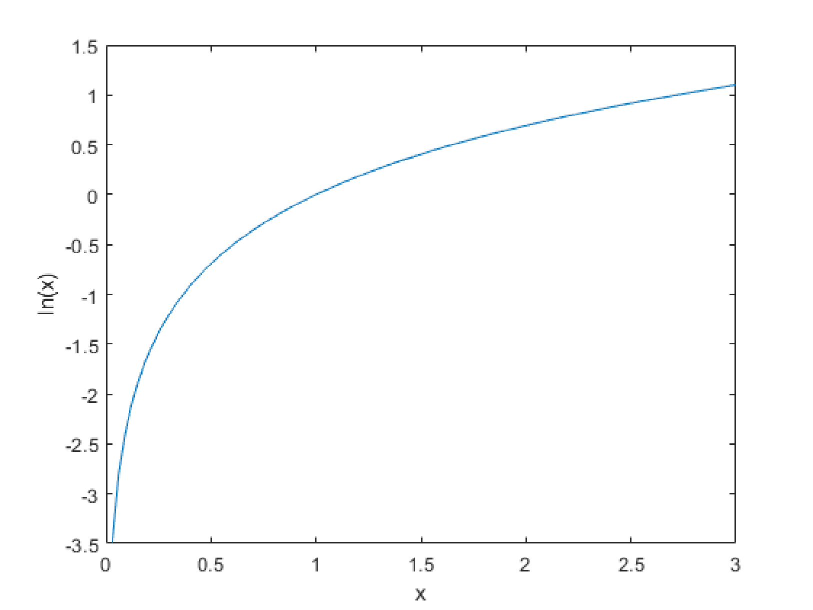
    $= a_1a_2 - b_1b_2 + (a_1b_2 + b_1a_2)i$
  - Definition of magnitude or absolute value:
    - $|z_1| = |a_1 + b_1 i| = abs(z_1) = \sqrt{a_1^2 + b_1^2}$

02 Basic Calculus

02 Exponential functions and logarithms

- The natural base: $e = 2.718 \ldots$
- The exponential function: $\exp(x) = e^x$
  - $e^x e^y = e^{x+y}$
  - $e^{-x} = \frac{1}{e^x}$
- The natural logarithm: $\ln(x) = \log_e x$
  - $\ln(e^x) = x$
  - $e^{\ln x} = x$
  - $\ln(1) = 0$
  - $\ln(0) = -\infty$





02

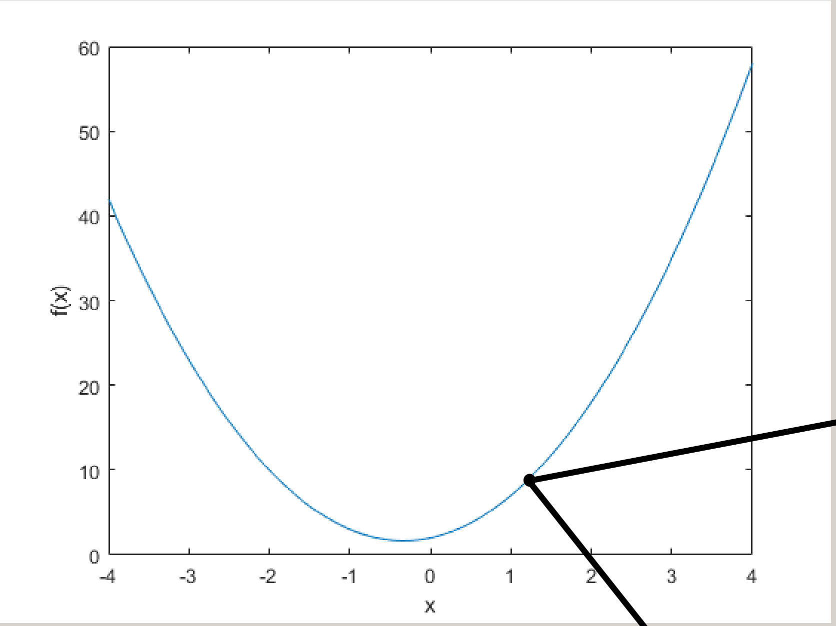
Laws of logarithms

- $\ln(xy) = \ln(x) + \ln(y)$
- $\ln\left(\frac{1}{x}\right) = -\ln(x)$
- $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$
- $\ln(x^y) = yln(x)$

- Examples:
  - $\ln(3a)$
  - $\ln\big((a+2)(x-u)\big)$
  - $\ln\left(\frac{1}{4q}\right)$
  - $\ln\left(\frac{w}{4q}\right)$
  - $\ln(34^5)$
  - $\ln(a^{b^c})$

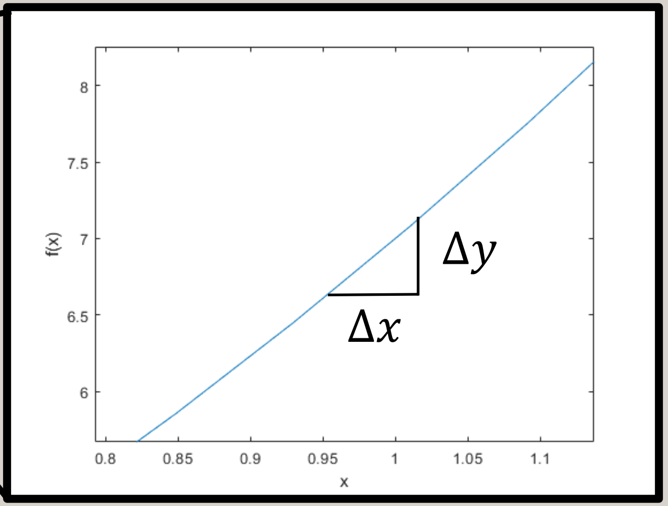
Try to solve by yourself !
The answers and the steps can be found in the canvas video “laws of logarithms”.

02 Derivative: the slope of arbitrary functions



- Derivative: $\frac{df(x)}{\mathrm{d}x} = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

The derivative is the local rate of change

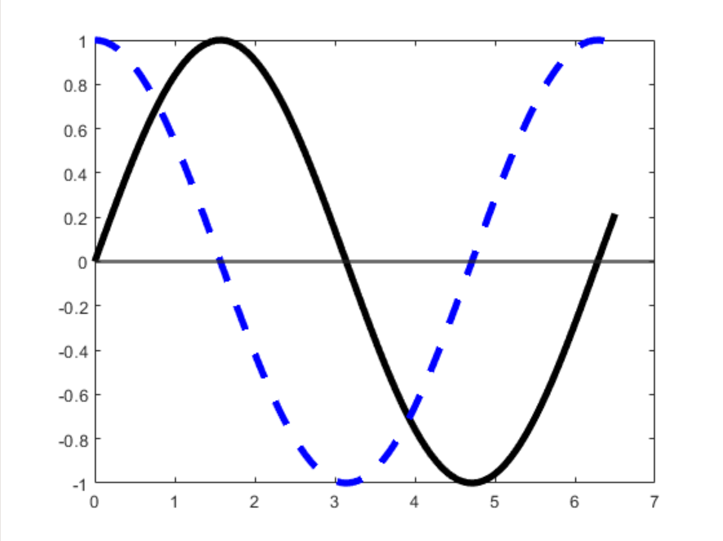


02 The derivatives of some common functions

	$f(x)$	$f'(x)$
• A constant :	c	0
• A straight line:	$m + kx$	k
• A power function:	x^a	ax^{a-1}
• The exponential function	e^x	e^x

02 The derivatives of some common functions

$f(x)$	$f'(x)$
$\ln(x)$	$\frac{1}{x}$
$sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$



02

Rules of differentiation

	$\boldsymbol{f(x)}$	$\boldsymbol{f'(x)}$
• Constant remain:	$\mathrm{c} \cdot h(x)$	$\mathrm{c} \cdot h'(x)$
• Sums :	$f(x) + g(x)$	$f'(x) + g'(x)$
• Products:	$f(x)g(x)$	$f'(x)g(x) + f(x)g'(x)$
• The reciprocal rule:	$\frac{1}{f(x)}$	$-\frac{f'(x)}{f(x)^2}$

02

Rules of differentiation

	$f(x)$	$f'(x)$
• The quotient rule:	$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$
• The chain rule:	$f(g(x))$	$f'\big(g(x)\big)g'(x)$

- Examples:
  - $f(x) = 3xe^x, f'(x) = ?$
  - $q(x) = \frac{x^2}{e^{x+1}}, q'(x) = ?$

02 Linearization

The introduction to linearization can be found in the canvas video "Linearization".

- When we want an approximation of a function $f(x)$ close to a specific point x^*:

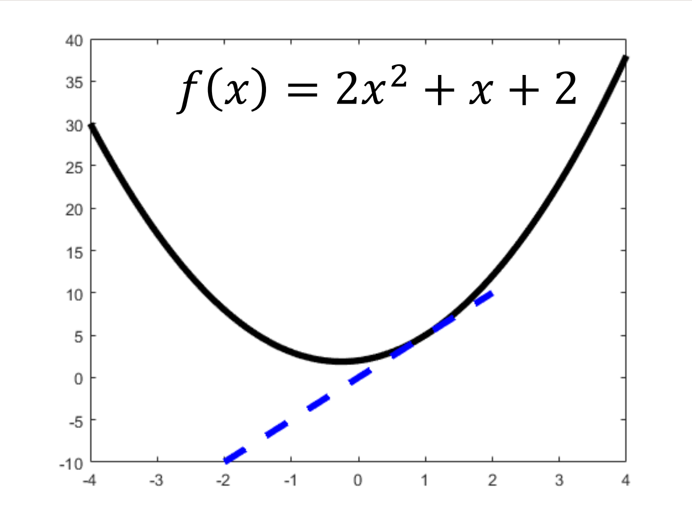
$$\frac{\Delta y}{\Delta x} \approx f'(x) \leftrightarrow \Delta y \approx f'(x)\Delta x$$

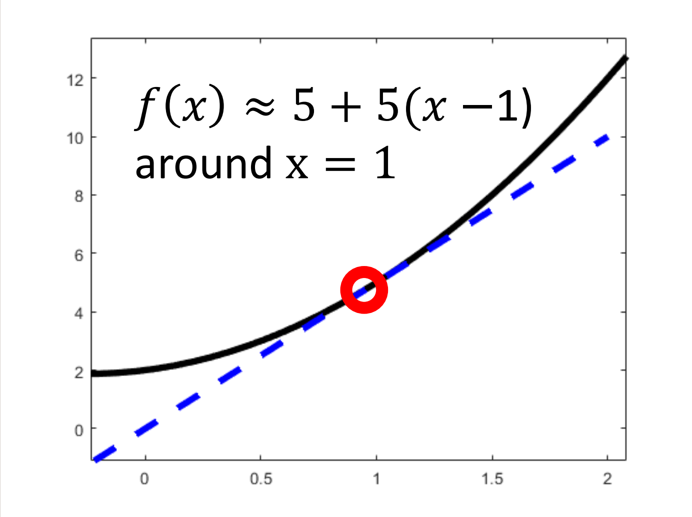
$$\leftrightarrow f(x) - f(x^*) \approx f'(x^*)(x - x^*)$$

$$\leftrightarrow f(x) \approx f(x^*) + f'(x^*)(x - x^*)$$

$$f(x^*) + f'(x^*)(x - x^*)$$

gives us a linear approximation of $f(x)$ close to x^*





02 Min/Max problems

- How to find the local maximal or minimal of a function?

  $f'(x) = 0 \leftrightarrow$ slope=0 Could be either local minimum or maximum

  1. Compare $f(x)$ for all the x that suffices $f'(x) = 0$ *or*
  2. Check the second derivative $f''(x)$ for all x that suffices $f'(x) = 0$

- Second derivative: $\frac{d}{dx}\left(\frac{df(x)}{dx}\right) = \frac{d^2f(x)}{dx^2} = f"(x)$ corresponds to the **curvature** or **concavity** of $f(x)$

  $f"(x) < 0$: local maximum

  $f"(x) > 0$: local minimum

02 Min/Max problems

- Example 1: *find the maximal value of* $f(x) = -x^2$ in the interval $-1 < x < 1$.
- Example 2: maximize $f(x) = x - e^x$

$f'(x) = 1 - e^x$

$f'(x) = 0 \leftrightarrow 1 - e^x = 0 \leftrightarrow x = 0$

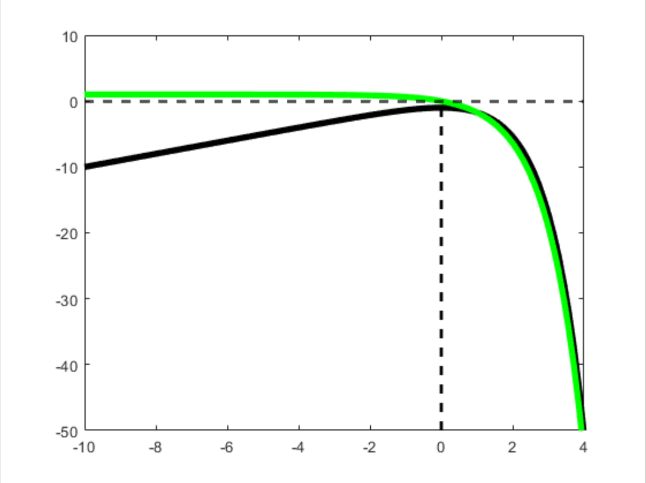
The example can be found in the canvas video "Min/Max problems".

- Examine the behaviour of $f(x)$ and $f'(x)$ from different sides of $x = 0$:

	$\boldsymbol{x < 0}$	$\boldsymbol{x = 0}$	$\boldsymbol{x > 0}$
$f'(x)$	$+$	$\mathbf{0}$	$-$
$f(x)$	Increase	---	Decrease

02 Min/Max problems

The example can be found in the canvas video “Min/Max problems”.



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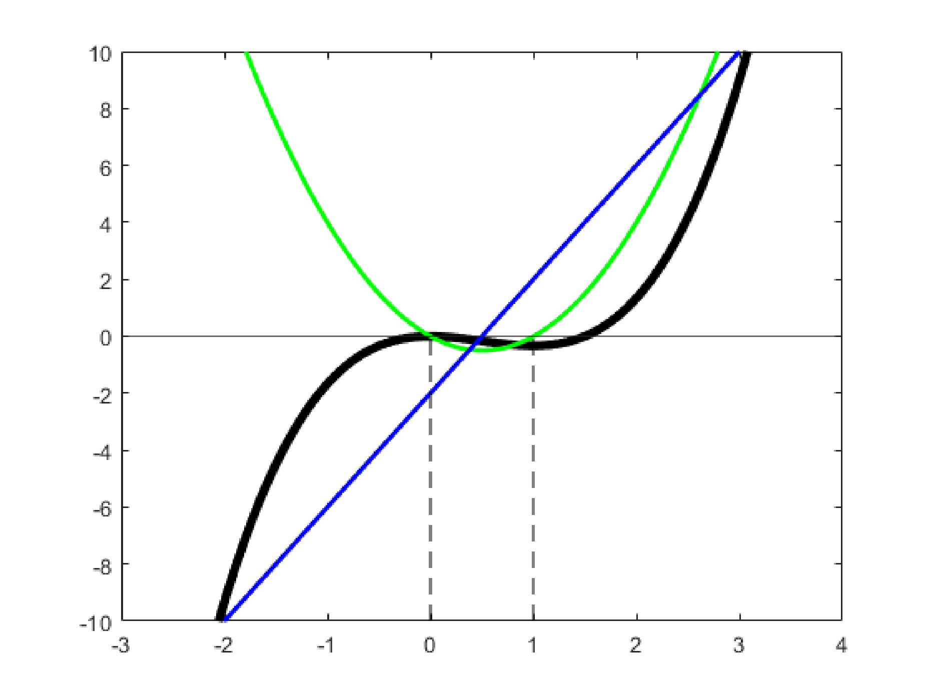
- Examine the behaviour of $f(x)$ and $f'(x)$ from different sides of $x = 0$:

	$\boldsymbol{x < 0}$	$\boldsymbol{x = 0}$	$\boldsymbol{x > 0}$
$f'(x)$	$+$	$\mathbf{0}$	$-$
$f(x)$	Increase	---	Decrease

$f'(x)$is only equal to zero in one single point. It can not change sign without passing zero. We are therefore sure that $f'(x)$ stays positive below $x = 0$ and stays negative above $x = 0$. It follows that $f(x)$ has a global maximum at $x = 0$.

02 Min/Max problems

- Example 3: $g(x) = \frac{2}{3}x^3 - x^2$ *is defined* $g: X \to \mathbb{R}$



Can you find the local minimum and maximum of $g(x)$?

The example can be found in the canvas video “Min/Max problems”.

02 Primitive functions

- The inverse of differentiation: the primitive function of $f(x)$ is any function which derivative if $f(x)$
- The primitive function is often denoted $F(x)$: $F'(x) = f(x)$
- If a particular $F(x)$ is a primitive function of $f(x)$, so is $F(x) + c, where\ c$ is a constant
- Some examples:

$f(x)$	$F(x)$
a	$ax + c$
x	$\frac{x^2}{2} + c$
e^x	$e^x + c$