In this PDF, I will list some errors that occur in the article or appendix. Some errors look insignificant. But for mathematical formulas, any tiny error may confuse. Anyway, these errors make the algorithm hard to be implemented for the reader.

1. In S1 Appendix S5, the tuple of the vector does not correspond:

$$y(t_j) = \big(Y(t_j), Y(t_j-\tau), \ldots, Y(t_j-(E-1)\tau)\big) = \big(y_j^{(1)}, \ldots, y_j^{(E+1)}\big), Y \in \mathbb{R}, \quad \text{(S5)}$$

(1) (2) ... (E) shoude be E

2. Due to the error of S5, in S4, s_j^y becomes an E-tuple vector.

$$s_j^y = \left(\frac{y_j^{(2)}-y_j^{(1)}}{y_j^{(1)}}, \ldots, \frac{y_j^{(E+1)}-y_j^{(E)}}{y_j^{(E)}}\right), y_j \in \mathbb{R}, \quad \text{(S4)}$$

1, ... E

But accroding to defination in p4 S1 Appendix, s should be an E-1 tuple vector:

> In our paradigm, the patterns we define for E=2 are: i) ↗: $X(t\text{-}\tau) < X(t)$, ii) →: $X(t\text{-}\tau) = X(t)$, iii) ↘: $X(t\text{-}\tau) > X(t)$ and these cover all the possible temporal patterns that characterize time series. Furthermore, for E=3 we define i) ↗↗: $X(t\text{-}2\tau) < X(t\text{-}\tau) < X(t)$, ii) →↗: $X(t\text{-}2\tau) = X(t\text{-}\tau) < X(t)$, iii) ↘↗: $X(t\text{-}2\tau) > X(t\text{-}\tau) < X(t)$, iv) ↗→: $X(t\text{-}2\tau) < X(t\text{-}\tau) = X(t)$, v) →→: $X(t\text{-}2\tau) = X(t\text{-}\tau) = X(t)$, vi) ↘→: $X(t\text{-}2\tau) > X(t\text{-}\tau) = X(t)$, vii) ↗↘: $X(t\text{-}2\tau) < X(t\text{-}\tau) > X(t)$, viii) →↘: $X(t\text{-}2\tau) = X(t\text{-}\tau) > X(t)$, ix) ↘↘: $X(t\text{-}2\tau) > X(t\text{-}\tau) < X(t)$. The expansion on E=4 and beyond is plausibly derived in the same rationale.

In fact, we can't get the Est element in s_j^y (S4) because $y_j^{(E+1)}$ actually does not exist (S5).

3. In S1 Appendix S6-S9:

> Then we estimate (i.e., predict) the mutual neighbors that correspond to $x(t)$ (contemporaneous to $y(t)$) by using the time indices of $y(t)$'s nearest neighbors: $\widehat{NN_{x(t)}} = x_{t_{y_1}}, \ldots, x_{t_{y_{E+1}}}$. We calculate similarly the "predicted" average pattern $\widehat{P_{x(t)}}$ as follows:
>
> $$\widehat{P_{x(t)}} = signature\big(\widehat{S_{x(t)}}\big), \widehat{S_{x(t)}} \in \mathbb{R}^E, \quad \text{(S6)}$$
>
> where
>
> $$\widehat{S_{x(t)}} = \sum_{j=1}^{E+1} w_j^y s_j^{\hat{x}}, w_j^y \in [0,1], s_j^{\hat{x}} \in \mathbb{R}^E, \text{ for all } \widehat{NN_{x(t)}}, \quad \text{(S7)}$$
>
> $$s_j^{\hat{x}} = \left(\frac{\hat{x}_j^{(2)}-\hat{x}_j^{(1)}}{\hat{x}_j^{(1)}}, \ldots, \frac{\hat{x}_j^{(E+1)}-\hat{x}_j^{(E)}}{\hat{x}_j^{(E)}}\right), \hat{x}_j \in \mathbb{R}, \quad \text{(S8)}$$
>
> $$\hat{x}(t_j) = \big(\hat{X}(t_{y_1}), \hat{X}(t_{y_2}), \ldots, \hat{X}(t_{y_{E+1}})\big) = \big(\hat{x}_j^{(1)}, \ldots, \hat{x}_j^{(E+1)}\big), \hat{X} \in \mathbb{R}, \quad \text{(S9)}$$

if the author doesn't wrong, S9 is not a counterpart of S5. Thus $\hat{x}(t_j)$ is an (E+1)-tuple vector with no problem. And $s_j^{\hat{x}}$ in S8 is an E-tuple vector. S7 calculates the weighted average of the pattern for all $\widehat{NN_{x(t)}}$, but according to S8 and S9 there is only one $s_j^{\hat{x}}$ or all $s_j^{\hat{x}}$ is equaled. S7 is meaningless.
even if we ignore the problem above, $\hat{S}_{x(t)}$ is an E-tuple vector, but $S_{x(t)}$ calculate from S2-S5 is an $E-1$-tuple vector according to points 1 and 2. tuple of $\hat{S}_{x(t)}$ and $S_{x(t)}$ is even not equaled. how to "predict"?

4. In the article and appendix, the author mentions a principle more than once, like "The strength of causality is determined by the overall **accuracy** percentage between the estimated $\hat{P}_x(t)$ and the actual $P_{x(t)}$". But in the actual calculations, it seems don't base on 'accuracy' (S15):

$$PC[P_X, P_Y] = \sum_t \mathrm{erf}\left(\frac{mean(|S_{y(t)}|)}{mean(|S_{x(t)}|)}\right), \text{ erf: error squashing function.} \quad \text{(S15)}$$

In conclusion, according to the information from the article and related appendix readers can't implement the algorithm which constitutes the key content of the paper. Thus the validity of the method and the authenticity of the result can't be verified.