Solutions to Exercises from
High-Dimensional Probability

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Exercise 2.1.

Solution. □

Exercise 2.2.

Solution.

$$f(x) := \max_i f_i(x) \quad f_i \text{ are convex functions}$$

$$f(\lambda x + (1-\lambda)y) = \max_i f_i(\lambda x + (1-\lambda)y) \leq$$

$$\max_i \lambda f_i(x) + (1-\lambda) f_i(y) \leq \max_i \lambda f_i(x) + \max_i (1-\lambda) f_i(y) = \lambda f(x) + (1-\lambda) f(y)$$

□

Exercise 2.3.

Solution.

1. induction on the definition
2. definition of expected value for discrete random variable

□

Exercise 2.4.

Solution. We know $T \subseteq \text{conv}(T)$, therefore $\sup_{x\in T} f(x) \leq \sup_{x\in\text{conv}(T)} f(x)$.
Also we have for some $\epsilon > 0$ and $S := \sup_{x\in\text{conv}(T)} f(x)$ that $\exists y \in \text{conv}(T)\ f(y) > S - \epsilon$. Now $y = \sum_{i=1}^m \lambda_i x_i$ for $x_i \in T, \lambda \geq 0, \sum_{i=1}^m \lambda_i = 1$ which also means $\lambda_i \leq 1$. By convexity of f and Jensen's inequality we have:

$$f(y) \leq \sum_{i=1}^m \lambda_i f(x_i) \leq \max_{1\leq i\leq m} f(x_i) \leq \sup_{x\in T} f(x)$$

So

$$[\forall \epsilon > 0 \quad S - \epsilon < f(y) \leq \sup_{x\in T} f(x)] \implies \sup_{x\in\text{conv}(T)} f(x) \leq \sup_{x\in T} f(x)$$

□

Exercise 2.5.

Solution. case $n=2$: $x=(x_1,x_2)$, $x_i \in [0,1]$:

$$x = (1-x_2)(x_1(1,0)+(1-x_1)(0,0)) + x_2(x_1(1,1)+(1-x_1)(0,1))$$
$$= (1-x_2)(x_1)(1,0)+(1-x_2)(1-x_2)(0,0)+x_2(x_1)(1,1)+x_2(1-x_1)(0,1)$$

conjecture: For each $v=(v_1,\cdots,v_n)\in\{0,1\}^n$ define:

$$\lambda_v := \prod_{i=1}^{n}[(v_i)(x_i)+(1-v_i)(1-x_i)]$$

proof by induction: We can verify for $n=1,2$ as above. Assume this works for $n=k-1$, meaning any point in $[0,1]^{k-1}$ can be written as $x=\sum_{v\in\{0,1\}^{k-1}}\lambda_v v$. We can prove now for $n=k$ by taking a chosing one of its dimensions and letting a hyper-surface orthogonal to that point create a $k-1$ dimensional cube. now we have □

Exercise 2.6.

Exercise 2.7.

Exercise 2.8.

Solution. We want to prove that with probability less than or equal to n there exists an independent group of size more than $2\log_2 n$. We have $n\geq 7$, therefore $\lfloor 2\log_2 n\rfloor > 4$. So we have the size $k\geq 5$.

□

Exercise 2.9.

Exercise 2.10.

Exercise 2.11.

Exercise 2.12.

Solution.

1. An inequality that relates to different norms is Holder's inequality so we try to use that. We define $r:=\frac{q}{p}$, where its conjugate is $r':=1/(1-\frac{p}{q})=\frac{q}{q-p}$. We take the r-norm of X^p and r'-norm of 1 to see:
$$\|X^p\|_1 \leq \|X^p\|_r\|1\|_{r'} = \|X^p\|_r$$
$$\implies \mathbb{E}|X^p| \leq (\mathbb{E}|X|^q)^{p/q} \implies \textit{answer}$$

   For $0\leq p<1$ we should have used Jensen instead of Holder.

2. Recall $\|X\|_{Lp}=(\mathbb{E}|X|^p)^{1/p}$. For continuous X with distribution f we have $\mathbb{E}|X|^p=\int_D |x|^p f(x)\ dx$, where D is the domain of f (range of X). Take $\alpha=\frac{p+q}{2}$. Now if we had $f(x)=|x|^{-1-\alpha}$ then $|x|^{p-1-\alpha}$ will converge (its exponent is ≤ -1) and $|x|^{q-1-\alpha}$ will diverge. But we need to make sure f is a valid density function, i.e. $\int_D f(x)\ dx=1$. To make things simpler we take $X\geq 1$ so we can remove the absolution value functions. Now we have $int_1^\infty x^{-1-\alpha}=[x^{-\alpha}/-\alpha]_1^\infty=1/\alpha$. So we must define $f_X(x):=\alpha x^{-1-\alpha}$.
   Also for a discrete example we could create an infinite sum that converges diverges when its elements are gone to the powers more than p and converges otherwise.

□

Exercise 2.13.

Solution. □

3 Chapter 2

Exercise 3.1. (Products of i.i.d. random variables do not concentrate)

$$X_1, \cdots, X_n \in \text{Uniform}([0,1])$$

$$Y_n := X_1 \cdots X_n$$

Solution. Since X_i are independent:

$$\mathbb{E}[Y_n] = \prod_{i=1}^{n} \mathbb{E}[X_i] = \prod_{i=1}^{n} \int_0^1 x \, dx = 2^{-n}$$

$$\mathbb{P}[Y_n \geq \mathbb{E}Y_n] = \mathbb{P}[Y_n \geq 2^{-n}] \geq \mathbb{P}(Y_1 \geq \frac{1}{2} \wedge \cdots \wedge Y_n \geq \frac{1}{2}) = (0.5)^n$$

We also have:

$$\log Y_n = \sum_{i=1}^{n} \log X_i, \quad \log X_i \in (-\infty, 0]$$

$$\mathbb{P}[Y_n \geq \mathbb{E}Y_n] = \mathbb{P}(\sum_{i=1}^{n} \log X_i \geq -n \log 2) \leq$$

??? □

Exercise 3.2. (Gaussian tails: a lower bound)

Solution.

$$\mathbb{P}(g \geq t) = \int_t^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx, \quad \frac{d}{dt}\mathbb{P}(g \geq t) = -\frac{1}{\sqrt{2\pi}} e^{-t^2/2}$$

$$\frac{d}{dt} \frac{t}{t^2+1} \cdot \frac{1}{\sqrt{2\pi}} e^{-t^2/2} = \frac{1}{\sqrt{2\pi}} \left(\frac{1-2t^2-t^4}{(t^2+1)^2}\right) e^{-t^2/2}$$

defining $h(t) := \mathbb{P}(g \geq t) - \frac{t}{t^2+1} \cdot \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$ we can see that

$$\frac{d}{dt} h(t) = \frac{-2}{(t^2+1)^2 \sqrt{2\pi}} e^{-t^2/2}$$

. We have $h'(t) < 0$ for all t. and $\lim_{t\to\infty} h'(t) = 0$. So the function $h(t)$ is strictly decreasing. We can also verify that $h(0) > 0$ and $\lim_{t\to\infty} h(t) = 0$, so the function cannot become negative at any point t.

□

Exercise 3.3. (Mills ratio)

Exercise 3.4. (Truncated Gaussian moments)

Solution.

1.

$$\mathbb{E}[g\mathbf{1}_{g>t}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x\mathbf{1}_{x>t} e^{-x^2/2}\, dx = \frac{1}{\sqrt{2\pi}} \int_t^{\infty} xe^{-x^2/2}\, dx = \frac{1}{\sqrt{2\pi}} [-e^{-x^2/2}]_t^{\infty} = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$$

2.

$$\mathbb{E}[g^2\mathbf{1}_{g>t}] = \frac{1}{\sqrt{2\pi}} \int_t^{\infty} x^2 e^{-x^2/2}\, dx$$

Integration by parts $u := x$ and $v := e^{-x^2/2}$ gives:

$$\leq \frac{1}{\sqrt{2\pi}} \big(te^{-t^2/2} + \int_t^{\infty} e^{-x^2/2}\, dx\big) \leq \frac{1}{\sqrt{2\pi}} \big(te^{-t^2/2} + \frac{1}{t} e^{-t^2/2}\big) = \frac{1}{\sqrt{2\pi}} (t + \frac{1}{t}) e^{-t^2/2}$$

□

Exercise 3.5. (Completing the proof of Hoeffding inequality)

Solution. Taylor expansion □

Exercise 3.6. (Gaussian tail by the exponential moment method)

Solution. for any $t \geq 0$:

$$\mathbb{P}(g \geq t) = \mathbb{P}(e^{\lambda g} \geq e^{\lambda t}) \leq e^{-\lambda t} \mathbb{E}[e^{\lambda g}]$$

$$= e^{-\lambda t} \int_{-\infty}^{\infty} e^{\lambda x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = e^{-\lambda t} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{\lambda^2 - (x-\lambda)^2}{2}}\, dx$$

$$= e^{-\lambda t + \lambda^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-(x-\lambda)^2}{2}}\, dx = e^{-\lambda t + \lambda^2/2}$$

$$\text{minimize by setting } \lambda = t \implies \mathbb{P}(g \geq t) \leq e^{-t^2/2}$$

□

Exercise 3.7. (Small ball probability)

Solution.

$$\mathbb{P}(\sum_{i=1}^{N} X_i \leq \epsilon N) = \mathbb{P}(e^{\lambda \sum_{i=1}^{N} X_i} \leq e^{\lambda \epsilon N}) \leq e^{-\lambda \epsilon N} \prod_{i=1}^{N} \mathbb{E}[e^{\lambda X_i}] = e^{-\lambda \epsilon N} \prod_{i=1}^{N}$$

□